Rotational Invariance of the Effective Refractive Index in Photonic Crystals

Ting-Hang Pei and Yang-Tung Huang

***Abstract*—The rotational invariance of the effective refractive index in the 2-D photonic crystal is discussed. It is proven that, in the case of the *n*-fold symmetrical *ω*–*k* equifrequency surface (EFS), there are two solutions that show the invariance of the effective refractive index after rotating a multiple of the basic repeated angle 2 π/n within the first Brillouin zone. For the noncircular solution, the behavior of the effective refractive index $N_{\text{eff}}(\theta)$ at normal incidence shows that $N_{\text{eff}}(0°)$ has an additional constant of two in the very high symmetric EFS, higher than in the low symmetric case.**

***Index Terms*—Effective refractive index, photonic crystal.**

I. Introduction

THE photonic crystal (PhC) is formed with periodic dielectric materials, which possesses the photonic band structure (PBS) and exhibits new electromagnetic phenomenon [1], [2]. Its structure is complicated and performs very abundant optical characteristics, such as the superprism and the negative-like diffraction [3], [4]. In such periodic structure, the Bloch theorem is an appropriate method to solve the PBS of the PhC [5], which is very analogous to the electron band structure in a solid.

Recently, several effective-medium approaches about disordered PhCs or metamaterials [6]–[9] reveal the possibility that treats some PhCs as simple media with effective refractive indices in certain frequency regions. Other method calculating homogenization of PhCs by well-known coherent-potential approximation (CPA) are proposed [10], [11]. The periodic defects (line defects) modes calculated by simulating the PhC as an effective photonic insulator is also presented [12]. In our previous works [13]–[15], the high-transmission PhC heterostructure Y-branch waveguide, splitter, and the periodic-defect PhC are proposed by using the concept of the effective refractive index. It is shown that variation of the air-hole radius result in different effective refractive indices. In the first two references [13], [14], the structure is composed of two different PhCs, which can be treated as two effective media like the core and cladding of the conventional waveguide. In the last one [15], we prove that the periodic-defect PhC can be replaced with an effective PhC, in which the defect-free PhC and periodic defects are treated as homogeneous medium and an array of effective dielectric rods, respectively. Both results tells us that the concept of the effective medium holds in certain frequency region without losing correctness of optical performances.

Usually, the effective refractive index is determined by the equifrequency surface (EFS) of the PBS in the first Brillouin zone [3]. Furthermore, all EFSs used in [13]–[15] are almost circular and they give constant effective refractive indices for all propagation angles, which are the same as real homogeneous media. It can be said that the effective refractive index calculated from the EFS inherits the symmetric characteristics of the EFS, that is, the rotational invariance under any angle. It makes us consider a problem: is there any EFS of the PhC except for the circular one also giving the characteristic of rotational invariance on the effective refractive index? In this letter, we look for the EFSs of the PhC which keep the effective refractive index invariant after rotating an angle of a multiple of $2\pi/n$ in the n-fold symmetric case. We prove that there are two solutions satisfying the above condition of rotational invariance, and one triangular PhC case is demonstrated.

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T.-H. Pei is with the Microelectronics and System Information Research Center, National Chiao Tung University, Hsinchu 300, Taiwan (e-mail: thp3000.ee88g@nctu.edu.tw).

Y.-T. Huang is with the Department of Electronics Engineering, National Chiao Tung University, Hsinchu 300, Taiwan (e-mail: huangyt@nctu.edu.tw).

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II. Simulated System

A. *Effective Refractive Index*

Before discussions, it needs to mention how to define the effective refractive index of a PhC. First of all, the propagation direction of light in the PhC has to be determined. It is the same as the direction of the group velocity in the PhC [16], which is normal to the EFS at a certain wave vector and defined as $\vec{v}_g = \nabla_{\vec{k}}\omega$ where $\vec{k}$ and ω are the wave vector and the frequency, respectively. According to the conservation rule, the incident wave vector and the refracted wave vector are continuous for the tangential components parallel to the interface between the incident medium and the PhC. Given the incident wave vector with a frequency and an incident angle, the refracted wave vector as well as the refracted angle can be determined. By using Snell's law, the effective refractive indices varied with the incident angles are obtained.

B. *Problem About Rotational Invariance*

The EFSs of the triangular PhC in the first Brillouin zone have a characteristic of 6-fold symmetry [5]. They own three mirror planes intersecting each other by angles of 60° as shown in Fig. 1. Under operations of σ_y, σ_y' and σ_y'' mirror reflections,

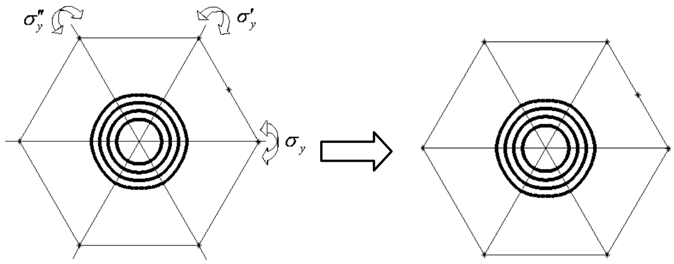

Fig. 1. Invariant characteristics of the EFS of the triangular PhC under operations of rotations and mirror reflections in the first Brillouin zone.

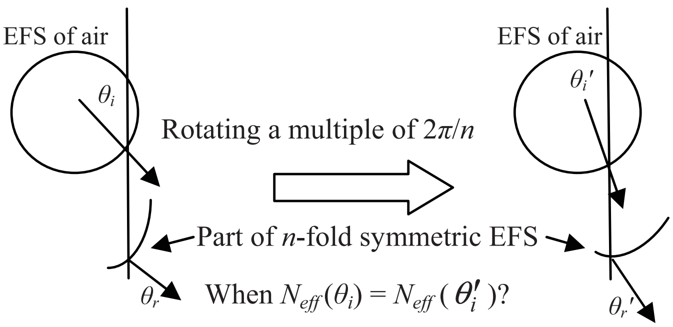

Fig. 2. Rotational invariance of the effective negative index obtained by rotating an angle of a multiple of $2\pi/n$ in the n-fold symmetric EFS.

all EFSs remain invariant [5]. They are also invariant after rotating an angle of a multiple of $\pi/3$. The angle $\pi/3$ is the basic angle of a hexagon. Form the mathematic viewpoint, above characteristics of EFSs can be extended to the n-fold symmetric structure. Since the effective refractive index of a PhC is defined from the EFS, it makes us arise curiosity about whether the effective refractive index inherits the characteristic of rotational invariance. In Fig. 2, the $2\pi/n$ rotational invariance of the effective refractive index is schematically shown. In the following, we prove that only two kinds of EFSs keep the effective refractive index invariant after rotating an angle of a multiple of $2\pi/n$ in the n-fold symmetric case.

III. Discussion and Results

Let us consider any shape of the n-fold symmetric EFS in the first Brillouin zone. The kind of EFS possesses a basic rotational-invariance angle $\theta_n = 2\pi/n$. All the following discussions are dealt with in the k-space. The relation between k_x and k_y is represented by a function $k_y = f(k_x)$. For convenience, light is incident from air with an incident angle θ_i into the PhC with a refracted angle θ_r. According to the conservation rule, an effective refractive index $N_{eff}(\theta_i)$ can be defined. The EFS in air forms a circle with a radius $k = \omega/c = \sqrt{k_x^2 + k_y^2}$. The component of the incident wave vector parallel to the interface is k_x, which is continuous when light is across the interface. The refracted direction is the direction of the group velocity $\vec{v}_g = \nabla_{\vec{k}}\omega$. The geometrical relationship between the EFS and the refracted direction is shown in Fig. 3, in which $\sin\theta_r$ is expressed as

$$\sin\theta_r = \sqrt{\frac{[f'(k_x)]^2}{1+[f'(k_x)]^2}}, \quad (1)$$

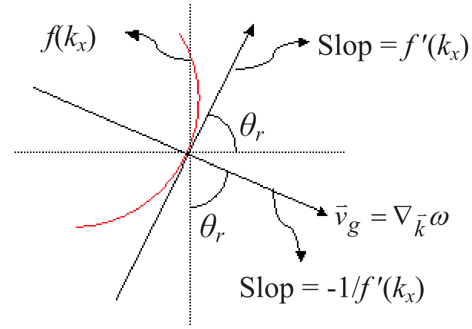

Fig. 3. Geometric relationship between the ω−k equifrequency curve and the refracted direction (which is also the direction of the group velocity).

where $f'(k_x) = df(k_x)/dk_x$. Because $\sin\theta_i$ is equal to k_x/k, then $N_{eff}(\theta_i)$ is expressed as

$$N_{eff}(\theta_i) = \frac{\sin\theta_i}{\sin\theta_r} = \frac{k_x\sqrt{1+[f'(k_x)]^2}}{k\,|f'(k_x)|}. \quad (2)$$

Here we only consider the case of the positive refraction. The result can also be applied to the case of the negative refraction. When we counterclockwise rotate two perpendicular axes of the k-space by an angle θ, an integer multiple of θ_n, the same shape is repeated in the first Brillouin zone. The operation can be expressed by a matrix representation with a rotational matrix $R(\theta)$:

$$\begin{pmatrix} K_x \\ g \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k_x \\ f \end{pmatrix}, \quad (3)$$

where K_x and g are the new two components of the wave vector k after rotation, respectively, which can be expressed in terms of k_x and f:

$$K_x = k_x \cdot \cos\theta + f \cdot \sin\theta, \quad (4)$$

$$g = -k_x \cdot \sin\theta + f \cdot \cos\theta \quad (5)$$

According to Eq. (1), the new effective refractive index after rotation is

$$N_{eff}(\theta_i') = \frac{K_x\sqrt{1+[g'(k_x)]^2}}{k\,|g'(k_x)|}, \quad (6)$$

where θ_i' is the new incident angle. In Eq. (5), g is expressed as the function of k_x, and its differential form in Eq. (6) is defined as the differentiation of g to K_x. We also express g' and K_x in terms of k_x and f in order to compare to the original N_{eff}. Using the chain rule, the differentiation of g is expressed as

$$\frac{dg}{dK_x} = \frac{dg}{dk_x}\cdot\frac{dk_x}{dK_x} = \frac{dg}{dk_x}\cdot\frac{1}{dK_x/dk_x} = \frac{-\sin\theta + f'(k_x)\cdot\cos\theta}{\cos\theta + f'(k_x)\cdot\sin\theta} \quad (7)$$

Substituting Eqs. (4) and (7) into Eq. (6), we obtain the new expression of $N_{eff}(\theta_i')$ in terms of k_x and f:

$$N_{eff}(\theta_i') = \frac{[k_x\cdot\cos\theta + f(k_x)\cdot\sin\theta]\sqrt{1+[f'(k_x)]^2}}{k\,|-\sin\theta + f'(k_x)\cdot\cos\theta|} \quad (8)$$

If we want the effective refractive index to be unchanged after rotation, the expression of Eq. (8) must equal to that of Eq. (2), which is

$$N_{eff}\left(\theta_i'\right) = N_{eff}\left(\theta_i\right). \quad (9)$$

From Eq. (9), we have

$$\frac{[k_x \cdot \cos(\theta) + f(k_x) \cdot \sin(\theta)]}{|-\sin(\theta) + f'(k_x) \cdot \cos(\theta)|} = \frac{k_x}{|f'(k_x)|}. \quad (10)$$

Here θ is less than $\pi/2$. After arranging Eq. (10), two relations are obtained.

Relation 1: if $f'(k_x) > 0$ and $\sin\theta < f'(k_x) \cdot \cos\theta$, or *if* $f'(k_x) < 0$, then

$$k_x = -f'(k_x) \cdot f(k_x) \quad (11)$$

Relation 2: if $f'(k_x) > 0$ and $\sin\theta > f'(k_x) \cdot \cos\theta$, then

$$2k_x \cdot f'(k_x) \cdot \cos\theta + f'(k_x) \cdot f(k_x) \cdot \sin\theta = k_x \cdot \sin\theta. \quad (12)$$

Eqs. (11) and (12) are two differential equations, and we solve them in the following. First, Eq. (11) can be rewritten as

$$\frac{df(k_x)}{dk_x} = -\frac{k_x}{f(k_x)}. \quad (13)$$

After integrating both sides, we obtain

$$k_x^2 + (f(k_x))^2 = (f(0))^2. \quad (14)$$

This is a solution of a circle in the two-dimensional plane. It can be seen that one solution reaching the effective refractive index invariant, after rotating an angle θ, is a circular EFS in the any n-fold symmetric case. Next, we solve Eq. (12) to obtain the second solution. Eq. (12) can be rewritten as

$$f'(k_x) = \frac{df(k_x)}{dk_x} = \frac{k_x \cdot \sin\theta}{2k_x \cdot \cos\theta + f(k_x) \cdot \sin\theta}. \quad (15)$$

If $f(k_x) = k_x$¡D$u(k_x)$ is used in Eq. (15), then we have

$$\frac{dk_x}{k_x} = \frac{a+u}{1-au-u^2}du, \quad (16)$$

where $a = 2\cos\theta/\sin\theta$. Then the second solution is obtained:

$$c = k_x\sqrt{u^2+au-1}\left[\frac{\left(2u+a-\sqrt{a^2+4}\right)}{\left(2u+a+\sqrt{a^2+4}\right)}\right]^{a/2\sqrt{a^2+4}}, \quad (17)$$

where $u = f/k_x$. c is given by the condition at $k_x \to 0°$, which means $\theta_i \to 0°$ and $u \to \infty$. No matter which a is, c in Eq. (17) approximates to $f(k_x = 0)$. Furthermore, the effective refractive index of the second solution is expressed as

$$N_{eff}(\theta_i) = k_x\sqrt{(a+u)^2+1}\Big/k. \quad (18)$$

We can easily check its behavior at $\theta_i \to 0°$. When n as well as a are finite, $N_{eff}(0°) \sim f(0)/k$. On the other hand, when n as well as $a$goes to infinite, we have

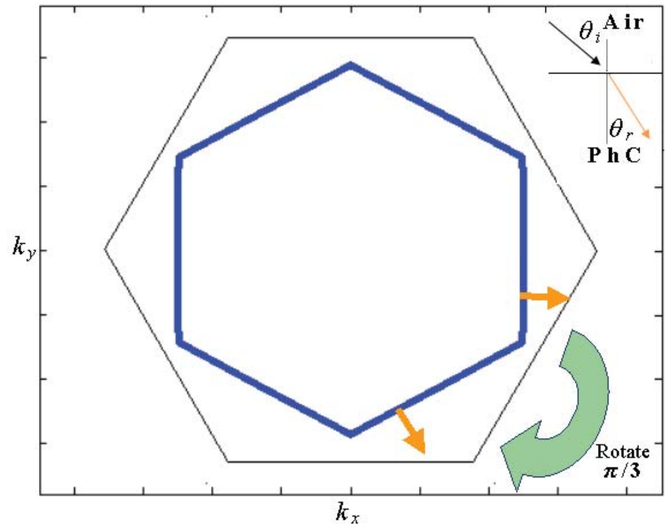

Fig. 4. Second solution of the triangular PhC with effective refractive index invariant after rotation by angle $\theta = \pi/3$.

$$\begin{aligned} N_{eff}(0°) &\approx \frac{k_x(u+a)}{k} \\ &\approx \frac{f(0)}{k} + \lim_{\substack{n\to\infty \\ \theta_i \to 0}} \frac{2\cos(2\pi/n)}{\sin(2\pi/n)}\sin\theta_i \\ &\approx \frac{f(0)}{k} + 2. \end{aligned} \quad (19)$$

It shows, in the very high symmetric case, $N_{eff}(0°)$ has an additional constant of 2.

Next, we demonstrate an EFS case of $n = 6$. In this case, Eq. (17) is hard to solve, so we go back to Eq. (15) and use the series expansion method to solve it. $f(k_x)$ can be expanded in infinite series of the power of k_x as follows:

$$f(k_x) = \sum_{m=0}^{\infty} C_m k_x^m. \quad (20)$$

Substituting Eq. (20) into Eq. (15), we obtain

$$\sum_{m=1}^{\infty} mC_m k_x^{m-1} = k_x\Big/\left(ak_x + \sum_{m=0}^{\infty} C_m k_x^m\right). \quad (21)$$

After rearranging Eq. (21) and comparing each term in both sides step by step, each coefficient C_m can be obtained. We use first 201 terms to demonstrate $f(k_x)$ in the case of the triangular PhC with $a = 1.1547$. The curve with blue color is shown in Fig. 4, which is very close to the shape of the first Brillouin zone. This shape can be found at the edge of the first photonic band or higher photonic bands.

Above discussions tell us that the effective refractive index from the EFS doesn't possess the same symmetrical characteristic in most situations, even though the EFS inherits the symmetrical characteristic of the periodic structure. Comparing the first solution with the second one, only this kind of EFS with a circular form has a constant effective refractive index like a homogeneous medium. The circular EFS possesses infinite number of rotational symmetries and can repeat its geometrical structure no matter what the rotational angle is.

In the homogeneous medium, the rotational invariance of the effective refractive index is obviously because the EFS is a circle. Although the EFS of the PhC cannot be perfectly circular, however, some EFSs fairly approximate to circles in a degree of 99.0% or more. In our previous works [13]–[15], all of them show that the effective refractive indices work very well for all angles when ω is below 0.36 $(2\pi c/a)$, especially in the lower frequency region $(\omega \leq 0.1\ (2\pi c/a))$.

IV. Conclusion

In conclusion, we have shown that two kinds of EFSs possess the invariance of the effective refractive index after rotating a multiple of the basic repeated angle $2\pi/n$ in the n-fold symmetric case. One is circular and the other is described by Eq. (17). The circular EFS has infinite number of rotational symmetries and a constant effective refractive index for any incident angle. For the second solution, the $n = 6$ case is demonstrated. Furthermore, the effective refractive index at $\theta_i \to 0°$ is considered, and the result tells us that $N_{eff}(0°)$ is $f(0)/k$ in the low symmetrical EFS, and has an additional constant of 2 in the very high symmetric case. All results are easily checked by experiments and useful when the optical characteristics of the PhC or other periodic structures are considered.

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