Nonlinear Dyn (2012) 70:2187–2202
DOI 10.1007/s11071-012-0609-6

ORIGINAL PAPER

Synchronization of chaotic system with uncertain variable parameters by linear coupling and pragmatical adaptive tracking

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Received: 5 March 2012 / Accepted: 10 September 2012 / Published online: 3 October 2012

Abstract We study the synchronization of general chaotic systems which satisfy the Lipschitz condition only, with uncertain *variable* parameters by linear coupling and pragmatical adaptive tracking. The uncertain parameters of a system vary with time due to aging, environment, and disturbances. A sufficient condition is given for the asymptotical stability of common zero solution of error dynamics and parameter update dynamics by the Ge–Yu–Chen pragmatical asymptotical stability theorem based on equal probability assumption. Numerical results are studied for a Lorenz system and a quantum cellular neural network oscillator to show the effectiveness of the proposed synchronization strategy.

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Keywords Quantum Cellular Neural Network (Quantum-CNN) · Lipschitz condition chaos synchronization · Pragmatical asymptotical stability · Uncertain variable parameter

1 Introduction

The idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [1]. Since then, there has been particular interest in chaotic synchronization, due to many potential applications in secure communication [2], and chemical and biological systems [3, 4]. There are many control methods to synchronize chaotic systems, such as linear coupling, for which the implementation is rather easy, adaptive control, impulsive control, sliding mode control, and other methods [5]. Most of them are based on the exact knowledge of the system structure and parameters. But in practice, some or all of the system parameters are uncertain. Moreover, these parameters may change from time to time and become chaotic because of chaotic disturbances. For uncertain parameters, a lot of works have proceeded to solve this problem by adaptive synchronization [6–12]. In the current scheme of adaptive synchronization [13–15], the traditional Lyapunov stability theorem and Barbalat lemma are used to prove that the error vector approaches zero as time approaches infinity. But the question, why the estimated parameters also approach the uncertain parameters, has remained without answer. From the (Ge–Yu–Chen) GYC pragmatical asymptotical stability theorem [16–18], the question is strictly answered. In this paper, the synchronization of general chaotic systems which satisfy the Lipschitz condition only, with unknown parameters which are altered under some *variable* disturbances, by linear coupling and GYC pragmatical adaptive tracking, is studied first.

As numerical examples, the Lorenz system and recently developed quantum cellular neural network Quantum-CNN chaotic oscillator are used. GYC pragmatical adaptive tracking is used to track *variable* parameters in unidirectional coupled systems. Two Lorenz systems and two Quantum-CNN systems by GYC pragmatical adaptive tracking are given as simulation examples. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dots cellular automata structures to which in the last decade a wide interest has been devoted with particular attention toward quantum computing [19–21].

This paper is organized as follows: In Sect. 2, by the GYC pragmatical asymptotical stability theorem and by using Lipschitz conditions, theoretical analysis of synchronization is given. In Sect. 3 linear feedback controllers are used. By GYC pragmatical adaptive tracking, chaos synchronization of two Lorenz systems and of two Quantum-CNN oscillator systems are achieved by numerical simulations. Conclusions are given in Sect. 4. The GYC pragmatical asymptotical stability theorem is presented in the Appendix. Intuitively, this theorem is different from the traditional Lyapunov stability theorem in that when the points in the neighborhood of zero solution initiating trajectories not approaching zero with time are "not too many," i.e., in a subset of Lebesque measure 0 in mathematical language, [22] we can neglect their existence, i.e., the zero solution is actually asymptotically stable.

2 Theoretical analyses

Consider a nonautonomous system in the form as follows:

$$\dot{x} = F\big(t, x, B(t)\big) \tag{1}$$

The slave system is given by

$$\dot{y} = F\big(t, y, \hat{B}(t)\big) + \hat{K}(x-y) \tag{2}$$

where $x \in R^n$, $y \in R^n$, $B \in R^M$ is a vector of uncertain *variable* coefficients in F, $\hat{B} \in R^M$ is a vector of estimated coefficients in F, $F : \Omega_1 \subset R_+ \times R^n \times R^M \to R^n$ satisfies Lipschitz conditions $\|F(t,x_1,B) - F(t,x_2,B)\| \le G\|x_1 - x_2\|$ and $\|F(t,x,B) - F(t,x,\hat{B})\| \le G\|B - \hat{B}\|$ in Ω_1 with Lipschitz constant G. $\hat{K} = \mathrm{diag}[\hat{K}_1, \ldots, \hat{K}_i, \ldots, \hat{K}_n]$, $\hat{K}_i : \Omega_2 \subset R_+ \times R^n \times R^n \to R$ $(i = 1, \ldots, n)$ is the estimated coupling strength entry. Ω_1 and Ω_2 are domains containing the origin. For given $(t_0, x_0, y_0, B_0) \in \Omega_1 \cap \Omega_2$, the solutions $[x^T(t,t_0,x_0,y_0,B_0), y^T(t,t_0,x_0,y_0,B_0)]^T$ of Eqs. (1) and (2) exist for $t \ge t_0$.

If the synchronization can be accomplished when $t \to \infty$, the limit of the error vector $e(t) = [e_1, e_2, \ldots, e_n]^T$ must approach zero:

$$\lim_{t\to\infty} e = 0 \tag{3}$$

where

$$e = x - y \tag{4}$$

From Eqs. (1), (2), and (4), we have

$$\dot{e} = \dot{x} - \dot{y} \tag{5}$$

$$\dot{e} = F(t,x,B) - F(t,x-e,\hat{B}) - \hat{K}(x-y) \tag{6}$$

A Lyapnuov function $V(e, \tilde{B}, \tilde{G})$ is chosen as a positive definite function

$$V(e, \tilde{B}, \tilde{G}) = \frac{1}{2}e^T e + \frac{1}{2}\tilde{B}^T\tilde{B} + \frac{1}{2}\tilde{G}^2 \tag{7}$$

where $\tilde{G} = G - \hat{G}$; $\hat{G}$ is the estimated Lipschitz constant, $\tilde{B} = B - \hat{B}$.

When $M = n$, the time derivative of V along any solution of the differential equation system consisting of Eq. (6) and update differential equations for $\tilde{B}$ and $\tilde{G}$ is

$$\begin{aligned}
&\dot{V}(e, \tilde{B}, \tilde{G}) \\
&\quad = e^T\big[F(t,x,B) - F(t,x-e,B) + F(t,x-e,B) \\
&\qquad - F(t,x-e,\hat{B}) - \hat{K}e\big] + \tilde{B}^T\dot{\tilde{B}} + \tilde{G}\dot{\tilde{G}} \\
&\quad = e^T\big[F(t,x,B) - F(t,x-e,B) - \hat{K}e\big] + \tilde{G}\dot{\tilde{G}} \\
&\qquad + e^T\big[F(t,x-e,B) - F(t,x-e,\hat{B})\big] + \tilde{B}^T\dot{\tilde{B}}
\end{aligned} \tag{8}$$

where $\tilde{B} = B - \hat{B}$.

By Lipschitz condition,

$$\begin{aligned}
\dot{V}(e, \tilde{B}, \tilde{G}) \le{}& G\|e\|^2 - e^T\hat{K}e + \tilde{G}\dot{\tilde{G}} \\
&+ e^T\big[F(t,x-e,B) - F(t,x-e,\hat{B})\big] \\
&+ \tilde{B}^T\dot{\tilde{B}}
\end{aligned} \tag{9}$$

Since

$$e^T\big[F(t,x-e,B)-F(t,x-e,\hat{B})\big] \le |e_1|\cdot\big|F_1(t,x-e,B)-F_1(t,x-e,\hat{B})\big|+\cdots + |e_n|\cdot\big|F_n(t,x-e,B)-F_n(t,x-e,\hat{B})\big| \quad (10)$$

by Schwarz inequality [23] and Lipschitz condition, it is obtained that

$$|e_1|\cdot\big|F_1(t,x-e,B)-F_1(t,x-e,\hat{B})\big|+\cdots + |e_n|\cdot\big|F_n(t,x-e,B)-F_n(t,x-e,\hat{B})\big| \le \|e\|\cdot\big\|F(t,x-e,B)-F(t,x-e,\hat{B})\big\| \le G\|e\|\cdot\|\tilde{B}\| \quad (11)$$

Therefore,

$$\dot{V}(e,\tilde{B},\tilde{G}) \le G\|e\|^2-e^T\hat{K}e+\tilde{G}\dot{\tilde{G}}+G\|e\|\cdot\|\tilde{B}\| + \tilde{B}_1\dot{\tilde{B}}_1+\cdots+\tilde{B}_n\dot{\tilde{B}}_n \quad (12)$$

Choose

$$\dot{\tilde{G}}=-e^Te, \qquad \hat{K}=\operatorname{diag}[\hat{G}+G] \quad (13)$$

and

$$\dot{\tilde{B}}_1=-G\tilde{B}_1\|e\|/\|\tilde{B}\|,\ldots,\dot{\tilde{B}}_N = -G\tilde{B}_n\|e\|/\|\tilde{B}\| \quad (14)$$

we have

$$\tilde{B}^T\dot{\tilde{B}}=-G\big(\tilde{B}_1^2+\cdots+\tilde{B}_N^2\big)\|e\|/\|\tilde{B}\| = -G\|\tilde{B}\|^2\cdot\|e\|/\|\tilde{B}\| = -G\|e\|\cdot\|\tilde{B}\| \quad (15)$$

Introducing Eqs. (15), (13) in and (12), we get

$$\dot{V}(e,\tilde{B},\tilde{G}) \le G\|e\|^2-\operatorname{diag}[\hat{G}+G]\|e\|^2-\tilde{G}\|e\|^2 + G\|e\|\cdot\|\tilde{B}\|-G\|e\|\cdot\|\tilde{B}\| = -G\|e\|^2=-G\big(e_1^2+\cdots+e_n^2\big) \quad (16)$$

$\dot{V}$ is a negative semidefinite of e, $\tilde{B}$, $\tilde{G}$, by the GYC pragmatical asymptotical stability theorem (see the Appendix), the solution $e=0$, $\tilde{B}=0$, $\tilde{G}=0$ is asymptotically stable.

When $M\neq n$, on the right-hand side of Eq. (9), the other terms remain unchanged, and we want only to reduce last two terms

$$e^T\big[F(t,x-e,B)-F(t,x-e,\hat{B})\big]+\tilde{B}^T\dot{\tilde{B}} \quad (17)$$

When $M>n$, we put

$$e^T=e^T=[e_1,\ldots,e_n,e_{n+1},\ldots,e_M]^T \quad (18)$$

where $e_{n+1}=e_{n+2}=\cdots=e_M=0$. The first term of Eq. (17) becomes

$$e^T\big[F(t,x-e,B)-F(t,x-e,\hat{B})\big] \le |e_1|\cdot\big|F_1(t,x-e,B)-F_1(t,x-e,\hat{B})\big| +\cdots+|e_n|\cdot\big|F_n(t,x-e,B)-F_n(t,x-e,\hat{B})\big| + |e_{n+1}|\cdot\big|F_{n+1}(t,x-e,B) - F_{n+1}(t,x-e,\hat{B})\big|+\cdots + |e_M|\cdot\big|F_M(t,x-e,B) - F_M(t,x-e,\hat{B})\big| \le G\|e_M\|\cdot\|\tilde{B}\| \quad (19)$$

In Eq. (19), the last term is obtained by Schwarz inequality. Similarly, we choose

$$\dot{\tilde{B}}_1=-G\tilde{B}_1\|e\|/\|\tilde{B}\|,\ldots,\dot{\tilde{B}}_M = -G\tilde{B}_M\|e\|/\|\tilde{B}\| \quad (20)$$

Then

$$\tilde{B}^T\dot{\tilde{B}}=-G\big(\tilde{B}_1^2+\cdots+\tilde{B}_M^2\big)\|e\|/\|\tilde{B}\| = -G\|\tilde{B}\|^2\|e\|/\|\tilde{B}\|=-G\|e\|\cdot\|\tilde{B}\| \quad (21)$$

Introducing Eqs. (19), (21), in Eq. (9), we can also get lastly

$$\dot{V}(e,\tilde{B},\tilde{G})\le -G\big(e_1^2+\cdots+e_n^2\big) \quad (22)$$

By the same reasoning as when $M=n$, the solution $e=0$, $\tilde{B}=0$, $\tilde{G}=0$ is asymptotically stable.

When $M<n$, we put

$$F_i(t,x-e,B)-F_i(t,x-e,\hat{B})=0 \quad i=M+1,\ldots,n \quad (23)$$

since $B_{M+1},\ldots,B_n$ do not exist,

$$\tilde{B}_{M+1}=\cdots=\tilde{B}_n=0 \quad (24)$$

$$\|\tilde{B}\|^2=\tilde{B}_1^2+\cdots+\tilde{B}_M^2+\tilde{B}_{M+1}^2+\cdots+\tilde{B}_n^2 \quad (25)$$

Then by the Schwarz inequality,

$$e^T\big[F(t,x-e,B)-F(t,x-e,\hat{B})\big] \le |e_1|\cdot\big|F_1(t,x-e,B)-F_1(t,x-e,\hat{B})\big| +\cdots+|e_M|\cdot\big|F_M(t,x-e,B) - F_M(t,x-e,\hat{B})\big|+|e_{M+1}| \cdot\big|F_{M+1}(t,x-e,B) - F_{M+1}(t,x-e,\hat{B})\big|+\cdots + |e_n|\cdot\big|F_n(t,x-e,B)-F_n(t,x-e,\hat{B})\big| \le G\|e\|\cdot\|\tilde{B}\| \quad (26)$$

Similarly, choose

$$\begin{aligned}\dot{\tilde{B}}_1 &= -G\tilde{B}_1\|e\|/\|\tilde{B}\|, \ldots, \dot{\tilde{B}}_M \\ &= -G\tilde{B}_M\|e\|/\|\tilde{B}\|, \dot{\tilde{B}}_{M+1} \\ &= -G\tilde{B}_{M+1}\|e\|/\|\tilde{B}\|, \ldots, \dot{\tilde{B}}_n \\ &= -G\tilde{B}_n\|e\|/\|\tilde{B}\|\end{aligned} \tag{27}$$

$$\begin{aligned}\tilde{B}^T\dot{\tilde{B}} &= -G(\tilde{B}_1^2 + \cdots + \tilde{B}_n^2)\|e\|/\|\tilde{B}\| \\ &= -G\|\tilde{B}\|^2\|e\|/\|\tilde{B}\| = -G\|e\| \cdot \|\tilde{B}\|\end{aligned} \tag{28}$$

Introducing Eqs. (26), (28) in Eq. (9), we can also get lastly

$$\dot{V}(e, \tilde{B}, \tilde{G}) \le -G(e_1^2 + \cdots + e_n^2) = -Ge^T e \tag{29}$$

By the same reasoning as the case $M = n$, the solution $e = 0$, $\tilde{B} = 0$, $\tilde{G} = 0$ is asymptotically stable.

Remark In the current scheme of adaptive synchronization [13–15], the traditional Lyapunov stability theorem and Barbalat lemma are used to prove the error vector approaches zero, as time approaches infinity. But the question, why the estimated parameters also approach uncertain parameters, remains no answer. By GYC pragmatical asymptotical stability theorem, the question can be answered strictly. Moreover, the asymptotical stability is global; see the Appendix.

3 Numerical examples

Case I Periodic parameters for Lorenz system, $M = n$
The master Lorenz system with uncertain variable parameters is

$$\begin{cases}\dot{x}_1 = -A_1(t)(x_1 - x_2) \\ \dot{x}_2 = A_2(t)x_1 - x_2 - x_1x_3 \\ \dot{x}_3 = x_1x_2 - A_3(t)x_3\end{cases} \tag{30}$$

where $A_1(t)$, $A_2(t)$ and $A_3(t)$ are uncertain parameters. In simulation, we take

$$\begin{aligned}A_1(t) &= \sigma(1 + d_1 \sin \varpi_1 t) \\ A_2(t) &= \gamma(1 + d_2 \sin \varpi_2 t) \\ A_3(t) &= b(1 + d_3 \sin \varpi_3 t)\end{aligned} \tag{31}$$

where σ, γ, b, d_1, d_2, d_3, ϖ_1, ϖ_2, and ϖ_3 are positive constants.

By Eq. (2), the slave Lorenz system is

$$\begin{cases}\dot{y}_1 = -\hat{A}_1(t)(y_1 - y_2) + (\hat{G} + G)(x_1 - y_1) \\ \dot{y}_2 = \hat{A}_2(t)y_1 - y_2 - y_1y_3 + (\hat{G} + G)(x_2 - y_2) \\ \dot{y}_3 = y_1y_2 - \hat{A}_3(t)y_3 + (\hat{G} + G)(x_3 - y_3)\end{cases} \tag{32}$$

where $\hat{K} = \hat{G} + G$. $\hat{G}$ is the estimated value of G.

Take $\sigma = 10$, $\gamma = 28$, $b = 8/3$, $d_1 = 0.05$, $d_2 = 0.01$, $d_3 = 0.1$, $\varpi_1 = 9$, $\varpi_2 = 15$, $\varpi_3 = 18$, and the initial condition is $[x_0^T \ y_0^T \ \hat{A}_0^T \ \hat{G}_0]^T = [111\,000\,000\,0]^T$.

Subtracting Eq. (32) from Eq. (30), we obtain an error dynamics.

$$\begin{aligned}\dot{e}_1 &= -A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) \\ &\quad - (\hat{G} + G)(x_1 - y_1) \\ \dot{e}_2 &= A_2(t)x_1 - x_2 - x_1x_3 - \hat{A}_2(t)y_1 + y_2 \\ &\quad + y_1y_3 - (\hat{G} + G)(x_2 - y_2) \\ \dot{e}_3 &= x_1x_2 - A_3(t)x_3 - y_1y_2 + \hat{A}_3(t)y_3 \\ &\quad - (\hat{G} + G)(x_3 - y_3)\end{aligned} \tag{33}$$

where $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, $e_3 = x_3 - y_3$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} (x_i - y_i) = 0, \quad i = 1, 2, 3 \tag{34}$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{35}$$

since G is constant, $\dot{G} = 0$. Define

$$\tilde{A}(t) = [\tilde{A}_1(t) \quad \tilde{A}_2(t) \quad \tilde{A}_3(t)]^T \tag{36}$$

$$\begin{aligned}\tilde{A}_1(t) &= A_1(t) - \hat{A}_1(t) \\ \tilde{A}_2(t) &= A_2(t) - \hat{A}_2(t) \\ \tilde{A}_3(t) &= A_3(t) - \hat{A}_3(t)\end{aligned} \tag{37}$$

then

$$\begin{aligned}\dot{\tilde{A}}_1(t) &= \sigma d_1 \varpi_1 \cos \varpi_1 t - \dot{\hat{A}}_1(t) \\ \dot{\tilde{A}}_2(t) &= \gamma d_2 \varpi_2 \cos \varpi_2 t - \dot{\hat{A}}_2(t) \\ \dot{\tilde{A}}_3(t) &= b d_3 \varpi_3 \cos \varpi_3 t - \dot{\hat{A}}_3(t)\end{aligned} \tag{38}$$

Choose $\dot{\tilde{A}}_1(t)$, $\dot{\tilde{A}}_2(t)$, and $\dot{\tilde{A}}_3(t)$ as

$$\begin{aligned}\dot{\tilde{A}}_1 &= -G\tilde{A}_1\|e\|/\|\tilde{A}\| \\ \dot{\tilde{A}}_2 &= -G\tilde{A}_2\|e\|/\|\tilde{A}\| \\ \dot{\tilde{A}}_3 &= -G\tilde{A}_3\|e\|/\|\tilde{A}\|\end{aligned} \tag{39}$$

Choose a Lyapunov function is given in the form of positive definite function:

$$V(e_1, e_2, e_3, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{G}) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + \tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{A}_3^2 + \tilde{G}^2\right) \tag{40}$$

Its time derivative along any solution of Eqs. (33), (35), and (39) is

$$\begin{aligned}
\dot{V} &= e_1\big[-A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) \\
&\quad - (\hat{G} + G)(x_1 - y_1)\big] + e_2\big[A_2(t)x_1 - x_2 - x_1x_3 \\
&\quad - \hat{A}_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2)\big] \\
&\quad + e_3\big[x_1x_2 - A_3(t)x_3 - y_1y_2 + \hat{A}_3(t)y_3 \\
&\quad - (\hat{G} + G)(x_3 - y_3)\big] + \tilde{A}_1\dot{\tilde{A}}_1 + \tilde{A}_2\dot{\tilde{A}}_2 + \tilde{A}_3\dot{\tilde{A}}_3 \\
&\quad - \tilde{G}\dot{\hat{G}} \\
\dot{V} &= e_1\big[-A_1(t)(x_1 - x_2) + A_1(t)(y_1 - y_2) \\
&\quad - (\hat{G} + G)(x_1 - y_1)\big] + e_2\big[A_2(t)x_1 - x_2 - x_1x_3 \\
&\quad - A_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2)\big] \\
&\quad + e_3\big[x_1x_2 - A_3(t)x_3 - y_1y_2 + A_3(t)y_3 \\
&\quad - (\hat{G} + G)(x_3 - y_3)\big] + \tilde{A}_1(y_1 - y_2)e_1 - \tilde{A}_2y_1e_2 \\
&\quad - \tilde{A}_3y_3e_3 - G\|e\|\big(\tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{A}_3^2\big)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}} \\
\dot{V} &\le G\|e\|^2 - (\hat{G} + G)\|e\|^2 + G\|e\|\|\tilde{A}\| \\
&\quad - G\|e\|\big(\tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{A}_3^2\big)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}
\end{aligned}$$

$\dot{V}$ can be rewritten as

$$\dot{V} \le -G\|e\|^2 \tag{41}$$

$\dot{V}$ is negative semidefinite function of e, $\tilde{A}$, $\tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (33), adaptive laws (35), and parameter dynamics (39) is asymptotically stable. Now, D is a 4-manifold, $n = 7$ and the number of error state variables $p = 3$. When $e_i = 0$ $(i = 1, 2, 3)$ and $\tilde{A}_i$, $\tilde{G}$ take arbitrary values, $\dot{V} = 0$, so X is a 4-manifold, $m = n - p = 7 - 3 = 4$. $m + 1 < n$ is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The GYC pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{A}_i = \tilde{G} = 0$ $(i = 1, 2, 3)$ is asymptotically stable. Moreover, the result is global asymptotically stable (see the Appendix). The numerical results are shown in Figs. 1, 2 and 3. The chaos synchronization is accomplished. The coupling strength required is $K = 2G = 38.26$.

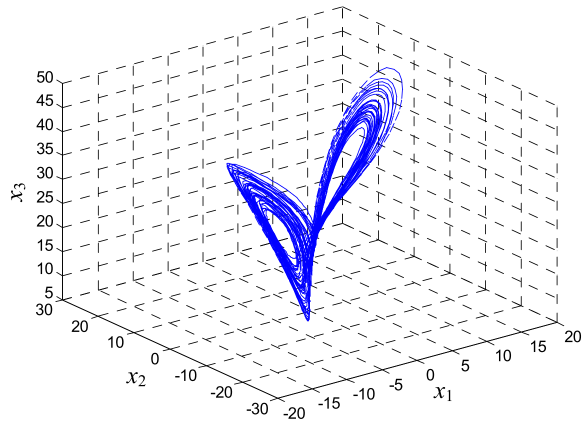

Fig. 1 Phase portrait for Lorenz with $\sigma = 10$, $\gamma = 28$, $b = 8/3$

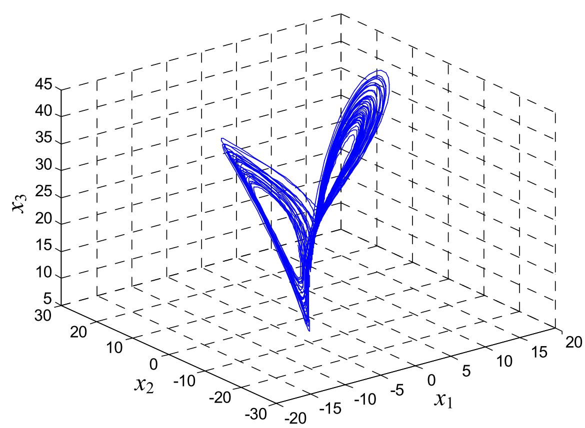

Fig. 2 Phase portrait for Eq. (30) with $A_1(t) = \sigma(1 + d_1 \sin \varpi_1 t)$, $A_2(t) = \gamma(1 + d_2 \sin \varpi_2 t)$ and $A_3(t) = b(1 + d_3 \sin \varpi_3 t)$

Case II Exponentially increasing and decreasing parameters for Quantum-CNN system, $M = n$ For a two-cell Quantum-CNN, the following differential equations are obtained [1–3]:

$$\begin{cases}
\dot{x}_1 = -2a_1\sqrt{1 - x_1^2}\sin x_2 \\
\dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1 - x_1^2}}\cos x_2 \\
\dot{x}_3 = -2a_2\sqrt{1 - x_3^2}\sin x_4 \\
\dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1 - x_3^2}}\cos x_4
\end{cases} \tag{42}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the interdot energy inside each cell, and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. When

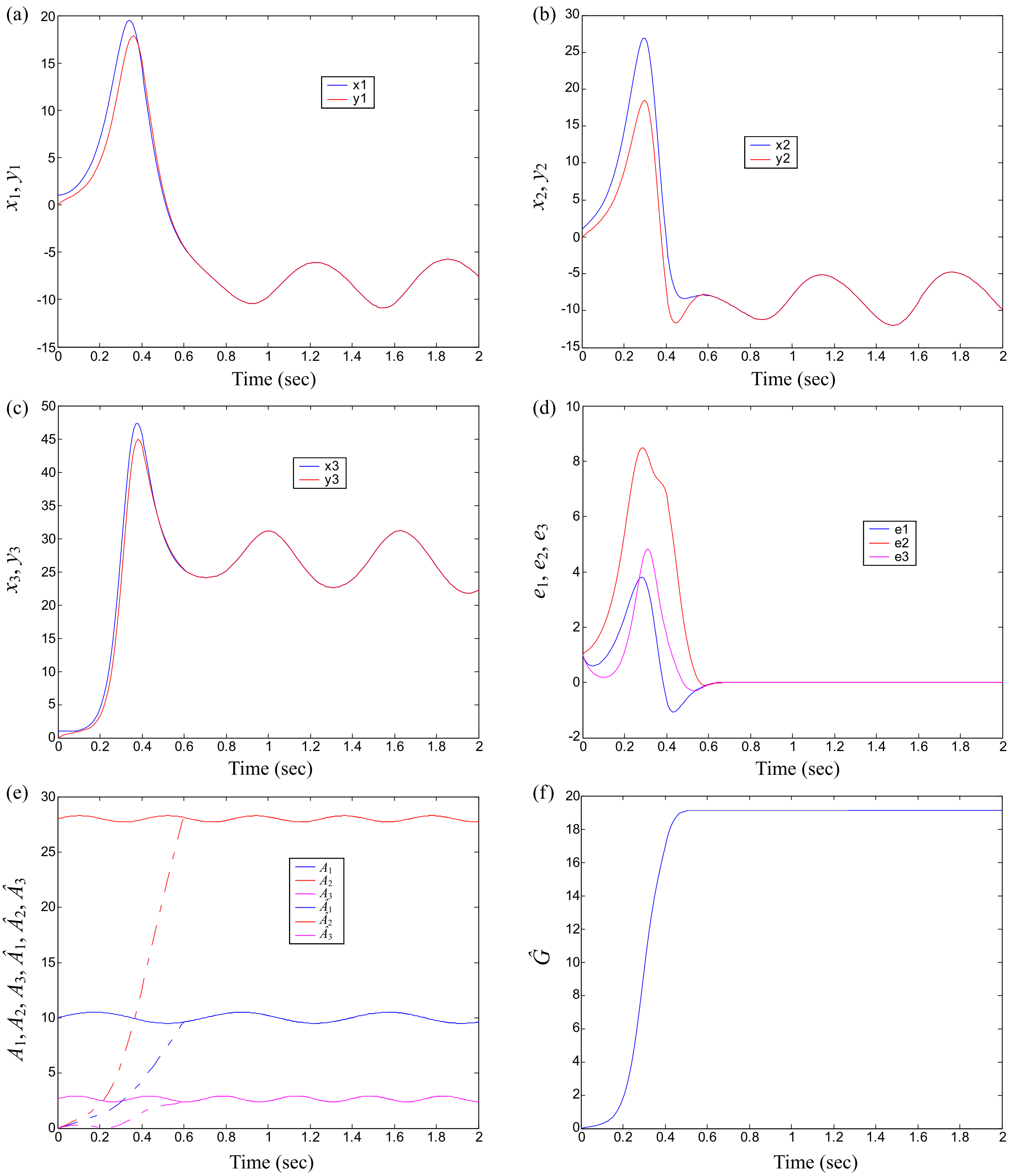

Fig. 3 Time histories of states, state errors, A_1, A_2, A_3, $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$, and estimated Lipschitz constant $\hat{G}$ for Case I

$a_1 = 6.8$, $a_2 = 4.3$, $\omega_1 = 4.7$, and $\omega_2 = 3.9$, the system is chaotic.

The master Quantum-CNN system with uncertain variable parameters is

$$\begin{cases} \dot{x}_1 = -2A_1(t)\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -A_3(t)(x_1 - x_3) + 2A_1(t)\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2A_2(t)\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -A_4(t)(x_3 - x_1) + 2A_2(t)\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{43}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$, and $A_4(t)$ are uncertain parameters. In simulation, we take

$$\begin{aligned} A_1(t) &= a_1\big[1 + c_1\big(1 - e^{-b_1 t}\big)\big] \\ A_2(t) &= a_2\big[1 + c_2\big(1 - e^{-b_2 t}\big)\big] \\ A_3(t) &= \omega_1\big[1 + c_3\big(1 - e^{-b_3 t}\big)\big] \\ A_4(t) &= \omega_2\big[1 + c_4\big(1 - e^{-b_4 t}\big)\big] \end{aligned} \tag{44}$$

where b_1, b_2, b_3, b_4, c_1, c_2, c_3, and c_4 are constants. Take $b_1 = 0.05$, $b_2 = 0.004$, $b_3 = 0.004$, $b_4 = 0.005$, $c_1 = -0.25$, $c_2 = 0.15$, $c_3 = -0.2$, and $c_4 = 0.1$.

By Eq. (2), the slave Quantum-CNN system is

$$\begin{cases} \dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 \\ \qquad + (\hat{G} + G)(x_1 - y_1) \\ \dot{y}_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\ \qquad + (\hat{G} + G)(x_2 - y_2) \\ \dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + (\hat{G} + G)(x_3 - y_3) \\ \dot{y}_4 = -\hat{\omega}_2(y_3 - y_1) + 2\hat{a}_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \\ \qquad + (\hat{G} + G)(x_4 - y_4) \end{cases} \tag{45}$$

where $\hat{K} = \hat{G} + G$. $\hat{G}$ is the estimated value of G. The initial values are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = -0.2$, $y_2(0) = 0.41$, $y_3(0) = 0.25$, $y_4(0) = -0.81$ and $[\hat{a}_{10}\ \hat{a}_{20}\ \hat{\omega}_{10}\ \hat{\omega}_{20}\ \hat{G}_0]^T = [0\ 0\ 0\ 0\ 0]^T$. The error dynamic is

$$\begin{aligned} \dot{e}_1 &= -2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 \\ &\quad - (\hat{G} + G)e_1 \\ \dot{e}_2 &= -A_3(t)(x_1 - x_3) + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ &\quad + \hat{\omega}_1(y_1 - y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\ &\quad - (\hat{G} + G)e_2 \\ \dot{e}_3 &= -2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 \\ &\quad - (\hat{G} + G)e_3 \\ \dot{e}_4 &= -A_4(t)(x_3 - x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \\ &\quad + \hat{\omega}_2(y_3 - y_1) - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \\ &\quad - (\hat{G} + G)e_4 \end{aligned} \tag{46}$$

where $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, $e_3 = x_3 - y_3$, $e_4 - x_4 - y_4$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} (x_i - y_i) = 0, \quad i = 1, 2, 3, 4 \tag{47}$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{48}$$

since G is constant, $\dot{G} = 0$. Define

$$\begin{aligned} \tilde{a}_1 &= A_1(t) - \hat{a}_1, \qquad \tilde{a}_2 = A_2(t) - \hat{a}_2 \\ \tilde{\omega}_1 &= A_3(t) - \hat{\omega}_1, \qquad \tilde{\omega}_2 = A_4(t) - \hat{\omega}_2 \end{aligned} \tag{49}$$

then

$$\begin{aligned} \dot{\tilde{a}}_1 &= a_1 b_1 c_1 e^{-b_1 t} - \dot{\hat{a}}_1 \\ \dot{\tilde{a}}_2 &= a_2 b_2 c_2 e^{-b_2 t} - \dot{\hat{a}}_2 \\ \dot{\tilde{\omega}}_1 &= \omega_1 b_3 c_3 e^{-b_3 t} - \dot{\hat{\omega}}_1 \\ \dot{\tilde{\omega}}_2 &= \omega_2 b_4 c_4 e^{-b_4 t} - \dot{\hat{\omega}}_2 \end{aligned} \tag{50}$$

Let

$$\tilde{A} = [\tilde{a}_1 \quad \tilde{a}_2 \quad \tilde{\omega}_1 \quad \tilde{\omega}_2] \tag{51}$$

Choose $\dot{\tilde{a}}_1$, $\dot{\tilde{a}}_2$, $\dot{\tilde{\omega}}_1$, and $\dot{\tilde{\omega}}_2$ as

$$\begin{aligned} \dot{\tilde{a}}_1 &= -G\tilde{a}_1\|e\|/\|\tilde{A}\| \\ \dot{\tilde{\omega}}_1 &= -G\tilde{\omega}_1\|e\|/\|\tilde{A}\| \\ \dot{\tilde{a}}_2 &= -G\tilde{a}_2\|e\|/\|\tilde{A}\| \quad \text{and} \\ \dot{\tilde{\omega}}_2 &= -G\tilde{\omega}_2\|e\|/\|\tilde{A}\| \end{aligned} \tag{52}$$

A Lyapunov function is given in the form of positive definite function:

$$V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{G}) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{G}^2\right) \tag{53}$$

Its time derivative along any solution of Eqs. (46), (48), and (52) is

$$\begin{aligned}
\dot{V} &= e_1\Big[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 \\
&\quad - (\hat{G}+G)e_1\Big] + e_2\Big[-A_3(t)(x_1 - x_3) \\
&\quad + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1 - y_3) \\
&\quad - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2\Big] \\
&\quad + e_3\Big[-2A_2(t)\sqrt{1-x_3^2}\sin x_4 \\
&\quad + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3\Big] \\
&\quad + e_4\Big[-A_4(t)(x_3 - x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \\
&\quad + \hat{\omega}_2(y_3 - y_1) - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \\
&\quad - (\hat{G}+G)e_4\Big] + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{\omega}_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\dot{\tilde{\omega}}_2 \\
&\quad - \tilde{G}\dot{\tilde{G}}
\end{aligned}$$

$$\begin{aligned}
\dot{V} &= e_1\Big[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 \\
&\quad + 2A_1(t)\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1\Big] \\
&\quad + e_2\Big[-A_3(t)(x_1 - x_3) + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\
&\quad + A_3(t)(y_1 - y_3) - 2A_1(t)\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\
&\quad - (\hat{G}+G)e_2\Big] + e_3\Big[-2A_2(t)\sqrt{1-x_3^2}\sin x_4 \\
&\quad + 2A_2(t)\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3\Big] \\
&\quad + e_4\Big[-A_4(t)(x_3 - x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \\
&\quad + A_4(t)(y_3 - y_1) - 2A_2(t)\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \\
&\quad - (\hat{G}+G)e_4\Big] + \tilde{a}_1\Big[2\sqrt{1-y_1^2}\sin y_2 e_1 \\
&\quad - \frac{2y_1}{\sqrt{1-y_1^2}}\cos y_2 e_2\Big] + \tilde{\omega}_1\big[(y_1 - y_3)e_2\big] \\
&\quad + \tilde{a}_2\Big[2\sqrt{1-y_3^2}\sin y_4 e_3 - \frac{2y_3}{\sqrt{1-y_3^2}}\cos y_4 e_4\Big] \\
&\quad + \tilde{\omega}_2\big[(y_3 - y_1)e_4\big] \\
&\quad - G\|e\|\left(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2\right)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}
\end{aligned}$$

$$\begin{aligned}
\dot{V} &\le G\|e\|^2 - (\hat{G}+G)\|e\|^2 + G\|e\|\|\tilde{A}\| \\
&\quad - G\|e\|\left(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2\right)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}
\end{aligned}$$

$\dot{V}$ can be rewritten as

$$\dot{V} \le -G\left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \tag{54}$$

$\dot{V}$ is a negative semidefinite function of $e_i, \tilde{a}_j, \tilde{\omega}_j, \tilde{G}$ ($i = 1, 2, 3, 4$; $j = 1, 2$). The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (46), adaptive laws (48), and parameter dynamics (52) is asymptotically stable. Now, D is a 5-manifold, $n = 9$ and the number of error state variables $p = 4$. When $e_i = 0$ ($i = 1, 2, 3, 4$) and $\tilde{a}_j, \tilde{\omega}_j, \tilde{G}$ ($j = 1, 2$) take arbitrary values, $\dot{V} = 0$, so X is a 5-manifold, $m = n - p = 9 - 4 = 5$. $m + 1 < n$ is satisfied. By the GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The GYC pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{a}_j = \tilde{\omega}_j = \tilde{G} = 0$ ($i = 1, 2, 3, 4$; $j = 1, 2$) is asymptotically stable. Moreover, the result is global asymptotically stable (see the Appendix). The numerical results are shown in Figs. 4, 5 and 6. The chaos synchronization is accomplished. The coupling strength required is $K = 2G = 5.54$.

Case III Periodically and exponentially increasing and decreasing parameters for Quantum-CNN system, $M < n$ The master Quantum-CNN system with un-

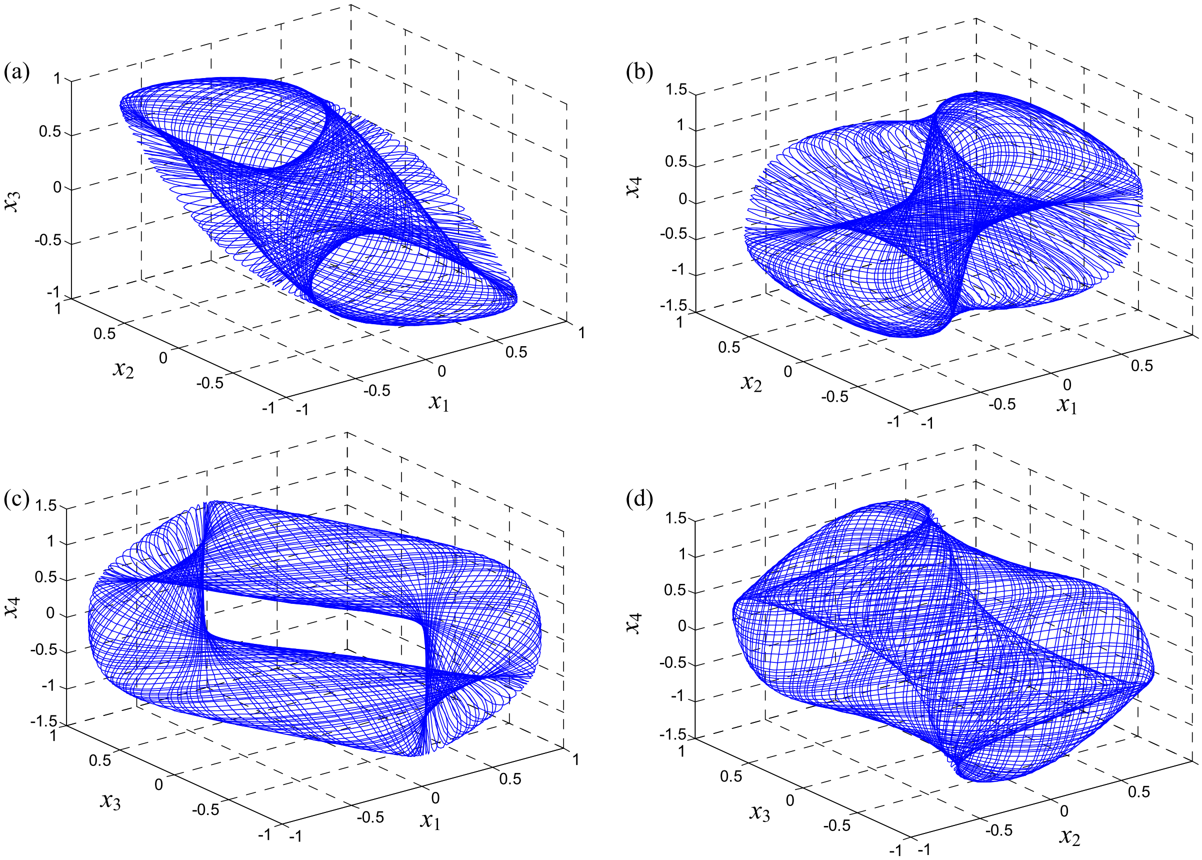

Fig. 4 Phase portrait for chaotic system (42)

certain variable parameters is

$$\begin{cases} \dot{x}_1 = -2A_1(t)\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -A_3(t)(x_1-x_3) + 2A_1(t)\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2A_2(t)\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -A_4(t)(x_3-x_1) + 2A_2(t)\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{55}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$, and $A_4(t)$ are uncertain parameters. In simulation, we take

$$\begin{aligned} A_1(t) &= a_1\bigl[1 + c_1\bigl(1 - e^{-b_1 t}\sin \varpi_1 t\bigr)\bigr] \\ A_2(t) &= a_2\bigl[1 + c_2\bigl(1 - e^{-b_2 t}\sin \varpi_2 t\bigr)\bigr] \\ A_3(t) &= \omega_1\bigl[1 + c_3\bigl(1 - e^{-b_3 t}\sin \varpi_3 t\bigr)\bigr] \\ A_4(t) &= \omega_2\bigl[1 + c_4\bigl(1 - e^{-b_4 t}\sin \varpi_4 t\bigr)\bigr] \end{aligned} \tag{56}$$

where b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, ϖ_1, ϖ_2, ϖ_3, and ϖ_4 are constants. Take $b_1 = 0.001$, $b_2 = 0.002$, $b_3 = 0.004$, $b_4 = 0.005$, $c_1 = -0.25$, $c_2 = 0.15$, $c_3 = -0.2$, $c_4 = 0.1$, $\varpi_1 = 5$, $\varpi_2 = 1$, $\varpi_3 = 3$, and $\varpi_4 = 6$. System (55) is chaotic.

By Eq. (2), the slave Quantum-CNN system is

$$\begin{cases} \dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)(y_1-x_1) \\ \dot{y}_2 = -\hat{\omega}_1(y_1-y_3) + 2\hat{a}_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\ \qquad - (\hat{G}+G)(y_2-x_2) \\ \dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)(y_3-x_3) \\ \dot{y}_4 = -A_4(t)(y_3-y_1) + 2\hat{a}_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \\ \qquad - (\hat{G}+G)(y_4-x_4) \end{cases} \tag{57}$$

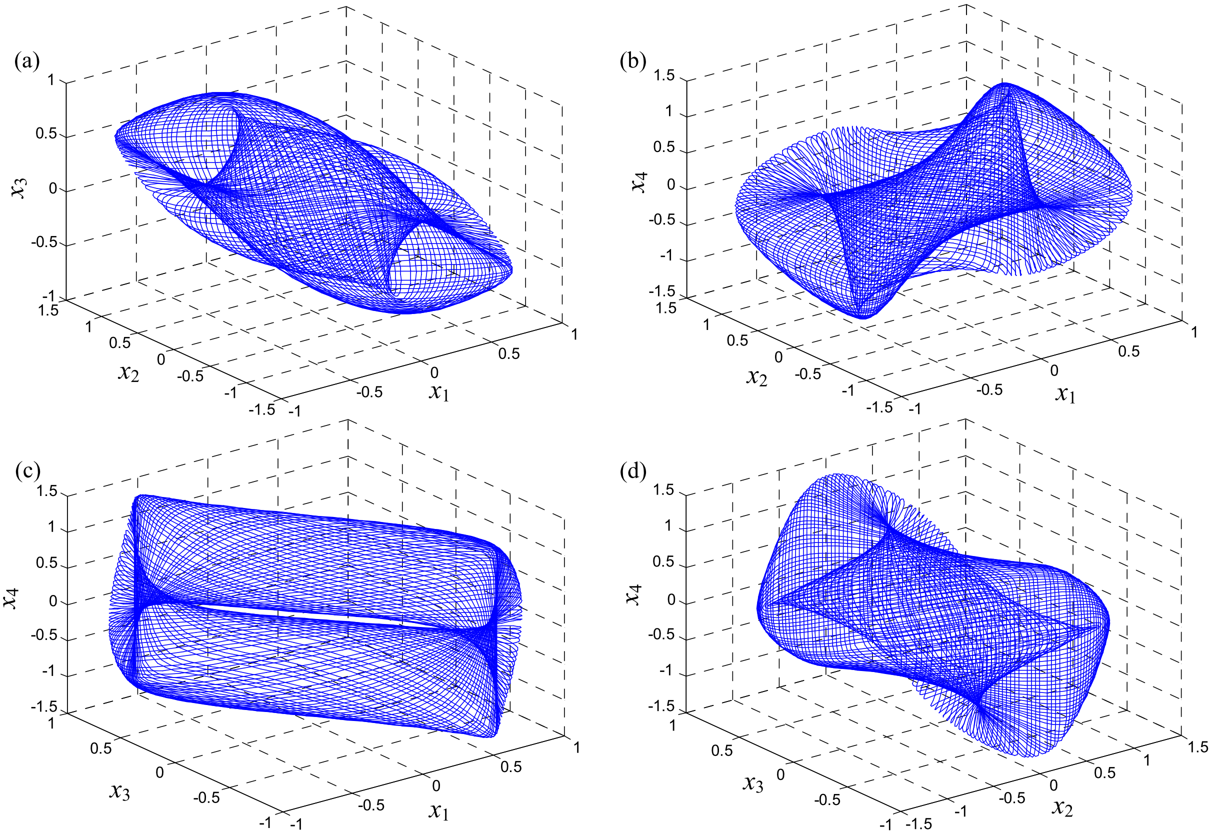

Fig. 5 Phase portrait for chaotic system (43)

where $\hat{K} = \hat{G} + G$. $\hat{G}$ is the estimated value of G. The error dynamic is

$$
\begin{aligned}
\dot{e}_1 &= -2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 \\
&\quad - (\hat{G}+G)e_1 \\
\dot{e}_2 &= -A_3(t)(x_1 - x_3) + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\
&\quad + \hat{\omega}_1(y_1 - y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\
&\quad - (\hat{G}+G)e_2 \\
\dot{e}_3 &= -2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 \\
&\quad - (\hat{G}+G)e_3 \\
\dot{e}_4 &= -A_4(t)(x_3 - x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \\
&\quad + A_4(t)(y_3 - y_1) \\
&\quad - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4
\end{aligned}
\tag{58}
$$

where $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, $e_3 = x_3 - y_3$, $e_4 - x_4 - y_4$.

Our aim is

$$
\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1,2,3,4 \tag{59}
$$

Let adaptive law be

$$
\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{60}
$$

since G is constant, $\dot{G} = 0$. Define

$$
\begin{aligned}
&\tilde{a}_1 = A_1(t) - \hat{a}_1, \qquad \tilde{a}_2 = A_2(t) - \hat{a}_2 \\
&\tilde{\omega}_1 = A_3(t) - \hat{\omega}_1
\end{aligned}
\tag{61}
$$

then

$$
\begin{aligned}
\dot{\tilde{a}}_1 &= a_1 b_1 c_1 e^{-b_1 t}\sin\varpi_1 t \\
&\quad - a_1 c_1 \varpi_1 e^{-b_1 t}\cos\varpi_1 t - \dot{\hat{a}}_1 \\
\dot{\tilde{\omega}}_1 &= \omega_1 b_3 c_3 e^{-b_3 t}\sin\varpi_3 t \\
&\quad - \omega_1 c_3 \varpi_3 e^{-b_3 t}\cos\varpi_3 t - \dot{\hat{\omega}}_1 \\
\dot{\tilde{a}}_2 &= a_2 b_2 c_2 e^{-b_2 t}\sin\varpi_2 t \\
&\quad - a_2 c_2 \varpi_2 e^{-b_2 t}\cos\varpi_2 t - \dot{\hat{a}}_2
\end{aligned}
\tag{62}
$$

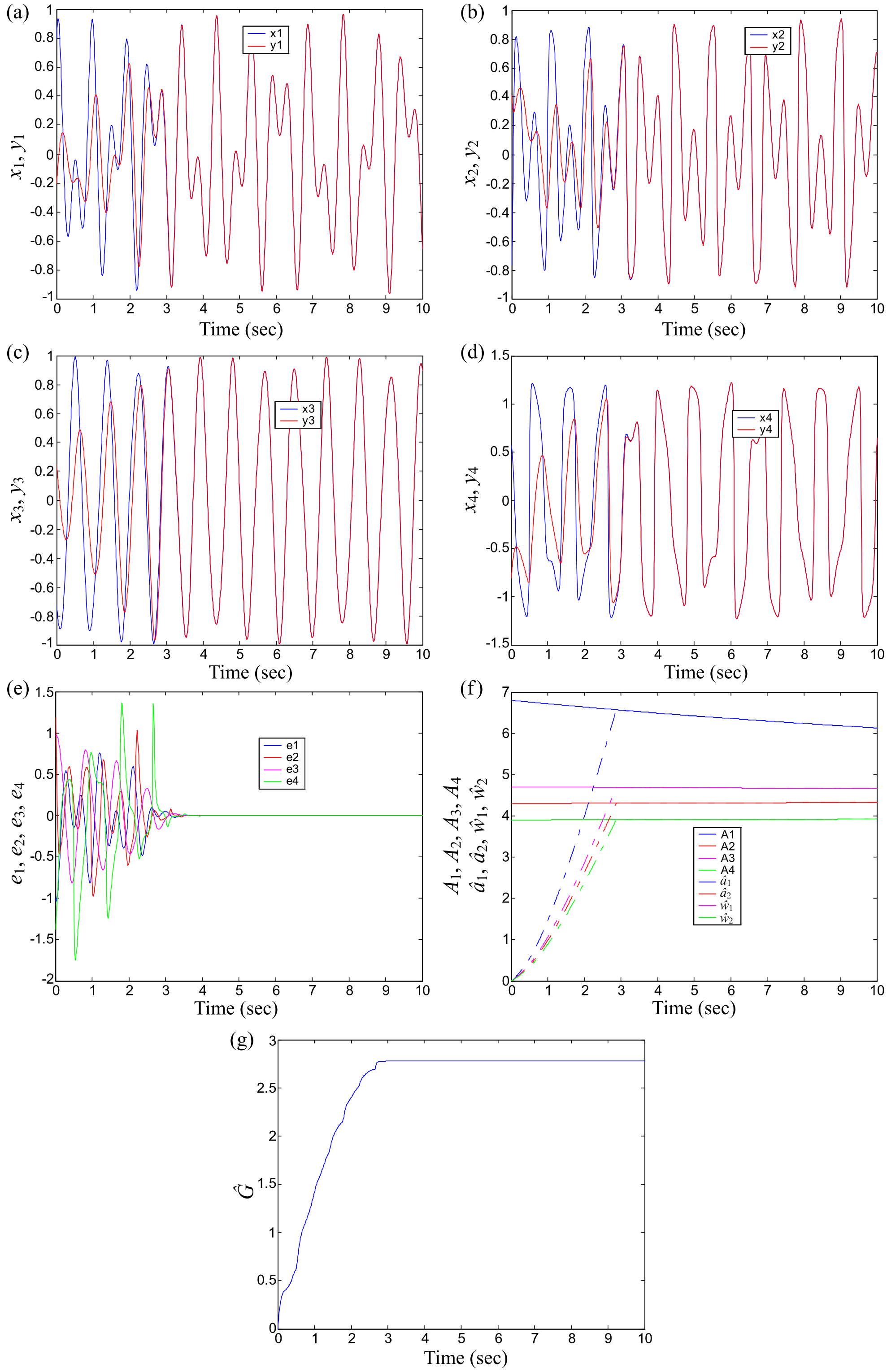

Fig. 6 Time histories of states, state errors, A_1, A_2, A_3, A_4, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$, $\hat{w}_2$, and estimated Lipschitz constant $\hat{G}$ for Case II

Let

$$\tilde{A} = [\tilde{a}_1 \quad \tilde{a}_2 \quad \tilde{\omega}_1] \tag{63}$$

Choose $\dot{\tilde{a}}_1$, $\dot{\tilde{a}}_2$, and $\dot{\tilde{\omega}}_1$ as

$$\begin{aligned} &\dot{\tilde{a}}_1 = -G\tilde{a}_1\|e\|/\|\tilde{A}\| \\ &\dot{\tilde{\omega}}_1 = -G\tilde{\omega}_1\|e\|/\|\tilde{A}\| \quad \text{and} \\ &\dot{\tilde{a}}_2 = -G\tilde{a}_2\|e\|/\|\tilde{A}\| \end{aligned} \tag{64}$$

A Lyapunov function is given in the form of a positive definite function:

$$\begin{aligned} &V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{G}) \\ &\quad = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{G}^2\right) \end{aligned} \tag{65}$$

Its time derivative along any solution of Eqs. (58), (60), and (64) is

$$\begin{aligned} \dot{V} = {} & e_1\left[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2\right. \\ & \left. - (\hat{G}+G)e_1\right] + e_2\left[-A_3(t)(x_1-x_3)\right. \\ & + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1-y_3) \\ & \left. - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2\right] \\ & + e_3\left[-2A_2(t)\sqrt{1-x_3^2}\sin x_4\right. \\ & + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 \\ & \left. - (\hat{G}+G)e_3\right] + e_4\left[-A_4(t)(x_3-x_1)\right. \\ & + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + A_4(t)(y_3-y_1) \\ & \left. - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4\right] \\ & + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{\omega}_1\dot{\tilde{\omega}}_1 - \tilde{G}\dot{\hat{G}} \\ \dot{V} = {} & e_1\left[-2A_1(t)\sqrt{1-x_1^2}\sin x_2\right. \\ & + 2A_1(t)\sqrt{1-y_1^2}\sin y_2 \\ & \left. - (\hat{G}+G)e_1\right] + e_2\left[-A_3(t)(x_1-x_3)\right. \\ & + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + A_3(t)(y_1-y_3) \\ & \left. - 2A_1(t)\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2\right] \\ & + e_3\left[-2A_2(t)\sqrt{1-x_3^2}\sin x_4\right. \\ & \left. + 2A_2(t)\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3\right] \\ & + e_4\left[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right. \\ & + A_4(t)(y_3-y_1) - 2A_2(t)\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \\ & \left. - (\hat{G}+G)e_4\right] + \tilde{a}_1\left[2\sqrt{1-y_1^2}\sin y_2 e_1\right. \\ & \left. - \frac{2y_1}{\sqrt{1-y_1^2}}\cos y_2 e_2\right] + \tilde{\omega}_1\left[(y_1-y_3)e_2\right] \\ & + \tilde{a}_2\left[2\sqrt{1-y_3^2}\sin y_4 e_3 - \frac{2y_3}{\sqrt{1-y_3^2}}\cos y_4 e_4\right] \\ & - G\|e\|\left(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2\right)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}} \\ \dot{V} \le {} & G\|e\|^2 - (\hat{G}+G)\|e\|^2 + G\|e\|\|\tilde{A}\| \\ & - G\|e\|\left(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2\right)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}} \end{aligned}$$

$\dot{V}$ can be rewritten as

$$\dot{V} \le -G\left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \tag{66}$$

$\dot{V}$ is a negative semidefinite function of e_i, $\tilde{a}_j$, $\tilde{\omega}_1$, $\tilde{G}$ ($i = 1, 2, 3, 4$; $j = 1, 2$). The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (58), adaptive laws (60) and parameter dynamics (64) is asymptotically stable. Now, D is a 4-manifold, $n = 8$ and the number of error state variables $p = 4$. When $e_i = 0$ ($i = 1, 2, 3, 4$) and $\tilde{a}_j$, $\tilde{\omega}_1$, $\tilde{G}$ ($j = 1, 2$) take arbitrary values, $\dot{V} = 0$, so X is a 4-manifold, $m = n - p = 8 - 4 = 4$. $m + 1 < n$ is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The GYC pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{a}_j = \tilde{\omega}_1 = \tilde{G} = 0$ ($i = 1, 2, 3, 4$; $j = 1, 2$) is asymptotically stable. Moreover, the result is global asymptotically stable (see the Appendix). The numerical results are shown in Figs. 7 and 8. The chaos synchronization is accomplished. The coupling strength required is $K = 2G = 6.34$.

4 Conclusions

Using the Lipschitz condition, the synchronization of Lorenz chaotic systems and of Quantum-CNN chaotic oscillator systems with uncertain *variable* parameters by linear coupling and GYC pragmatical adaptive tracking is implemented by the GYC pragmatical asymptotical stability theorem. Tracking uncertain *variable* parameters is firstly studied in this pa-

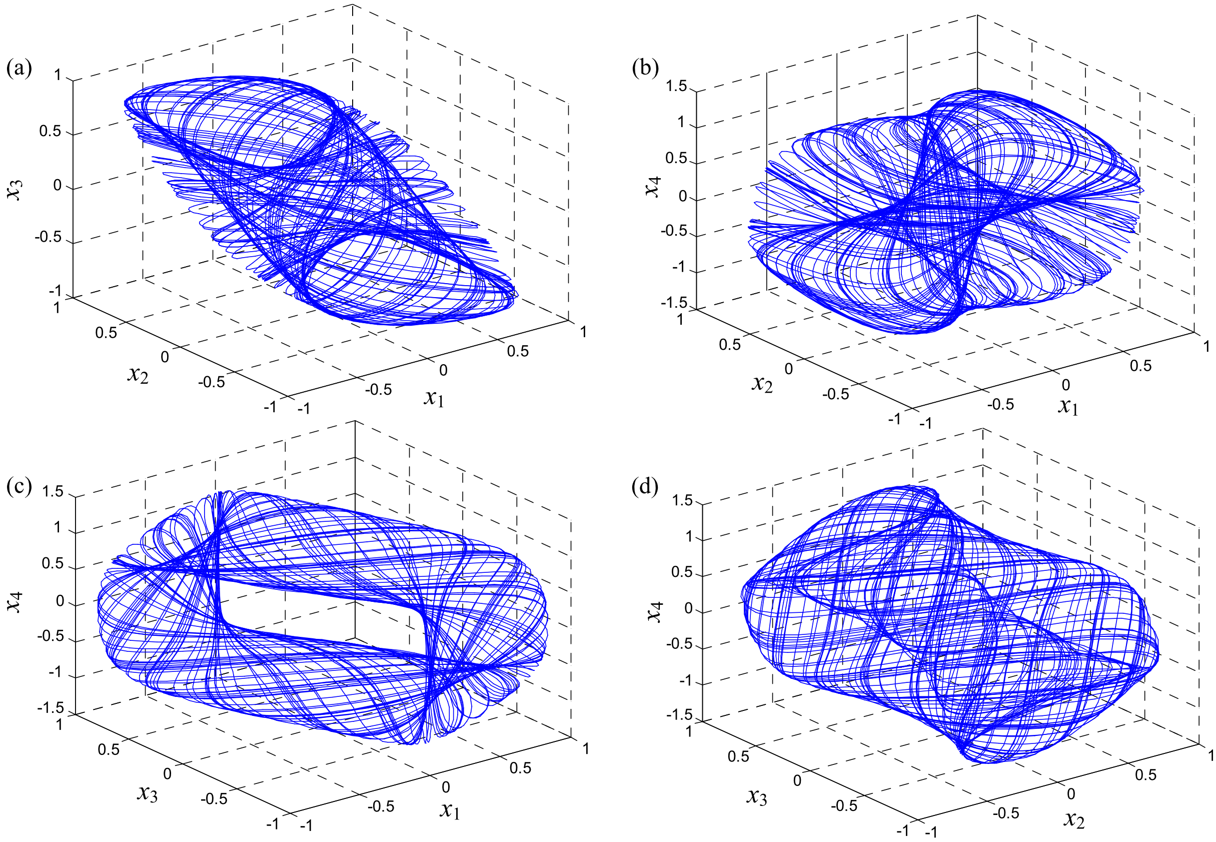

Fig. 7 Phase portrait for chaotic system (54)

per. This is more reasonable, because system parameters always vary due to aging, environment, and disturbances. Two Lorenz systems are synchronization in one case: with oscillating parameters. Two Quantum-CNN systems are the synchronization in two cases: (1) with exponentially increasing and decreasing parameters (2) with periodically and exponentially increasing and decreasing parameters. The computer simulation results imply that the present scheme is very satisfactory.

Acknowledgements This research was supported by the National Science Council, Republic of China, under Grant Number NSC 98-2218-E-011-010.

Appendix: GYC pragmatical asymptotical stability theorem

The stability for many problems in real dynamical systems is actual asymptotical stability, although may not be mathematical asymptotical stability. The mathematical asymptotical stability demands that trajectories from all initial states in the neighborhood of zero solution must approach the origin as $t \to \infty$. If there are only a small part or even a few of the initial states from which the trajectories do not approach the origin as $t \to \infty$, the zero solution is not mathematically asymptotically stable. However, when the probability of occurrence of an event is zero, it means the event does not occur actually. If the probability of occurrence of the event that the trajectories from the initial states are that they do not approach zero when $t \to \infty$, i.e., these trajectories are not asymptotical stale for the zero solution is zero, the stability of the zero solution is actual asymptotical stability though it is not mathematical asymptotical stability. In order to analyze the asymptotical stability of the equilibrium point of such systems, the GYC pragmatical asymptotical stability theorem is used.

Let X and Y be two manifolds of dimensions m and n $(m < n)$, respectively, and φ be a differentiable map

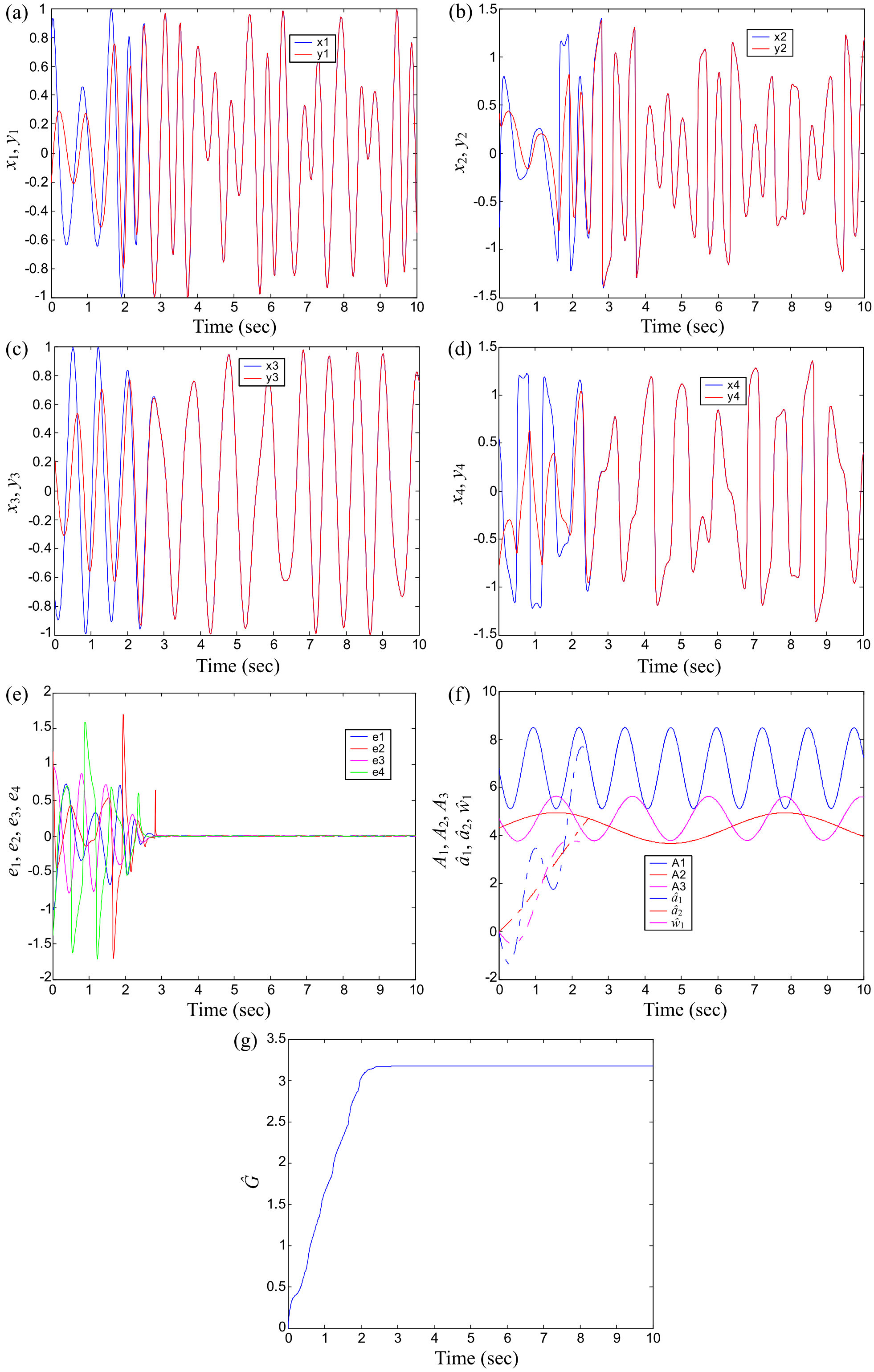

Fig. 8 Time histories of states, state errors, A_1, A_2, A_3, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$, and estimated Lipschitz constant $\hat{G}$ for Case III

from X to Y; then $\varphi(X)$ is subset of Lebesque measure 0 of Y [22]. For an autonomous system

$$\dot{x} = f(x_1, \ldots, x_n) \tag{67}$$

where $x = [x_1, \ldots, x_n]^T$ is a state vector, the function $f = [f_1, \ldots, f_n]^T$ is defined on $D \subset R^n$, an n-manifold.

Let $x = 0$ be an equilibrium point for the system (67). Then

$$f(0) = 0 \tag{68}$$

For a nonautonomous system,

$$\dot{x} = f(x_1, \ldots, x_{n+1}) \tag{69}$$

where $x = [x_1, \ldots, x_{n+1}]^T$, the function $f = [f_1, \ldots, f_n]^T$ is defined on $D \subset R^n \times R_+$, here $t = x_{n+1} \subset R_+$. The equilibrium point is

$$f(0, x_{n+1}) = 0 \tag{70}$$

Definition The equilibrium point for the system is pragmatically asymptotically stable provided that with initial points on C which is a subset of Lebesque measure 0 of D, the behaviors of the corresponding trajectories cannot be determined, while with initial points on $D - C$, the corresponding trajectories behave as that agree with traditional asymptotical stability [19, 20].

Theorem *Let $V = [x_1, x_2, \ldots, x_n]^T : D \to R_+$ be positive definite and analytic on D, where $x_1, x_2, \ldots, x_n$ are all space coordinates such that the derivative of V through Eqs.* (67) *or* (69), *$\dot{V}$, is negative semidefinite of $[x_1, x_2, \ldots, x_n]^T$.*

For the autonomous system, let X be the m-manifold consisting of the point set for which $\forall x \neq 0$, $\dot{V}(x) = 0$ and D is a n-manifold. If $m + 1 < n$, then the equilibrium point of the system is pragmatically asymptotically stable.

For the nonautonomous system, let X be the $m + 1$-manifold consisting of the point set for which $\forall x \neq 0$, $\dot{V}(x_1, x_2, \ldots, x_n) = 0$ and D is an $n + 1$-manifold. If $m + 1 + 1 < n + 1$, i.e., $m + 1 < n$, then the equilibrium point of the system is pragmatically asymptotically stable. Therefore, for both the autonomous and nonautonomous system, the formula $m + 1 < n$ is universal. So, the following proof is only for the autonomous system. The proof for the nonautonomous system is similar.

Proof Since every point of X can be passed by a trajectory of Eq. (67), which is one-dimensional, the collection of these trajectories, C, is a $(m + 1)$-manifold [16, 17].

If $m + 1 < n$, then the collection C is a subset of Lebesque measure 0 of D. By the above definition, the equilibrium point of the system is pragmatically asymptotically stable.

If an initial point is ergodicly chosen in D, the probability of that the initial point falls on the collection C is zero. Here, equal probability is assumed for every point chosen as an initial point in the neighborhood of the equilibrium point. Hence, the event that the initial point is chosen from collection C does not occur actually. Therefore, under the equal probability assumption, pragmatical asymptotical stability becomes actual asymptotical stability. When the initial point falls on $D - C$, $\dot{V}(x) < 0$, the corresponding trajectories behave as that agree with traditional asymptotical stability because by the existence and uniqueness of the solution of initial-value problem, these trajectories never meet C.

For Eq. (7), the Lyapunov function is a positive definite function of n variables, i.e., p error state variables and $n - p = m$ differences between unknown and estimated parameters, while $\dot{V} = e^T C e$ is a negative semidefinite function of n variables. Since the number of error state variables is always more than one, $p > 1$, $m + 1 < n$ is always satisfied, by pragmatical asymptotical stability theorem we have

$$\lim_{t \to \infty} e = 0 \tag{71}$$

and the estimated parameters approach the uncertain parameters. The pragmatical adaptive control theorem is obtained. Therefore, the equilibrium point of the system is pragmatically asymptotically stable. Under the equal probability assumption, it is actually asymptotically stable for both error state variables and parameter variables. □

References

1. Pecora, L.-M., Carroll, T.-L.: Synchronization in chaotic systems. Phys. Rev. Lett. **64**, 821–824 (1990)
2. Terry, J.R., Vanwiggeren, G.-D.: Chaotic communication using generalized synchronization. Chaos Solitons Fractals **12**, 145–152 (2001)
3. Guo, X.-W., Shu, L.-Q.: Chemical chaotic schemes derived from NSG system. Chaos Solitons Fractals **15**, 663–671 (2003)

4. Petrovskii, S., Li, B.-L., Malchow, H.: Quantification of spatial aspect of chaotic dynamics in biological and chemical systems. Bull. Math. Biol. **65**, 425–446 (2003)
5. Chen, S., Zhang, Q., Xie, J., Wang, A.: A stable-manifold-based method for chaos control and synchronization. Chaos Solitons Fractals **20**, 947–954 (2004)
6. Chen, S., Lü, J.: Synchronizing of an uncertain unified chaotic system via adaptive control. Chaos Solitons Fractals **14**, 643–647 (2002)
7. Park, J.H.: Adaptive synchronization of hyperchaotic Chen system with uncertain parameters. Chaos Solitons Fractals **26**, 959–964 (2005)
8. Park, J.H.: Adaptive synchronization of Rossler system with uncertain parameters. Chaos Solitons Fractals **25**, 333–338 (2005)
9. Elabbasy, E.-M., Agiza, H.-N., El-Desoky, M.-M.: Adaptive synchronization of a hyperchaotic system with uncertain parameter. Chaos Solitons Fractals **30**, 1133–1142 (2006)
10. Ge, Z.-M., Yang, C.-H., Chen, H.-H., Lee, S.-C.: Nonlinear dynamics and chaos control of a physical pendulum with vibrating and rotation support. J. Sound Vib. **242**(2), 247–264 (2001)
11. Fotsin, H., Bowong, S.: Adaptive control and synchronization of chaotic systems consisting of Van der Pol oscillators coupled to linear oscillators. Chaos Solitons Fractals **27**, 822–835 (2006)
12. Ge, Z.-M., Chen, Y.-S.: Adaptive synchronization of unidirectional and mutual coupled chaotic systems. Chaos Solitons Fractals **26**, 881–888 (2005)
13. Liao, T.: Adaptive synchronization of two Lorenz systems. Chaos Solitons Fractals **9**(9), 1555–1561 (1998)
14. Yang, X., Xu, W., Sun, Z.: Estimating model parameters in nonautonomous chaotic systems using synchronization. Phys. Lett. A **364**, 378–388 (2007)
15. Ge, Z.-M., Yang, C.-H.: The symplectic synchronization of different chaotic systems. Chaos Solitons Fractals **40**(5), 2532–2543 (2007)
16. Ge, Z.-M., Yu, J.-K., Chen, Y.-T.: Pragmatical asymptotical stability theorem with application to satellite system. Jpn. J. Appl. Phys. **38**, 6178–6179 (1999)
17. Ge, Z.-M., Yu, J.-K.: Pragmatical asymptotical stability theorem partial region and for partial variable with applications to gyroscopic systems. Chin. J. Mech. **16**(4), 179–187 (2000)
18. Ge, Z.-M., Yang, C.-H.: Pragmatical generalized synchronization of chaotic systems with uncertain parameters by adaptive control. Physica D, Nonlinear Phenom. **231**, 87–94 (2007)
19. Fortuna, L., Porto, D.: Quantum-CNN to generate nanoscale chaotic oscillator. Int. J. Bifurc. Chaos **14**(3), 1085–1089 (2004)
20. Ge, Z.-M., Yang, C.-H.: Synchronization of complex chaotic systems in series expansion form. Chaos Solitons Fractals **34**, 1649–1658 (2007)
21. Ge, Z.-M., Yang, C.-H.: Generalized synchronization of Quantum-CNN chaotic oscillator with different order systems. Chaos Solitons Fractals **35**, 980–990 (2008)
22. Matsushima, Y.: Differentiable Manifolds. Dekker, New York (1972)
23. Vidyasagar, M.: Nonlinear System Analysis, 2nd edn., Prentice Hall, New Jersey, p. 15 (1993)