On the Duality of MIMO Transceiver Designs With Bit Allocation

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***Abstract*—Bit rate and power are two commonly used optimality criteria for MIMO transceiver design. In the literature, bit rate maximization and power minimization problems are viewed as different problems and solved independently. In this paper, we derive the duality between these two problems for both the cases with and without integer constraint on bit allocation. We will show that if a transceiver is optimal for the power-minimizing problem, it is also optimal for the rate maximizing problem, and the converse is true. Such a duality has not been stated and proved in the literature to the best of our knowledge. The derivation does not involve any existing optimal solution and we can establish duality result even for the rate maximization problem with integer bit constraint, which the optimal solution is not known. The duality also allows us to develop an algorithm for finding the rate-maximizing transceiver with integer bit allocation using the solution of power-minimizing system. We will also consider some possible generalizations of the problem, for example, when there is a constraint on the maximal constellation size and when the subchannel bit error rates (BERs) are constrained. For each of these cases, we will see that the duality between the two problems continued to hold. In the simulations, we will compute the optimal solutions for these two problems and demonstrate the duality between these two.**

***Index Terms*—Communication systems, MIMO systems, receivers, transceivers, transmitters.**

I. Introduction

Multiple-input multiple-output (MIMO) channels arise in applications such as wireless communication systems that use multiple antennas and also telephone cables that consist of many twisted wire pairs. The information capacity for MIMO transmissions was analyzed in [1] and [2]. In the literature, many criteria have been considered for designing MIMO transceivers, e.g., [3]–[25]. Transceiver design that minimizes mean-squared error (MMSE) is considered in [3]. Optimal transceiver that maximizes mutual information is proposed in [4] and [5]. Optimal transceivers for two design criteria: maximum signal-to-noise ratio (SNR) under zero-forcing (ZF) constraint and MMSE, are developed in [6]. The optimal ZF transceiver that minimizes the bit error rate (BER) is derived in [7]. A minimum-BER design with a channel independent transmitter is considered in [8]. MMSE criterion and minimum error rate criterion for a given power constraint are proposed in [9]. To incorporate quality of service criterion in the design, a weighted MMSE criterion subject to a transmit power constraint is proposed in [10].

In the context of transceiver design, there have been many studies on power minimization [11]–[18] and rate maximization [19]–[25]. ZF solutions with the aim of minimizing the total transmit power for a given BER are developed in [11]. Bit allocation is incorporated in the design of ZF transceiver for minimizing transmit power with quality of service constraint in [12] and [13]. Optimal ZF transceiver with bit allocation for minimizing transmit power for a target bit rate was considered in [14] and [15]. In [16], the transmit power is minimized with different quality of service requirements using the MMSE criterion. Optimal transceiver that minimizes transmit power with a global quality of service constraint is proposed in [17]. A two step transmitter design for minimizing the total transmit power for a target bit rate and BER constraint is proposed in [18]. ZF transceiver design with bit allocation for maximizing bit rate is proposed in [19] and [20]. An unified framework for designing MMSE MIMO systems with a power constraint is proposed in [21]. A number of useful objective functions can be considered in this framework, for example, the optimal MMSE transceivers that maximize the bit rate and mutual information. Based on the results in [21], the transceiver design with bit allocation for maximizing bit rate is proposed in [22]. Designs of discrete multitone transceivers for maximizing bit rates or minimizing transmit power are proposed in [23]. Transceiver designs for a number of design criteria are proposed in [24] and [25]. For example, power-minimizing transceiver, rate-maximizing transceiver, and BER-minimizing transceiver for a given constellation can be obtained. In the earlier works that considered bit allocation [3]–[25], the transceiver are either designed for a given bit allocation or designed with real-valued bit allocation. For the power minimization problem with integer bit allocation, an exhaustive search has been proposed in [22] to find the optimal solution. The problem of jointly designing transceiver and integer bit allocation for maximizing bit rate is still open.

Designs of integer bit allocation have been considered in [26]–[33] for a given ZF transceiver. It is shown in [26] that a greedy algorithm can be used to find the optimal integer bit allocation for maximizing a concave function. The greedy algorithm is used in [27] to solve the power-minimizing problem and rate-maximizing problem when a ZF multicarrier transceiver is given. Algorithms for allocating integer bits in discrete multitone systems to minimize transmit power is proposed in [28] and [29]. In [30], an optimal integer bit loading algorithm is presented for minimizing BER in multicarrier system. A bit

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loading algorithm for satisfying the quality of service constraint is proposed in [31]. In [32], an efficient bit loading algorithm is proposed to minimize an arbitrary convex objective function. An integer bit allocation is proposed in [33] to maximize the transmission bit rate in the presence of intercarrier interference. The algorithms in [26]–[33] can be used to find integer bit allocation if the transceiver is given and only bit allocation is to be determined. They cannot be applied if the transceiver is to be designed together with integer bit allocation.

In this paper, we consider the duality between the problem of maximizing bit rate with bit allocation and the problem of minimizing power with bit allocation. These two problems have been treated as different problems in the literature and they have been addressed independently, power minimization problem in [11]–[17] and rate maximization in [19]–[22]. We will show that these two are actually dual problems. In particular, the optimal solution obtained in either one problem is also optimal for the other. We will consider both cases when there is no integer constraint on bit allocation and when there is an integer constraint. The latter turns out to be a nontrivial generalization of the former and separate proofs are needed. The duality will be derived without using any existing optimal solution of power minimizing or rate maximizing transceiver. As a result, the duality can be obtained even for the rate maximizing problem with integer bit allocation, which the optimal solution has not been solved yet in the literature. Furthermore, the duality result can be applied to develop an algorithm to find the optimal solution of the rate maximization problem with integer bit constraint using the solution of the power-minimizing problem. We will also consider four possible generalizations: 1) the case symbol error rate constraints are different for all the subchannels; 2) the case with an additional constraint on the maximal constellation size; 3) the case when the subchannel BER are constrained, and 4) the case when the averaged BER is constrained. For each of these generalizations, we can also establish the duality between the power minimization and rate maximization problems. The duality between the two problems will also be demonstrated through simulation examples.

The rest of the paper is organized as follows. In Section II, we will introduce the MIMO system model with bit allocation. In Section III, we will formulate the power-minimizing and rate maximizing problems and derive the duality for these two problems when there is no integer constraint on bit allocation. In Section IV, we will consider the duality between these two problems when integer bit constraint is imposed. Duality for generalizations of these two problems are discussed in Section V. In Section VI, we will present algorithms for finding the solutions of the two transceiver design problems with integer bit allocation. In Section VII, we will demonstrate the results by simulations.

II. System Model

A generic MIMO communication system is shown in Fig. 1. The MIMO channel is modeled by a $P \times N$ memoryless matrix $\mathbf{H}$. The $P \times 1$ channel noise $\mathbf{q}$ is additive white Gaussian noise with variance N_0. The transmitter matrix $\mathbf{F}$ is of size $N \times M$ with $M \leq \min(P, N)$. The receiver matrix $\mathbf{G}$ is of size $M \times P$. The input of the transmitter is $\mathbf{s}$, an $M \times 1$ vector of modulation symbols. The symbols are assumed to be zero mean and unit variance, i.e., $\mathrm{E}[s_k] = 0$ and $\sigma_{s_k}^2 = 1$ for $k = 0, 1, \ldots, M-1$. The autocorrelation matrix of $\mathbf{s}$ is assumed to be $\mathrm{E}[\mathbf{s}\mathbf{s}^\dagger] = \mathbf{I}_M$, where $\dagger$ denotes the transpose conjugate and the notation $\mathbf{I}_M$ is used to represent the $M \times M$ identity matrix. Hence the total transmit power P is

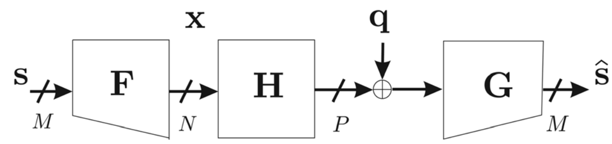

Fig. 1. MIMO communication system.

$$P = \mathrm{E}\{\mathbf{x}^\dagger \mathbf{x}\} = \mathrm{Tr}(\mathbf{F}\mathbf{F}^\dagger) = \sum_{k=0}^{M-1} [\mathbf{F}^\dagger \mathbf{F}]_{kk} \tag{1}$$

where $\mathbf{x}$ is the transmitter output indicated in Fig. 1 and the notation $[\mathbf{A}]_{kl}$ denotes the (k, l)th element of matrix $\mathbf{A}$. The output of the receiver is given by

$$\hat{\mathbf{s}} = \mathbf{GHFs} + \mathbf{Gq}. \tag{2}$$

The error vector $\mathbf{e}$ is defined as

$$\mathbf{e} = \mathbf{s} - \hat{\mathbf{s}}. \tag{3}$$

The MMSE and ZF receiver are given, respectively, by [25]

$$\mathbf{G} = \begin{cases} \mathbf{F}^\dagger \mathbf{H}^\dagger [\mathbf{H}\mathbf{F}\mathbf{F}^\dagger \mathbf{H}^\dagger + N_0 \mathbf{I}_P]^{-1}, & \text{MMSE receiver;} \\ (\mathbf{F}^\dagger \mathbf{H}^\dagger \mathbf{H}\mathbf{F})^{-1} \mathbf{F}^\dagger \mathbf{H}^\dagger, & \text{ZF receiver.} \end{cases} \tag{4}$$

The mean-squared error (MSE) matrix $\mathbf{E} = \mathrm{E}[\mathbf{e}\mathbf{e}^\dagger]$ is given by [24], [25]

$$\mathbf{E} = \begin{cases} [N_0^{-1} \mathbf{F}^\dagger \mathbf{H}^\dagger \mathbf{H}\mathbf{F} + \mathbf{I}_M]^{-1}, & \text{MMSE receiver;} \\ N_0 (\mathbf{F}^\dagger \mathbf{H}^\dagger \mathbf{H}\mathbf{F})^{-1}, & \text{ZF receiver.} \end{cases} \tag{5}$$

The kth subchannel error variance $\sigma_{e_k}^2 = [\mathbf{E}]_{kk}$.

For QAM modulation, the symbol error rate ϵ_k of the kth subchannel is well approximated by [37]

$$\epsilon_k \approx 4 \left(1 - \frac{1}{2^{b_k/2}}\right) Q\left(\sqrt{\frac{3\beta_k}{(2^{b_k} - 1)}}\right) \tag{6}$$

where b_k is the number of bits loaded on the kth subchannel, and β_k is the signal to interference-plus-noise ratio (SINR) [24], [34]. For the MMSE receiver $\beta_k = 1/\sigma_{e_k}^2 - 1$ and for the ZF receiver $\beta_k = 1/\sigma_{e_k}^2$. The function $Q(x)$ is the area under a Gaussian tail, i.e., $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-u^2/2} du$. The total number of bits that can be transmitted in one block is $B = \sum_{k=0}^{M-1} b_k$. In this paper, the derivation is given for the MMSE case. The results for the ZF case can be obtained in a similar way.

III. Power-Minimizing and Rate-Maximizing Problems With Non-Integer Bit Allocation

In this section, we consider the power-minimizing and rate-maximizing problems when bit allocation is not integer constrained. The optimal solutions of both problems are available in the literature [11]–[25], and the solutions look similar. However, it is not obvious that they are the identical. We will establish the connection between these two and show that they are actually dual problems. The results are derived without using the existing optimal solutions. For a given symbol error rate constraint ϵ and target bit rate B_0, the power-minimizing problem $\mathsf{A}_{\rm pow}$ with real bit allocation can be formulated as [11]–[17]

$$(\mathsf{A}_{\rm pow}) \quad \begin{array}{ll} \underset{\mathbf{F},\{b_k\}}{\text{minimize}} & P=\sum_{k=0}^{M-1}[\mathbf{F}^\dagger\mathbf{F}]_{kk} \\ \text{subject to} & \begin{cases} \sum_{k=0}^{M-1} b_k \geq B_0 \\ \epsilon_k \leq \epsilon,\ 0\leq k\leq M-1 \\ b_k \in R^+,\ 0\leq k\leq M-1 \end{cases} \end{array} \tag{7}$$

where ϵ_k is the symbol error rate of the kth subchannel and R^+ is the set of nonnegative real numbers. Given a symbol error rate constraint ϵ and power constraint P_0, the rate-maximizing problem $\mathsf{A}_{\rm rate}$ with real bit allocation is [19]–[22]

$$(\mathsf{A}_{\rm rate}) \quad \begin{array}{ll} \underset{\mathbf{F},\{b_k\}}{\text{maximize}} & B=\sum_{k=0}^{M-1} b_k \\ \text{subject to} & \begin{cases} \sum_{k=0}^{M-1}[\mathbf{F}^\dagger\mathbf{F}]_{kk} \leq P_0, \\ \epsilon_k \leq \epsilon,\ 0\leq k\leq M-1, \\ b_k \in R^+,\ 0\leq k\leq M-1. \end{cases} \end{array} \tag{8}$$

In either problem, we need to design the transmit matrix $\mathbf{F}$ and bit allocation $\{b_k\}$ jointly to maximize bit rate or minimize power. The following lemmas will be useful for subsequent discussion.

Lemma 1: Given a channel matrix $\mathbf{H}$, consider a system with a fixed target error rate, i.e., $\epsilon_k=\epsilon$ for all k. Suppose the transmitter of the system is equal to $\alpha\mathbf{F}$, where $\mathbf{F}$ is some $N\times M$ matrix such that $\mathbf{HF}\neq\mathbf{0}$ and α is a positive real number. Then the transmit power and the achievable bit rate of the system are continuous and strictly increasing functions of α.

Proof: See Appendix A. □

Lemma 1 implies that if we increase the transmit power by choosing $\alpha>1$, the bit rate will always be increased. Next we will show that if we decrease the transmit power for some k_0th subchannel and keep the others unchanged, the error variance of the k_0th subchannel will be increased while the error variances of the other subchannels will be decreased.

Lemma 2: For the MIMO transceiver in Fig. 1, suppose the channel matrix $\mathbf{H}$ is given and the transmitter of the system is replaced by $\mathbf{FD}$, where $\mathbf{F}$ is an $N\times M$ matrix and $\mathbf{D}$ is a diagonal matrix. The diagonal elements of $\mathbf{D}$ are given by

$$[\mathbf{D}]_{ll}=\begin{cases} 1, & l\neq k_0; \\ \mu, & l=k_0 \end{cases} \tag{9}$$

for some k_0, where μ is a positive real number. Then for $l\neq k_0$, the error variances $\sigma_{e_l}^2$ are increasing and continuous functions of μ. For $l=k_0$ the error variances $\sigma_{e_{k_0}}^2$ is a decreasing and continuous function of μ.

Proof: See Appendix B. □

In the following lemma, we will show that inequalities in the power-minimizing problem (7) and the rate-maximizing problem (8) become equalities when optimal designs are used.

Lemma 3: If $(\mathbf{F}^*,\{b_k^*\})$ is optimal for the power-minimizing problem $\mathsf{A}_{\rm pow}$ in (7), the transmission bit rate B is equal to the target bit rate B_0 and all the error rate ϵ_k are equal to ϵ. Similarly, for the rate-maximizing problem $\mathsf{A}_{\rm rate}$ in (8), the transmit power P of the optimal solution is equal to P_0 and all the error rates ϵ_k are equal to ϵ.

Proof: See Appendix C. □

Combining Lemma 1 and Lemma 3, we can show that, for the problem $\mathsf{A}_{\rm rate}$ the maximal bit rate is a strictly increasing function of the power constraint. That is, $B^*(P_1)<B^*(P_2)$ whenever $P_1<P_2$, where $B^*(x)$ denotes the maximal bit rate for $\mathsf{A}_{\rm rate}$ when the power constraint is x. To see this, let $P_1<P_2$. It follows that $B^*(P_1)\leq B^*(P_2)$. So we only need to show that $B^*(P_1)\neq B^*(P_2)$. Suppose $B^*(P_1)=B^*(P_2)$ for $P_1<P_2$. By Lemma 3, the transmit power of the optimal solution that achieves $B^*(P_1)$ is equal to P_1. Using Lemma 1, we can always find a new system that achieves bit rate $\tilde{B}>B^*(P_1)$ using power $\tilde{P}=P_2$, which contracts the definition of $B^*(P_2)$. This completes the proof.

Remarks:

1) Based on (6), given the symbol error rate ϵ_k and β_k, the number of bits that can be loaded on the kth subchannel is well approximated by [38]

$$b_k=\log_2\left(1+\frac{\beta_k}{\Gamma_k}\right) \tag{10}$$

where Γ_k is the so-called SNR gap. If the symbol error rate is approximated in (6), then the resulting gap is $\Gamma_k=[Q^{-1}(\epsilon_k/4)]^2/3$. Another possible approximation of symbol error rate is [37] $\epsilon_k\approx 2\exp(-1.5\beta_k/(2^{b_k}-1))$, where $\exp(x)$ denotes the exponential function of x. In this case, $\Gamma_k=(2/3)\log_e(2/\epsilon_k)$. In general, the bit allocation obtained in (10) is not integer.

2) Lemma 3 shows that all the inequalities in constraints of $\mathsf{A}_{\rm pow}$ and $\mathsf{A}_{\rm rate}$ become equalities when the solutions are optimal. This means that when the optimal transmitter $\mathbf{F}^*$ is given for $\mathsf{A}_{\rm pow}$ or $\mathsf{A}_{\rm rate}$, the bit allocation can be obtained directly by substituting $\epsilon_k=\epsilon$ in (10). Therefore, we only need to design $\mathbf{F}$ directly but not bit allocation in these two problems.

3) When the error rate is constrained to be equal to ϵ for all subchannels, it has been shown that equality in the power and bit rate constraints will hold [17], [22], [25] using majorization theorem [41] and optimization theorem [42]. Lemma 3 is more general in that we have considered inequality constraint on the error rate in addition to power and bit rate constraints.

The results in Lemma 1 and Lemma 3 allow us to establish the duality between $\mathsf{A}_{\rm pow}$ and $\mathsf{A}_{\rm rate}$ in Theorem 1 and Theorem 2.

Theorem 1: Given a target transmission rate B_0 and symbol error rate constraint ϵ, suppose the transmitter $\mathbf{F}^*$ is an optimal solution for $\mathsf{A}_{\rm pow}$, and the minimized power is P^*. Now, given transmit power constraint $P_0=P^*$ and symbol error rate constraint ϵ, the same $\mathbf{F}^*$ also maximizes the bit rate for the problem in $\mathsf{A}_{\rm rate}$. Furthermore, the maximized rate in this case is equal to B_0.

Proof: As $\mathbf{F}^*$ is optimal for $\mathsf{A}_{\rm pow}$, the minimized transmit power is $P^*=\sum_{k=0}^{M-1}[\mathbf{F}^{*\dagger}\mathbf{F}^*]_{kk}$. By Lemma 3, the total bit rate is equal to the target rate B_0 and all the symbol error rates ϵ_k^* are equal to ϵ. Now, let us consider the problem in $\mathsf{A}_{\rm rate}$ with power constraint $P_0=P^*$ and error rate constraint ϵ. Suppose

$\tilde{\mathbf{F}}$ is optimal for $\mathcal{A}_{\rm rate}$. By Lemma 3, the transmit power used in this case is equal to P^* and symbol error rates are equal to ϵ. Since we already know $\mathbf{F}^*$ can achieve bit rate B_0 with transmit power P^*, the maximal bit rate $\tilde{B}$ achieved in $\mathcal{A}_{\rm rate}$ must be larger than or equal to B_0, i.e., $\tilde{B} \geq B_0$. If $\tilde{B} = B_0$, we get the desired result that $\mathbf{F}^*$ is also optimal for $\mathcal{A}_{\rm rate}$. Suppose $\tilde{B} > B_0$, i.e., more than B_0 bits can be transmitted when P^* is given. Consider a new transceiver with transmitter $\mathbf{F}' = \alpha\tilde{\mathbf{F}}$, where $0 < \alpha < 1$. By Lemma 1 we know the bit rate of such a system is a strictly increasing function of α and is continuous on α. So we can always find $\alpha < 1$ such that $B' = B_0$. Since $\alpha < 1$, the required power is smaller than P^*. This is a contradiction to the assumption that P^* is the minimal transmit power when B_0 is given in the power-minimizing problem. Therefore, the maximal bit rate is B_0. Since $\mathbf{F}^*$ can achieve bit rate B_0 with power P^*, it is an optimal solution for $\mathcal{A}_{\rm rate}$. $\triangle\triangle\triangle$

Theorem 2: Given a transmit power constraint P_0 and symbol error rate constraint ϵ, suppose the transmitter $\mathbf{F}^*$ is an optimal solution for the rate-maximizing problem $\mathcal{A}_{\rm rate}$, and the maximized rate is B^*. Then the same $\mathbf{F}^*$ also minimizes the transmit power for the problem $\mathcal{A}_{\rm pow}$ when the target bit rate B_0 is equal to B^* and symbol error rate constraint is ϵ. Furthermore, the minimized power in this case is equal to P_0.

Proof: As $\mathbf{F}^*$ is optimal for the problem $\mathcal{A}_{\rm rate}$, by Lemma 3, the transmit power used in this case is equal to the constraint P_0 and the error rate is $\epsilon_k^* = \epsilon$ for $k = 0, \ldots, M-1$. Consider the problem $\mathcal{A}_{\rm pow}$ with target bit rate $B_0 = B^*$ and error rate constraint ϵ. Suppose $\tilde{\mathbf{F}}$ is an optimal solution for $\mathcal{A}_{\rm pow}$ and the minimized power is $\tilde{P}$. By Lemma 3, the transmitted bit rate is equal to the target B^* and all the error rates are ϵ. Also the minimal power $\tilde{P}$ in $\mathcal{A}_{\rm pow}$ must be smaller than or equal to P_0 since we already know $\mathbf{F}^*$ can achieve bit rate B^* with transmit power P_0, i.e., $\tilde{P} \leq P_0$. If the minimized transmit power $\tilde{P}$ is equal to P_0, we get the desired result that $\mathbf{F}^*$ is also optimal for $\mathcal{A}_{\rm pow}$. Suppose $\tilde{P} < P_0$, i.e., transmit power smaller than P_0 can be achieved when target rate B_0 is B^*. Consider a new system with transmitter $\mathbf{F}' = \alpha\tilde{\mathbf{F}}$, where $\alpha = \sqrt{P_0/\tilde{P}} > 1$. Then the transmit power of the new system is $P' = P_0$. Using Lemma 1 we know the bit rate of the new system will be larger than B^* for the same error rate constraint ϵ. This is a contradiction to the assumption that B^* is the maximal bit rate for $\mathcal{A}_{\rm rate}$ when P_0 is the constraint. Therefore, the minimal power $\tilde{P}$ is equal to P_0 and $\mathbf{F}^*$ is an optimal solution for $\mathcal{A}_{\rm pow}$. $\triangle\triangle\triangle$

Theorems 1 and 2 together show that if a transceiver is optimal in the power-minimizing problem, it is also optimal in the rate-maximizing problem, and vice versa. In the above discussion, the bits assigned to the subchannels are not constrained to be integers. Such a duality also exists for the case when bit allocation is constrained to be integer. However, there are some subtle differences as we will see in the next section.

IV. Transceiver Design With Integer Bit Allocation

In this section, we consider the power-minimizing problem and rate-maximizing problem with integer bit allocation. With the constraint of integer bit allocation, the power-minimizing problem becomes

$$(\mathcal{A}_{\rm pow,int}) \quad \begin{array}{ll} \underset{\mathbf{F},\{b_k\}}{\text{minimize}} & P = \sum_{k=0}^{M-1}[\mathbf{F}^\dagger\mathbf{F}]_{kk} \\ \text{subject to} & \begin{cases} \sum_{k=0}^{M-1} b_k \geq B_0 \\ \epsilon_k \leq \epsilon,\ 0 \leq k \leq M-1, \\ b_k \in Z^+,\ 0 \leq k \leq M-1 \end{cases} \end{array} \tag{11}$$

where Z^+ denotes the set of nonnegative integers. The rate-maximizing problem with integer bit allocation is formulated as

$$(\mathcal{A}_{\rm rate,int}) \quad \begin{array}{ll} \underset{\mathbf{F},\{b_k\}}{\text{maximize}} & B = \sum_{k=0}^{M-1} b_k \\ \text{subject to} & \begin{cases} \sum_{k=0}^{M-1}[\mathbf{F}^\dagger\mathbf{F}]_{kk} \leq P_0, \\ \epsilon_k \leq \epsilon,\ 0 \leq k \leq M-1, \\ b_k \in Z^+,\ 0 \leq k \leq M-1. \end{cases} \end{array} \tag{12}$$

The following lemma shows that for the power-minimizing problem with integer bit constraint, the inequalities in the bit rate constraint and error rate constraint become equalities when the solution is optimal. This is similar to the case of power minimization problem without integer constraint. Such a property does not hold for the rate maximization problem with integer constraint as we will see later.

Lemma 4: For the power-minimizing problem $\mathcal{A}_{\rm pow,int}$ in (11), the bit rate of the optimal solution is equal to B_0 and the symbol error rates $\epsilon_k = \epsilon$ for all k.

Proof: See Appendix D. $\square$

Lemma 4 leads to the following result that states that if a solution is optimal for $\mathcal{A}_{\rm pow,int}$, it is also optimal for $\mathcal{A}_{\rm rate,int}$.

Theorem 3: Consider the power-minimizing problem $\mathcal{A}_{\rm pow,int}$ with a target transmission rate B_0 and symbol error rate constraint ϵ. Suppose $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathcal{A}_{\rm pow,int}$, and in this case the minimized power is P^*. Now for the problem $\mathcal{A}_{\rm rate,int}$ with transmit power constraint $P_0 = P^*$ and error rate constraint ϵ, the same $(\mathbf{F}^*, \{b_k^*\})$ also maximizes the transmission rate and the maximized rate is equal to B_0.

Proof: As $(\mathbf{F}^*, \{b_k^*\})$ is optimal for the problem $\mathcal{A}_{\rm pow,int}$, by Lemma 4 the bit rate is $B^* = \sum_{k=0}^{M-1} b_k^* = B_0$, and all the symbol error rates satisfy $\epsilon_k^* = \epsilon$. Now, let us consider the problem $\mathcal{A}_{\rm rate,int}$ with power constraint $P_0 = P^*$ and error rate constraint ϵ. Suppose $(\tilde{\mathbf{F}}, \{\tilde{b}_k\})$ is optimal for the problem $\mathcal{A}_{\rm rate,int}$ and the maximal bit rate is $\tilde{B} = \sum_{k=0}^{M-1} \tilde{b}_k$. All the corresponding error rates $\tilde{\epsilon}_k$ satisfy $\tilde{\epsilon}_k \leq \epsilon$ and the transmit power $\tilde{P}$ satisfies the power constraint, i.e., $\tilde{P} \leq P^*$. Since we already know the solution of $\mathcal{A}_{\rm pow,int}$ can achieve bit rate B_0 with power P^*, the maximal bit rate $\tilde{B}$ in $\mathcal{A}_{\rm rate,int}$ must be larger than or equal to B_0, i.e., $\tilde{B} \geq B_0$. We will prove the theorem by showing: (i) the transmit power $\tilde{P}$ is equal exactly to P^*; and (ii) the maximized rate $\tilde{B}$ is in fact equal to B_0.

(i) $\tilde{P} = P^*$: Suppose $\tilde{P} < P^*$. This means $\tilde{\mathbf{F}}$ and $\{\tilde{b}_k\}$ can achieve a smaller transmit power and still satisfy all the constraints in $\mathcal{A}_{\rm pow,int}$. This contradicts the assumption that $\mathbf{F}^*$ and $\{b_k^*\}$ are optimal for $\mathcal{A}_{\rm pow,int}$. So we have $\tilde{P} = P^*$.

(ii) $\tilde{B} = B_0$: If $\tilde{B} = B_0$, we get the desired result that $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathcal{A}_{\rm rate,int}$. Suppose $\tilde{B} > B_0$. Using a procedure similar to that in Lemma 4, we can find another system that achieves bit rate $B' = \tilde{B} - 1 \geq B_0$, with transmit power $P' < P^*$, and error rate $\epsilon'_k \leq \epsilon$. This contradicts the assumption that $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathcal{A}_{\text{pow,int}}$. Therefore, we conclude that the maximized bit rate for the problem $\mathcal{A}_{\text{rate,int}}$ is B_0 and the power used is P^*. Therefore, the solution $(\mathbf{F}^*, \{b_k^*\})$ of $\mathcal{A}_{\text{pow,int}}$ is also optimal for the problem $\mathcal{A}_{\text{rate,int}}$. △△△

When the symbol error rate constraint ϵ is fixed, the maximal rate for $\mathcal{A}_{\text{rate,int}}$ is a function of the power constraint P_0. Similarly, for a fixed ϵ, the minimal power of $\mathcal{A}_{\text{pow,int}}$ is a function of target rate B_0. For convenience, we use $P^*_{\text{int}}(x)$ to denote the minimal transmit power for $\mathcal{A}_{\text{pow,int}}$ when the target bit rate x is given and $B^*_{\text{int}}(x)$ to denote as the maximal bit rate for $\mathcal{A}_{\text{rate,int}}$ when the power constraint is x.

The Functions $B^*_{\text{int}}(x)$ and $P^*_{\text{int}}(x)$: Using theorem 3, we will see that $B^*_{\text{int}}(x)$ is not continuous. It is a staircase-like function as shown in Fig. 2(a). This means a nonzero increase in the power constraint does not necessarily implies a nonzero increase in the maximized bit rate. This is different from the case without integer constraint in Section III. To explain this, consider the problem $\mathcal{A}_{\text{pow,int}}$ with two target bit rates B_1 and $B_1 + 1$. Let $P_1 = P^*_{\text{int}}(B_1)$ and $P_2 = P^*_{\text{int}}(B_1 + 1)$. We can plot the minimal transmit power as a function of target bit rate as in Fig. 2(b). By Theorem 3, we know $B^*_{\text{int}}(P_1) = B_1$ and $B^*_{\text{int}}(P_2) = B_1 + 1$. Now suppose the power constraint P_0 for $\mathcal{A}_{\text{rate,int}}$ is such that $P_1 < P_0 < P_2$. Then the maximal bit rate $B^*_{\text{int}}(P_0)$ for $\mathcal{A}_{\text{rate,int}}$ is equal to B_1 as we will see next. Since we already know that the maximal bit rate is B_1 when the power constraint is P_1, we have $B^*_{\text{int}}(P_0) \geq B_1$. Suppose $B^*_{\text{int}}(P_0) > B_1$. This contradicts the fact that P_2 is the minimal power for $\mathcal{A}_{\text{pow,int}}$ when the target bit rate is $B_1 + 1$. Hence we have $B^*_{\text{int}}(P_0) = B_1$. This implies that for any power constraint P that satisfies $P_1 \leq P_0 < P_2$, the maximal bit rate is $B^*_{\text{int}}(P_0) = B_1$. When the power constraint $P_0 = P_2$, the maximal bit rate is increased to $B_1 + 1$. Therefore, $B^*_{\text{int}}(x)$ is the staircase-like function in Fig. 2(a).

From the plot of $B^*_{\text{int}}(P_0)$ in Fig. 2(a) we can see that for $\mathcal{A}_{\text{rate,int}}$ there can be many solutions that achieve the same maximal bit rate, but with transmit power smaller than P_0. Hence for the problem $\mathcal{A}_{\text{rate,int}}$, the result in Lemma 3 is not true any more and the results of the real bit allocation case do not carry over to the integer bit allocation case. To establish the duality with $\mathcal{A}_{\text{pow,int}}$, we will consider the solution with the smallest transmit power among all possible solutions. Using the technique in the proof of Lemma 3, we can show that when the solution of $\mathcal{A}_{\text{rate,int}}$ with the smallest transmit power is used, the actual symbol error rate is equal exactly to ϵ.

Theorem 4: Consider the problem $\mathcal{A}_{\text{rate,int}}$ with power constraint P_0 and symbol error rate constraint ϵ. Suppose $(\mathbf{F}^*, \{b_k^*\})$ is the solution that has the smallest transmit power P^* among all possible solutions. Let the maximized rate be B^*. Given target rate $B_0 = B^*$ and error rate constraint ϵ for the problem $\mathcal{A}_{\text{pow,int}}$, the same solution also minimizes the transmit power and the minimal power is P^*.

Proof: As $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathcal{A}_{\text{rate,int}}$, the maximized rate is $B^* = \sum_{k=0}^{M-1} b_k^*$. The transmit power is $P^* \leq P_0$, and all the error rates satisfy $\epsilon_k^* \leq \epsilon$. Consider the power minimizing problem $\mathcal{A}_{\text{pow,int}}$ with target bit rate $B_0 = B^*$ and the same error rate constraint ϵ. Suppose $(\tilde{\mathbf{F}}, \{\tilde{b}_k\})$ is optimal for $\mathcal{A}_{\text{pow,int}}$ and the minimized power is $\tilde{P}$. By Lemma 4, the bit rate $\sum_{k=0}^{M-1} \tilde{b}_k$ is equal to the target bit rate B^*. Since we already know $(\mathbf{F}^*, \{b_k^*\})$ can achieve bit rate B^* with transmit power P^*, the minimal power $\tilde{P}$ must be smaller than or equal to P^*, i.e., $\tilde{P} \leq P^*$. If $\tilde{P}$ is equal to P^*, we get the desired result that $(\mathbf{F}^*, \{b_k^*\})$ is an optimal solution for $\mathcal{A}_{\text{pow,int}}$. Assume $\tilde{P}$ is smaller, i.e., $\tilde{P} < P^*$. This means $(\tilde{\mathbf{F}}, \{\tilde{b}_k\})$ can achieve bit rate B^* with a smaller power $\tilde{P}$. It contradicts the assumption that $(\mathbf{F}^*, \{b_k^*\})$ is the optimal solution for the problem $\mathcal{A}_{\text{rate,int}}$ that has the smallest transmit power. Hence we have $\tilde{P} = P^*$ and the solution $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathcal{A}_{\text{pow,int}}$. △△△

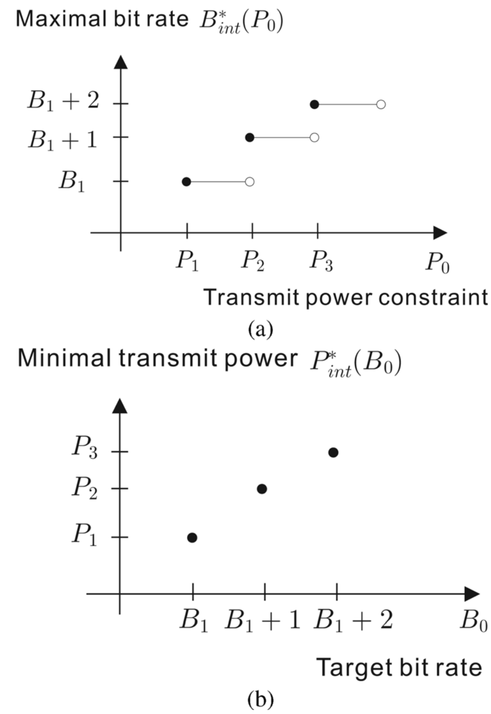

Fig. 2. (a) Maximal bit rate as a function of power constraint P_0 for $\mathcal{A}_{\text{rate,int}}$. (b) Minimal transmit power as a function of target bit rate B_0 for $\mathcal{A}_{\text{pow,int}}$.

Theorem 3 shows that the optimal solution obtained in the power-minimizing problem is also an optimal solution in the rate-maximizing problem. Theorem 4 shows that the solution with the smallest transmit power in the rate-maximizing problem is also optimal in the power-minimizing problem.

Remark on ZF Receiver: The derivations in Sections III and IV are given for the MMSE receiver. It can be shown that the duality between the power minimization and rate maximization problems also hold for the ZF case. For the MMSE case, we have used the results in Lemmas 1, 3, and 4 to prove the main results in Theorems 1–4. Lemma 2 is used in the proof of Lemmas 3 and 4. For the ZF case, if we decrease the transmit power for some k_0 subchannel, the error variance of the k_0 subchannel will be increased and the error variances of the other subchannels will be unchanged. Thus for the ZF case, Lemma 2 is no needed. Using the methods of MMSE case, we can prove the results in Lemmas 1, 3, 4, and also Theorems 1–4 for the ZF case.

V. Generalizations

The power minimization and rate maximization can be generalized in a number of ways. In this section we will discuss four possible generalizations: (1) the case when the error rate constraints of the subchannels are different; (2) the case when there is a constraint on the maximal constellation size for each subchannel; (3) the case when the subchannel BER constraint is used; and (4) the case when the averaged BER is constrained. For each case, we will see that the duality between the power minimization and rate maximization problems can be established with and without integer bit constraint.

Different Symbol Error Rate Constraints: In Sections 3 and 4, the error constraints of all the subchannels are the same. Suppose the error rate constraints of the subchannels are different, i.e., $\epsilon_k \leq \zeta_k$, where ϵ_k is the actual symbol error rate and ζ_k is the error rate constraint of the kth subchannel. Using the techniques in the proof of Lemma 3–4, we can show that the symbol error rate ϵ_k^* of the optimal solution is equal to ζ_k. Then it follows that the results in Theorem 1–4 continue to hold.

Constraint on the Constellation Size: Suppose there is a constraint on the maximal constellation size of each subchannel (cap constraint), i.e., $b_k \leq b_{\max}$. $b_{\max}$ is the maximal number of bits that can be loaded. First, let us consider the power minimization problem A_{pow} and the rate maximization problem A_{rate} (with real-valued bit allocation). Using the steps in the proof of Lemma 3, it can be verified that the results in Lemma 3 still hold for A_{pow}, but not for A_{rate}. Because of the constraint $b_k \leq b_{\max}$, the maximal bit rate of A_{rate} must be smaller than or equal to $Mb_{\max}$. Suppose the maximal bit rate is $Mb_{\max}$ when the power constraint is P_0. Then the maximal bit rate cannot be increased further even if we increase the power constraint. This means there could be many solutions that achieve the same maximal bit rate, but with different transmit power. For this reason, we will consider the solution for A_{rate} with the smallest transmit power. This is similar to the rate maximization problem with integer bit allocation discussed in Section IV. Following the technique used in Section IV, we can show that the solution of A_{pow} is equivalent to the solution of A_{rate} with the smallest transmit power.

For the two problems with integer bit allocation, $\mathsf{A}_{\text{pow,int}}$ and $\mathsf{A}_{\text{rate,int}}$, we can also impose a cap constraint. Using the steps in the proof of Lemma 4, we can show that the results in Lemma 4 still hold for $\mathsf{A}_{\text{pow,int}}$. For $\mathsf{A}_{\text{rate,int}}$, we also consider the solution with the smallest transmit power and hence we can obtain the duality described in Theorem 3-4 when the constellation size is constrained.

Subchannel Bit Error Rate Constraints: In Sections III and 4, the dualities are derived with symbol error rate constraint for QAM symbols. More generally, it can be shown that Lemmas 1, 3, and 4, are still valid if the error rate is a decreasing and continuous function of the so-called rate-normalized SNR $\gamma_k = \beta_k/(2^{b_k} - 1)$ [35]. Let us consider the case when BERs of the QAM symbols are constrained, i.e., $P_{e,k} \leq P_{e,\max}$ for $0 \leq k \leq M-1$. When $b_k \geq 2$ and the BER is smaller than 10^{-3}, it is demonstrated in [36] that the BER $P_{e,k}$ is well approximated by

$$P_{e,k} \approx 0.2 \exp\left(\frac{-1.6\beta_k}{2^{b_k} - 1}\right). \tag{13}$$

Since the BER function in (13) is a decreasing and continuous function of the rate-normalized SNR, the results in Lemmas 1, 3, and 4 continue to hold. Then following the same techniques in the proof of Theorem 1–4, we can reach the duality results given in Theorem 1–4 when the BER constraint is used.

Averaged BER Constraint: In Sections III–IV and the above discussions, error rate constraints are imposed on individual subchannels. Now, let us consider the design with a constraint on the maximal value of the averaged BER. The averaged bit error constraint is given by

$$P_{e,\text{avg}} = \frac{1}{B}\sum_{k=0}^{M-1} b_k P_{e,k} \leq P_{e,\max} \tag{14}$$

where $P_{e,\max}$ is the maximal value of the averaged BER. Using (13), we know $P_{e,k}$ is a decreasing and continuous function of the rate-normalized SNR and hence the results in Lemma 1 are valid. The results in Lemma 3–4 are valid if: (i) given bit allocation $\{b_k\}$, $P_{e,\text{avg}}$ is a decreasing and continuous function of $\{\beta_k\}$; and (ii) given $\{\beta_k\}$, a nonzero decrease in b_{k_0} for some k_0th subchannel implies a nonzero decrease in $P_{e,\text{avg}}$. Using (13), we can see that $P_{e,\text{avg}}$ satisfies these two conditions and hence the results in Lemmas 3 and 4 continue to hold and the same duality results in Theorem 1–4 can be obtained.

VI. Optimal Solution for Transceiver Design With Bit Allocation

There have been many papers on the topics of power minimization [11]–[17] and bit rate maximization [19]–[25]. For the power minimization problem with integer bit allocation, the solution has been found in [22]. There is no solution yet for the rate maximization problem with integer bit allocation. In Section VI-A, we will review the solution of A_{pow} and A_{rate} (no integer constraint on bit allocation). In Section VI-B, we will review the solution of $\mathsf{A}_{\text{pow,int}}$ and show how to find the solution of $\mathsf{A}_{\text{rate,int}}$ by the solution of $\mathsf{A}_{\text{pow,int}}$ using the duality derived in Section IV.

A. Review of the Optimal Solution for A_{pow} and A_{rate} [17], [22], [24]

Let the singular value decomposition of the $P \times N$ channel matrix $\mathbf{H}$ be

$$\mathbf{H} = \mathbf{U}\begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{V}^{\dagger} \tag{15}$$

where $\mathbf{\Lambda}$ is diagonal that contains the nonzero singular values of $\mathbf{H}$. The elements of $\mathbf{\Lambda}$ are in nonincreasing order. The $P \times P$ matrix $\mathbf{U}$ and the $N \times N$ matrix $\mathbf{V}$ are unitary. For the power-minimizing problem A_{pow} with target bit rate B_0 and error rate constraint ϵ, the solution is given by [17], [24]

$$\mathbf{F} = \mathbf{V}_1\mathbf{D}^{1/2} \tag{16}$$

where $\mathbf{V}_1$ contains the first M columns of $\mathbf{V}$ and $\mathbf{D}$ is a diagonal matrix with diagonal element $[\mathbf{D}]_{kk} = (\alpha - [\mathbf{\Lambda}]_{kk}^{-2} N_0 \Gamma)^+$, where $(x)^+ = \max(x, 0)$. The constant α is chosen such that $\sum_{k=0}^{M-1} b_k = B_0$.

The solution of rate-maximizing problem $\mathcal{A}_{\text{rate}}$ with power constraint P_0 and error rate constraint ϵ is given by [22], [24]

$$\mathbf{F} = \mathbf{V}_1 \mathbf{D}^{1/2} \tag{17}$$

where $\mathbf{D}$ is a diagonal matrix with diagonal element $[\mathbf{D}]_{kk} = (\beta - [\mathbf{\Lambda}]_{kk}^{-2} N_0 \Gamma)^+$. The constant β is chosen such that $\sum_{k=0}^{M-1} [\mathbf{D}]_{kk} = P_0$.

Note that the overall transfer function $\mathbf{T} = \mathbf{GHF}$ of the optimal MMSE solutions of $\mathcal{A}_{\text{pow}}$ and $\mathcal{A}_{\text{rate}}$ is a diagonal matrix that can be singular. Let $\mathbf{\Sigma}$ be a diagonal matrix, where $[\mathbf{\Sigma}]_{ii} = [\mathbf{T}]_{ii}^{-1}$ for $[\mathbf{T}]_{ii} \neq 0$ and $[\mathbf{\Sigma}]_{ii} = 0$ for $[\mathbf{T}]_{ii} = 0$. Then we have the ZF system if the receiver is given by $\mathbf{G}_{zf} = \mathbf{\Sigma G}$. It can be verified that the SNR of this ZF receiver is the same as that of the optimal MMSE solution. Thus the bit rate achieved by the ZF receiver $\mathbf{G}_{zf}$ is the same as that achieved by the optimal MMSE receiver. Therefore, for either $\mathcal{A}_{\text{pow}}$ or $\mathcal{A}_{\text{rate}}$ the optimal solution of the MMSE transceiver is the same as the ZF transceiver.

B. Optimal Solution of $\mathcal{A}_{\text{pow,int}}$ and $\mathcal{A}_{\text{rate,int}}$

First, we will review the optimal solution for $\mathcal{A}_{\text{pow,int}}$.

Review of Optimal Solution for $\mathcal{A}_{\text{pow,int}}$ [22]: Let us consider the case when MMSE receiver is used. The optimal power-minimizing transceiver can be found using [16] if the optimal integer bit allocation is given to us. However, we do not know optimal bit allocation beforehand. Nonetheless, for a given target bit rate B_0, there are only a finite number of possible integer bit allocation. In particular, $\{b_k\}$ is such that $b_k \in Z^+$ and

$$b_0 + b_1 + \cdots + b_{M-1} = B_0. \tag{18}$$

Let L be the number of such integer bit allocations $\{b_k\}$. We can compute L using [43], i.e., $L = (B_0 + M - 1)!/(B_0!(M-1)!)$. For each integer bit allocation $\{b_k\}$ that satisfies the condition in (18), the result in [16] can be used to find the transceiver that minimizes the transmit power. The optimal solution of $\mathcal{A}_{\text{pow,int}}$ can be obtained by choosing the integer bit allocation and transceiver that have the minimal transmit power among all the possible solutions. The optimal solution for $\mathcal{A}_{\text{pow,int}}$ can be obtained in no more than $LM^2(M+1)/6$ iterations [16]. The optimal transmitter is of the form $\mathbf{F} = \mathbf{V}_1 \mathbf{D}^{1/2} \mathbf{Q}$, where $\mathbf{Q}$ is an unitary matrix. Note that the optimal solution of $\mathcal{A}_{\text{pow,int}}$ does not have the diagonal structure as the case of real-valued bit allocation in Section VI-A. Thus the solution for the MMSE receiver is different from that for the ZF case.

Given any integer bit allocation, from [24] we know that the optimal ZF transceiver for $\mathcal{A}_{\text{pow,int}}$ has the same form as in (16). In this case, the error variance is $\sigma_{e_k}^2 = (N_0^{-1} [\mathbf{D}]_{kk} [\mathbf{\Lambda}]_{kk}^2 + 1)^{-1}$. Substituting the expression of $\sigma_{e_k}^2$ into (10), we have $[\mathbf{D}]_{kk} = N_0 \Gamma_k [\mathbf{\Lambda}]_{kk}^{-2} (2^{b_k} - 1)$, and hence the transmit power becomes

$$\sum_{k=0}^{M-1} [\mathbf{D}]_{kk} = \sum_{k=0}^{M-1} N_0 \Gamma_k [\mathbf{\Lambda}]_{kk}^{-2} (2^{b_k} - 1). \tag{19}$$

To minimize the transmit power in (19), it is shown in [26] that greedy algorithm can be used to find the optimal integer bit allocation.

Optimal Solution for $\mathcal{A}_{\text{rate,int}}$: For the rate-maximizing problem $\mathcal{A}_{\text{rate,int}}$ with power constraint P_0, if the maximal rate $B_{\text{int}}^*(P_0)$ is known, we can solve it using the solution of $\mathcal{A}_{\text{pow,int}}$ based on Theorem 3. We can find $B_{\text{int}}^*(P_0)$ using an iterative search. For example, starting from $B_0 = 1$ we compute $P_{\text{int}}^*(B_0)$. If $P_{\text{int}}^*(B_0) \le P_0$, we increase B_0 by one and compute $P_{\text{int}}^*(B_0)$ again until $P_{\text{int}}^*(B_0) > P_0$. Then $B_{\text{int}}^*(P_0) = B_0 - 1$. To reduce the number of iterations we note that $B_{\text{int}}^*(P_0) \le B^*(P_0)$, where $B^*(P_0)$ is the maximal bit rate of the rate maximization problem $\mathcal{A}_{\text{rate}}$ without integer bit constraint. As a result, $B_{\text{int}}^*(P_0) \le \lfloor B^*(P_0) \rfloor$, where the notation $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x. Using this property and Theorem 3 we have the following algorithm.

Algorithm for Finding the Solution of $\mathcal{A}_{\text{rate,int}}$:

1) Initially, given the power constraint P_0, compute the maximal bit rate $B^*(P_0)$ for $\mathcal{A}_{\text{rate}}$. Then set $B_0 = \lfloor B^*(P_0) \rfloor$.
2) Given the target bit rate B_0, find the optimal bit allocation and transceiver for minimizing transmit power in $\mathcal{A}_{\text{pow,int}}$. Compute the minimal power $P_{\text{int}}^*(B_0)$.
3) If $P_{\text{int}}^*(B_0) > P_0$, set $B_0 = B_0 - 1$ and go to step 2. If $P_{\text{int}}^*(B_0) \le P_0$, then the maximal bit rate $B_{\text{int}}^*(P_0) = B_0$.

In this algorithm, the number of iterations is equal to $\lfloor B^*(P_0) \rfloor - B_{\text{int}}^*(P_0)$. This number is in fact less than M as we explain below. Suppose $\lfloor B^*(P_0) \rfloor - B_{\text{int}}^*(P_0) \ge M$. Let $\{b_k^*\}$ be the optimal real-valued bit allocation of $\mathcal{A}_{\text{rate}}$, i.e., $B^*(P_0) = \sum_{k=0}^{M-1} b_k^*$. Then $\{\lfloor b_k^* \rfloor\}$ is also a valid integer bit allocation that satisfies the error rate constraint. Since $\{b_k^*\}$ is real, we have $\lfloor B^*(P_0) \rfloor - \sum_{k=0}^{M-1} \lfloor b_k^* \rfloor \le B^*(P_0) - \sum_{k=0}^{M-1} \lfloor b_k^* \rfloor < M$. This implies $\sum_{k=0}^{M-1} \lfloor b_k^* \rfloor > B_{\text{int}}^*(P_0)$, which contradicts the definition of $B_{\text{int}}^*(P_0)$. Therefore we have $\lfloor B^*(P_0) \rfloor - B_{\text{int}}^*(P_0) < M$. Note the number M is an upper bound of the number of iterations. As the optimal solution for $\mathcal{A}_{\text{rate,int}}$ is obtained using the solution of $\mathcal{A}_{\text{pow,int}}$, the optimal solution of the MMSE receiver is different from that of the ZF receiver.

VII. Simulation

In the simulations, we will demonstrate the duality between the power-minimizing problem and rate-maximizing problem. In the following examples, the number of subchannels M is 4. The noise vector $\mathbf{q}$ is assumed to be complex white Gaussian with $\mathrm{E}[\mathbf{q}\mathbf{q}^\dagger] = \mathbf{I}_4$. The symbol error rate constraint ϵ is assumed to be 10^{-4}. In examples 1, we use a fixed 4×4 MIMO channel. In examples 2–3, the results are averaged over random channels. For the problems $\mathcal{A}_{\text{pow}}$ and $\mathcal{A}_{\text{rate}}$, we use the solutions in Section VI-A. For $\mathcal{A}_{\text{pow,int}}$ and $\mathcal{A}_{\text{rate,int}}$, we use the solutions in Section VI-B.

TABLE I
(A) MINIMAL POWER $P(B_0)$ FOR $\mathcal{A}_{\text{pow}}$ WHEN $B_0 = 2, 4, 6, 8, 10$ BITS. (B) MAXIMAL BIT RATE FOR $\mathcal{A}_{\text{rate}}$ WHEN THE POWER CONSTRAINT $P_0 = P(B_0)$

(a)

B_0 (bits)	$P^*(B_0)$ (dB)
2	3.2833
4	8.0001
6	11.1790
8	13.7891
10	16.1370

(b)

$P_0 = P^*(B_0)$ (dB)	$B^*(P_0)$ (bits)
3.2833	2
8.0001	4
11.1790	6
13.7891	8
16.1370	10

TABLE II
(A) MAXIMAL BIT RATE FOR $\mathcal{A}_{\text{rate}}$ WHEN $P_0 = 2, 4, 8, 16, 32$ DB. (B) MINIMAL POWER $P(B_0)$ FOR $\mathcal{A}_{\text{pow}}$ WHEN $B_0 = B(P_0)$

(a)

P_0 (dB)	$B^*(P_0)$ (bits)
2	1.6108
4	2.2447
8	3.9999
16	9.8792
32	28.7068

(b)

$B_0 = B^*(P_0)$ (bits)	$P^*(B_0)$ (dB)
1.6108	2
2.2447	4
3.9999	8
9.8792	16
28.7068	32

Example 1. Duality Between $\mathcal{A}_{\text{pow}}$ and $\mathcal{A}_{\text{rate}}$: In this example, we will demonstrate the results in Theorem 1 and Theorem 2. Consider a 4×4 channel $\mathbf{H}$ that is given by

$$\begin{pmatrix} -0.5+0.6i & -0.5-1.1i & 0.2-0.2i & 0.4-0.5i \\ -0.3+0.6i & -0.2+1.4i & -0.4+0.9i & 0.8-0.5i \\ -0.1+0.5i & -0.4-0.3i & 0.9+0.3i & 0.1+0.2i \\ 1.1+0.6i & -0.5+0.4i & 0.0-0.2i & -0.3+1.4i \end{pmatrix}. \tag{20}$$

As we have discussed in Section VI-A, for both $\mathcal{A}_{\text{pow}}$ and $\mathcal{A}_{\text{rate}}$, the solution for the MMSE receiver is the same as that for the ZF receiver. Given target bit rate B_0, we use (16) to find the optimal transceiver, and (1) to compute the corresponding transmit power $P^*(B_0)$ for the problem $\mathcal{A}_{\text{pow}}$. Table I(a) shows the minimal transmit power $P^*(B_0)$ when the target bit rates are $B_0 = 2, 4, 6, 8, 10$ bits. Using the minimized power in Table I(a) as power constraint, Table I(b) shows the maximal bit rate for the rate maximizing problem $\mathcal{A}_{\text{rate}}$. The rates are computed using II for the optimal transceiver in (17). We can see that $B^*(P^*(B_0)) = B_0$ and the solution of the power-minimizing problem is also optimal for the rate-maximizing problem as we have shown in Theorem 1.

Table II(a) shows the maximal bit rate $B^*(P_0)$ for $\mathcal{A}_{\text{rate}}$ when the power constraints are $P_0 = 2, 4, 8, 16, 32$ dB. Table II(b) shows the minimal power $P^*(B_0)$ for the problem $\mathcal{A}_{\text{pow}}$ when the target bit rates are equal to the maximized rate in Table II(a). We can see that $P^*(B^*(P_0)) = P_0$ and the solution of rate-maximizing problem is also a solution of the power-minimizing problem as we have shown in Theorem 2.

Example 2. Duality Between $\mathcal{A}_{\text{pow}}$ and $\mathcal{A}_{\text{rate}}$: In this example, we use random channels to demonstrate the connections between power minimization and rate maximization problems. The channel is of size 4×4 and the elements are complex Gaussian random variables whose real and imaginary parts are independent with zero mean and variance 1/2. The following numerical results are generated by averaging the maximal bit rate and minimal transmit power over 10^6 channel realizations. For each channel realization, we compute the optimal solutions of $\mathcal{A}_{\text{pow}}$ and $\mathcal{A}_{\text{rate}}$ using (16) and (17) in Section VI-A. Fig. 3 shows the averaged maximal transmission rates $B^*(P_0)$ of $\mathcal{A}_{\text{rate}}$ as a function of power constraint. Fig. 4 shows

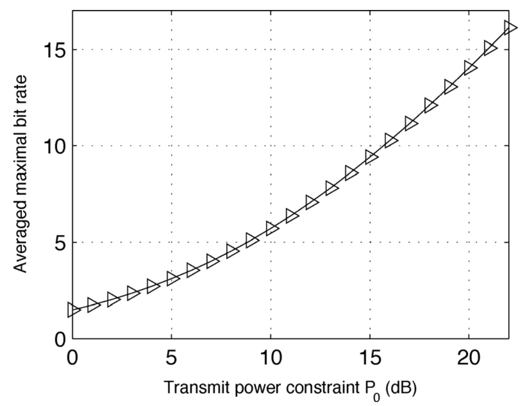

Fig. 3. Averaged maximal bit rate $B^*(P_0)$ for $\mathcal{A}_{\text{rate}}$ as a function of power constraint P_0 without integer constraint.

the averaged minimal transmit power $P^*(B_0)$ of $\mathcal{A}_{\text{pow}}$ as a function of target bit rate. We can observe the duality between that the power-minimizing and rate-maximizing problems from Figs. 3 and 4. For example, the minimal power of $\mathcal{A}_{\text{pow}}$ is 9 dB when the target bit rate is 5 bits. When we set the power constraint in $\mathcal{A}_{\text{rate}}$ to be 9 dB, the maximal bit rate is 5 bits. On the other hand, the maximal bit rate of $\mathcal{A}_{\text{rate}}$ is 9 bits when the power constraint is 15 dB. When we set the target bit rate in $\mathcal{A}_{\text{pow}}$ to be 9 bits, the minimal power is 15 dB.

Example 3. Minimal Power for $\mathcal{A}_{\text{pow,int}}$ and Maximal Bit Rate for $\mathcal{A}_{\text{rate,int}}$: We use the same random channel as in example 2. In Table III, we compute the minimal transmit power of $\mathcal{A}_{\text{pow,int}}$ with MMSE and ZF receivers. When the target bit rate is B_0, the minimal transmit powers of the MMSE case and the ZF case are denoted by $P^*_{\text{int,mmse}}(B_0)$ and $P^*_{\text{int,zf}}(B_0)$, respectively. For comparison, we also show the transmit power $P^*(B_0)$ of $\mathcal{A}_{\text{pow}}$ (without integer constraint). The notation $P^*(B_0)$ does not have a subscript (MMSE or ZF) because the solutions for $\mathcal{A}_{\text{pow}}$ using the MMSE receiver and ZF receiver are the same. We can see that the gap between $P^*_{\text{int,mmse}}(B_0)$ and $P^*_{\text{int,zf}}(B_0)$ is small. Also, the difference between $P^*_{\text{int,mmse}}(B_0)$ and $P^*(B_0)$ is smaller than 0.21 dB. In Table IV, we compute the maximal bit rate of $\mathcal{A}_{\text{rate,int}}$ for the MMSE and ZF receivers. The maximal bit rate for the MMSE and ZF cases are denoted, respectively, by $B^*_{\text{int,mmse}}(P_0)$ and

TABLE III
TRANSMIT POWER OF $\mathcal{A}_{\text{pow}}$ (WITHOUT INTEGER BIT ALLOCATION), $\mathcal{A}_{\text{pow,int}}$ (ZF), AND $\mathcal{A}_{\text{pow,int}}$ (MMSE) WHEN THE TARGET BIT RATE IS $B_0 = 2, 4, 6, 8, 10, 12$ BITS

B_0 (bits)	$P^*(B_0)$ (dB)	$P^*_{int,zf}(B_0)$ (dB)	$P^*_{int,mmse}(B_0)$ (dB)
2	2.1167	2.3322	2.3254
4	7.1634	7.3107	7.3078
6	10.6614	10.7832	10.7826
8	13.4708	13.5796	13.5788
10	15.9114	16.0175	16.0170
12	18.1311	18.2314	18.2312
14	20.1976	20.2972	20.2970

TABLE IV
BIT RATE OF $\mathcal{A}_{\text{rate}}$ (WITHOUT INTEGER BIT ALLOCATION), $\mathcal{A}_{\text{rate,int}}$ (ZF), AND $\mathcal{A}_{\text{rate,int}}$ (MMSE) WHEN THE POWER CONSTRAINT IS $P_0 = 2, 4, 6, 8, 10, 12, 14, 16$ dB

P_0 (dB)	$B^*(P_0)$ (bits)	$B^*_{int,zf}(P_0)$ (bits)	$B^*_{int,mmse}(P_0)$ (bits)
2	2.0305	1.4549	1.4572
4	2.7096	2.1549	2.1557
6	3.5429	2.9858	2.9865
8	4.5391	3.9689	3.9700
10	5.7103	5.1326	5.1333
12	7.0629	6.4788	6.4794
14	8.5888	8.0033	8.0037
16	10.2714	9.6811	9.6815

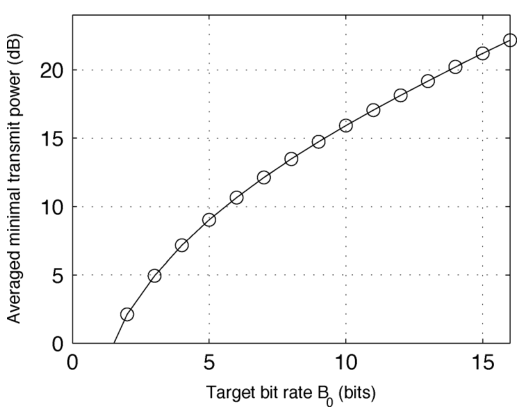

Fig. 4. Averaged minimal transmit power $P^*(B_0)$ for $\mathcal{A}_{\text{pow}}$ as a function of target bit rate B_0 without integer constraint.

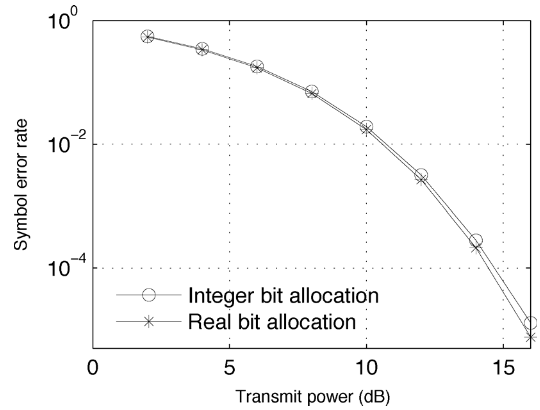

Fig. 5. Symbol error rate versus the transmit power.

$B^*_{\text{int,zf}}(P_0)$. Also shown in Table IV is the maximal bit rate $B^*(P_0)$ of $\mathcal{A}_{\text{rate}}$ (without integer constraint). The notation $B^*(P_0)$ does not have a subscript because the solutions for $\mathcal{A}_{\text{rate}}$ using the MMSE receiver and ZF receiver are the same. We can see that $B^*_{\text{int,zf}}(P_0)$ is close to $B^*_{\text{int,mmse}}(P_0)$. The difference between $B^*_{\text{int,mmse}}(P_0)$ and $B^*(P_0)$ is smaller than 0.6 bits. This gap is less than 0.15 bits per symbol.

Example 4: In this example, we use the same random channel as in example 2. Fig. 5 shows the symbol error rates versus the transmit power. The results are averaged over 10^6 channel realizations. The transmission rate B_0 is equal to 8 bits. There are two curves shown in Fig. 5. One is the case with real bit allocation (Section VI-A) and another is the case when integer bit constraint is considered (Section VI-B). For the case with integer bit allocation, we have used the MMSE receiver. We can see that the gap between these two curves is very small, which means the use of integer bit allocation only leads to a minor performance degradation.

VIII. CONCLUSION

In this paper, we considered two commonly used transceiver design criteria: power minimization criterion and rate maximization criterion. The duality between these two problems was derived without using any existing solutions. As a result, we can also establish the duality for the rate maximization problem with integer bit allocation, which the optimal solution has not been found yet. Using the duality, the optimal solution of the rate maximization problem with integer bit constraint can be found. We have also considered four possible generalizations: 1) the symbol error rate constraints are different for all the subchannels; 2) there is a constraint on the maximal constellation size; 3) subchannel BER constraints are used; and 4) averaged BER constraint is used. For all the generalizations, we have established the duality between these two problems. In the simulations, the duality between these two problems has been demonstrated.

APPENDIX A
PROOF OF LEMMA 1

Consider the system with the transmitter $\tilde{\mathbf{F}} = \alpha\mathbf{F}$, where $\mathbf{F}$ is some $N \times M$ matrix such that $\mathbf{HF} \neq \mathbf{0}$. Suppose the error rate is fixed to be ϵ for all the subchannels. The transmit power is given by

$$\mathrm{Tr}(\tilde{\mathbf{F}}\tilde{\mathbf{F}}^\dagger) = \alpha^2\mathrm{Tr}(\mathbf{F}\mathbf{F}^\dagger). \tag{21}$$

So the transmit power is a continuous and strictly increasing function of α. Next we will show the achievable bit rate is also a continuous and strictly increasing function in terms of α. The MSE matrix of the system is $\tilde{\mathbf{E}} = [\alpha^2 N_0^{-1}\mathbf{F}^\dagger\mathbf{H}^\dagger\mathbf{HF} + \mathbf{I}_M]^{-1}$. It follows that $\tilde{\sigma}_{e_k}^2 = [\tilde{\mathbf{E}}]_{kk}$ is a continuous function of α and so is $\tilde{\beta}_k = 1/\tilde{\sigma}_{e_k}^2 - 1$. Using the expression of the symbol error rate in (6), we can see $\tilde{b}_k$ is also a continuous function of α. Suppose $\alpha_1 < \alpha_2$. Let $\tilde{\mathbf{E}}_1$ be the MSE matrix when α_1 is used and $\tilde{\mathbf{E}}_2$ be the MSE matrix when α_2 is used. Since $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$ are positive definite matrices and $\tilde{\mathbf{E}}_2^{-1} - \tilde{\mathbf{E}}_1^{-1} > 0$, then we have $\tilde{\mathbf{E}}_1 - \tilde{\mathbf{E}}_2 > 0$ [40]. As a result, the total bit rate achieved by α_2 is larger than that achieved by α_1. Hence the bit rate of the system is a continuous and strictly increasing function of α. $\triangle\triangle\triangle$

APPENDIX B
PROOF OF LEMMA 2

Given the channel matrix $\mathbf{H}$ and the transmitter $\mathbf{FD}$, the MSE matrix becomes $\tilde{\mathbf{E}} = (N_0^{-1}\mathbf{D}\mathbf{F}^\dagger\mathbf{H}^\dagger\mathbf{HFD} + \mathbf{I}_M)^{-1}$. The noise variance $\tilde{\sigma}_{e_l}^2$ is given by

$$\tilde{\sigma}_{e_l}^2 = [\tilde{\mathbf{E}}]_{ll} = \left[\mathbf{D}^{-1}\left(N_0^{-1}\mathbf{F}^\dagger\mathbf{H}^\dagger\mathbf{HF} + \mathbf{D}^{-2}\right)^{-1}\mathbf{D}^{-1}\right]_{ll}. \tag{22}$$

The derivative of $\tilde{\sigma}_{e_l}^2$ with respect to μ is

$$\frac{\partial\tilde{\sigma}_{e_l}^2}{\partial\mu} = \frac{\partial[\tilde{\mathbf{E}}]_{ll}}{\partial\mu}. \tag{23}$$

Define $\mathbf{B} = (N_0^{-1}\mathbf{F}^\dagger\mathbf{H}^\dagger\mathbf{HF} + \mathbf{D}^{-2})^{-1}$. For $l \neq k_0$, we have

$$\frac{\partial[\tilde{\mathbf{E}}]_{ll}}{\partial\mu} = \left[\frac{\partial\mathbf{B}}{\partial\mu}\right]_{ll}. \tag{24}$$

The derivative of $\mathbf{B}$ with respect to μ is [39]

$$\frac{\partial\mathbf{B}}{\partial\mu} = -\mathbf{B}\frac{\partial\mathbf{B}^{-1}}{\partial\mu}\mathbf{B}^\dagger. \tag{25}$$

Using the definition of $\mathbf{B}$, we have

$$\frac{\partial\mathbf{B}^{-1}}{\partial\mu} = \begin{bmatrix} 0 & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & -2\mu^{-3} & & \vdots \\ & & & \ddots & \\ 0 & & \cdots & & 0 \end{bmatrix}. \tag{26}$$

So we can obtain

$$\frac{\partial\mathbf{B}}{\partial\mu} = 2\mu^{-3}\mathbf{b}_{k_0}\mathbf{b}_{k_0}^\dagger \tag{27}$$

where $\mathbf{b}_{k_0}$ is the k_0th column of $\mathbf{B}$. Using (24) and (27), for $l \neq k_0$ we have

$$\frac{\partial\tilde{\sigma}_{e_l}^2}{\partial\mu} = \left[2\mu^{-3}\mathbf{b}_{k_0}\mathbf{b}_{k_0}^\dagger\right]_{ll} = 2\mu^{-3}|[\mathbf{B}]_{lk_0}|^2 \geq 0. \tag{28}$$

Thus we conclude that $\tilde{\sigma}_{e_l}^2$ is an increasing function of μ for $l \neq k_0$. For $l = k_0$, we have

$$\frac{\partial\tilde{\sigma}_{e_{k_0}}^2}{\partial\mu} = \left[\frac{\partial(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1})}{\partial\mu}\right]_{k_0k_0} = \left[\frac{\partial(\mu^{-2}\mathbf{B})}{\partial\mu}\right]_{k_0k_0}. \tag{29}$$

Using chain rule and $\tilde{\sigma}_{e_{k_0}}^2 = \mu^{-2}[\mathbf{B}]_{k_0k_0}$, we can obtain

$$\frac{\partial\tilde{\sigma}_{e_{k_0}}^2}{\partial\mu} = -2\mu^{-3}[\mathbf{B}]_{k_0k_0} + 2\mu^{-5}[\mathbf{B}]_{k_0k_0}^2 \tag{30}$$

$$= -2\mu^{-1}\tilde{\sigma}_{e_{k_0}}^2\left(-1 + \tilde{\sigma}_{e_{k_0}}^2\right). \tag{31}$$

Since $\tilde{\mathbf{E}}^{-1} - \mathbf{I}_M \geq 0$, from [40] we know $\mathbf{I}_M - \tilde{\mathbf{E}} \geq 0$. Then we have $\tilde{\sigma}_{e_{k_0}}^2 \leq 1$ and thus $\frac{\partial\tilde{\sigma}_{e_{k_0}}^2}{\partial\mu} \leq 0$. As a result, we can conclude that $\tilde{\sigma}_{e_{k_0}}^2$ is a decreasing function of μ. $\triangle\triangle\triangle$

APPENDIX C
PROOF OF LEMMA 3

Equalities Hold in the Power-Minimizing Problem: Suppose $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathcal{A}_{\text{pow}}$ and the minimized power is P^*. Let $\{\epsilon_k^*\}$ and B^* be the symbol error rates and bit rate achieved by the optimal solution. Then we have $\epsilon_k^* \leq \epsilon$ and $B^* \geq B_0$. First we show that $\epsilon_k^* = \epsilon$ for all k. Suppose the error rate of the k_0th subchannel is $\epsilon_{k_0}^* < \epsilon$. Consider a new system with the same bit allocation $\{b_k^*\}$, but transmitter is changed to

$$\tilde{\mathbf{F}} = \mathbf{F}^*\mathbf{D} \tag{32}$$

where $\mathbf{D}$ is a diagonal matrix as defined in Lemma 2. k_0 is some subchannel index and $0 < \mu < 1$, to be chosen later. Using (1), the new transmit power $\tilde{P}$ is

$$\tilde{P} = \sum_{k=0}^{M-1}[\mathbf{F}^{*\dagger}\mathbf{F}^*]_{kk}[\mathbf{D}]_{kk}^2 < P^*. \tag{33}$$

The bit rate of the new system is still B^* as bit allocation is not changed. Next we will show that there always exists $0 < \mu < 1$ such that all the error rate constraints will be satisfied, i.e.

$$\tilde{\epsilon}_k = 4\left(1 - \frac{1}{2^{b_k^*/2}}\right)Q\left(\sqrt{\frac{3}{(2^{b_k^*} - 1)\tilde{\sigma}_{e_k}^2}}\right) \leq \epsilon \quad \text{for } 0 \leq k \leq M-1 \tag{34}$$

where $\tilde{\sigma}^2_{e_k}$ is the error variance for the kth subchannel of the new system. Using Lemma 2, when $\mu < 1$ we have

$$\tilde{\sigma}^2_{e_k} \leq \sigma^{*2}_{e_k}, \quad \text{for } k \neq k_0 \tag{35}$$

which implies $\tilde{\epsilon}_k \leq \epsilon^*_k \leq \epsilon$ for $k \neq k_0$. For $k = k_0$, we rearrange the inequality in (34), and the error rate constraint for the k_0th subchannel can be rewritten as

$$\tilde{\sigma}^2_{e_{k_0}} \leq \gamma \tag{36}$$

where

$$\gamma = 1 \Bigg/ \left(\left(\frac{2^{b^*_{k_0}} - 1}{3} \right) \left[Q^{-1} \left(\frac{\epsilon}{4\left(1 - 2^{b^*_{k_0}/2}\right)} \right) \right]^2 \right).$$

Choose $\mu = \sqrt{\sigma^{*2}_{e_{k_0}}/\gamma}$, where $\sigma^{*2}_{e_{k_0}}$ is the error variance for the k_0th subchannel of the optimal solution. Since $\epsilon^*_{k_0} < \epsilon$, we have $\sigma^{*2}_{e_{k_0}} < \gamma$, which implies that $\mu < 1$. The noise variance $\tilde{\sigma}^2_{e_{k_0}}$ is given by

$$\tilde{\sigma}^2_{e_{k_0}} = [\tilde{\mathbf{E}}]_{k_0 k_0} = [\mathbf{B}]_{k_0 k_0}/\mu^2 = \gamma [\mathbf{B}]_{k_0 k_0}/\sigma^{*2}_{e_{k_0}} \tag{37}$$

where $\mathbf{B} = (N_0^{-1}\mathbf{F}^\dagger \mathbf{H}^\dagger \mathbf{H}\mathbf{F} + \mathbf{D}^{-2})^{-1}$. Since $\mathbf{B}^{-1} - \mathbf{E}^{*-1} \geq 0$, from [40] we have $\mathbf{E}^* - \mathbf{B} \geq 0$ and hence $\sigma^{*2}_{e_{k_0}} \geq [\mathbf{B}]_{k_0 k_0}$. Therefore, we have $\tilde{\sigma}^2_{e_{k_0}} \leq \gamma$, which implies $\tilde{\epsilon}_{k_0} \leq \epsilon$. With this choice of α, the new system $(\tilde{\mathbf{F}}, \{b^*_k\})$ can achieve a smaller transmit power $\tilde{P} < P^*$ and still satisfy all the constraints in A_{pow}. This contradicts the assumption that P^* is the minimal power when B_0 is given. Hence we have that $\epsilon^*_k = \epsilon$ for all k. Next we prove that the bit rate B^* is equal to B_0. Suppose $B^* > B_0$. Consider a new system with transmitter $\tilde{\mathbf{F}} = \alpha \mathbf{F}^*$, where $\alpha > 0$ is a scalar. For the target error rate ϵ, we know from Lemma 1 that the bit rate of the new system is a strictly increasing and continuous function of α. So we can properly choose $\alpha < 1$ such that the new bit rate $\tilde{B} = B_0$. In this case, the required power is smaller than P^*. This contradicts the assumption that P^* is the minimal power when B_0 is given, so $B^* = B_0$.

Equalities Hold in the Rate-Maximizing Problem: Suppose $(\mathbf{F}^*, \{b^*_k\})$ is optimal for A_{rate} and the maximized bit rate is B^*. Let P^* and $\{\epsilon^*_k\}$ be the transmit power and error rates of the optimal solution. Then we have $P^* \leq P_0$ and $\epsilon^*_k \leq \epsilon$. First we show that $\epsilon^*_k = \epsilon$ for all k. Suppose for the k_0th subchannel, $\epsilon^*_{k_0} < \epsilon$. From (6), we know that the error rate ϵ_k is a continuous and increasing function of the number of bits allocated when β_k is fixed. For the same $\mathbf{F}^*$, we can increase the number of bits allocated to the k_0th subchannel such that the new error rate $\tilde{\epsilon}_{k_0}$ satisfies

$$\epsilon^*_{k_0} < \tilde{\epsilon}_{k_0} \leq \epsilon. \tag{38}$$

The error rates of other subchannels are not affected while a higher bit rate be achieved. This contradicts the assumption that B^* is the maximal bit rate when the power constraint P_0 is given. Hence we have that $\epsilon^*_k = \epsilon$ for all k. Now let us show $P^* = P_0$. Suppose $P^* < P_0$. Consider the new system with transmitter $\tilde{\mathbf{F}} = \alpha \mathbf{F}^*$, where $\alpha = \sqrt{P_0/P^*} > 1$. The power of the new system is $\tilde{P} = P_0$. From Lemma 1, the bit rate of the new system is a strictly increasing function of α, and we have $\tilde{B} > B^*$. This contradicts the assumption that B^* is the maximal bit rate when the power constraint P_0 is given, so we obtain the conclusion that $P^* = P_0$. $\triangle\triangle\triangle$

Appendix D
Proof of Lemma 4

Suppose $(\mathbf{F}^*, \{b^*_k\})$ is optimal for $\mathsf{A}_{\text{pow,int}}$. Let ϵ^*_k be the error rate on the kth subchannel of the optimal system. Then ϵ^*_k is given by

$$\epsilon^*_k = 4\left(1 - \frac{1}{2^{b^*_k/2}}\right) Q\left(\sqrt{\frac{3\beta^*_k}{(2^{b^*_k} - 1)}}\right) \tag{39}$$

where $\beta^*_k = 1/\sigma^{*2}_{e_k} - 1$ as the receiver is MMSE. The minimized power P^* is given by $P^* = \sum_{k=0}^{M-1} [\mathbf{F}^{*\dagger}\mathbf{F}^*]_{kk}$. The bit rate B^* is $B^* = \sum_{k=0}^{M-1} b^*_k$. Using the technique in the proof of Lemma 3, we can show that $\epsilon^*_k = \epsilon$ for all k. But the technique does not work for property $B^* = B_0$ and a different proof is needed. Suppose

$$B^* > B_0 \tag{40}$$

Suppose $b^*_{k_0} > 0$ for some k_0th subchannel. Consider a new system with the bit allocation changed to

$$\tilde{b}_k = \begin{cases} b^*_{k_0} - 1, & k = k_0, \\ b^*_k, & \text{otherwise,} \end{cases} \tag{41}$$

and the transmitter changed to $\tilde{\mathbf{F}} = \mathbf{F}^*\mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. $[\mathbf{D}]_{ll} = 1$ for $l \neq k_0$ and $[\mathbf{D}]_{ll} = \mu$ for $l = k_0$. $0 < \mu < 1$ is a positive real number to be chosen later. The bit rate of the new system is $\tilde{B} = B^* - 1 \geq B_0$. The transmit power $\tilde{P}$ of the new system is smaller than P^* because

$$\tilde{P} = \sum_{k=0}^{M-1} [\tilde{\mathbf{F}}^\dagger \tilde{\mathbf{F}}]_{kk} < \sum_{k=0}^{M-1} [\mathbf{F}^{*\dagger}\mathbf{F}^*]_{kk} = P^*. \tag{42}$$

Next, we will show that with appropriate choice of α, the error rate $\tilde{\epsilon}_k$ of the new system still satisfies the error rate constraint in $\mathsf{A}_{\text{pow,int}}$. Using (6), $\tilde{\epsilon}_k$ can be expressed as

$$\tilde{\epsilon}_k = 4\left(1 - \frac{1}{2^{\tilde{b}_k/2}}\right) Q\left(\sqrt{\frac{3\tilde{\beta}_k}{(2^{\tilde{b}_k} - 1)}}\right) \tag{43}$$

$$\leq 4\left(1 - \frac{1}{2^{b^*_k/2}}\right) Q\left(\sqrt{\frac{3\tilde{\beta}_k}{(2^{\tilde{b}_k} - 1)}}\right) \tag{44}$$

where $\tilde{\beta}_k = 1/\tilde{\sigma}^2_{e_k} - 1$. Observe that the symbol error rate $\tilde{\epsilon}_k$ of the new system will be smaller than ϵ^*_k if the quantity in the Q function of (44) is larger than or equal to that in the Q function of (39), i.e.

$$\frac{1}{2^{\tilde{b}_k} - 1}\left(\frac{1}{\tilde{\sigma}^2_{e_k}} - 1\right) \geq \frac{1}{2^{b^*_k} - 1}\left(\frac{1}{\sigma^{*2}_{e_k}} - 1\right), \quad \forall k. \tag{45}$$

When $\mu < 1$, using Lemma 2 we have

$$\tilde{\sigma}_{e_k}^2 \leq \sigma_{e_k}^{*2}, \quad \text{for } k \neq k_0. \tag{46}$$

Using (41) and (46), we have

$$\frac{1}{2^{\tilde{b}_k} - 1}\left(\frac{1}{\tilde{\sigma}_{e_k}^2} - 1\right) \geq \frac{1}{2^{b_k^*} - 1}\left(\frac{1}{\sigma_{e_k}^{*2}} - 1\right) \quad \text{for } k \neq k_0 \tag{47}$$

which implies $\tilde{\epsilon}_k < \epsilon_k^* = \epsilon$ for $k \neq k_0$. For $k = k_0$, we can always find $\mu < 1$ such that (45) is satisfied. For example, we can choose

$$\mu = \sqrt{\frac{1}{\beta_{k_0}^* + 1}\left(1 + \frac{2^{\tilde{b}_{k_0}} - 1}{2^{b_{k_0}^*} - 1}\beta_{k_0}^*\right)}. \tag{48}$$

It can be verified that $1 - \mu^2 = \frac{\beta_{k_0}^*}{\beta_{k_0}^* + 1}(1 - \frac{2^{\tilde{b}_{k_0}} - 1}{2^{b_{k_0}^*} - 1}) > 0$, and thus $\mu < 1$. In this case, we have

$$\frac{1}{\tilde{\sigma}_{e_{k_0}}^2} = \frac{\mu^2}{[\mathbf{B}]_{k_0 k_0}} = \frac{\sigma_{e_{k_0}}^{*2}}{[\mathbf{B}]_{k_0 k_0}}\left(1 + \frac{2^{\tilde{b}_{k_0}} - 1}{2^{b_{k_0}^*} - 1}\beta_{k_0}^*\right) \tag{49}$$

$$\geq \left(1 + \frac{2^{\tilde{b}_{k_0}} - 1}{2^{b_{k_0}^*} - 1}\beta_{k_0}^*\right) \tag{50}$$

where $\mathbf{B} = (N_0^{-1}\mathbf{F}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{F} + \mathbf{D}^{-2})^{-1}$. The last inequality comes from the fact that $\sigma_{e_{k_0}}^{*2} \geq [\mathbf{B}]_{k_0 k_0}$. Rearranging (50), we can see that (45) is satisfied for $k = k_0$. Therefore, we have $\tilde{\epsilon}_k < \epsilon_k^* = \epsilon$ for all k. This means $(\tilde{\mathbf{F}}, \{\tilde{b}_k\})$ can achieve a smaller transmit power and still satisfies all the constraints in $\mathsf{A}_{\text{pow,int}}$. This contradicts the assumption that $(\mathbf{F}^*, \{b_k^*\})$ is optimal for $\mathsf{A}_{\text{pow,int}}$. Hence the total bit rate B^* of the optimal solution must be equal to B_0. $\triangle\triangle\triangle$

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