HYDROLOGICAL PROCESSES
Hydrol. Process. **27**, 319–323 (2013)
Published online 13 March 2012 in Wiley Online Library
(wileyonlinelibrary.com) DOI: 10.1002/hyp.8447

Variability of volume strain in bounded heterogeneous media

Ching-Min Chang and Hund-Der Yeh*
Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan

Abstract:

The deformation of the solid matrix affects the fluid pore pressure and flow by altering the pore volume. Such interaction in turn affects the storage of groundwater in the void space. Obviously, this subject is of interest in groundwater hydrology. This paper describes an investigation of the effect of aquifer heterogeneity on the variability of the fluid pressure head and solid's volume strain, where the assumption of a constant vertical total stress leads to a relatively simple relationship between changes in solid's volume strain and fluid pressure head. To solve the problem analytically, focus is placed on the one-dimensional models. It is found from our closed-form solutions that the variance and correlation length of the log hydraulic conductivity are important in increasing the variability of pressure head and solid's volume strain. It is hoped that our findings will provide a basic framework for understanding and quantifying field-scale volume strain processes and be useful in stimulating further research in this area.

KEY WORDS stochastic analysis; volume strain; bounded heterogeneous media

Received 13 July 2011; Accepted 16 November 2011

INTRODUCTION

Fluctuations in pore groundwater pressure in response to the changes in imposed stresses are often encountered in many practical problems of subsurface flow. In general, the porous medium is deformable. Such interaction will cause the deformation of the solid matrix, which in turn affects the storage of groundwater in the void space. In addition, under certain conditions, the deformation may reach an extent that it manifests itself as land subsidence (e.g. Yeh *et al*., 1996). Thus, the assessment of the variability of the poroelastic response of the medium is essential for the planning and management of groundwater resources in aquifers.

It is well known that the heterogeneity of the medium plays an important role in influencing the behavior of groundwater flow at field scale. Many practical problems of subsurface flow require predictions over relative large space scale, where a wide range of formation heterogeneities are included in the flow domain. Therefore, there arises a need to incorporate the influence of natural heterogeneity into the poroelastic response of the medium. Motivated by that, the purpose of this paper is to assess the influence of heterogeneity, related to the random spatial variability of hydraulic conductivity, on the earthquake-induced spatial variations in pore pressure and volume strain.

Toward this goal, the spectral representation techniques (Bakr *et al*., 1978; Gelhar and Axness, 1983; Li and McLaughlin, 1991) posed in the framework of stochastic analysis will be adopted. The results of this work may serve as rough estimates of the uncertainty of prediction of field-scale poroelastic response of the aquifer and provide a basis for the planning and management of groundwater resources in aquifers.

PROBLEM FORMULATION

The behavior of fluid flow in fully elastic porous materials cannot be fully described except through coupling the two-way interaction between fluids and deformation, according to the articles reported by Verruij*t* (1969) and others (e.g. Van Der Kamp and Gale, 1983; Green and Wang, 1990)

$$\frac{1}{\alpha}\nabla^2\varepsilon_b - \nabla^2 P = 0 \tag{1}$$

$$\nabla \bullet \left[\frac{K}{\gamma_w}\nabla P\right] = n\beta\frac{\partial P}{\partial t} + \frac{\partial \varepsilon_b}{\partial t} \tag{2}$$

where $\alpha = (\lambda+2\mu)^{-1}$, λ and μ are macroscopic constant coefficients called Lame's coefficients for a porous medium, ε_b is the solid's volume strain, K is the hydraulic conductivity, γ_w is the unit weight of water, n is the porosity, β is the coefficient of fluid compressibility, and P is the pressure increment. The term involving the time derivative of volume strain couples the solid-matrix deformation and the fluid flow. Rigorous application of full coupling is often difficult because they cannot be solved analytically for a general flow configuration.

However, under the assumptions of zero horizontal strain and a constant vertical total stress, which admit a relatively simple relationship between changes in solid's volume strain and fluid pressure, namely

*Correspondence to: Hund-Der Yeh, Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan.
E-mail: hdyeh@mail.nctu.edu.tw

$$\varepsilon_b = \alpha P \tag{3}$$

Equations (1) and (2) can then reduce to a diffusion type of equation in P as (e.g. Verruijt, 1969; Van Der Kamp and Gale, 1983; Green and Wang, 1990)

$$\frac{\partial}{\partial X_i}\left[\frac{K}{\gamma_w}\frac{\partial P}{\partial X_i}\right] = S_s \frac{\partial}{\partial t}\frac{P}{\gamma_w} \tag{4}$$

Or

$$\frac{\partial^2}{\partial X_i^2}\frac{P}{\gamma_w} + \frac{\partial \ln K}{\partial X_i}\frac{\partial}{\partial X_i}\frac{P}{\gamma_w} = \frac{S_s}{K}\frac{\partial}{\partial t}\frac{P}{\gamma_w} \tag{5}$$

where $S_s = \gamma_w\,(n\beta + \alpha)$.

Groundwater levels in aquifers may fluctuate in response to the passage of seismic waves from distant earthquakes. Seismic waves cause spatial variations in volume strain and hence spatial variations in pore pressure. Equation (4) has been widely used to explain the post-seismic groundwater level changes in terms of the pressure diffusion induced by the coseismic strain (e.g. Kumpei, 1992; Rojstaczer *et al*., 1995; Roeloffs, 1996, 1998; Ge and Stover, 2000; Ohno *et al*., 2006). Comprehensive overviews of earthquake-related hydrologic phenomena are given by Roeloffs (1996, 1998).

The relationship (3) with (4) is of particular interest regarding a simplification of hydromechanical coupling, which allows for analyzing the effect of aquifer heterogeneity on the volume strain in randomly heterogeneous aquifers analytically, and this is the task undertaken here.

Invoking the perturbation approximation and spectral representation techniques (Bakr *et al*., 1978; Gelhar and Axness, 1983; Li and McLaughlin, 1991), we seek the solution of pressure head perturbation expressed in terms of ln$K$ perturbation. As such, the variability of pressure head and hence the variability of solid's volume strain can be related to the statistical properties of ln$K$.

MATHEMATICAL DEVELOPMENT

The starting point is Equation (5). The pressure head and the log hydraulic conductivity in Equation (5) are regarded as realizations of random fields. The Lame's coefficients do not vary significantly in space compared with the spatial variation of hydraulic conductivity (e.g. Frias *et al*., 2004; Wang and Hsu, 2009). The effects of variations of Lame's coefficients are therefore neglected in this analysis.

Express P and K as the sum of an ensemble and a zero-mean perturbation, respectively:

$$P(X,t)/\gamma_w = \Phi(X,t) + \phi(X,t) \tag{6}$$

$$\ln K(X) = F + f(X) \tag{7}$$

Upon insertion of Equations (6) and (7), the mean equation which corresponding to Equation (5) is

$$\frac{\partial^2 \Phi}{\partial X_i^2} = \frac{1}{D}\frac{\partial \Phi}{\partial t} \tag{8}$$

and the first-order mean removed equation for the pressure head perturbation is

$$\frac{\partial^2 \phi}{\partial X_i^2} + \frac{\partial f}{\partial X_i}\frac{\partial \Phi}{\partial X_i} = \frac{1}{D}\left[\frac{\partial \phi}{\partial t} + f\frac{\partial \Phi}{\partial t}\right] \tag{9}$$

where $D = e^F / S_s$.

The approach followed is to solve the perturbation Equation (9) to characterize the variability of pressure head. Consider that the flow is only in the horizontal direction and thus Equation (9) can be simplified to the one-dimensional case. The one-dimensional results can provide a clear insight of the impact of natural heterogeneity on the variability of pressure head.

First-order mean pressure head and pressure head perturbations

To solve Equation (9), the mean Equation (8) must be solved first in order to know the space and time derivatives of the mean pressure head in Equation (9). The boundary conditions needed in the solution of Equation (8) in the case of bounded one-dimensional flow are

$$\Phi(0,t) = \Phi_1 \tag{10a}$$

$$\Phi(L,t) = \Phi_2 \tag{10b}$$

where L is the bounded domain size. The initial condition used by Ohno *et al*. (2006) for the mean diffusion model is

$$\Phi(X,0) = \Phi_0 \cos\left(\frac{\pi X}{L}\right) \tag{10c}$$

The boundary of prescribed pressure head (Equations 10(a) or 10(b)), known as the Dirichlet boundary condition or the first-type boundary condition, occurs when the aquifer flow domain is in contact with a surface water body (such as a lake, river, or reservoir etc.). It is assumed that one of the boundaries is a recharge boundary, and the other is a discharge boundary. Suppose that at time $t = 0$, the mean pressure head in the aquifer fluctuates (as in response to the passage of seismic waves) according to Equation 10(c).

Note that the boundary conditions are assumed to be deterministic and the only source of uncertainty is the variability of log conductivity. The solution is found by the method of eigenfunction expansions (e.g. Farlow, 1993; Haberman, 1998) to be

$$\Phi(X,t) = \Phi_1 - JX + \frac{4}{\pi}\Phi_0 \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)} \exp\left[-(n+1)^2\pi^2\tau\right] \sin[(n+1)\pi\xi] - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\Phi_1 + (-1)^{n+1}\Phi_2}{n}\exp\left[-n^2\pi^2\tau\right]\sin(n\pi\xi) \tag{11}$$

where $J = (\Phi_1 - \Phi_2)/L$, $\tau = Dt/L^2$, and $\xi = X/L$.

From Equation (11), we immediately have

$$\frac{\partial \Phi}{\partial t} = -\frac{4\pi D}{L^2}\Phi_0 \sum_{n=1}^{\infty} \frac{(n+1)^3}{n(n+2)} \exp\left[-(n+1)^2\pi^2\tau\right] \sin[(n+1)\pi\xi] + \frac{2\pi D}{L} \sum_{n=1}^{\infty} n\left[\Phi_1 + (-1)^{n+1}\Phi_2\right] \exp\left[-n^2\pi^2\tau\right] \sin(n\pi\xi) \tag{12}$$

$$\frac{\partial \Phi}{\partial X} = -J + \frac{4}{L}\Phi_0 \sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)} \exp\left[-(n+1)^2\pi^2\tau\right] \cos\left[(n+1)\pi\xi\right] - \frac{2}{L} \sum_{n=1}^{\infty} \left[\Phi_1 + (-1)^{n+1}\Phi_2\right] \exp\left[-n^2\pi^2\tau\right] \cos(n\pi\xi) \tag{13}$$

We find that for $\tau >> 1/\pi^2$, corresponding to ranges of e^F, S_s, and L likely to be interest (e.g. Freeze and Cherry, 1979), the transient terms in Equations (12) and (13) become negligibly small. To simplify the analysis, we assume that sufficient time has elapsed since the initial condition. As such, Equation (13) leaves only the steady component and the pressure head perturbation. Equation (9) in the case of one-dimensional flow reduces to

$$D\frac{\partial^2 \phi}{\partial X^2} - DJ\frac{\partial f}{\partial X} = \frac{\partial \phi}{\partial t} \tag{14}$$

with boundary and initial conditions

$$\phi(0,t) = 0 \tag{15a}$$

$$\phi(L,t) = 0 \tag{15b}$$

$$\phi(X,0) = 0 \tag{15c}$$

The solution to Equations (14) and (15) can be determined using Fourier-Stieltjes representations for the perturbed quantities (Bakr *et al.*, 1978; Gelhar and Axness, 1983; Li and McLaughlin, 1991). By using this approach, the ln$K$ perturbation field f is assumed to be a second-order stationary random field and represented by the following wave number integral:

$$f(X) = \int_{-\infty}^{\infty} e^{iRX} dZ_f(R) \tag{16}$$

where $dZ_f(R)$ is the complex Fourier amplitude of ln$K$ process and R is the wave number. However, due to the effect of a finite bounded flow domain, the pressure head perturbed quantities in Equation (14) is presented using the nonstationary spectral representation (Li and McLaughlin, 1991) as

$$\phi(X,t) = \int_{-\infty}^{\infty} \Theta_{pf}(X,t,R) dZ_f(R) \tag{17}$$

where $\Theta_{pf}(X, t, R)$ is a transfer function to be given.

Based on Equations (16) and (17), the perturbation Equation (14) can be expressed in spectral space as

$$D\frac{\partial^2 \Theta_{pf}}{\partial X^2} - iRDJe^{iRX} = \frac{\partial \Theta_{pf}}{\partial t} \tag{18}$$

with

$$\Theta_{pf}(0,t) = 0 \tag{19a}$$

$$\Theta_{pf}(L,t) = 0 \tag{19b}$$

$$\Theta_{pf}(X,0) = 0 \tag{19c}$$

Equations (18) and (19) admit the following solution (e.g. Farlow, 1993; Haberman, 1998):

$$\Theta_{pf}(X,t) = i\frac{2}{\pi}JR \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-(-1)^n \exp[iRL]}{R^2 - n^2\pi^2/L^2} \times \left[1 - \exp\left(-n^2\pi^2\tau\right)\right] \sin(n\pi\xi) \tag{20}$$

In the case when τ is large ($\pi^2\tau >> 1$) (e.g. Haberman, 1998),

$$\Theta_{pf}(X,t) \approx i\frac{2}{\pi}J\left[1 - \exp\left(-\pi^2\tau\right)\right] \frac{R[1+\exp(iRL)]}{R^2 - \pi^2/L^2} \sin(\pi\xi) \tag{21}$$

Variances of pressure head and volume strain

By virtue of Equations (17) and (21), we obtain

$$\phi(X,t) = i\frac{2}{\pi}J\left[1 - \exp\left(-\pi^2\tau\right)\right] \sin(\pi\xi) \int_{-\infty}^{\infty} \frac{R[1+\exp(iRL)]}{R^2 - \pi^2/L^2} dZ_f(R) \tag{22}$$

Taking the expected value of the product of Equation (22) and its complex conjugate and making use of the spectral representation theorem gives the variance of pressure head as

$$\sigma_\phi^2(X,t) = \frac{8}{\pi^2}J^2\left[1 - \exp\left(-\pi^2\tau\right)\right]^2 \sin^2(\pi\xi) \int_{-\infty}^{\infty} \frac{R^2[1+\cos(RL)]}{\left[R^2 - \pi^2/L^2\right]^2} S_{ff}(R) dR \tag{23}$$

Note that from Equation (3), the variance of solid's strain can be related to that of pressure head by

$$\sigma_{\varepsilon b}^2 = \left(\frac{\gamma_w}{\lambda + 2\mu}\right)^2 \sigma_\phi^2 = \alpha^2\gamma_w^2\sigma_\phi^2 \tag{24}$$

To proceed with the development of the variances of pressure head Equation (23) and solid's volume strain Equation (24), one must select the form of the ln$K$ spectrum. The random ln$K$ perturbation field f under consideration is characterized by the following spectral density function (Bakr *et al.*, 1978)

$$S_{ff}(R) = \frac{2\eta^3 R^2}{\pi(1+\eta^2R^2)^2}\sigma_f^2 \tag{25}$$

where η is the correlation length of ln$K$ and σ_f^2 is the variance of ln$K$.

Substituting Equation (25) into Equation (23) leads to the following result for the variance of pressure head

$$\sigma_\phi^2(X,t) = \frac{8}{\pi^2}J^2L^2\sigma_f^2\left[1-\exp\left(-\pi^2\tau\right)\right]^2\sin^2(\pi\xi)\frac{\ell}{\left(1+\pi^2\ell^2\right)^2}\left[(\ell+1)\exp(-1/\ell)+\ell\left(1-\pi^2\ell\right)\right] \tag{26}$$

where $\ell = \eta/L$. The variance of solid's volume strain results from Equations (24) and (26) as follows:

$$\sigma_{\varepsilon_b}^2(X,t) = \frac{8}{\pi^2}J^2L^2\alpha^2\gamma_w^2\sigma_f^2\left[1-\exp\left(-\pi^2\tau\right)\right]^2\sin^2(\pi\xi)\frac{\ell}{\left(1+\pi^2\ell^2\right)^2}\left[(\ell+1)\exp(-1/\ell)+\ell\left(1-\pi^2\ell\right)\right] \tag{27}$$

Note that based on Equations (3) and (11), the mean solid's volume strain and its large-time solution take the forms, respectively, as

$$\overline{\varepsilon_b}(X,t) = \alpha\gamma_w\left\{\Phi_1 - JX + \frac{4}{\pi}\Phi_0\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}\exp\left[-(n+1)^2\pi^2\tau\right]\sin[(n+1)\pi\xi] - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\Phi_1+(-1)^{n+1}\Phi_2}{n}\exp\left[-n^2\pi^2\tau\right]\sin(n\pi\xi)\right\} \tag{28a}$$

$$\bar{\varepsilon_b}(X,t) \approx \alpha\gamma_w\left[\Phi_1 - JX - \frac{2}{\pi}(\Phi_1+\Phi_2)\exp\left[-\pi^2\tau\right]\sin(\pi\xi)\right] \tag{28b}$$

The linear relationship between the variation of the solid's volume strain and σ_f^2 in Equation (27) suggests that the variation of the solid's volume strain increases linearly with the heterogeneity of the medium. The result of Equation (27) is presented graphically in Figure 1, which shows the increase in the variance of solid's volume strain with time. It also indicates that the variance of the solid's volume strain increases with the correlation length of ln$K$ η. An increase in η produces more persistence of volume strain fluctuation around the mean, which, in turn, leads to larger deviations from the mean solid's volume strain. This behavior is due to the impact of η on the variance of pressure head (Equations (24) and (26)). A larger η results in an increase in the variation of pressure head and, consequently, results in higher variation of the volume strain. The pressure head profile with a smaller correlation length η will be rougher, while that with a larger η will be smoother. This implies that the pressure head fluctuations are either consistently above or below the mean pressure head for a larger η and thus leads to larger deviations of the pressure head from the mean pressure head level. This result of the increase in the head variance with the correlation length of ln$K$ is similar to those reported by Bakr *et al.* (1978) and Mizell *et al.* (1982) for the case of groundwater flow in a heterogeneous non-deforming medium. Figure 2 depicts the behavior of the dimensionless solid's volume strain as a function of dimensionless position for various values of τ.

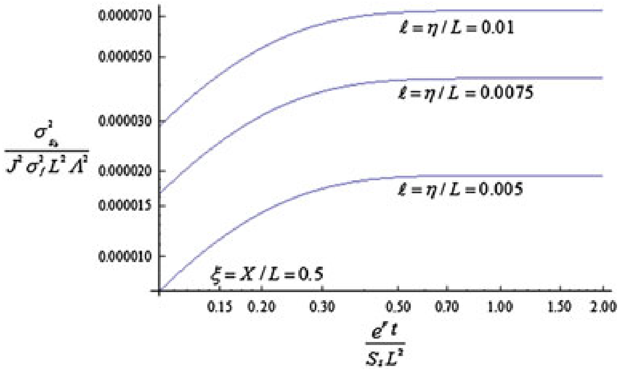

Figure 1. Dimensionless variance of solid's volume strain *versus* dimensionless time for various correlation scale of ln$K$, where $\Lambda = \alpha\gamma_w$

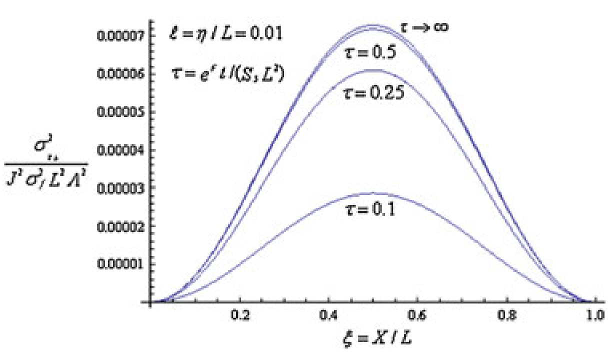

Figure 2. Dimensionless variance of solid's volume strain *versus* dimensionless position for various values of τ, where $\Lambda = \alpha\gamma_w$

CONCLUSION

This paper addresses the problem of the variability of poroelastic response (namely, the changes in fluid pressure head and solid's volume strain) to the passage of seismic waves from distant earthquakes. The closed-form expressions for the means and variances of pressure head and solid's volume strain, and the mean specific discharge, which are expressed in terms of the statistical properties of the log hydraulic conductivity field, are reported in a bounded one-dimensional heterogeneous medium. These expressions are developed directly from the nonstationary representation for head perturbation (Li and McLaughlin, 1991). It is found that the heterogeneity of the medium and the correlation length of the log hydraulic conductivity are important in enhancing the variability of pressure head and solid's volume strain. It is hoped that our findings will be useful in stimulating further research in this area.

ACKNOWLEDGEMENTS

Research leading to this work has been partially supported by 'Aim for the Top University Plan' of the National Chiao Tung University and Ministry of Education, Taiwan, and the grants from Taiwan National Science Council under the contract numbers NSC 99-2221-E-009-062-MY3, NSC100-2221-E-009-106, and NSC101-3113-E-007-008. We are grateful to the anonymous reviewers for constructive comments that improved the quality of the work.

REFERENCES

Bakr AA, Gelhar LW, Gutjahr AL, MacMillan JR. 1978. Stochastic analysis of spatial variability in subsurface flows: 1. Comparison of one- and three-dimensional flows. *Water Resources Research* **14**(2): 263–271.

Farlow SJ. 1993. *Partial Differential Equations for Scientists and Engineers*. Dover: New York, N. Y.

Freeze RA, Cherry JA. 1979. *Groundwater*. Prentice Hall: Englewood Cliffs, N. J.

Frias DG., Murad MA, Pereira F. 2004. Stochastic computational modeling of highly heterogeneous poroelastic media with long-range correlations. *International Journal for Numerical and Analytical Methods in Geomechanics* **28**(1): 1–32.

Ge S, Stover SC. 2000. Hydrodynamic response to strike- and dip-slip faulting in a half-space. *Journal of Geophysical Research* **105**(B11): 25513–25524.

Gelhar LW, Axness CL. 1983. Three-dimensional stochastic analysis of macrodispersion in aquifers. *Water Resources Research* **19**(1): 161–180.

Green DH, Wang HF. 1990. Specific storage as a poroelastic coefficient. *Water Resources Research* **26**(7): 1631–1637.

Haberman R. 1998. *Elementary Applied Partial Differential Equations: With Fourier Series and Boundary Value*. Prentice Hall: Upper Saddle River, New Jersey.

Kumpei H-J. 1992. About the potential of wells to reflect stress variations within inhomogeneous crust. *Tectonophysics* **211**: 317–336.

Li S-G, McLaughlin D. 1991. A nonstationary spectral method for solving stochastic groundwater problems: Unconditional analysis. *Water Resources Research* **27**(7): 1589–1605.

Mizell SA, Gutjahr AL, Gelhar LW. 1982. Stochastic analysis of spatial variability in two-dimensional steady groundwater flow assuming stationary and non-stationary heads. *Water Resources Research* **18**(4): 1053–1067.

Ohno M, Sato T, Notsu K, Wakita H, Ozawa K. 2006. Groundwater-level changes due to pressure gradient induced by nearby earthquakes off Izu Peninsula, 1997. *Pure and Applied Geophysics* **163**: 647–655.

Roeloffs EA. 1996. Poroelastic techniques in the study of earthquake-related hydrologic phenomena. *Advances in Geophysics* **37**: 135–195. Acadimic Press, San Diego, C. A.

Roeloffs EA. 1998. Persistent water level changes in a well near Parkfield, California, due to local and distant earthquakes. *Journal of Geophysical Research* **103**(B1): 869–889.

Rojstaczer S, Wolf S, Michel R. 1995. Permeability enhancement in the shallow crust as a cause of earthquake-induced hydrological changes. *Nature* **373**: 237–239.

Van Der Kamp G, Gale JE. 1983. Theory of earth tide and barometric effects in porous formations with compressible grains. *Water Resources Research* **19**(2): 538–544.

Verruijt A. 1969. Elastic storage in aquifers. In *Flow through Porous Media*, De Wiest RJM (ed.). Academic Press: New York, N. Y.

Wang S-J, Hsu K-C. 2009. The application of the first-order second-moment method to analyze poroelastic problems in heterogeneous porous media. *Journal of Hydrology* **369**: 209–221.

Yeh H-D, Lu R-H, Yeh G-T. 1996. Finite element modeling for land displacements due to pumping. *International Journal for Numerical and Analytical Methods in Geomechanics* **20**(2): 79–99.