Observation of disordered wave functions with conical second-harmonic generation and verification of transition from extended to prelocalized states in weak localization

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We experimentally acquire the disordered wave functions from the conical second-harmonic generation to explore the variation of weak localization from extended to prelocalized states. We numerically verify that the experimental density distributions with different extents of weak localization can be excellently analyzed with a reduced version of the nonlinear σ model (RV-NLS model). Moreover, we demonstrate that the χ-square distributions with fractional degrees of freedom are practically equivalent to the density distributions of the RV-NLS model. Our finding indicates that the concept of fractional degrees of freedom can be applied to the statistical properties of disordered wave functions.

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I. INTRODUCTION

Wave localization, which results from the peculiar interference of waves scattered by disorders, is an intriguing phenomenon beyond diffusion theory and transfer treatment.[1–3] Since the fundamental processes of scattering and interference are identical for classical and quantum waves, the phenomena of wave localization have been extensively investigated in different physical systems.[4–7] Recent developments have led to much interest in various disordered media specified by weak (WL)[8–14] or strong localization (SL).[5,15–17] It could be found[1–17] that the localization phenomenon is still an important issue and deserves further investigations.

Theoretical analyses and experimental observations for the disordered wave functions are the straightforward procedures to determine the extent of wave localization. Numerous theoretical models[18–22] have been constructed to explore the extent of wave localization. Recently, the nonlinear σ models based on the supersymmetry theory have been employed to investigate the statistical properties of disordered wave functions.[22] The zero-dimensional (0D) nonlinear σ model has been shown to be equivalent to the random matrix method[22] in the diffusive limit of disordered systems. In the weakly disordered systems, the wave functions are widely spread over space, corresponding to the so-called *extended* states. With the one-dimensional (1D) nonlinear σ model, the density distributions of the extended states can be expressed as an analytical formula related to the well-known Porter-Thomas (P-T) distribution.[23] On the other hand, the wave functions of the strongly disordered systems display log-normal asymptotic forms and long-tail characteristics in the density distributions,[24,25] corresponding to the so-called *prelocalized* states. Fal′ko and Efetov [20] developed the reduced version of the nonlinear σ model (RV-NLS model) to analyze the long-tail density distributions of the prelocalized states. Although the RV-NLS model seems to be applicable in order to quantify the varying extent of WL, detailed comparisons with experimental observations have not been performed as yet.

In experiments the disordered wave functions were measured in a microwave cavity to show the influence of chaos and localization in disordered quantum billiards.[13] In 2006, Chen *et al.*[26] demonstrated the spatial structure of two-dimensional (2D) disordered wave functions from exploring the near-field patterns of conical second-harmonic generation (SHG) in a $GdCa_4O(BO)$ (GdCOB) nonlinear crystal with moderate defect domains. So far, experimental results for the disordered wave functions only covered a partial WL regime and did not provide a comprehensive analysis of the transition from extended to prelocalized states.

In this work we experimentally generate the 2D disordered wave functions by systematically scanning a GdCOB nonlinear crystal in the conical SHG process to explore the characteristics of WL. We numerically confirm that the RV-NLS model can provide statistical analyses to agree very well with the experimental wave functions with various localizations. Furthermore, we find that the density distributions of the disordered wave functions can be analytically expressed as the χ-square distributions with fractional parameters. Since the parameters in the formal expression of χ-square distributions are only integers for the integral degrees of freedom,[27] we use the terminology of *fractional* χ-square distribution to distinguish the difference. Finally, we construct the relationship between the RV-NLS model and the fractional χ-square distributions to reveal the characteristics of the fractional degrees of freedom in the disordered wave functions. Although the present results focus on the regime of WL, the fractional χ-square distribution might be useful for the full crossover of localization. We also believe that the present model can be employed to study the degree of localization in various disordered systems[8–14] such as scattering powder, cold atoms, randomized laser materials, liquid crystal, scattered systems, microcavities, and graphene.

II. EXPERIMENTAL SETUP AND RESULTS

Figure 1 shows the experimental setup that is a diode-pumped actively Q-switched Nd:YAG laser with intracavity SHG in the GdCOB crystal. The gain medium is a 0.8 at. % Nd^{3+}:YAG crystal with a length of 10 mm. The GdCOB crystal was cut for type I frequency doubling in the XY planes ($\theta = 90^\circ$, $\varphi = 46^\circ$) with a length of 2 mm and a cross section of 3 mm × 3 mm. All crystals were coated for antireflection ($R < 2\%$) at 1064 nm on both sides. The radius of curvature of the concave-front mirror is 50 cm with a coating of antireflection ($R < 0.2\%$) at 808 nm, high reflection ($R > 99.8\%$) at 1064 nm,

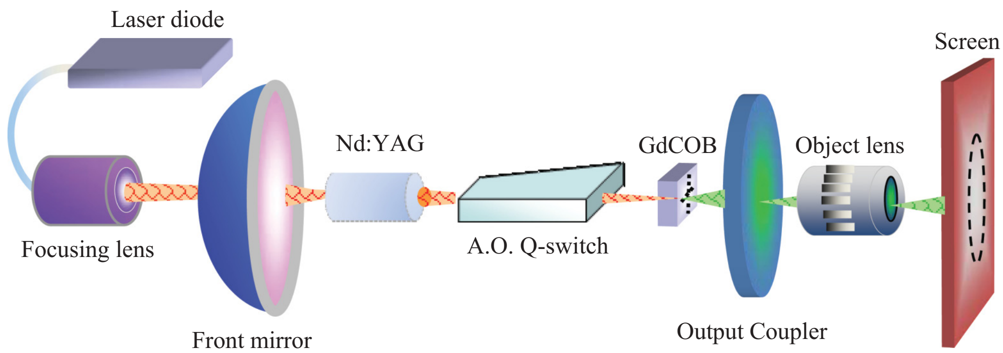

FIG. 1. (Color online) Experimental setup for the generation of disordered wave functions with the diode-pumped Q-switched Nd:YAG laser of intracavity SHG in the GdCOB crystal.

and 532 nm on the entrance side, and high transmission ($T > 90\%$) at 808 nm on the other side. The output coupler is a plant mirror with a coating of high reflection ($R > 99.8\%$) at 1064 nm and high transmission at 532 nm ($T > 85\%$). The pump source is a 10 W 808 nm fiber-coupled laser diode with a core diameter of 800 μm. A focusing lens with a focal length of 2.5 cm and 90% coupling efficiency was employed to reimage the pump beam into the laser gain medium. The acoustic-optic Q switch with a length of 30 mm has a coating with antireflection at 1064 nm on both sides and is driven at a 27.12-MHz center frequency with 15.0 W of rf power. An object lens was used to reimage the near-field patterns on the screen.

It has been shown that GdCOB crystals possess various random defect domains which can be used to generate the intensities $|\Psi(\vec{r})|^2$ of 2D disordered wave functions in the SHG process.[26] Here we find that the extent of random defect domains significantly depend on the transverse position of the GdCOB crystal. With this feature, we can scan all transverse positions of the GdCOB crystal to generate a variety of disordered wave functions from extended to prelocalized states as shown in Figs. 2(a)–2(f).

III. THEORETICAL ANALYSIS

To determine the extent of localization, the density probability distribution $P(|\Psi(\vec{r})|^2)$ is illustrated to specify the localization of wave functions. For extended states in quantum chaotic systems, a random-matrix method and an equivalent 0D nonlinear σ model have been verified to give good explanations of universal statistic behaviors with the P-T distribution.[22] For weakly disordered systems, density probability of the normalized disordered wave functions can be expressed with the 1D nonlinear σ model as[19,22,26]

$$P_{1D}(I) = P_{P-T}(I)\left[1 + (\mathrm{IPR} - 3)\left(\tfrac{1}{8} - \tfrac{1}{4}I + \tfrac{1}{24}I^2\right)\right], \quad (1)$$

where $I = |\Psi(\vec{r})|^2$, $P_{P-T}(I) = \exp(-I/2)/\sqrt{2\pi I}$ is the P-T distribution, and $\mathrm{IPR} = \int I^2 d^2r$ is the inverse participation ratio associated with the extent of localization. For P-T distribution, the IPR can be directly achieved to be $\mathrm{IPR} = \int_0^\infty I^2 P_{P-T}(I)\,dI = 3.0$, indicating the chaotic systems. The larger the IPR value, the stronger the extent of localization. The IPR values of the disordered systems are definitely greater than 3.0. However, Eq. (1) reveals that the density distribution of the 1D nonlinear σ model will be a negative value at $I = 3$ for the disordered wave function with $\mathrm{IPR} > 7.0$. We numerically

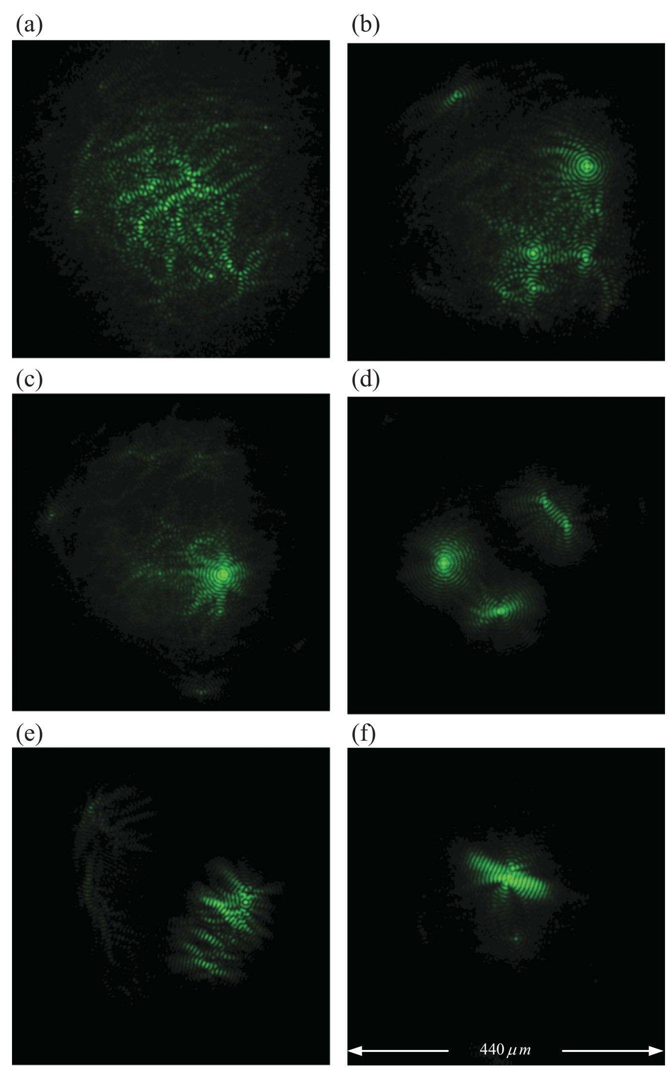

FIG. 2. (Color online) (a)–(f) Experimental observation of near-field wave patterns measured at different transverse positions of the GdCOB crystal.

confirm that the 1D nonlinear σ model is only appropriate for the disordered wave function with IPR < 5.5. For stronger disorder, higher densities of the distribution functions decay more slowly in the region where the 1D nonlinear σ models break down. Therefore, a more appropriate model should be given to clarify the varying extent of localization.

In the following we employ the experimental data to testify the RV-NLS model that is developed to quantitatively specify different regimes of localization. The RV-NLS model indicated by a dimensionless parameter g is given by[20,22]

$$P_\sigma(I;g) = \frac{A}{\sqrt{I}} \exp\left[-g\left(\frac{z(I)}{2} + \frac{z(I)^2}{4}\right)\right], \qquad (2)$$

where A is a normalized constant, $z(I)$ could be solved numerically according to the relation $z\,e^z = I/g$, and g is the dimensionless conductance[2,3] used to identify the degree of localization. The parameter g is also called the "Thouless number" which was first proposed by Thouless in the discussion on the scaling theories of localization.[2,3] The dimensionless conductance g is adopted by the scaling theory as its only parameter and depends on the dimensionality of the system. For the 2D case, $g \sim kl/\ln(L/l)$,[2] where k is the wave vector, $k = 2\pi/\lambda$, l signifies the value of mean free path, and L denotes the size of the system. The formal definition of g is $g \equiv G(L)/(e^2/2\hbar)$ (Ref. 2), where $G(L)$ is the conductance of a hypercube of size L^d, d relates to the dimensionality, $\hbar$ is Plank's constant, and e is the electronic charge. In the diffusive limit of $g \gg 1$, the density distribution reveals a universality of the statistics of localized waves. The value of g is substantially decreased due to WL which is the precursor of Anderson localization (SL) of $g \approx 1$.[28] In other words, the scaling parameter g can be exploited to specify the extent of localization for the experimental results. Figures 3(a)–3(f) depict the numerical results of the RV-NLS model for the

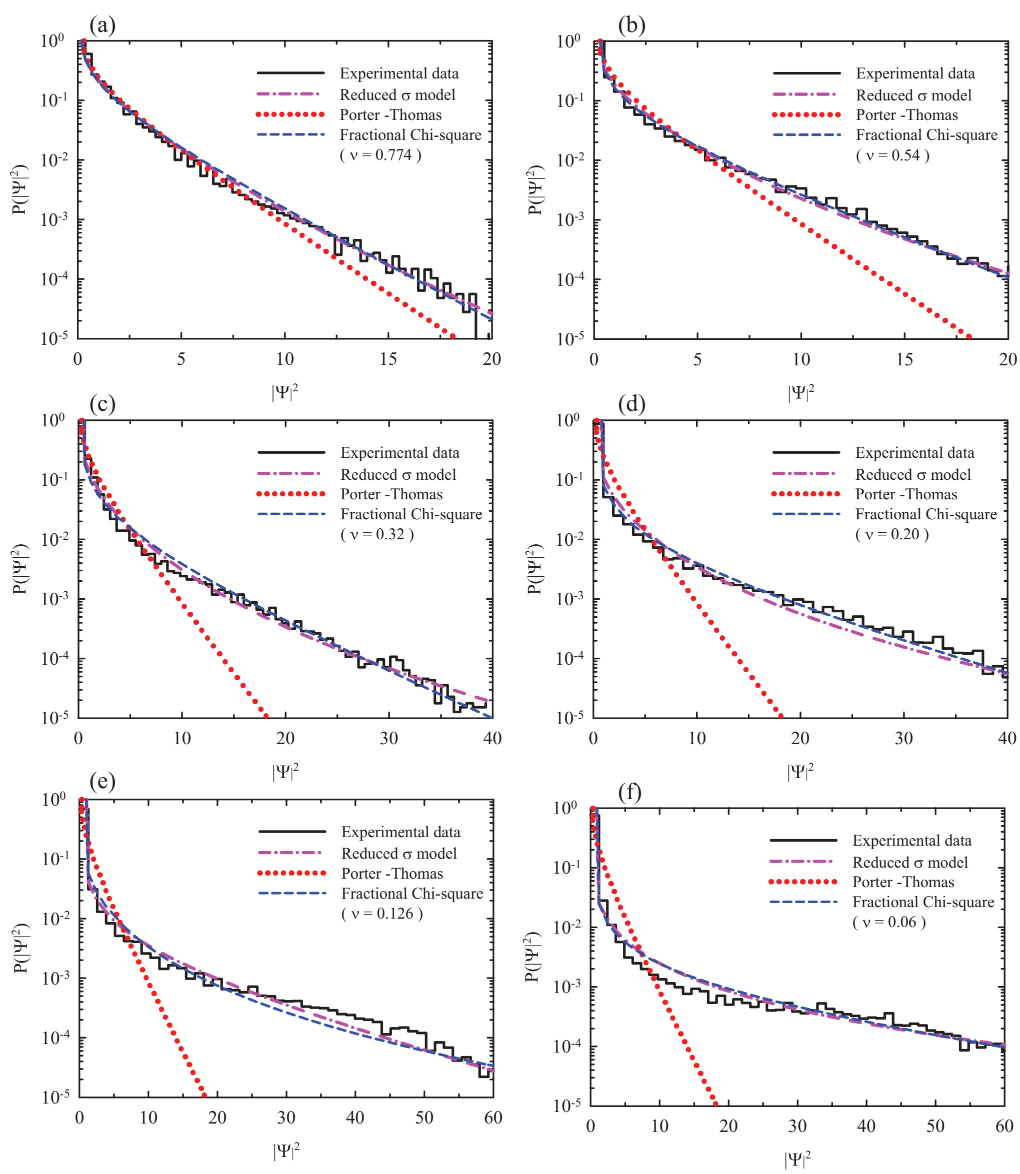

FIG. 3. (Color online) (a)–(f) Experimental and theoretical density distributions $P(I)$ corresponding to experimental data in Fig. 2(a)–2(f), respectively.

best fits to the wave patterns shown in Figs. 2(a)–2(f), where the values of g are found to be 33, 11, 5.5, 3.5, 2.3, and 1.1, respectively. It can be seen that the density distributions generated with the RV-NLS model agree very well with the experimental results for all cases. Actually, Efetov[22] has once bought up the idea that the RV-NLS model can be applied to explain the statistical behavior for the disordered wave functions in a microwave cavity.[13] Employing the laser system with the conical SHG operation, we have verified here the practicability of the RV-NLS model in another disordered system. The fact implies possible extension of the RV-NLS model on the studies of different extents of localization in various kinds of disordered systems.

Besides the verification of the RV-NLS model, we originally find that the χ-square distributions with fractional parameters can satisfactorily describe the experimental results. The analytic expression of the χ-square distributions is given by[27]

$$P_{CS}(I;v) = \Gamma(v/2)^{-1}(vI/2)^{v/2}I^{-1}e^{-vI/2}, \quad (3)$$

where $v > 0$ is a parameter referred to the number of degrees of freedom and $\Gamma(v/2)^{-1}$ is the γ function which serves to normalize the density distributions $P_{CS}(I;v)$. The P-T distribution $P_{P-T}(I)$ is the χ-square distribution with one degree of freedom, i.e., $P_{CS}(I;v=1)$.[23] In addition, the exponential distribution $\exp(-I)$ can be referred to the χ-square distribution with two degrees of freedom, i.e., $P_{CS}(I;v=2)$. Even though there is no conceptual difficulty in extending an integer value of v to a noninteger, it has not been confirmed whether noninteger degrees of freedom have any applications in nature. As shown in Figs. 3(a)–3(f), the χ-square distributions with $0.06 \leqslant v < 1$, almost identical to the features of the RV-NLS model, can excellently illustrate the experimental results. The values of v for experimental wave patterns in Figs. 2(a)–2(f) are 0.774, 0.54, 0.32, 0.20, 0.126, and 0.06, respectively. The evidence shows that the tails of the density distribution decay more slowly at small values of v and the degree of localization becomes larger while the values of v decrease rapidly. The investigation yields a clear result that the fractional χ-square distribution could be a powerful procedure for analyzing the statistical properties of the localization phenomena. It is well known that the noninteger dimensionality is an important property of most fractals. Our exploration reveals that noninteger or fractional parameters are also valid concepts in statistical distributions of disordered wave functions.

The validity and equivalence between the density distributions $P_{CS}(I;v)$ and $P_\sigma(I;g)$ imply that the two parameters v and g are related. The relationship between v and g according to the experimental results is marked with blue dots in Fig. 4. We employ an empirical form of $v = 1 - \exp(-0.08\, g^{0.85})$ to express the relationship between v and g, as depicted with a solid line in Fig. 4. The empirical expression indicates the two properties: One is $v \to 1$ as $g \to \infty$ to indicate no WL effects and the other is $v_c \approx 0.06$ with $g \approx 1$ to signify the SL threshold. In other words, the statistical properties for the WL and SL effects can be manifested with the χ-square distributions with the parameters in the region of $v_c \leqslant v < 1$ and $0 \leqslant v < v_c$, respectively. Taking the familiar parameter g as a standard of scaling, the careful mapping of g and v of the two models helps to clarify the regime of different extents of localization with the new parameter v. Though our experimental results cover only the regime of WL, the fractional χ-square distributions might be extended to the study of other disordered systems that reveal the full crossover of wave localization.

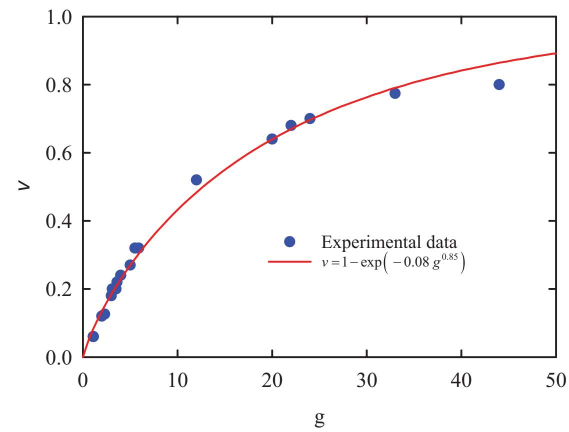

FIG. 4. (Color online) Blue dots: The relation between v and g according to the experimental data. Red line: Empirical form for the relationship between v and g.

IV. CONCLUSION

In summary, we have experimentally generated the optical patterns from the conical SHG process to investigate the disordered wave functions with different extents of WL from extended to prelocalized states. It has been numerically confirmed that the statistical characteristics of experimental disordered wave functions can be explained very well with the RV-NLS model. Furthermore, we have found that the fractional χ-square distributions are nearly equivalent to the distributions of the RV-NLS model. With this result, the concept of the fractional degrees of freedom can be used to manifest the extent of localization for the disordered wave functions. It is believed that the present work can bring more insight into the localization phenomena of diverse disordered systems.

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