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# Classical $N = 2$ $W$-superalgebras from superpseudodifferential operators

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**Abstract.** We study the supersymmetric Gelfand–Dickey algebras associated with the superpseudodifferential operators of positive, as well as negative, leading order. We show that, upon the usual constraint, these algebras contain the $N = 2$ super Virasoro algebra as a subalgebra when the leading order is odd. The decompositions of the coefficient functions into $N = 1$ primary fields are then obtained by covariantizing the superpseudodifferential operators. We discuss the problem of identifying $N = 2$ supermultiplets and work out a couple of supermultiplets by explicit computations.

## 1. Introduction

The relevance of the study of $W$-algebras in two-dimensional conformal field theory is now quite clear. The quantum $W$-algebras were first introduced by Zamolodchikov as extensions of conformal symmetry [1]. Soon after this work, it was realized that the classical $W_n$-algebras arise quite naturally as the exotic Hamiltonian structures for the generalized KdV hierarchies [2–7]. These Hamiltonian structures can be elegantly expressed by the second Gelfand–Dickey bracket defined by differential operators [8–10]. Extensions of the Gelfand–Dickey bracket for pseudodifferential operators give a class of $W$-type algebras called $W_{\mathrm{KP}}^{(n)}$ which are the Hamiltonian structures of the KP hierarchy [11–15]. Recently, the supersymmetric version of the second Gelfand–Dickey brackets were constructed [16–19]. A series of $N = 1$ and $N = 2$ $W$-superalgebras have been obtained from the brackets defined by superdifferential operators.

In this paper, we study the superalgebras arising from the second Gelfand–Dickey brackets defined by superpseudodifferential operators [20–22]. These superalgebras, to our knowledge, are still unexplored. Our main motivation comes from the fact that, in the bosonic case, all hitherto known $W_\infty$-type algebras can be obtained from $W_{\mathrm{KP}}^{(n)}$ and its 'analytic continuation' $W_{\mathrm{KP}}^{(q)}$ [23] via reductions, contractions or truncations [24]. Thus, we believe that the superalgebras from superpseudodifferential operators could possibly lead to an interesting super version of $W_\infty$-type algebras. The first aim of this paper is, therefore, to find the $N = 2$ analogue of $W_{\mathrm{KP}}^{(n)}$. To this purpose, we consider the usual reduction of these superalgebras. We find that it is possible when the leading order is an (positive or negative) odd integer. In other words, in this case, these superalgebras contain the $N = 2$ super Virasoro algebra as a subalgebra. In order to see whether these superalgebras are genuine $N = 2$ $W$-superalgebras or not, we need to identify the required $N = 2$ supermultiplets. This is a very difficult task. We know that in the case of ordinary (pseudo)differential operators, the desired primary fields can easily be obtained by putting the operators into a conformally covariant form [25,26]. However, the superconformally covariant form of superdifferential operators can only give us the decompositions of coefficient functions into $N = 1$ primary fields due to the fact that these superoperators are defined on the $(1|1)$ superspace [27,28]. The $N = 2$ supermultiplets can be identified only if we further compute the Hamiltonian flow defined by the spin-1 current and redefine the $N = 1$ primary fields properly. The last step is where the difficulty lies since there is so far no systematic way of handling the spin-1 flow. Therefore, these $N = 2$ supermultiplets have never been completely identified. Despite this, we have still carried out the superconformal covariantization program for the superpseudodifferential operators to obtain the series of $N = 1$ primary fields. Then, we discuss the problem of identifying $N = 2$ supermultiplets. In fact, we show that the identification problem for the case of leading order $2m + 1$ is equivalent to that for the case of leading order $-2m - 1$. Moreover, two supermultiplets are identified by explicit computations.

We organize this paper as follows. In section 2, we introduce the second Gelfand–Dickey bracket for superpseudodifferential operators and show that a reduction yields the $N = 2$ super Virasoro algebra if the leading order is odd. In section 3, we prove that the action of a superconformal transformation on the superpseudodifferential operator is nothing but a Hamiltonian flow defined by the second Gelfand–Dickey bracket. The superconformally covariant form of the superpseudodifferential operators is obtained. In section 4, we identify the first two $N = 2$ supermultiplets of the negative part of an odd-order superpseudodifferential operator. We present our concluding remarks in section 5.

## 2. Superpseudodifferential operators and the second Gelfand–Dickey bracket

We consider the superdifferential operators on a $(1|1)$ superspace with coordinate $(x, \theta)$. These operators are polynomials in the supercovariant derivative $D = \partial_\theta + \theta\partial_x$, whose coefficients are $N = 1$ superfields, i.e.

$$L = D^n + U_1 D^{n-1} + U_2 D^{n-2} + \cdots + U_n + U_{n+1} D^{-1} + \cdots \tag{2.1}$$

where $n$ is a non-zero integer (can be negative). As usual, we assume that they are homogeneous under the usual $Z_2$ grading; that is, $|U_i| = i \pmod 2$. The bracket will involve functionals of the form

$$F[U] = \int_B f(U) \tag{2.2}$$

where $f(U)$ is a homogeneous (under $Z_2$ grading) differential polynomial of the $U_i$'s and $\int_B = \int dx\, d\theta$ is the Berezin integral which is defined in the usual way, namely, if we write $U_i = u_i + \theta v_i$ and $f(U) = a(u, v) + \theta b(u, v)$, then $\int_B f(U) = \int dx\, b(u, v)$. The multiplication is given by the super Leibniz rule

$$D^k \Phi = \sum_{i=0}^{\infty} \begin{bmatrix} k \\ k-i \end{bmatrix} (-1)^{|\Phi|(k-i)} \Phi^{[i]} D^{k-i} \tag{2.3}$$

where $k$ is an arbitrary integer and $\Phi^{[i]} = (D^i \Phi)$ and the superbinomial coefficients $\begin{bmatrix} k \\ i \end{bmatrix}$ are defined by

$$\begin{bmatrix} k \\ k-i \end{bmatrix} = \begin{cases} 0 & \text{for } i < 0 \text{ or } (k, i) \equiv (0, 1) \pmod 2 \\ \begin{pmatrix} [\frac{1}{2}k] \\ [\frac{1}{2}(k-i)] \end{pmatrix} & \text{otherwise} \end{cases} \tag{2.4}$$

where $\binom{p}{q}$ is the ordinary binomial coefficient. Next, we introduce the notion of super-residue and supertrace. Given a superpseudodifferential operator $P = \sum p_i D^i$, we define its super-residue sres $P = p_{-1}$ and its supertrace as $\operatorname{Str} P = \int_B \operatorname{sres} P$. In the usual manner, it can be shown that the supertrace of a supercommutator vanishes, i.e. $\operatorname{Str}[P, Q] = 0$, where $[P, Q] \equiv PQ - (-1)^{|P||Q|}QP$. Finally, for a given functional $F[U] = \int_B f(U)$, we define its gradient $\mathrm{d}F$ by

$$\mathrm{d}F = \sum_{k=1}^{n}(-1)^{n+k} D^{-n+k-1}\frac{\delta f}{\delta U_k} \tag{2.5}$$

where

$$\frac{\delta f}{\delta U_k} = \sum_{i=0}^{\infty}(-1)^{|U_k|i+i(i+1)/2} D^i \frac{\partial f}{\partial U_k^{[i]}}. \tag{2.6}$$

Equipped with these notions, we now define the supersymmetric second Gelfand–Dickey bracket as

$$\{F, G\} = (-1)^{|F|+|G|+n} \operatorname{Str}[L(\mathrm{d}F\, L)_+\, \mathrm{d}G - (L\, \mathrm{d}F)_+ L\, \mathrm{d}G] \tag{2.7}$$

where $()_+$ denotes the differential part of a superpseudodifferential operator. It has been shown that (2.7) indeed defines a Hamiltonian structure: it is antisupersymmetric and satisfies the super-Jacobi identity [20–22].

When $n$ is positive and when $U_{n+1} = U_{n+2} = \cdots = 0$ (i.e. when $L$ is a superdifferential operator), it can be shown that when the constraint $U_1 = 0$ is imposed, the induced bracket is well defined only when $n$ is odd [17]. The reason is that this constraint is second class when $n$ is odd, while it becomes first class for even $n$. To compute these induced brackets, we need to modify at least one of $\mathrm{d}F$ and $\mathrm{d}G$ defined by (2.5) due to the absence of $U_1$. The prescription is to add a term $D^{-n}V$ to, say, $\mathrm{d}G$ in such a way that

$$\operatorname{sres}[L, D^{-n}V + \mathrm{d}G] = 0. \tag{2.8}$$

We shall denote $X_G = D^{-n}V + \mathrm{d}G$ for this choice of $V$. Replacing $\mathrm{d}G$ in (2.7) by $X_G$ then gives the induced bracket. It has been shown that if we define (of course, only when $n \geqslant 3$)

$$T = U_3 - \tfrac{1}{2}U_2' \qquad J = U_2 \tag{2.9}$$

where $V' = (DV)$, $V'' = (D^2V), \ldots$ etc, then $T$ and $J$ obey the $N = 2$ super Virasoro algebra

$$\begin{aligned}
\{T(X), T(Y)\} &= \left[\tfrac{1}{4}m(m+1)D^5 + \tfrac{3}{2}T(X)D^2 + \tfrac{1}{2}T'(X)D + T''(X)\right]\delta(X-Y)\\
\{T(X), J(Y)\} &= \left[-J(X)D^2 + \tfrac{1}{2}J'(X)D - \tfrac{1}{2}J''(X)\right]\delta(X-Y)\\
\{J(X), T(Y)\} &= \left[J(X)D^2 - \tfrac{1}{2}J'(X)D + J''(X)\right]\delta(X-Y)\\
\{J(X), J(Y)\} &= -\left[m(m+1)D^3 + 2T(X)\right]\delta(X-Y)
\end{aligned} \tag{2.10}$$

where we have written $n = 2m + 1$ and $\delta(X - Y) = \delta(x - y)(\theta - w)$.

The first natural question one can think of is whether or not the above result remains true when the superpseudodifferential operators are used instead. By straightforward calculations, we can show that the answer is yes. In other words, as long as $n$ is an odd integer, $T$ and $J$, defined by (2.9), together obey the $N = 2$ super Virasoro algebra. What remains to be checked is if the required $N = 2$ supermultiplets can be defined as differential polynomials in the coefficient functions $U_k$. To this end, we need to consider the Hamiltonian flows defined by the two linear functionals

$$\begin{aligned} G &= \int_B T\xi = \int_B (U_3\xi + \tfrac{1}{2}U_2\xi') \\ H &= \int_B J\zeta = \int_B U_2\zeta \end{aligned} \tag{2.11}$$

where $|\xi(x,\theta)| = |\zeta(x,\theta)| = 0$. We find that the transformations of $L$ under the Hamiltonian flows defined by $G$ and $H$ are

$$\begin{aligned} J(X_G) &\equiv (LX_G)_+L - L(X_GL)_+ \\ &= \left[\xi D^2 + \frac{1}{2}\xi' D + \frac{(m+1)}{2}\xi''\right]L - L\left[\xi D^2 + \frac{1}{2}\xi' D - \frac{m}{2}\xi''\right] \\ J(X_H) &\equiv (LX_H)_+L - L(X_HL)_+ \\ &= \left[-\zeta D - (m+1)\zeta'\right]L - L\left[-\zeta D + m\zeta'\right]. \end{aligned} \tag{2.12}$$

Since $T$ is the super Virasoro generator, $J(X_G)$ is called the super Virasoro flow. If the expicit forms of (2.12) are known, one can read off the corresponding brackets at once by using the formula

$$J(X_F) = \sum_{k=2}^{\infty}(-1)^{k|F|+1}\{U_k, F\}D^{n-k}. \tag{2.13}$$

We shall prove in the next section that $J(X_G)$ in (2.12) is the infinitesimal form of the superconformal covariance of $L$.

## 3. The superconformally covariant form of $L$

In this section we give the super Virasoro flow $J(X_G)$ a geometrical interpretation and put $L$ into a superconformally covariant form. We shall follow the construction established in [27, 28]. Let us recall that on the $(1|1)$ superspace with coordinate $X = (x, \theta)$, the most general superdiffeomorphism has the form

$$\tilde{x} = g(x) + \theta\kappa(x) \qquad \tilde{\theta} = \chi(x) + \theta B(x) \tag{3.1}$$

where $|g| = |B| = 0$ and $|\kappa| = |\chi| = 1$. The superdiffeomrphism (3.1) is a superconformal transformation if

$$D = (D\tilde{\theta})\tilde{D}. \tag{3.2}$$

A function $f(X)$ is called a superconformal primary field of spin $h$ if, under superconformal transformation, it transforms as

$$f(\tilde{X}) = (D\tilde{\theta})^{-2h} f(X). \tag{3.3}$$

We shall denote by $F_h$ the space of all superconformal primary fields of spin $h$. As usual, a superpseudodifferential operator $\Delta$ is called a covariant operator if it maps $F_h$ to $F_l$ for some $h$ and $l$.

We study the covariance property of

$$L = D^n + U_2 D^{n-2} + U_3 D^{n-3} + \cdots + U_n + U_{n+1} D^{-1} + \cdots \tag{3.4}$$

where we have set $U_1$ to be zero. Our aim is to see if some $h$ and $l$ can be found so that, under superconformal transformation $X \longrightarrow \tilde{X}$,

$$L(\tilde{X}) = (D\tilde{\theta})^{-2l} L(X) (D\tilde{\theta})^{2h}. \tag{3.5}$$

As in the case of superdifferential operators, the constraint $U_1 = 0$ determines both $h$ and $l$ [27, 28]. In fact, simple algebras give (for any non-zero $n$)

$$(\tilde{D})^n (D\tilde{\theta})^{-2h} = (D\tilde{\theta})^{-2h-n} \left( D^n + A_{n-1} \frac{D^2\tilde{\theta}}{D\tilde{\theta}} D^{n-1} + \cdots \right) \tag{3.6}$$

where

$$A_{n-1} = \begin{cases} m & (n = 2m) \\ -2h - m & (n = 2m + 1). \end{cases} \tag{3.7}$$

Thus, $U_1 = 0$ can be preserved under superconformal transformation only when

$$n = 2m + 1 \qquad h = -\tfrac{1}{2} m \qquad l = \tfrac{1}{2}(m + 1). \tag{3.8}$$

In summary, we have the covariance condition

$$L(\tilde{X}) = (D\tilde{\theta})^{-(m+1)} L(X) (D\tilde{\theta})^{-m}. \tag{3.9}$$

The transformation laws for $U_k$'s are then completely determined by (3.9). For example, simple computations yield the expected transformation laws of $J = U_2$ and $T = U_3 - \frac{1}{2} U_2'$:

$$\begin{aligned} J(X) &= J(\tilde{X})(D\tilde{\theta})^2 \\ T(X) &= T(\tilde{X})(D\tilde{\theta})^3 + \tfrac{1}{2} m(m+1) S(\tilde{X}, X) \end{aligned} \tag{3.10}$$

where $S(\tilde{X}, X)$ is the super Schwarzian defined by

$$S(\tilde{X}, X) = \frac{D^4\tilde{\theta}}{D\tilde{\theta}} - 2 \left( \frac{D^3\tilde{\theta}}{D\tilde{\theta}} \right) \left( \frac{D^2\tilde{\theta}}{D\tilde{\theta}} \right). \tag{3.11}$$

It is interesting to note that the 'central charge' $c_m = \frac{1}{2} m(m + 1)$ in (3.10) does not change sign under the sign change of the leading order $n = 2m + 1$: $m \longrightarrow -m - 1$. To understand this point, let us consider the pair of superpseudodifferential operators

$$L^{\pm} = D^{\pm 2m \pm 1} + U_2^{\pm} D^{\pm 2m \pm 1 - 2} + U_3^{\pm} D^{\pm 2m \pm 1 - 3} + \cdots. \tag{3.12}$$

We shall take $L^-$ to be the formal inverse of $L^+$, i.e.

$$L^+L^- = 1. \tag{3.13}$$

The most important point here is that (3.13) is invariant under superconformal transformation (3.9). Equality (3.13) has fixed the functional relations between the $U_k^+$ and $U_k^-$. In fact, expanding the left-hand side of (3.13) yields

$$U_2^- = -U_2^+ \qquad U_3^- = U_3^+ - (U_2^+)'. \tag{3.14}$$

As a consequence,

$$\begin{aligned} J^- &\equiv U_2^- = -U_2^+ \equiv -J^+ \\ T^- &\equiv U_3^- - \tfrac{1}{2}(U_2^-)' = U_3^+ - \tfrac{1}{2}(U_2^+)' \equiv T^+. \end{aligned} \tag{3.15}$$

It is clear now why the central charge remains unchanged under $n \longrightarrow -n$. We point out that the brackets (2.10) are invariant under $J \longrightarrow -J$. So, the first of (3.15) would not harm these brackets.

We show that the infinitesimal form of covariance condition (3.9) is nothing but the Hamiltonian flow $J(X_G)$, defined by (2.11) and (2.12). First, we recall the most general infinitesimal form of superconformal transformation:

$$\begin{aligned} \tilde{x} &= x - \epsilon(x) - \theta\eta(x) \\ \tilde{\theta} &= \theta - \tfrac{1}{2}\partial_x\epsilon(x)\theta - \eta(x) \end{aligned} \tag{3.16}$$

where $|\epsilon| = 0$ and $|\eta| = 1$. Defining $\xi(x) = \frac{1}{2}\epsilon(x) + \theta\eta(x)$, we can show by induction that, for non-negative integer $k$ [28],

$$(\tilde{D})^k = D^k + D[D^k, \xi]D + [D^k, \xi]D^2 + \mathrm{O}(\xi^2). \tag{3.17}$$

If one re-examines the proof for this equivalence, in the case of the superdifferential operator given in [28], one easily recognizes that (3.17) is the key formula. Therefore, to generalize this proof to the present case we need only to prove the validity of (3.17) when $k$ is a negative integer. To check the validity, we start with $k = -1$. From $D\tilde{\theta} = 1 - \xi''$, we have

$$\begin{aligned} \tilde{D}^{-1} &= D^{-1}(D\tilde{\theta}) \\ &= D^{-1} - D^{-1}\xi'' \\ &= D^{-1} - D^{-1}[D^2, \xi] \\ &= D^{-1} - D\xi + D^{-1}\xi D^2 \\ &= D^{-1} + D[D^{-1}, \xi]D + [D^{-1}, \xi]D^2 \end{aligned}$$

as desired. For $k < -1$, we can easily prove the validity by induction. With the validity of (3.17) for arbitrary integer $k$, the desired proof follows the proof of [28] *mutatis mutandi*. We therefore conclude that the infinitesimal form of (3.9) is indeed the super Virasoro flow $J(X_G)$.

To covariantize the superpseudodifferential operators, we briefly review the necessary set-up [27, 28]. First, we introduce a Grassmanian odd function $B(X)$ which transforms under superconformal transformation as

$$B(\tilde{X}) = (D\tilde{\theta})B(X) + \frac{D^2\tilde{\theta}}{D\tilde{\theta}}. \tag{3.18}$$

We then make the following identification:

$$T(X) = \frac{m(m+1)}{2}\left[D^2B(X) - (DB(X))B(X)\right]. \tag{3.19}$$

Clearly, (3.19) defines nothing when $m = 0, -1$. This means that the covariantization program used here is not applicable to these two cases. As a matter of fact, it reflects that when the leading order is $\pm 1$, no $N = 1$ primary basis can be defined. We can actually verify this claim via the direct method of construction used in [18]. One should note that different $B(X)$'s may actually define the same $T(X)$, as long as its variation $\delta B$ satisfies

$$(\delta B)'' - (\delta B)'B - B'\delta B = 0. \tag{3.20}$$

The transformation law of $B(X)$ enables us to introduce a covariant superderivative defined by

$$\hat{D}_{2k} \equiv D - 2kB(X). \tag{3.21}$$

One can verify easily that $\hat{D}_{2k}$ maps from $F_k$ to $F_{k+\frac{1}{2}}$. Hence, the operator

$$\begin{aligned} \hat{D}^l_{2k} &\equiv \hat{D}_{2k+l-1}\hat{D}_{2k+l-2}\cdots\hat{D}_{2k} \qquad (l > 0) \\ &= [D-(2k+l-1)B][D-(2k+l-2)B]\cdots[D-2kB] \end{aligned} \tag{3.22}$$

maps from $F_k$ to $F_{k+\frac{l}{2}}$. Obviously, we also need the inverse operators of $\hat{D}^l_{2k}$ $(l > 0)$, which are defined as

$$\begin{aligned} \hat{D}^{-1}_{2k} &\equiv (\hat{D}_{2k-1})^{-1} = [D-(2k-1)B]^{-1} \\ \hat{D}^{-l}_{2k} &\equiv \hat{D}^{-1}_{2k-l-1}\hat{D}^{-1}_{2k-l-2}\cdots\hat{D}^{-1}_{2k} \qquad (l > 0). \end{aligned} \tag{3.23}$$

With these definitions, we have the following formulae:

$$\hat{D}_{2k}\delta B = -\delta B\hat{D}_{2k-1} + \triangle B \tag{3.24}$$

where $\delta B$ is an *arbitrary* variation and $\triangle B \equiv D(\delta B) - B\delta B$

$$\begin{aligned} \hat{D}_{2k+1}\hat{D}_{2k}\delta B &= \delta B\hat{D}_{2k}\hat{D}_{2k-1} \\ \hat{D}^{-1}_{2k-1}\hat{D}^{-1}_{2k}\delta B &= \delta B\hat{D}^{-1}_{2k-2}\hat{D}^{-1}_{2k-1} \end{aligned} \tag{3.25}$$

where $\delta B$ is subjected to (3.20). By using (3.21)–(3.25), we can derive (which were derived in [27, 28], only for positive $m$)

$$\delta_B\hat{D}^{2m}_{2k} = -\delta B(m\hat{D}^{2m-1}_{2k}) - \triangle B\left[m(2k+m-1)\hat{D}^{2m-2}_{2k}\right] \tag{3.26}$$

and

$$\delta_{\mathrm{B}}\hat{D}_{2k}^{2m+1} = -\delta B\left[(2k+m)\hat{D}_{2k}^{2m}\right] - \triangle B\left[m(2k+m)\hat{D}_{2k}^{2m-1}\right]. \tag{3.27}$$

Here, $\delta B$ is subject to constraint (3.20).

We now write the covariant form of $L$

$$\begin{aligned} L &= D^{2m+1} + U_2 D^{2m-1} + U_3 D^{2m-2} + \cdots \\ &= \hat{D}_{-m}^{2m+1} + \Delta_2^{(2m+1)}(U_2, T) + \sum_{k=4}^{\infty} \Delta_k^{(2m+1)}(W_k, T) \end{aligned} \tag{3.28}$$

where $W_k$ is a superconformal primary field of spin $\frac{k}{2}$ and

$$\Delta_p^{(2m+1)}(W_p, T) = \sum_{i=0}^{\infty} \alpha_{p,i}^{(2m+1)} (\hat{D}_p^i W_p) \hat{D}_{-m}^{2m+1-p-i} \qquad \alpha_{p,0} = 1. \tag{3.29}$$

The coefficients $\alpha_{p,i}^{(2m+1)}$ are determined by requiring that the right-hand side of (3.29) depends on $B$ only through $T$. In other words, they are solved from the recursion relations arising from the equations $\delta_{\mathrm{B}}\Delta_p^{(2m+1)} = 0$. However, since (3.26) and (3.27) are valid for all integers $m$, we expect that the recursion relations obtained for positive $m$ [28] remain valid for non-positive $m$. As a result, the formulae of $\alpha_{p,i}^{(2m+1)}$ for positive $m$ remain valid for non-positive $m$. Therefore, without any calculations, we have

$$\begin{aligned} \alpha_{2p,2l}^{(2m+1)} &= (-1)^l \frac{\binom{l+p-m-1}{l}\binom{p+l-1}{l}}{\binom{2p+l-1}{l}} \\ \alpha_{2p,2l+1}^{(2m+1)} &= \frac{(-1)^l}{2} \frac{\binom{p+l-m-1}{l}\binom{p+l}{l}}{\binom{2p+l}{l}} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} \alpha_{2q+1,2l}^{(2m+1)} &= (-1)^l \frac{\binom{q+l-m-1}{l}\binom{q+l}{l}}{\binom{2q+l}{l}} \\ \alpha_{2q+1,2l+1}^{(2m+1)} &= (-1)^l \frac{m-q}{2q+1} \frac{\binom{q+l-m}{l}\binom{q+l}{l}}{\binom{2q+l+1}{l}}. \end{aligned} \tag{3.31}$$

Substitutions of (3.29)–(3.31) back into (3.28) give the desired decompositions of the coefficient functions $U_k$ into differential polynomials in $T$ and the $N = 1$ primary fields $W_k$.

We have seen in this section that the generalization of the covariantization program established in [27, 28] to the case of superpseudodifferential operators is quite straightforward. Key formulae like (3.17), (3.26)–(3.31) remain unchanged.

## 4. $N=2$ Supermultiplets

The existence of the $N=2$ super Virasoro algebra (2.10) leads naturally to the conjecture that the $N=1$ primary fields $W_k$ can be redefined in such a way that $W_{2k}$ and $W_{2k+1}$ ($k \geqslant 2$) together form a $N=2$ supermultiplet, i.e. under the spin-1 flow $J(X_H)$, defined by (2.11), they transform as

$$\begin{aligned}\delta_\zeta W_{2k} &= 2W_{2k+1}\zeta \\ \delta_\zeta W_{2k+1} &= -kW_{2k}\zeta'' + \tfrac{1}{2}W'_{2k}\zeta' - \tfrac{1}{2}W''_{2k}\zeta.\end{aligned} \tag{4.1}$$

Since there is no simple way to handle this flow, it is not even clear whether or not this conjecture holds in general for superdifferential operators. Hence, we shall restrict ourselves to a very limited goal. We shall just consider the negative part of a superpseudodifferential operator of positive leading order $2m+1$ ($m>0$) and present a general observation on this problem.

First, we observe that

$$[J(X_H)]_\pm = \left[-\zeta D - (m+1)\zeta'\right] L_\pm - L_\pm \left[-\zeta D + m\zeta'\right] \tag{4.2}$$

that is, the positive part $L_+$ and the negative part $L_-$ transform independently under spin-1 flow. Therefore, it is possible to consider only the negative part. Secondly, since for a given $k>1$, $U_{2m+k}$ is a function of $T$ and $W_{2m+l}$ ($k \geqslant l$), and since

$$\begin{aligned}\delta_\zeta T &= \left[-JD^2 + \tfrac{1}{2}J'D - \tfrac{1}{2}J''\right]\zeta \\ \delta_\zeta J &= \left[m(m+1)D^3 + 2T\right]\zeta\end{aligned} \tag{4.3}$$

$\delta_\zeta W_{2m+k}$ must depend only on $J$, $T$ and $W_{2m+l}$ ($k \geqslant l$). As a result, the possible redefinition of $W_{2m+k}$ is of the form

$$\bar{W}_{2m+k} = W_{2m+k} + f_{2m+k}(J, W_{2m+1}, W_{2m+2}, \ldots, W_{2m+k-1}) \tag{4.4}$$

where $f_{2m+k}$ is a differential polynomial. For instance, based on the dimensional consideration, we have

$$\begin{aligned}&\bar{W}_{2m+2} = W_{2m+2} \qquad \bar{W}_{2m+3} = W_{2m+3} \\ &\bar{W}_{2m+4} = W_{2m+4} + aJW_{2m+2} \qquad \bar{W}_{2m+5} = W_{2m+5} + bJW_{2m+3}.\end{aligned} \tag{4.5}$$

It follows immediately from (4.5) that $W_{2m+2}$ and $W_{2m+3}$ must form a $N=2$ supermultiplet if it exists at all. In the following, we verify that this is indeed true and determine the values of $a$ and $b$ which make $\bar{W}_{2m+4}$ and $\bar{W}_{2m+5}$ form a $N=2$ supermultiplet.

Using (3.30), (3.31) and the following identities:

$$\begin{aligned}\hat{D}^2_{2k} &= D^2 - BD - 2kB' \\ \hat{D}^3_{2k} &= D^3 - (2k+1)BD^2 - (2k+1)B'D - 2kB'' + 4k(k+1)BB'\end{aligned} \tag{4.6}$$

$$\begin{aligned}\hat{D}^{-1}_{-m} &= D^{-1} + (m+1)BD^{-2} - (m+1)B'D^{-3} - \left[(m+1)B'' + (m+1)^2B'B\right]D^{-4} + \cdots \\ \hat{D}^{-2}_{-m} &= D^{-2} + BD^{-3} - (m+2)B'D^{-4} + \cdots \\ \hat{D}^{-3}_{-m} &= D^{-3} + (m+2)BD^{-4} + \cdots \\ \hat{D}^{-4}_{-m} &= D^{-4} + \cdots\end{aligned} \tag{4.7}$$

we easily compute

$$
\begin{aligned}
\Delta^{(2m+1)}_{2m+2}(W_{2m+2},T) &= W_{2m+2}D^{-1} + \tfrac{1}{2}W'_{2m+2}D^{-2} - \tfrac{1}{2}W''_{2m+2}D^{-3} \\
&\quad - \left[\frac{m+2}{2(2m+3)}W'''_{2m+2} + \frac{2(m+1)}{m(2m+3)}TW_{2m+2}\right]D^{-4} + \cdots \\
\Delta^{(2m+1)}_{2m+3}(W_{2m+3},T) &= W_{2m+3}D^{-2} - \frac{1}{2m+3}W'_{2m+3}D^{-3} - \frac{m+2}{2m+3}W''_{2m+3}D^{-4} + \cdots \\
\Delta^{(2m+1)}_{2m+4}(W_{2m+4},T) &= W_{2m+4}D^{-3} + \tfrac{1}{2}W'_{2m+4}D^{-4} + \cdots \\
\Delta^{(2m+1)}_{2m+5}(W_{2m+5},T) &= W_{2m+5}D^{-4} + \cdots .
\end{aligned}
\tag{4.8}
$$

The desired decompositions that can then be read off from (4.8) are

$$
\begin{aligned}
U_{2m+2} &= W_{2m+2} \\
U_{2m+3} &= W_{2m+3} + \tfrac{1}{2}W'_{2m+2} \\
U_{2m+4} &= W_{2m+4} - \frac{1}{2m+3}W'_{2m+3} - \tfrac{1}{2}W''_{2m+2} \\
U_{2m+5} &= W_{2m+5} + \tfrac{1}{2}W'_{2m+4} - \frac{m+2}{2m+3}W''_{2m+3} - \frac{m+2}{2(2m+3)}W'''_{2m+2} - \frac{2(m+1)}{m(2m+3)}TW_{2m+2} .
\end{aligned}
\tag{4.9}
$$

Next, we find the spin-1 transformations of $U_{2m+2}, \ldots, U_{2m+5}$:

$$
\begin{aligned}
[J(X_H)]_- &= [-\zeta D - (m+1)\zeta']L_- - L_-[-\zeta D + m\zeta'] \\
&\equiv (\delta_\zeta U_{2m+2})D^{-1} + (\delta_\zeta U_{2m+3})D^{-2} + (\delta_\zeta U_{2m+4})D^{-3} + (\delta_\zeta U_{2m+5})D^{-4} + \cdots
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
\delta_\zeta U_{2m+2} &= [-U_{2m+2}D + 2U_{2m+3}]\zeta \\
\delta_\zeta U_{2m+3} &= [-(m+1)U_{2m+2}D^2 + U_{2m+3}D - U'_{2m+3}]\zeta \\
\delta_\zeta U_{2m+4} &= [-(m+1)U_{2m+2}D^3 - U_{2m+3}D^2 - (U'_{2m+4} - 2U_{2m+5})]\zeta \\
\delta_\zeta U_{2m+5} &= [(m+1)U_{2m+2}D^4 + mU_{2m+3}D^3 - (m+2)U_{2m+4}D^2 + U_{2m+5}D - U'_{2m+5}]\zeta .
\end{aligned}
\tag{4.11}
$$

Combining (4.9) and (4.11), we finally obtain

$$
\begin{aligned}
\delta_\zeta W_{2m+2} &= 2W_{2m+3}\zeta \\
\delta_\zeta W_{2m+3} &= \left[-(m+1)W_{2m+2}D^2 + \tfrac{1}{2}W'_{2m+2}D - \tfrac{1}{2}W''_{2m+2}\right]\zeta \\
\delta_\zeta W_{2m+4} &= 2W_{2m+5}\zeta - \frac{2(m+1)}{m(2m+3)}W_{2m+2}\left[m(m+1)D^3 + 2T\right]\zeta \\
\delta_\zeta W_{2m+5} &= \left[-(m+2)W_{2m+4}D^2 + \tfrac{1}{2}W'_{2m+4}D - \tfrac{1}{2}W''_{2m+4}\right]\zeta \\
&\quad + \frac{2(m+1)}{m(2m+3)}W_{2m+3}\left[m(m+1)D^3 + 2T\right]\zeta \\
&\quad + \frac{2(m+1)}{m(2m+3)}W_{2m+2}\left[-JD^2 + \tfrac{1}{2}J'D - \tfrac{1}{2}J''\right]\zeta .
\end{aligned}
\tag{4.12}
$$

As expected, $W_{2m+2}$ and $W_{2m+3}$ indeed form an $N=2$ supermultiplet, while $\delta_\zeta W_{2m+4}$ and $\delta_\zeta W_{2m+5}$ both contain some unwanted terms. Therefore, we have to consider the redefinitions (4.5). In fact, we find

$$\delta_\zeta \bar{W}_{2m+4} = 2\bar{W}_{2m+5}\zeta + 2(a-b)JW_{2m+3}\zeta + \left[a - \frac{2(m+1)}{m(2m+3)}\right](\delta_\zeta J)W_{2m+2}. \tag{4.13}$$

Hence, the only choice is

$$a = b = \frac{2(m+1)}{m(2m+3)}. \tag{4.14}$$

With this choice, we verify

$$\delta_\zeta \bar{W}_{2m+5} = \left[-(m+2)\bar{W}_{2m+4}D^2 + \tfrac{1}{2}\bar{W}'_{2m+4}D - \tfrac{1}{2}\bar{W}''_{2m+4}\right]\zeta \tag{4.15}$$

as we wished.

We have thus identified the first two $N=2$ supermultiplets in the negative part of $L$. It is natural to expect that all the desired supermultiplets actually exist.

Finally, we present an observation on this identification problem. We shall show that, if all the required $N=2$ supermultiplets can be defined when the leading order is $2m+1$ ($m$ can be either positive or negative), then they can also be defined when the leading order is $-2m-1$. For definiteness, we assume, for a moment, that $m>0$. We use the notation defined by (3.12) and (3.15) and impose condition (3.13). We have observed in the previous section that (3.13) is invariant under superconformal transformation. We now recast this statement by means of the super Virasoro flows defined by the second Gelfand–Dickey bracket. Let $\delta_\xi^\pm L^\pm$ denote the super Virasoro flows generated by $T^\pm$ via the respective second Gelfand–Dickey bracket. Then (3.13) implies

$$\begin{aligned}
\delta_\xi^+ L^- &= -L^-(\delta_\xi^+ L^+)L^- \\
&= -L^-\left[\left(\xi D^2 + \tfrac{1}{2}\xi' D + \frac{m+1}{2}\xi''\right)L^+ - L^+\left(\xi D^2 + \tfrac{1}{2}\xi' D - \frac{m}{2}\xi''\right)\right]L^- \\
&= \left[\xi D^2 + \tfrac{1}{2}\xi' D + \frac{(-m-1)+1}{2}\xi''\right]L^- - L^-\left[\xi D^2 + \tfrac{1}{2}\xi' D - \frac{(-m-1)}{2}\xi''\right] \\
&= \delta_\xi^- L^-.
\end{aligned} \tag{4.16}$$

The fact that $T^- = T^+$, together with (4.16), leads to the statement that, under identification (3.13), the $N=1$ primary fields which appear in the superconformally covariant form of $L^+$ are still primary fields, even when the second Gelfand–Dickey bracket of $L^-$ is used instead. As a consequence, decompositions of the coefficients $U_k^+$ into $N=1$ primary fields immediately induce decompositions of $U_k^-$ by the use of (3.13). Next we consider the spin-1 flows which we shall denote by $\delta_\zeta^\pm L^\pm$. Repeating the above steps yields

$$\begin{aligned}
\delta_\zeta^+ L^- &= -L^-(\delta_\zeta^+ L^+)L^- \\
&= \left[-\zeta D + m\zeta'\right]L^- - L^-\left[-\zeta D - (m+1)\zeta'\right] \\
&= \delta_\zeta^- L^-.
\end{aligned} \tag{4.17}$$

Now, since $J^- = -J^+$, we conclude that the second Gelfand–Dickey brackets of $L^+$ and $L^-$ both lead to the same spin-1 flow (up to an overall sign) when the functional

$H^+ = \int_B J^+\zeta$ is used in either bracket. More explicitly, what we have shown so far is that, for any functional $F$,

$$\begin{aligned} \{F, T^-(X)\}^- &= \{F, T^+(X)\}^+ = \{F, T^-(X)\}^+ \\ \{F, J^-(X)\}^- &= \{F, J^+(X)\}^+ = -\{F, J^-(X)\}^+ \end{aligned} \tag{4.18}$$

where $\{\,,\}^\pm$ denotes the second Gelfand–Dickey bracket of $L^\pm$, respectively. It is clear now that if $W_{2k}$ and $W_{2k+2}$ form an $N = 2$ supermultiplet with respect to $\{\,,\}^+$, then they will also with respect to $\{\,,\}^-$. Therefore, once the required $N = 2$ supermultiplets have been identified for $L^+$, the corresponding task for $L^-$ is automatically achieved. Interchanging the roles of $L^+$ and $L^-$ obviously gives the proof for $m < 0$. This completes the proof for the above claim.

## 5. Concluding remarks

In this paper, we have discussed the $N = 2$ superalgebras arising from the second Gelfand–Dickey bracket of superpseudodifferential operators. We find that the forms of several formulae, derived previously for the case of superdifferentials, remain unchanged in this case. In other words, the generalization is straightforward. For example, formulae (3.30) and (3.31), obtained in [27, 28], immediately give us the superconformally covariant form of superpseudodifferential operators. Hence, the biggest problem regarding the spectrum of these superalgebras is still the identification of $N = 2$ supermultiplets. Since the positive and negative parts of a superpseudodifferential operator transform independently under the super Virasoro flow as well as the spin-1 flow, unless the identification problem can be solved for pure superdifferentials, the resolution of this problem in the present case is not possible. We remark that, in [27, 29], it is observed that when $L = D^5 + U_2D^3 + \cdots + U_5$, the $N = 1$ primary fields arising from the Drinfeld–Sokolov-type matrix formalation [29, 30] form precisely the desired $N = 2$ supermultiplets. One might suspect that the matrix formulation might be helpful in this problem. Hence, it seems worthwhile to discuss the spin-1 flow in the context of matrix formulation. Finally, we remark that it would be interesting to investigate all the possible reductions, contractions and trunctions of these $W_{\mathrm{KP}}^{(n)}$-type superalgebras. Hopefully, some interesting $W_\infty$-type superalgebras will emerge. Work in this direction is in progress.

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