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A note on first passage functionals for hyper-exponential jump-diffusion processes

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Abstract

This investigation concerns the hyper-exponential jump-diffusion processes. Following the exposition of the two-sided exit problem by Kyprianou [10] and Asmussen and Albrecher [1], this study investigates first passage functionals for these processes. The corresponding boundary value problems are solved to obtain an explicit formula for the first passage functionals.

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1 Introduction

This investigation concerns the hyper-exponential jump-diffusion processes. Owing to its analytical tractability, such processes have become popular among practitioners and academicians who work in mathematical finance and insurance. See, for example, the work of Jeanblanc et al.[7], and Asmussen and Albrecher[1] and the references therein. Following the exposition of the two-sided exit problem by Kyprianou[10] and Asmussen and Albrecher[1], the first passage functional of the following form is studied herein:

$$\Phi(x) = \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right] \tag{1.1}$$

where $r \geq 0$, g is a nonnegative bounded measurable function, $X_0 = x$ a.s. under $\mathbb{P}_x$ and τ is the exit time of X from a finite interval $I = (h_1, h_2)$. The function $\Phi(x)$, given in Eq.(1.1), is typically referred to as the Gerber-Shiu function. For recent works on this topic, see [11], [3], [9] and Chapter XII of [1].

Section 2 describes the hyper-exponential jump-diffusion processes that are considered herein. To find an explicit formula for the function $\Phi(x)$ in Eq.(1.1), the corresponding two-sided boundary value problem is considered. By direct calculation, the associated integro-differential equation is transformed into a homogeneous ODE of higher order, which is then solved. In fact, in Theorem 2.5 below, this ODE is solved

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in closed form and its solution equals the first passage functional Φ. Theorem 2.5 is of interest, in itself, (see also Remark 2.6,) and it can be utilized to solve other problems in mathematical finance and insurance. (See, for example, [5] and [6].)

2 Main Results

Given a constant $r \geq 0$ and a nonnegative bounded measurable function g, the first passage functional $\Phi(x)$, defined in Eq.(1.1), is computed. X is assumed to be given by a jump-diffusion process,

$$X_t = ct + \sigma W_t - \sum_{n=1}^{N_t} Y_n, \qquad t \geq 0. \tag{2.1}$$

Here, $c \in \mathbb{R}$; $\sigma > 0$; $W = (W_t; t \geq 0)$ is a standard Brownian motion; $N = (N_t; t \geq 0)$ is a compound Poisson process with rate $\lambda > 0$, and the jump sizes $(Y_n, n \geq 1)$ are independent and identically distributed. All of the aforementioned objects are mutually independent. The distribution F of Y_1 is assumed to have the probability density function

$$f(y) = \begin{cases} \sum_{j=1}^{m_{(+)}} p_j \eta_j^+ e^{-\eta_j^+ y}, & y > 0, \\ 0, & y = 0, \\ \sum_{j=1}^{m_{(-)}} q_j \eta_j^- e^{\eta_j^- y}, & y < 0. \end{cases} \tag{2.2}$$

where $\sum_{j=1}^{m_{(+)}} p_j + \sum_{j=1}^{m_{(-)}} q_j = 1$, $p_j, q_j, \eta_j^{\pm} > 0$, and $\eta_i^{\pm} \neq \eta_j^{\pm}$ for $i \neq j$. $\mathbb{P}_x$ denotes the law of $X + x$ under $\mathbb{P}$. By Dynkin's formula and the theorem of Feynman and Kac, the following boundary value problem which admits at most one solution, must be solved: find $\Phi \in \mathcal{C}([h_1, h_2]) \cap \mathcal{C}^2((h_1, h_2))$ such that

$$\begin{cases} (\mathcal{L} - r)\Phi = 0, & \text{in } (h_1, h_2), \\ \Phi = g, & \text{on } (-\infty, h_1] \cup [h_2, \infty), \end{cases} \tag{2.3}$$

where $\mathcal{L}$ is the infinitesimal generator of X that acts on $h \in \mathcal{C}_0^2(\mathbb{R})$ by

$$\mathcal{L}h(x) = \frac{\sigma^2}{2} h''(x) + ch'(x) + \lambda \int h(x-y) f(y) dy - \lambda h(x). \tag{2.4}$$

For details, see [2]. $\mathcal{L}h(x)$ is defined by the Eq.(2.4) for all functions h on $\mathbb{R}$ such that h', h'' and the integral in Eq.(2.4) exist at x. Notably, the characteristic exponent $\psi(\zeta)$ of X is given by

$$\psi(\zeta) = \frac{\sigma^2}{2}\zeta^2 + c\zeta + \lambda\Big(\sum_{j=1}^{m_{(+)}} \frac{p_j \eta_j^+}{\zeta + \eta_j^+} + \sum_{j=1}^{m_{(-)}} \frac{-q_j \eta_j^-}{\zeta - \eta_j^-}\Big) - \lambda, \qquad \zeta \in i\mathbb{R}.$$

Accordingly, ψ is an analytic function on $\mathbb{C}$ except at a finite number of poles. Also, the equation $\psi(\zeta) - r = 0$ yields $m_{(+)} + m_{(-)} + 2$ distinct real zeros. (If $r = 0$, $c \neq 0$ or $m_{(+)} + m_{(-)} > 0$ is further assumed.) Set $S = m_{(+)} + m_{(-)} + 2$ and let $\rho_1, \rho_2, \cdots, \rho_S$ be the distinct real zeros of the equation $\psi(\zeta) - r = 0$.

Let $\mathcal{P}_0(\zeta) = \prod_{j=1}^{m_{(+)}} (\zeta + \eta_j^+) \prod_{j=1}^{m_{(-)}} (\zeta - \eta_j^-)$. Now, $\mathcal{P}_1(\zeta) = \mathcal{P}_0(\zeta)(\psi(\zeta) - r)$ is a polynomial whose zeros coincide with those of $\psi(\zeta) - r$. Denote by D the differential operator such that its characteristic polynomial is $\mathcal{P}_1(\zeta)$.

Proposition 2.1. *Suppose a bounded solution Φ defined on $\mathbb{R}$ to the boundary value problem (2.3) exists. Then, on (h_1, h_2), Φ is infinitely differentiable and satisfies the ODE,*

$$D\Phi \equiv 0, \text{ on } (h_1, h_2). \tag{2.5}$$

Hence, on (h_1, h_2), $\Phi(x) = \sum_{i=1}^{S} \mathbf{Q}_i e^{\rho_i x}$ for some constants $\mathbf{Q}_i$.

Proof. This proposition is proved by direct computation. Plugging the density function f, given by Eq.(2.2), into Eq.(2.4), yields the generator $\mathcal{L}$ that acts on Φ:

$$\begin{aligned}\mathcal{L}\Phi(x) = \quad & \tfrac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda\left(\textstyle\sum_{j=1}^{m_{(+)}} p_j\eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^{x}\Phi(y)e^{\eta_j^+ y}dy\right. \\ & \left.+\textstyle\sum_{j=1}^{m_{(-)}} q_j\eta_j^- e^{\eta_j^- x}\int_{x}^{\infty}\Phi(y)e^{-\eta_j^- y}dy\right) - \lambda\Phi(x) \qquad .\end{aligned}$$

From this equation and the fact that $\sigma > 0$ and $(\mathcal{L}-r)\Phi \equiv 0$, Φ is infinitely differentiable on (h_1, h_2), as can be established by induction, as in the work of Chen et al.[4].

Next, Φ will be shown to satisfy an ODE. Consider the differentiation rule,

$$\begin{aligned}&\left(\frac{d}{dx}+\eta_j^+\right)p_j\eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^{x}\Phi(y)e^{\eta_j^+ y}dy \\ &= p_j\eta_j^+\left[\left(-\eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^{x}\Phi(y)e^{\eta_j^+ y}dy + \Phi(x)\right) + \eta_j^+ e^{-\eta_j^+ x}\int_{-\infty}^{x}\Phi(y)e^{\eta_j^+ y}dy\right] \\ &= p_j\eta_j^+\Phi(x),\end{aligned}$$

and similarly, $\left(\frac{d}{dx}-\eta_j^-\right)q_j\eta_j^- e^{\eta_j^- x}\int_x^\infty \Phi(y)e^{-\eta_j^- y}dy = -q_j\eta_j^-\Phi(x)$. Since Φ is infinitely differentiable on (h_1, h_2) and $(\mathcal{L}-r)\Phi \equiv 0$ on (h_1, h_2),

$$\begin{aligned}0 &= \left(\frac{d}{dx}+\eta_1^+\right)\cdots\left(\frac{d}{dx}+\eta_{m_{(+)}}^+\right)\left(\frac{d}{dx}-\eta_1^-\right)\cdots\left(\frac{d}{dx}-\eta_{m_{(-)}}^-\right)(\mathcal{L}-r)\Phi(x) \\ &= \prod_{j=1}^{m_{(+)}}\left(\frac{d}{dx}+\eta_j^+\right)\prod_{j=1}^{m_{(-)}}\left(\frac{d}{dx}-\eta_j^-\right)\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\lambda-r\right)\Phi(x) \\ &+ \sum_{j=1}^{m_{(+)}}\prod_{k=1,k\neq j}^{m_{(+)}}\left(\frac{d}{dx}+\eta_k^+\right)p_j\eta_j^+\Phi(x) - \sum_{j=1}^{m_{(-)}}\prod_{k=1,k\neq j}^{m_{(-)}}\left(\frac{d}{dx}-\eta_k^-\right)q_j\eta_j^-\Phi(x). \end{aligned} \tag{2.6}$$

Hence, Eq.(2.6) transforms the integro-differential equation $(\mathcal{L}-r)\Phi \equiv 0$ into an ODE: $\widetilde{D}\Phi \equiv 0$, where $\widetilde{D}$ is a high order differential operator.

To complete the proof, $\widetilde{D}$ must be shown to coincide with D. (See the definition of D in the paragraph above Proposition 2.1.) To show that the characteristic polynomials of D and $\widetilde{D}$ coincide will suffice. Write $\widetilde{\mathcal{P}}(\zeta)$ as the characteristic polynomial of $\widetilde{D}$. Then, by Eq.(2.6), $\widetilde{\mathcal{P}}$ is given by

$$\begin{aligned}\widetilde{\mathcal{P}}(\zeta) = \quad & \prod_{j=1}^{m_{(+)}}(\zeta+\eta_j^+)\prod_{j=1}^{m_{(-)}}(\zeta-\eta_j^-)\left[\frac{\sigma^2}{2}\zeta^2 + c\zeta + \lambda\left(\sum_{j=1}^{m_{(+)}}\frac{p_j\eta_j^+}{\zeta+\eta_j^+} + \sum_{j=1}^{m_{(-)}}\frac{-q_j\eta_j^-}{\zeta-\eta_j^-}\right) - (\lambda+r)\right] \\ = \quad & \mathcal{P}_0(\zeta)(\psi(\zeta)-r).\end{aligned}$$

This equation reveals that the characteristic polynomial $\mathcal{P}_1(\zeta)$ of D equals that, $\widetilde{\mathcal{P}}(\zeta)$, of $\widetilde{D}$. The proof is complete. □

Proposition 2.2. *Suppose that Φ is a bounded solution to the boundary value problem (2.3) and, on (h_1, h_2), $\Phi(x) = \sum_{i=1}^{S}\mathbf{Q}_i e^{\rho_i x}$ for some constants $\mathbf{Q}_i$. Then the constant vector $\mathbf{Q}$ satisfies the equation*

$$A\mathbf{Q} = V_g \tag{2.7}$$

where

$$A = \begin{bmatrix} \frac{1}{\rho_1+\eta_1^+}e^{\rho_1 h_1} & \cdots & \frac{1}{\rho_S+\eta_1^+}e^{\rho_S h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\rho_1+\eta_{m_{(+)}}^+}e^{\rho_1 h_1} & \cdots & \frac{1}{\rho_S+\eta_{m_{(+)}}^+}e^{\rho_S h_1} \\ \frac{1}{\rho_1-\eta_1^-}e^{\rho_1 h_2} & \cdots & \frac{1}{\rho_S-\eta_1^-}e^{\rho_S h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\rho_1-\eta_{m_{(-)}}^-}e^{\rho_1 h_2} & \cdots & \frac{1}{\rho_S-\eta_{m_{(-)}}^-}e^{\rho_S h_2} \\ e^{\rho_1 h_1} & \cdots & e^{\rho_S h_1} \\ e^{\rho_1 h_2} & \cdots & e^{\rho_S h_2} \end{bmatrix} \tag{2.8}$$

and V_g is a column vector whose components $V_g(i)$ are given by the formula

$$V_g(i) = \begin{cases} \int_{-\infty}^{h_1} g(y)e^{\eta_i^+(y-h_1)}dy, & \textit{if } 1 \le i \le m_{(+)} \\ -\int_{h_2}^{\infty} g(y)e^{-\eta_{i-m_{(+)}}^-(y-h_2)}dy, & \textit{if } m_{(+)}+1 \le i \le m_{(+)}+m_{(-)} \\ g(h_1), & \textit{if } i = m_{(+)}+m_{(-)}+1 \\ g(h_2), & \textit{if } i = m_{(+)}+m_{(-)}+2 \end{cases} \tag{2.9}$$

Proof. Let $m = m_{(+)} + m_{(-)}$. Since $(\mathcal{L} - r)\Phi = 0$ on (h_1, h_2), for $x \in (h_1, h_2)$,

$$\begin{aligned} 0 = &\quad \tfrac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda \textstyle\int \Phi(x-y)f(y)dy - (\lambda + r)\Phi(x) \\ = &\quad \textstyle\sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x}(\tfrac{\sigma^2}{2}\rho_i^2 + c\rho_i - (\lambda + r)) + \lambda \int \Phi(x-y)f(y)dy. \end{aligned} \tag{2.10}$$

Furthermore,

$$\begin{aligned} &\int \Phi(x-y)f(y)dy = \left(\int_{-\infty}^{h_1} + \int_{h_2}^{\infty}\right) g(y)f(x-y)dy + \int_{x-h_2}^{x-h_1} \Phi(x-y)f(y)dy \\ &= \sum_{j=1}^{m_{(+)}} p_j e^{-\eta_j^+ x} \int_{-\infty}^{h_1} g(y)\eta_j^+ e^{\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}} q_j e^{\eta_j^- x} \int_{h_2}^{\infty} g(y)\eta_j^- e^{-\eta_j^- y}dy \\ &+ \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(-)}} q_j \eta_j^- \int_{x-h_2}^{0} e^{-\rho_i y} e^{\eta_j^- y}dy + \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(+)}} p_j \eta_j^+ \int_{0}^{x-h_1} e^{-\rho_i y} e^{-\eta_j^+ y}dy \\ &= \sum_{j=1}^{m_{(+)}} p_j e^{-\eta_j^+ x} \int_{-\infty}^{h_1} g(y)\eta_j^+ e^{\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}} q_j e^{\eta_j^- x} \int_{h_2}^{\infty} g(y)\eta_j^- e^{-\eta_j^- y}dy \\ &+ \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(-)}} \frac{q_j \eta_j^-}{\eta_j^- - \rho_i} \left(1 - e^{-(\eta_j^- - \rho_i)(h_2 - x)}\right) \\ &+ \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(+)}} \frac{p_j \eta_j^+}{\rho_i + \eta_j^+} \left(1 - e^{-(\rho_i + \eta_j^+)(x - h_1)}\right) \end{aligned} \tag{2.11}$$

Now, by Eqs.(2.10) and (2.11) and the fact $\psi(\rho_i) - r = 0$ for all i, we have

$$\begin{aligned} 0 = &\sum_{j=1}^{m_{(+)}} p_j e^{-\eta_j^+ x} \int_{-\infty}^{h_1} g(y)\eta_j^+ e^{\eta_j^+ y}dy + \sum_{j=1}^{m_{(-)}} q_j e^{\eta_j^- x} \int_{h_2}^{\infty} g(y)\eta_j^- e^{-\eta_j^- y}dy \\ &+ \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(-)}} \frac{-q_j \eta_j^-}{\eta_j^- - \rho_i} e^{-(\eta_j^- - \rho_i)(h_2 - x)} + \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(+)}} \frac{-p_j \eta_j^+}{\rho_i + \eta_j^+} e^{-(\rho_i + \eta_j^+)(x - h_1)}. \end{aligned}$$

Comparing $e^{-\eta_j^+ x}$ and $e^{\eta_j^- x}$ yields (2.7). The proof is complete. $\square$

Remark 2.3. *Consider the function* $V(x) = \sum_{i=1}^{S} \mathbf{Q}_i e^{\rho_i x}$ *for* $x \in (h_1, h_2)$*, and* $V(x) = g(x)$ *otherwise, where* g *is a bounded function on* $(h_1, h_2)^c$ *and* $\mathbf{Q}_i$*'s are given constants. Now,*

$$(\mathcal{L} - r)V(x) = \sum_{j=1}^{m_{(+)}} p_j e^{-\eta_j^+ x} \int_{-\infty}^{h_1} g(y)\eta_j^+ e^{\eta_j^+ y} dy$$
$$+ \sum_{j=1}^{m_{(-)}} q_j e^{\eta_j^- x} \int_{h_2}^{\infty} g(y)\eta_j^- e^{-\eta_j^- y} dy + \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(-)}} \frac{-q_j \eta_j^-}{\eta_j^- - \rho_i} e^{-(\eta_j^- - \rho_i)(h_2 - x)}$$
$$+ \sum_{i=1}^{m+2} \mathbf{Q}_i e^{\rho_i x} \sum_{j=1}^{m_{(+)}} \frac{-p_j \eta_j^+}{\rho_i + \eta_j^+} e^{-(\rho_i + \eta_j^+)(x - h_1)}. \tag{2.12}$$

Lemma 2.4. *For any* $h_1 < h_2$*, the matrix* A *given by Eq.(2.8) is invertible.*

Proof. Let $m = m_{(+)} + m_{(-)}$ and assume $AC = 0$ for some vector $C = [C_1, C_2, \cdots, C_{m+2}]^T$. Consider the function $V(x) = \sum_{i=1}^{m+2} C_n e^{\rho_n x}$ for $x \in (h_1, h_2)$, and $V(x) = 0$ otherwise. Since $AC = 0$ and by Eq.(2.12), $V(x)$ is a solution to the boundary value problem (2.3) with $g(x) \equiv 0$. From the uniqueness of solutions to the boundary value problem (2.3), $V(x) = \sum_{n=1}^{m+2} C_n e^{\rho_n x} = 0$ for all $x \in (h_1, h_2)$. Now consider the Wronskian

$$W(e^{\rho_1 x}, \cdots, e^{\rho_{m+2} x}) \equiv det \begin{bmatrix} e^{\rho_1 x} & \cdots & e^{\rho_{m+2} x} \\ \rho_1 e^{\rho_1 x} & \cdots & \rho_{m+2} e^{\rho_{m+2} x} \\ \vdots & \ddots & \vdots \\ \rho_1^{m+1} e^{\rho_1 x} & \cdots & \rho_{m+2}^{m+1} e^{\rho_{m+2} x} \end{bmatrix}.$$

Then

$$\begin{aligned} W(e^{\rho_1 x}, \cdots, e^{\rho_{m+2} x}) &= exp\left((\textstyle\sum_{n=1}^{m+2} \rho_n) x \right) det \begin{bmatrix} 1 & \cdots & 1 \\ \rho_1 & \cdots & \rho_{m+2} \\ \vdots & \ddots & \vdots \\ \rho_1^{m+1} & \cdots & \rho_{m+2}^{m+1} \end{bmatrix} \\ &= exp\left((\textstyle\sum_{n=1}^{m+2} \rho_n) x \right) \prod_{1 \le i < j \le m+2} (\rho_i - \rho_j) \neq 0. \end{aligned} \tag{2.13}$$

(The matrix in Eq.(2.13) is a Vandermonde matrix.) This inequality implies that $\{e^{\rho_n x} | 1 \le n \le m+2\}$ are linearly independent and so $C = 0$, which implies that A is invertible. □

In the following, for a given function g on $(h_1, h_2)^c$, $\mathbf{Q}(g) = A^{-1} V_g$ is set where A and V_g are defined as in Eqs.(2.8) and (2.9), respectively. Also, $\mathbf{Y} \bullet \mathbf{Z}$ is written for the usual inner product of the vectors $\mathbf{Y}$ and $\mathbf{Z}$ in $\mathbb{R}^S$ and for every real value x, $\mathbf{e}^{\rho}(x) = [e^{\rho_1 x}, \cdots, e^{\rho_S x}]$. Our main result is as follows.

Theorem 2.5. *Given a constant* $r \ge 0$ *and a nonnegative bounded function* g *on* $(h_1, h_2)^c$*, the function* $\Phi(x)$*, defined by the formula*

$$\Phi(x) = \begin{cases} \mathbf{Q}(g) \bullet \mathbf{e}^{\rho}(x), & \text{if } x \in (h_1, h_2) \\ g(x), & \text{if } x \notin (h_1, h_2) \end{cases}, \tag{2.14}$$

solves the boundary value problem (2.3). Additionally, $\Phi(x) = \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right]$*, where* $\tau = \inf\{t \ge 0 | X_t \notin (h_1, h_2)\}$.

Proof. The first statement follows by direct calculation using Eqs. (2.12) and (2.7). The proof of the second statement(concerning the uniqueness of solutions of the boundary value problem (2.3)) is the same as that of Proposition 4.1 in the work of [4] if $\mathbb{R}_+$ is replaced by $\overline{(h_1, h_2)^c}$. This proof is omitted here. □

Remark 2.6. *When X is a spectrally negative Lévy process, the Laplace transform of the two-sided exit times can be expressed in terms of scale functions. See, for example, Theorem 8.1 in the work of Kyprianou[10], Chapter XI Theorems 3.2-3.3 in the work of Asmussen and Albrecher[1], or the work of Rogers[12]. See also Theorem 5.3 in Chapter XI of the work of Asmussen and Albrecher[1] in which the process X has two-sided phase-type jumps. Kuznetsov et al in [9] took a completely different approach to obtain the law of the discounted overshoot for meromorphic Lévy processes. From [9], the formula for the function Φ is obtained in an integral form. In Theorem 2.5, thus obtained, $\Phi(x)$ is expressed as a linear combination of known functions and the coefficients are uniquely determined. It is worth noting that, by the same approach, similar results as in Theorem 2.5 can be obtained for the case (h_1, ∞) or $(-\infty, h_2)$. (See, also, the work of Chen et al.[4].)*

3 Examples

As an illustration of the main result(Theorem 2.5), we consider the Kou model, that is, $m_{(-)} = m_{(+)} = 1$. We write $\eta^{\pm}$ for $\eta_1^{\pm}$ and assume $\sigma > 0$. Fix (h_1, h_2) and write $\tau_{(h_1,h_2)}$ for the first exit time of X from (h_1, h_2). Note that the matrix A is given by the formula in Eq.(2.8).

Example 3.1. *Consider the case, $g(x) = 1$. Then $\Phi(x) = \mathbb{E}_x\left[e^{-r\tau_{(h_1,h_2)}}\right]$ and by direct calculation, $V_g = [\frac{1}{\eta^+}, -\frac{1}{\eta^-}, 1, 1]^T$. Simple algebraic calculation shows that $AQ = V_g$ is equivalent to $\widehat{A}\widehat{D}Q = \widehat{V}_g$ where $\widehat{V}_g = [0,0,0,1]^T$, $\widehat{D} = diag(\rho_1, \cdots, \rho_4)$ and*

$$\widehat{A} = \begin{bmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_1 h_2-\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_2 h_2-\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_3 h_2-\rho_3 h_1}}{\rho_3} & \frac{e^{\rho_4 h_2-\rho_4 h_1}}{\rho_4} \\ \frac{e^{\rho_1 h_2}}{\rho_1} & \frac{e^{\rho_2 h_2}}{\rho_2} & \frac{e^{\rho_3 h_2}}{\rho_3} & \frac{e^{\rho_4 h_2}}{\rho_4} \end{bmatrix}.$$

(Note that for the vector $\widehat{V}_g$, all components except the last one are zeros.) Therefore, $\mathbb{E}_x\left[e^{-r\tau_{(h_1,h_2)}}\right] = \sum_{i=1}^4 Q_i e^{\rho_i x}$ for $x \in (h_1, h_2)$ where $Q_i = (\rho_i det\widehat{A})^{-1} Y_i$,

$$Y_1 = -\begin{vmatrix} \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_2 h_2-\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_3 h_2-\rho_3 h_1}}{\rho_3} & \frac{e^{\rho_4 h_2-\rho_4 h_1}}{\rho_4} \end{vmatrix}, Y_2 = \begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_1 h_2-\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_3 h_2-\rho_3 h_1}}{\rho_3} & \frac{e^{\rho_4 h_2-\rho_4 h_1}}{\rho_4} \end{vmatrix},$$

$$Y_3 = -\begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_1 h_2-\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_2 h_2-\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_4 h_2-\rho_4 h_1}}{\rho_4} \end{vmatrix}, Y_4 = \begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} \\ \frac{e^{\rho_1 h_2-\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_2 h_2-\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_3 h_2-\rho_3 h_1}}{\rho_3} \end{vmatrix},$$

and $det\widehat{A} = \sum_{i=1}^4 \frac{e^{\rho_1 h_2}}{\rho_i} Y_i$.

Example 3.2. *Consider the case, $g(x) = \mathbf{1}_{\{x \ge h_2\}}$. Then $\Phi(x) = \mathbb{E}_x\left[e^{-r\tau_{(h_1,h_2)}} \mathbf{1}_{\{X_{\tau_{(h_1,h_2)}} \ge h_2\}}\right]$ and $V_g = [0, \frac{1}{\eta^-}, 0, 1]$. Now $AQ = V_g$ is equivalent to $\widehat{A}\widehat{D}Q = \widehat{V}_g$ where $\widehat{V}_g = [0,0,0,1]^T$, $\widehat{D} = diag(\rho_1, \cdots, \rho_4)$ and*

$$\widehat{A} = \begin{bmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_3 h_1}}{\rho_3} & \frac{e^{\rho_4 h_1}}{\rho_4} \\ \frac{e^{\rho_1 h_2}}{\rho_1} & \frac{e^{\rho_2 h_2}}{\rho_2} & \frac{e^{\rho_3 h_2}}{\rho_3} & \frac{e^{\rho_4 h_2}}{\rho_4} \end{bmatrix}.$$

Therefore, $\mathbb{E}_x\left[e^{-r\tau_{(h_1,h_2)}}\mathbf{1}_{\{X_{\tau_{(h_1,h_2)}}\geq h_2\}}\right] = \sum_{i=1}^{4} Q_i e^{\rho_i x}$ *for* $x \in (h_1,h_2)$. *Here* $Q_i = (\rho_i det\widehat{A})^{-1}Y_i$,

$$Y_1 = -\begin{vmatrix} \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_3 h_1}}{\rho_3} & \frac{e^{\rho_4 h_1}}{\rho_4} \end{vmatrix}, Y_2 = \begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_3 h_1}}{\rho_3} & \frac{e^{\rho_4 h_1}}{\rho_4} \end{vmatrix},$$

$$Y_3 = -\begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_4 h_2}}{\rho_4-\eta^-} \\ \frac{e^{\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_4 h_1}}{\rho_4} \end{vmatrix}, Y_4 = \begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta^-} & \frac{e^{\rho_2 h_2}}{\rho_2-\eta^-} & \frac{e^{\rho_3 h_2}}{\rho_3-\eta^-} \\ \frac{e^{\rho_1 h_1}}{\rho_1} & \frac{e^{\rho_2 h_1}}{\rho_2} & \frac{e^{\rho_3 h_1}}{\rho_3} \end{vmatrix},$$

and $det\widehat{A} = \sum_{i=1}^{4} \frac{e^{\rho_1 h_2}}{\rho_i} Y_i$.

Example 3.3. *Consider the case,* $g(x) = (x-h_2)^+$. *Then* $\Phi(x) = \mathbb{E}_x\left[e^{-r\tau_{(h_1,h_2)}}(X_{\tau_{(h_1,h_2)}} - h_2)^+\right]$ *and* $V_g = [0, \frac{-1}{\eta^-} - \frac{1}{(\eta^-)^2}, 0, 0]$. *Therefore,* $\mathbb{E}_x\left[e^{-r\tau_{(h_1,h_2)}}(X_{\tau_{(h_1,h_2)}} - h_2)^+\right] = \sum_{i=1}^{4} Q_i e^{\rho_i x}$ *where* $Q_i = (\frac{-1}{\eta^-} - \frac{1}{(\eta^-)^2})detA^{-1}Y_i$,

$$Y_1 = -\begin{vmatrix} \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ e^{\rho_2 h_1} & e^{\rho_3 h_1} & e^{\rho_4 h_1} \\ e^{\rho_2 h_2} & e^{\rho_3 h_2} & e^{\rho_4 h_2} \end{vmatrix}, Y_2 = \begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ e^{\rho_1 h_1} & e^{\rho_3 h_1} & e^{\rho_4 h_1} \\ e^{\rho_1 h_2} & e^{\rho_3 h_2} & e^{\rho_4 h_2} \end{vmatrix},$$

$$Y_3 = -\begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_4 h_1}}{\rho_4+\eta^+} \\ e^{\rho_1 h_1} & e^{\rho_2 h_1} & e^{\rho_4 h_1} \\ e^{\rho_1 h_2} & e^{\rho_2 h_2} & e^{\rho_4 h_2} \end{vmatrix}, Y_4 = \begin{vmatrix} \frac{e^{\rho_1 h_1}}{\rho_1+\eta^+} & \frac{e^{\rho_2 h_1}}{\rho_2+\eta^+} & \frac{e^{\rho_3 h_1}}{\rho_3+\eta^+} \\ e^{\rho_1 h_1} & e^{\rho_2 h_1} & e^{\rho_3 h_1} \\ e^{\rho_1 h_2} & e^{\rho_2 h_2} & e^{\rho_3 h_2} \end{vmatrix},$$

and $detA = \sum_{i=1}^{4} \frac{e^{\rho_n h_2}}{\rho_n - \eta^-} Y_i$.

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