Recent advances in modeling of well hydraulics

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ABSTRACT

Well hydraulics is a discipline to understand the process of flow to the well in an aquifer which is regarded as a source of groundwater. A variety of analytical and numerical models have been developed over the last few decades to provide a framework for understanding and quantifying the flow behavior in aquifer systems. In this review, we first briefly introduce the background of the theory of well hydraulics and the concepts, methodologies, and applications of analytical, semi-analytical, numerical and approximate methods in solving the well-hydraulic problems. We then address the subjects of current interests such as the incorporation of effects of finite well radius, wellbore storage, well partial penetration, and the presence of skin into various practical problems of groundwater flow. Furthermore, we also summarize recent developments of flow modeling such as the flow in aquifers with horizontal wells or collector wells, the capture zone delineation, and the non-Darcian flow in porous media and fractured formations. Finally, we present a comprehensive review on the numerical calculations for five well functions frequently appearing in well-hydraulic literature and suggest some topics in groundwater flow for future research.

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1. Introduction

1.1. Background

Groundwater is an important alternative resource to surface water for agriculture, industry and domestic use. Studies in groundwater hydrology are therefore devoted to evaluate the occurrence, availability and quality of groundwater. Two inherent characteristics of the aquifer, referred to as the specific storage and hydraulic conductivity, generally provide the fundamental bases for the quantitative studies on groundwater hydrology/ hydraulics. These two parameters are commonly determined from the field data of aquifer tests. Many analytical solutions have been developed rapidly and steadily since Theis published the solution of transient flow equation, a milestone in well hydraulics, in 1935 [1]. Some analytical solutions were used to develop the type curves so that the hydraulic parameters can be determined by the graphical fitting of the observed data to the type curves [2]. The analytical solutions are however restricted to the ideal cases, which should be under the conditions of simple aquifer boundary and homogeneous formation properties. Yet, the real-world well-hydraulic problems are often with complicated boundary

and/or heterogeneous aquifer properties. Under these circumstances, numerical approaches such as finite-difference methods, finite element methods, or boundary element methods are then employed to develop the numerical solutions. Numerical approaches have been extensively used in studying groundwater problems since the mid-1960s [3]. Often, newly developed numerical models are verified with analytical models to check their correctness or accuracy. Analytical models are often developed for better modeling the aquifer systems in different subjects by accounting for the effects of wellbore storage, skin zone, and well partial penetration on the groundwater flow. In last two decades, many analytical solutions arisen from various types of aquifer tests with considering those effects have been developed. Some groundwater issues, such as pollution and water resources management, also attract considerable concern of scientists and engineers. Many analytical models for describing groundwater flow induced by horizontal and collecting wells and for delineating the capture zone in a contaminated site are then developed in accordance with these issues. Many of the analytical solutions are in terms of complicated forms which may contain integrals of some special functions, e.g., Bessel functions or trigonometric functions, and these integrals are difficult to compute accurately because of the oscillatory nature and slow convergence in computing the integrand. Therefore, various techniques are proposed to numerically calculate the solutions with high accuracy.

In this review, we address the subjects of current interests in regard to modeling groundwater flow behavior in well hydraulics.

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For each of these subjects, we provide a brief and historical overview, summarize the recent developments, and suggest some topics for future research as well. In this chapter, we first present the groundwater flow equation and its associated boundary conditions in well hydraulics, then introduce the definitions of different types of aquifers, and finally review various aquifer tests in some detail.

1.2. Basic principle, flow equations and initial and boundary conditions

1.2.1. Darcian flow

Darcian flow, which obeys Darcy's law and describes laminar flow behavior in a porous medium, can be expressed as $q = -Ki$ in which q is the discharge velocity or Darcy velocity, K is the hydraulic conductivity, h is the hydraulic head, and $i = dh/dl$ is the hydraulic gradient. Instead of describing the flow state within individual pores, Darcy's law represents the statistical macroscopic equivalent of the Navier–Stokes equations for viscous flow in porous media.

1.2.2. Three dimensional groundwater flow equation

Based on Darcy's law, a three-dimensional (3-D) equation in Cartesian coordinates for groundwater flow in a heterogeneous and anisotropic aquifer can be written as

$$\frac{\partial}{\partial x}\left[K_x(x,y,z)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y(x,y,z)\frac{\partial h}{\partial y}\right] + \frac{\partial}{\partial z}\left[K_z(x,y,z)\frac{\partial h}{\partial z}\right]$$
$$= S_s\frac{\partial h}{\partial t} + R \tag{1}$$

where K_x, K_y, and K_z are the main components of the hydraulic conductivity tensor, S_s is the specific storage, R is a recharge term which is positive and defined as the volume of inflow to the flow system per unit volume of aquifer per unit of time, and t is the time from the start of test.

1.2.3. Radial flow equation

Radial flow toward the well or from the well occurs when a vertical extraction or injection is performed at the well in the aquifer. The differential equation for flow in aquifers in cylindrical coordinates can be written as

$$K_r\frac{\partial^2 h}{\partial r^2} + \frac{K_r}{r}\frac{\partial h}{\partial r} + K_z\frac{\partial^2 h}{\partial z^2} = S_s\frac{\partial h}{\partial t} \tag{2}$$

where r is the radial distance from the center of the pumping well.

1.2.4. Initial and boundary conditions

The initial condition should be specified when solving transient flow problems in which the hydraulic head changes with time. The mathematical expression for initial condition is denoted as

$$h = f(x,y,z,0) \tag{3}$$

It is important to select appropriate boundary conditions in developing a mathematical model for an aquifer flow system. Boundary conditions for groundwater flow problems are of three types, i.e., the first-type (Dirichlet), the second-type (Neumann), and the third-type (Robin) boundary conditions.

A constant hydraulic head is the first-type boundary which can be mathematically expressed as

$$h(x,y,z,t)|_\Gamma = f_1(x,y,z,t) \tag{4}$$

where f_1 is a known function. Physically, a surface water body, e.g., river, lake, or reservoir could be treated as a constant head boundary if they hydraulically connect with the aquifer.

The mathematical expression of the water flux at the boundary Γ is

$$q_n(x,y,z,t)|_\Gamma = f_2(x,y,z,t) \tag{5}$$

where q_n is a flux component normal to the boundary surface. An example of the second-type boundary can be selected at the top of an aquifer where there is recharge or discharge. The no-flow boundary ($q_n = 0$) is a special case of the second-type boundary condition.

The third-type boundary condition is a linear combination of Dirichlet and Neumann boundary conditions, which relates hydraulic head to the flux and can be mathematically expressed as

$$ah(x,y,z,t)|_\Gamma + bq_n(x,y,z,t)|_\Gamma = g(x,y,z,t), (x,y,z) \subset \Gamma \tag{6}$$

In well hydraulics, an example of this kind of boundary is the case when taking into account of the skin effect in the wellbore as mentioned in Section 3.2.

It is worthy to mention that the mixed boundary value problem arise when a boundary condition of different types specified on different subsets of the boundary Γ. If Γ is divided into two subsets Γ_1 and Γ_2, this mixed kind of boundary condition may be written as

$$h(x,y,z,t)|_{\Gamma_1} = g_1(x,y,z,t) \tag{7}$$

$$q_n(x,y,z,t)|_{\Gamma_2} = g_2(x,y,z,t) \tag{8}$$

The mixed boundary value problem is commonly encountered in the physical problems of fluid dynamics, electricity, heat flow, and others. In well hydraulics, it may appear in the case of constant head pumping in a well under partial penetration condition. Some issues relative to the mixed boundary value problem can refer to Section 3.3.

1.2.5. Aquifer of finite or infinite extent in horizontal direction

Analytical solutions can be used to predict the temporal or spatial drawdown distribution or to determine aquifer parameters. For instance, the most well-known transient and steady-state drawdown solutions for pumping with a constant rate conducted in confined aquifers are Theis and Thiem solutions, respectively. As indicated in Wang and Yeh [4], it is inappropriate to apply Theis equation to the case where an aquifer has a horizontally finite boundary or the pumping time tends to infinity. Most of existing analytical solutions in the well-hydraulic literature are developed by assuming that the aquifer is of infinite extent in horizontal direction. Studies on the groundwater problems in an aquifer with a horizontally finite boundary are very limited [5].

It is very common to consider that the aquifer is of finite extent in vertical direction in modeling groundwater problems. Some studies however treated the aquifer thickness as infinite because the aquifer thickness is relatively large when compared with the screen depth of the well [6].

The analytical solutions may be developed for flow in finite aquifers with a regular shape such as rectangular and wedge-shaped configuration or irregular shape such as step change configuration. Several studies provided analytical solutions for groundwater flow problems in aquifers whose plan views are rectangular [7–9], wedge-shaped [10–15] and triangular [16], and in aquifers whose cross sectional views are step-like [17,18].

1.3. Types of aquifer

An aquifer is a geological formation that is sufficiently permeable to store and transmit groundwater. Related terminologies include *aquitard* (also called as semipervious layer), defined as a geologic unit that has relatively low permeability compared with aquifers, and *aquiclude*, which is a formation not capable of transmitting a significant amount of groundwater. A groundwater flow system can be composed of different types of aquifers.

A confined aquifer, also known as an artesian or a pressure aquifer, is bounded both at the top and bottom by impervious or semipervious strata and thus hydraulically isolated from other geological formations. A piezometer or observation well installed in the confined aquifer has a hydrostatic pressure level which forms an imaginary surface called a piezometric surface. Generally, water in the piezometer or well rises above the top of the aquifer.

An unconfined aquifer is also called as a water table or phreatic aquifer. The upper boundary of the unconfined aquifer is a free surface called the water table.

An aquifer that loses or gains significant water through the overlain, underlain, or both semipermeable strata is called leaky aquifers. It is common that an aquifer hydraulically connects with other formations in nature. A system of multilayered aquifers (or multiple aquifers) is composed by a series of aquifers separated from each other by semipervious layers. The mechanics of fluid flow through a multiple aquifer system becomes complicated because of the hydraulic connection between individual aquifers. For groundwater flow associated with leaky and multilayered aquifers, the reader is referred to Section 3.8.

1.4. Aquifer tests

Two different types of pumping test are commonly used in the field for estimating aquifer and well characteristics; they are aquifer test and well test [19]. The aquifer test is used to determine the aquifer hydraulic parameters. On the other hand, the well test is utilized to provide the information about the specific capacity or the discharge–drawdown ratio of the well [20]. The aquifer test generally needs at least one observation well or piezometer to record the water level response to the pumping, while the well test may require only the analysis of pumped well data. Kruseman and de Ridder [20] presented comprehensive discussion on the general set-up regarding the selection of test site, the design and construction of discharging well, and the performance including the water level and discharge rate measurements of the aquifer test. In addition, Driscoll's book [21] provides a good reference on some practical well-hydraulic issues as well as some aspects involving well design, drilling, construction, maintenance and rehabilitation. In the petroleum literature, Gringarten [22] and others [23–25] gave the review of the evolution of aquifer test analysis over the past half century. The aquifer test may be further divided into four groups: constant rate test, constant head test, slug test, and recovery test.

1.4.1. Constant rate test

The constant rate test (CRT) (or constant discharge test) is the most common form of aquifer test in which a test well is pumped at a constant rate over a certain period of time. At the same time, the drawdown is measured in one or more nearby observation wells [19]. The flow rate to the well is general assumed constant and equal to the average value of the pumping rates which indeed vary with time and are difficult to maintain constant in the field tests. In the early stage of development for flow induced by pumping in confined aquifers, there are two classic articles devoted to that in well hydraulic literature. Thiem [26] was the first to derive a steady-state expression for a fully penetrating well with a constant rate pumping in a confined aquifer. Theis [1] analyzed the groundwater flow in a homogeneous, isotropic and infinitely radial extent confined aquifer with a fully penetrating well pumped at a constant rate. The diameter of the pumping well is infinitesimal and the well storage is negligible. In recent years, Yeh et al. [27] presented a mathematical model for an aquifer having a fully penetrating well with a finite-thickness skin and constant pumping rate. Later, Yang et al. [28] developed an analytical solution for transient flow in a confined aquifer with a partially penetrating

well pumped at a constant rate. For unconfined aquifers, Jacob [29] adopted the Dupuit assumption of no vertical flow to derive the steady-state drawdown in CRT. Boulton [30] introduced the drawdown equation of the water table near a pumped well based on the delayed response concept of unconfined aquifers. Neuman [31] obtained a transient drawdown solution by shifting the boundary condition from the free surface to the horizontal plane at a fixed position. Moench [32] developed a semianalytical solution by combining the Boulton and Neuman models for flow to a partially penetrating well in unconfined aquifers. Tartakovsky and Neuman [33] presented an analytical expression for drawdown in an unconfined aquifer caused by the pumping at a constant rate from a partially penetrating well. Pasandi et al. [34] provided an analytical model for CRT conducted at a partially penetrating well with a finite thickness skin in an unconfined aquifer. For leaky and multilayered aquifers, for instance, Hantush and Jacob [35] developed a mathematical model for aquifer dynamics under a transient CRT by assuming that the confined aquifer is overlain everywhere by a semipervious layer which has constant vertical hydraulic conductivity and a constant thickness. Moench [36] took into account the effect of large-diameter well and developed mathematical models for flow in an aquifer system, where semipervious layers are located above and below the main pumped aquifer. In addition, there are some other studies associated with the CRT in confined aquifers [17,37–40], in unconfined aquifers [41,42], and in leaky and multilayered aquifers [39,43–46].

1.4.2. Constant head test

For a constant head test (CHT), a constant water level in the test well is maintained, while the discharge from the wellbore is monitored throughout the time. The CHT is commonly adopted to perform in low permeability aquifers [47]. Advantage of CHT over CRT is that the effect of wellbore storage at the test well is minimized [47]. Note that the articles on CHT mentioned below are associated with the transient flow because the drawdown solutions tend to be identical in both CHT and CRT at sufficiently large time [48].

The literature on the solutions of CHT performed in confined aquifers is given in the following. Jacob and Lohman [48] considered a radial flow problem in a confined aquifer during the CHT and gave the solution of transient flux across the wellbore. Hantush [2] presented a transient drawdown solution in response to a fully penetrating well extracting groundwater from confined aquifers. Cassiani et al. [6] proposed a semianalytical expression for a flowing partially penetrating well with infinitesimal skin situated in an anisotropic confined aquifer. Other researches with regard to CHT conducted in confined aquifers can also be found in literatures [5,49,50]. For unconfined aquifers, for example, Chen and Chang [51] considered the skin effect and developed a well-hydraulic theory for CHT in unconfined aquifers. Chang et al. [52,53] gave hydraulic head solutions for CHT performed at a partially penetrating well in unconfined aquifers. The review of the recent development in the literature shows that there are a limited number of researches on the CHT in leaky and multilayered aquifers. Hantush [54] gave the transient solutions of drawdown and flux from the wellbore for leaky aquifers. Wen et al. [55] considered the effect of finite-thickness skin and presented a mathematical model for radial groundwater flow to a pumping well in a leaky aquifer during CHT.

1.4.3. Slug test

A slug test is a particular type of aquifer test in which a small amount of water in the control well is instantaneously added/removed to/from the well and the response in drawdown is monitored through time in the control well or the surrounding observation wells. The slug test has become popular in groundwater investigations because of its logistical and economic advantages

over other aquifer tests. The advantages include that the cost of test is low, the procedure is relatively simple and rapid, little or no water needs to be handled during the test, and the test provides information on local hydraulic properties [56]. The disadvantages include that the aquifer properties obtained from this test represent the formation only near the test wellbore and the result of the test may be influenced by gravel or sand pack material in the borehole adjacent to the well screen.

The slug test is usually performed in confined or unconfined aquifers for most engineering practices. A number of mathematical models for analyzing slug test data have been published. For confined aquifers, for example, Cooper et al. [57] assumed the well diameter is finite and provided an analytical solution for the analysis of slug tests conducted in fully penetrating wells. Faust and Mercer [58] simulated four slug test runs with a well skin hydraulic conductivity of 0.01, 0.1, 1.0, and 10 times the hydraulic conductivity of the formation. According to their calculation, the skin with a much lower permeability than that of the surrounding formation can lead to an incorrect estimate of the true hydraulic conductivity. Yeh et al. [59] presented a semi-analytical solution for a slug test performed at a partially penetrating well in a confined aquifer, accounting for the effect of finite-thickness skin. Other researches on the slug test conducted in confined aquifers can also be found in literatures [60–62]. For unconfined aquifers, Bouwer and Rice [63] proposed a procedure for calculating the hydraulic conductivity of aquifers with partially or completely penetrating wells. Dagan [64] presented simple numerical schemes for determining the hydraulic conductivity of unconfined formations from the analysis of recovery, packer, and slug test data. Butler et al. [65] presented a procedure for analyzing data from slug tests performed at partially penetrating wells in highly permeable formations which could be confined or unconfined. For multilayered aquifers, Butler et al. [66] employed a numerical model to evaluate the capability of multilevel slug tests in providing information about vertical variation in hydraulic conductivity in multi-layered aquifers.

1.4.4. Recovery test

At the end of a pumping/injection test, the water levels in the test and observation wells begin to increase/decrease. The process involved in the change in water levels are referred to as recovery, and the observed drawdown remaining during the recovery period is named as the residual drawdown. The equations describing the residual drawdown in a piezometer [67] and an observation well [67] during the recovery period after constantly pumping at a partially penetration well in a homogeneous confined aquifer have also been developed. The analysis of residual drawdown data can estimate the transmissivity and provide an independent check on the hydrogeological parameters found from the previous drawdown data analysis. There are many studies associated with the data analyses of water level recovery after a pumping at a constant rate in confined aquifers with neglecting well radius [68,69] or considering well radius [70], leaky aquifers [71], unconfined aquifers [72], as well as for head recovery after a packer test in unsaturated fractured media [73]. Shapiro et al. [74] presented a model based on the solutions of Papadopulos and Cooper [75] and Cooper et al. [57] to determine the early-time recovering water level following the shut down of a constant pumping in wells with turbulent head losses. Recently, Samani et al. [76] used derivative analysis of pumping and recovery test data to estimate the parameters of Shiraz aquifer in Fars province, Iran. Yeh and Wang [77] developed a mathematical model to describe the residual drawdown with consideration of the wellbore storage effect and the drawdown distribution occurring at the end of a previous CHT.

2. Solution methodology

A partial differential equation for describing groundwater flow can be solved either analytically or numerically. In this section, we summarize the methodologies and application of analytical, semi-analytical, numerical and approximate approaches which are often made in the groundwater area.

2.1. Analytical methods

As mentioned in Section 1, the phenomena of groundwater flow can be mathematically described by a partial differential equation (PDE) or a system of several PDEs, with associated boundary and initial conditions. Analytical methods or numerical techniques can be used to solve the flow equations for problems arisen in well hydraulics. In this section, we introduce some mathematical approaches commonly used in solving the groundwater problems analytically, including Laplace transform, Hankel transform, Fourier transform, Mellin transform, Green's function, dual integral/series equation method, and Boltzmann transform. Basically, the first four methods can be considered as different types of integral transform, which offers an easy and effective way in solving a variety of problems arising in engineering and physical science [78]. The concept of integral transform is somewhat analogous to that of logarithmic transform. Their main aim is to transform the given problem into one with reduced number of dimensions. By taking an integral transform, a PDE can be reduced to an ordinary differential equation (ODE) in the transformed domain. The solution can then be obtained by solving the ODE along with the associated initial or boundary conditions.

2.1.1. Laplace transform

Laplace transform $F(s)$ of a function $f(t)$ can reduce initial value problems with certain types of ODEs to the algebraic solutions. It can also be used to transform initial boundary value problems with certain classes of PDEs to ODEs [79]. Generally, Laplace transform is proved to be most useful in reducing the "time-like" variables. Note that a necessary condition for the existence of the Laplace transform is that the value $e^{-st}f$ vanishes when $t \to \infty$, otherwise the integral defined as the Laplace transform does not converge. In well hydraulics, Laplace transform is a powerful method to deal with transient groundwater problems when a first-order time derivative of function, i.e., the drawdown or hydraulic head, is involved. After taking Laplace transform with respect to the time variable, the original governing equation describing the transient groundwater flow becomes a boundary value problem. Notice that the transform procedure needs the information of the function value at the initial stage, i.e., the initial condition. Plenty of researches were conducted using Laplace transform for problems in the well hydraulic literature [55,60,80–83].

2.1.2. Fourier transform

The existence of the exponential Fourier transform $F(w)$ of a function $f(x)$ presumes that $f(x)$ vanishes when $|x|$ is infinity. Normally, such condition at either end of an infinite interval occurs when x is a "space-like" variable [79]. Hence, the exponential Fourier transform turns out to be helpful in reducing the "space-like" variables and it can be used to solve the boundary value problems for PDEs which describe groundwater flow problems with infinite boundaries. On the other hand, the Fourier sine (or cosine) transforms are well suited for solving flow problem of a semi-infinite domain represented by a second-order PDE when the function value f (or its derivative f_x) is known at the origin and the function and its derivatives are required to vanish as $x \to \infty$ [79]. Moreover, the finite Fourier sine and cosine transforms are particularly useful

in solving the problem of a finite domain. The Fourier transforms and finite Fourier transforms are appropriate in solving the problems involved the Dirichlet or Neumann boundary conditions. Generally, the governing equation describing the transient groundwater flow after taking the Fourier transforms reduces to an initial value problem. Applications of these transforms can be found, for example, in Tsou et al. [84] who used exponential Fourier, Fourier sine, and finite Fourier cosine transforms to solve a 3-D groundwater flow problem for a confined aquifer with a slanted well near a stream, in Huang et al. [85] who applied exponential Fourier, Fourier sine, and Laplace transforms to obtain an analytical solution for describing the head distribution in an unconfined aquifer with a single pumping horizontal well parallel to a fully penetrating stream, and in other literatures [10,86] employing the finite Fourier sine or cosine or both finite Fourier sine and cosine transforms to solve a variety of groundwater flow problems.

2.1.3. Hankel transform

The Hankel transform $F(\rho)$ of a function $f(r)$ has been found to be the most appropriate for problems formulated with equations expressed in polar and cylindrical coordinates. In applying the Hankel transform to such equations involving the radial variable ranging from zero to infinity, it is required that the value of f and f_r are bounded at the origin and $\sqrt{r}f$ and $\sqrt{r}f_r$ vanish at $r \to \infty$ [87]. Yeh and Chang [10] presented analytical solutions, developed via finite Fourier sine transform and Hankel transform, for describing hydraulic head due to pumping in a wedge-shaped aquifer with an infinite radius. With the aid of Hankel transform and Laplace transform, Malama et al. [46] presented a semi-analytical solution for the transient streaming potential response of an unconfined aquifer to the continuous pumping at a constant rate.

2.1.4. Mellin transform

The Mellin transform is closely related to the Fourier transform. Morse and Feshbach [88] demonstrated an example on the relation between the Mellin transform and Fourier transform, and mentioned that the theorems given for the Fourier transform theory can be rewritten and applied to Mellin transforms. Sneddon [89] showed that the Mellin transform can be expressed in a natural manner in the solution of boundary value problems regarding an infinite wedge. Chan et al. [90] used Mellin transform to find the solution for steady-state drawdown due to a single well pumping at a constant rate from a wedge-shaped confined aquifer. Craster [91] solved several free boundary conditions encountered in groundwater flow problem by the use of Mellin transform.

2.1.5. Green's function

Instead of infinite series or eigenfunction expansions, solutions of ODEs and PDEs can be expressed in terms of an integral with a fundamental function. The fundamental function, usually named as Green's function, is a part of a closed-form solution for linear models and represents the solution as the result of the influences of a system to a unit point load or source. The total responses are the linear superposition throughout the system; hence, the total solution can be obtained by adding the responses of influences in all parts of the domain [92]. Indelman and Zlotnik [93] used the mean Green function to develop a mathematical model of average nonuniform flow in heterogeneous stratified media. Using perturbation methods coupled with Green's function techniques, Axness and Carrera [94] presented a second-order analytical solution for the steady hydraulic head flow in a two-dimensional (2-D) confined heterogeneous aquifer. Asadi-Aghbolaghi and Seyyedian [16] developed Green's function solution for groundwater flow to a vertical well in a triangle-shaped aquifer using image well theory and double Fourier series.

2.1.6. Dual integral/series equation

A boundary value problem under a mixed boundary condition can be converted to dual integral/series equations using integral or finite transforms. The resulting dual equations are in the form of integral or series depending on whether the problem is infinite or finite. Once the dual integral/series equations are solved, the solutions of the mixed boundary value problems are obtained. More detailed development of the theory of dual integral/series equations is provided in Sneddon [95]. Cassiani and Kabala [96] used the method of dual integral equations to develop semi-analytical solutions for describing the well response to the pumping test and slug test performed at a partially penetrating well in a confined aquifer of semi-infinite vertical extent. Their solution is applicable for aquifers of a semi-infinite vertical extent or the situations where the bottom boundary of the aquifer is far enough from the tested area. Since the thickness of aquifer is generally finite in real world, Chang and Yeh [97] used the method of dual series equations to obtain a drawdown solution to the CHT performed at a partially penetrating well in an aquifer with a finite thickness. Chang and Yeh [98] further extended the work of Chang and Yeh [97] to more general problems that the screen of a partially penetrating well is allowed to install at varied depth of the aquifer. The way they solved the problem was to employ the method of triple series equations, which bears the similar concept to the method of dual series equations.

2.1.7. Boltzmann transform

The Boltzmann transform is a special case of a general method named similarity method. One can transfer a nonlinear diffusion equation into an ordinary differential equation by introducing a similarity variable, i.e., $x/\sqrt{t}$, which is a combination of independent variables. In the well hydraulics, Boltzmann transform is commonly used in solving two kinds of problems, i.e., Boussinesq equation for unconfined aquifers and non-Darcian flow problem. In solving the Boussinesq equation of unconfined aquifer, Boltzmann transform was employed, for example, in Wang and Zhan [99] who presented a solution for transient confined–unconfined flow due to a well of infinitesimal radius at a constant rate pumping. Among the published works in using Boltzmann transform to solve non-Darcian flow problems we may mention as follows. Sen [100,101] solved non-Darcian radial flow with and without considering the effect of wellbore storage. Wen et al. [102] derived the solutions for non-Darcian flows in a single vertical fracture to a well on the basis of Izbash power-law and Wen et al. [103] investigated non-Darcian flow toward a finite-diameter pumping well with and without considering the wellbore storage and derived the solutions of non-Darcian well functions at the wellbore.

2.2. Semi-analytical methods (Laplace inversion)

The transient groundwater flow equation, Eq. (1), is a diffusion type of PDEs with a first order differential in time. In most groundwater problems, Laplace transform is suitable to apply to Eq. (1) for removing the time variable t and transform PDEs into ODEs in Laplace domain. The Laplace-domain solutions may be obtained after solving the resulting ODEs with associated Laplace-domain boundary conditions. To acquire the inverse Laplace transform, one may use tables [104,105] together with rules or methods such as the shift theorem, partial fractions, and convolution theorem. The time-domain solutions (i.e., exact solutions or analytical solutions) can also be obtained by complex variable theory referred to as complex inversion integral or Bromwich's integral [106], occasionally along with the residue theorem and/or Jordan's lemma. However, the inversion of Laplace transform is generally rather complicated and may not be tractable in some groundwater flow problems. Moreover, the time-domain solutions are usually in

terms of improper integrals with limits of integration from zero to infinity and theirs integrands comprise singularity at the origin [27]. The integrands of those time-domain solutions are in terms of oscillatory functions comprised of terms of the product of the Bessel functions of the first and second kinds of zero and first orders. The numerical calculations for those solutions are therefore time-consuming and very difficult to perform accurately. Therefore, numerous methods have been devoted to the numerical inversion of the Laplace transform. For comprehensive bibliographies, the reader may refer to [107,108]. In the following we present a brief introduction to three inversion methods, namely Stehfest, Crump, and modified Crump methods, which are the most widely used approaches for numerical Laplace inversion in well hydraulic problems. Detailed algorithms of each can be found in Cheng [109] and de Hoog et al. [110].

2.2.1. Stehfest method

Since the complex analysis in Laplace inversion is difficult to implement, Stehfest [111] proposed a series to approximate the Laplace transform of a real-valued function $f(t)$ using the following formula

$$f(t) \approx \frac{\ln 2}{t} \sum_{n=1}^{N} c_n F\left(\frac{n \ln 2}{t}\right) \qquad (9)$$

where

$$c_n = (-1)^{n+N/2} \sum_{m=(n+1)/2}^{\min(n,N/2)} \frac{m^{N/2}(2m)!}{(N/2 - m)!m!(m - 1)!(n - m)!(2m - n)!} \qquad (10)$$

The numbers of terms N in the series must be even. According to Stehfest [111], the accuracy can be improved by increasing N terms. However, round off error limits the value of N. The value of N is suggested to be in the range $6 \leqslant N \leqslant 20$ for most of engineering purpose [109].

2.2.2. Crump method

Based on the trapezoidal rule, Crump [112] approximated the Laplace inversion by the following equation

$$f(t) \approx \left(\frac{e^{\varphi t}}{T_p}\right) \left\{ \frac{F(\varphi)}{2} + \sum_{k=1}^{\infty} \mathrm{Re}\left[F\left(\varphi + \frac{k\pi i}{T_p}\right) \cos\left(\frac{k\pi t}{T_p}\right) \right] \right.$$
$$\left. - \mathrm{Im}\left[F\left(\varphi + \frac{k\pi i}{T_p}\right) \sin\left(\frac{k\pi t}{T_p}\right) \right] \right\} \qquad (11)$$

A parameter ϕ is introduced as

$$\phi = \varphi - \frac{\ln E}{2T_p} \qquad (12)$$

where E is the error tolerance, φ is the real part of the leading pole of the function $F(p)$, and T_p is a periodic function to approximate $f(t)$. The default value of φ equals zero for functions without poles. Crump [112] used the epsilon-algorithm of Wynn [113] (also called the Shanks method) to accelerate the convergence of the sequence in Eq. (9). The Shanks method developed by Shanks [114] is a nonlinear sequence-to-sequence transformation. This transform is effective in accelerating the convergence of slowly convergent sequences and inducing convergence in divergent sequences.

2.2.3. Modified Crump method

A significant improvement over the Crump method is developed by de Hoog et al. [110]. They applied the Pade approximations to improve the acceleration procedure for approximating the transform of the sequence in Eq. (9). The Pade approximation can be expressed by the quotient of two polynomials as [115]

$$f(t) \cong R_N(t) = \frac{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n}{1 + b_1 t + b_2 t^2 + \cdots + b_m t^m}, \quad N = n + m \qquad (13)$$

where the parameters $a_0, a_1, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$, are available for the approximation of $f(t)$, by $R_N(t)$. This method is designed to reduce truncation error that always occurs due to the fact that the Fourier series is not an infinite series. A useful Fortran routine INLAP of IMSL [116] developed according to the work of de Hoog et al. [110] is available for Laplace inversion.

It has been realized by researchers that it is impossible to devise a universal algorithm that performs accurately for all types of functions. Neither Stehfest method nor Crump method can perfectly implement in problems of well hydraulics. Each of them has their merit or defect in some particular problems. The execution time is less in Stehfest method, while the result from Crump might be more accurate if more terms are given. It can be easily found from the literatures that used Stehfest method [11–13,34,36,53,117], Crump method [118], and modified Crump method [59,115,119] to numerically inverse the Laplace-domain solution into time-domain.

2.3. Numerical methods

In many practical problems of groundwater flow, analytical solutions are not possible due to the facts that the shapes of the boundaries might be irregular, the coefficients appearing in the governing equations and in the boundaries conditions might be space-variant, the dependent variables in the initial conditions might be non-uniformly distributed, and the source/sink term might be in nonanalytic forms. To deal with realistic situations, numerical techniques provide convenient, flexible, and powerful tools for solving groundwater flow problems in complex field situations as discussed above. Four widely used numerical methods in well hydraulics are the finite difference, finite element, boundary element and analytic element methods addressed below.

2.3.1. Finite difference method

The finite difference method is probably the first numerical approach used to solve PDEs. A PDE describing the head or drawdown distribution can be approximated by a single difference equation over a time interval or a system of algebraic difference equations over one or two time intervals at predetermined, finite number of discrete grid nodes in the problem domain. Forward, backward, and central difference schemes are commonly used for the finite difference approximation. The monographs of applications of finite difference methods to the problems in well hydraulics have been published and the interested reader may be referred to the book by Wang and Anderson [120] or Remson et al. [121] for the technical details.

2.3.2. Finite element method

Another powerful numerical technique, known as the finite element method, has been widely applied to numerous groundwater flow problems as well. While the finite difference method is usually implemented with rectangular cells, the finite element method is regarded as a flexible approach which can handle almost any shape of flow boundary and any combination of boundary conditions, inhomogeneous and anisotropic media, moving boundaries, free surfaces and interfaces, and multiphase flows [122]. For a concise yet comprehensive discussion of the application of finite element method in groundwater hydrology, the reader may be referred to Huyakorn and Pinder [123] or Lee [124]. The book by Yeh [125] is an advanced treatise on computational subsurface hydrology.

2.3.3. Boundary element method

The boundary element method or the boundary integral equation method consists of formulating boundary value problems in terms of an integral equation. The solution of the integral equation can be obtained by approximating the boundary as a series of straight lines or elementary curves with the simplified assumptions to the behavior of the solution along the boundary segments [126]. In other words, the solution exactly satisfies the governing differential equation, but approximately meets the boundary conditions. Application of the boundary integral equation method in groundwater flow is discussed in detail by Liggett and Liu [127].

2.3.4. Analytic element method

The analytic element method is usually applied to the aquifers of an infinite extent and uses superposition to generate the analytical solution of a certain problem, expressed as a sum of basic solutions, each with a number of possibly unknown parameters. These parameters are then determined from the boundary conditions. One of the primary distinctions between analytic and boundary element methods is that the boundary elements are always given in the form of integrals, which are often calculated numerically in boundary element method and analytically in analytic element method. The book associated with the analytic element method by Strack [126] provided a basic theoretical framework for the understanding of the analytic element method and the detailed mathematical descriptions of the analytic elements and their numerical implementation.

2.4. Approximate methods

The analytical solutions obtained from groundwater flow problems are often in the form of infinite series for aquifers of a finite domain [4] or integrals for aquifers of an infinite domain [128]. Those solutions in terms of infinite series may converge slowly and are not suitable for numerical computation for very small values of time [129]. On the other hand, the solutions are in the form of an integral mostly with the integration limits from zero to infinity and with the integration variable in the denominator of the integrand, posing the problem of singularity at the origin of the integration [50]. Due to the presence of singular point, the numerical calculation of those solutions is generally very time-consuming and rather difficult to achieve accurate results. Therefore, for solving the problems in well hydraulics there is a need to develop approximate solutions which have simpler forms than the analytical solutions and are much easier in describing the transient behavior of groundwater flow with desired accuracy. Without the aid of the numerical techniques such as finite difference methods and finite element methods, the approaches in the development of approximate solutions may be divided to three different categories. The first is to solve the flow equation slightly different from the original one for finding the approximate solution. The second is to derive the approximate solution based on the Laplace domain solution along with the relationship of large p (Laplace variable) versus small t (hereinafter referred to as LPST) or small p versus large t (hereinafter referred to as SPLT). The last is to apply various types of approximate techniques into the time domain solution.

2.4.1. Approximation in governing equation

In the first category, the perturbation method is commonly used to find an approximate solution to nonlinear equations with variable coefficients or an irregular domain which cannot be solved exactly. By adding a "small" parameter to the mathematical description of the problem, perturbation method decomposes a tough problem into an infinite number of relatively easy ones. Hence, the perturbation method is most useful when the few first-order perturbation terms reveal the important features of the solution and the remaining ones give small corrections of the approximation. The researchers who used the perturbation method to solve the problems in well hydraulics are, for example, Batu and van Genuchten [130] and Moutsopoulos and Tsihrintzis [131]. The former adopted a singular perturbation method to solve the Boussinesq equation for the problem of a constant injection into a radial aquifer while the latter derived approximate analytical solutions for transient state, nonlinear flows through porous media based on the perturbation method. The other approach used to develop an approximate solution of the diffusion equation was presented by Fang et al. [132] for a problem in electrochemical. They extended the solution of the diffusion equation of steady-state process to the non-steady-state solution. Perrochet [133] used a similar concept to develop an approximation solution, which is exactly the same as that given in Fang et al. [132], for transient wellbore flux to a well subject to a constant drawdown. The approximate solution obtained from this approach is generally applicable for all values of the time.

2.4.2. Approximation in Laplace domain solution

The Laplace domain solutions of groundwater flow equation for describing flow in unbounded aquifers may be in a form with a quotient of the modified Bessel functions of the second kind of order zero or one [28]. For example, the Laplace domain solution for a CHT in aquifers of an infinite extent can be expressed, in our notation, as $h(r,p) = h_w K_0(\eta r)/[p K_0(\eta r_w)]$ [49] in which r_w is the well radius, p is the Laplace variable, h_w is the hydraulic head at the wellbore, $\eta = \sqrt{p/D}$, D is the hydraulic diffusivity, and $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero. Generally speaking, the approximate time domain solutions can be obtained by first using series expansion of the Bessel functions and then inverting the quotients associated with polynomials in p based on the methods of partial fractions and table of Laplace transforms given in, for example, Spiegel [106]. Carslaw and Jaeger [129] gave a systematic discussion of the methods in finding solutions useful for small or large values of the time. They mentioned that the method of Heaviside's series expansion can be used to expand the Laplace domain solution denoted as $\bar{x}(p)$ in a series of power of $(1/p)$, and with this the corresponding time domain solution $x(t)$ can then be obtained as a power series in t. Additionally, they also pointed out that some Laplace domain solutions after taking the asymptotic expansions may result in a series of negative exponentials and these results generally lead to the solutions useful for relatively small values of the time. Carslaw and Jaeger [134] presented the analytical solutions to describe the temperature distributions for a wide variety of heat flow problems. Their solutions can be applied to groundwater flow problems based on physical analogy. For the CHT, they obtained a drawdown solution for small values of time and both small-time and large-time solutions of wellbore flux as well by using asymptotic expansions of the Bessel functions. Furthermore, they also gave a large-time solution for drawdown distribution due to CRT at a pumping well. van Everdingen and Hurst [135] introduced the concept of symbolic relation between the derivative operator of time, i.e., d/dt, and p and inferred that p must be small if t is large, or inversely, p should be large if t is small. Accordingly, one might obtain a small time or a large time solution based on the LPST or SPLT relationship, respectively, to the Laplace domain solution. Yeh and Wang [136] presented a short review on the application of LPST [137], SPLT [38], and both [138,139] to groundwater flow problems for obtaining the approximate solutions. In addition to those articles, a few works have been carried out to develop the approximate solutions in the groundwater literature. For example, among the published works on applying the relationships of LPST and SPLT we may mention the following. Chen and Chang [51] developed the early- and late-time solutions for a CHT in a homogeneous and anisotropic

unconfined aquifer with the skin effect. Hunt and Scott [140] presented two approximate solutions for flow to a well in a leaky confined aquifer overlain by two layers; one is an unconfined aquifer on the top and the other is an aquitard in between. Pasandi et al. [34] also provided early time and late time drawdown solutions in addition to the Laplace domain solution for CRTs at a partially penetrating well in a phreatic aquifer with considering the effects of wellbore storage and finite thickness skin. Tsai and Yeh [141] developed a large time solution for the wellbore flow rate induced by a constant-head test in two-zone finite confined aquifers by employing the relationship of SPLT and L'Hospital's rule. It is of interest to mention that some approximate solutions had been developed in other areas such as the solute transport problems based on the relationships of LPST [142] and SPLT [143] and the dual-porosity media problem using the relationships of both LPST and SPLT [144]. Note that care must be taken when applying the relationship of SPLT to derive the large time approximation to the Laplace domain solution. The large time solution of the wellbore flux for the constant-head test in a homogenous confined aquifer can be written as $\overline{Q}(r_w, p) \sim 1/[p \ln(p/\lambda)]$ [49] where $\lambda = 4T/(e^{2\gamma} r_w^2 S)$ and $\gamma = 0.57722...$ is Euler's constant. This approximate solution introduces a pole at Re $p = \lambda$ which is artificial and must be excluded [136,145].

2.4.3. Approximation in time domain solution

A large amount of methods had been proposed to develop the approximation of the time domain solutions for various problems in well hydraulics. Tseng and Lee [146] reviewed six approximate methods used to estimate Theis well function and gave a concise introduction to each method. Those six methods are series expansion, asymptotic expansion, continued fraction, polynomial, rational and Chebyshev polynomial approximations, recurrence relations, numerical quadrature and inverse Laplace transform techniques. The application of each of the first four methods to a time domain solution yields an explicit mathematical expression, while that of the last two methods will give a procedure for the numerical computation of the time domain solution. Among those four methods, the first two, series expansion and asymptotic expansion may be the most widely used techniques to find approximate solutions in groundwater problems. The Cooper–Jacob equation [147] is a good example of the application of the series expansion. This equation results from truncating the high order terms in the infinite series obtained after applying the series expansion to the Theis solution [1]. Some studies can be found in the well hydraulic literature that adopts the method of series expansion [148] or asymptotic expansion [149] to develop the approximate solutions. Interestingly, Barry et al. [150] presented an analytical expression for the Theis well function based on the interpolation between the series expansion of small u and the asymptotic expansion of large u.

3. Subjects in modeling groundwater flow

Due to the increasing needs of irrigation, industrial, urban and suburban expansion, various subjects concerning groundwater flow attract scientists and engineers' attentions. In this section, we first address topics of current interests regarding the effects of finite well radius, wellbore storage, well partial penetration, and the presence of skin zone on the groundwater flow problems. We then introduce the recent developments concerning the flow in aquifers with horizontal wells or collector wells, the capture zone delineation, and the non-Darcian flow in porous media and fractured formations. Finally, we give a critical synthesis of the body of work on numerical computations for five well functions often presented in the well-hydraulic literature.

3.1. Effects of finite well radius and wellbore storage

For small-diameter wells with radius varies between 0.05 m and 0.25 m, the groundwater flow is often modeled by neglecting the effect of well radius. In reality, large-diameter wells with radius ranges from 0.5 m to 2 m are commonly installed in many countries to meet a large demand for domestic and irrigation water uses. The behavior of drawdown solution in wells with a finite well radius is different from that in wells with an infinitesimal radius because of the effect of well radius and the contribution of wellbore storage. Papadopulos and Cooper [75] presented an exact solution for the drawdown due to pumping in a large-diameter well. Their solution took into account the effects of finite well radius and the water stored inside the wellbore, which were neglected in the Theis equation. Theoretical models are developed for incorporating the effect of finite well radius [151,152] and both effects of finite well radius and wellbore storage into models [34,153]. Other issues of large-diameter wells such as seepage face, an application of discrete kernel approach and well function approximations are widely discussed in the literatures. For example, Ojha and Gopal [154] proposed a model based on flow velocity to describe the seepage face variation for large-diameter wells in an unconfined aquifer. Sakthivadivel and Rushton [155] considered the dynamic seepage face and gave a methodology to determinate the aquifer parameters from pumping tests with large-diameter wells. Mishra and Chachadi [156] extended the discrete kernel theory for analyzing flow behavior toward a large-diameter well during the recovery phase. Chachadi and Mishra [157] used discrete kernel approach to analyze the unsteady flow toward a large-diameter well in a confined aquifer while determining well loss component. Since the well function for large-diameter wells proposed by Papadopulos and Cooper [75] is an integral function of transmissivity and storativity, it is difficult to explicitly determine those two aquifer parameters from pumping test data. The development of well function approximation for large-diameter wells has been the subject of many studies [158].

3.2. Skin effect

The drawdown induced by pumping conducted in aquifers may be influenced by a region near the well referred to as the skin zone, or simply the skin. This zone has a lower or higher permeability than the adjacent formation. The skin is introduced during the well installation process including the drilling, construction, and installation of annular fill of the pumping well. Two types of the well skins are classified by their permeability. If the permeability of the skin zone is less than that of the formation zone, this kind of skin will be called a positive skin. On the contrary, if the permeability of the skin zone is larger than that of the formation zone, the skin will be a negative skin. The approaches of quantifying the skin effect in the well hydraulic modeling by researchers may be grouped into two ways; one assumes the skin to be infinitesimally thin, while the other considers it to be of finite thickness.

3.2.1. Infinitesimal skin

Studies of infinitesimal skin have been widely investigated in the areas of petroleum industry and well hydraulics. The skin effect is regarded as an additional pressure drop near the wellbore and proportional to the wellbore flow rate. Under this consideration, both the skin thickness and storativity are neglected in the skin zone. As such, a skin factor is introduced to represent the lumped properties of the skin and used as an energy loss term at the wellbore for mathematical convenience. The published works assuming the skin to be infinitesimal are, for example, Cassiani et al. [6], Park and Zhan [159], Moench [160], and Chen and Chang [51].

3.2.2. Finite thickness of skin

Novakowski [161] mentioned that the thickness of the skin zone may range from a few millimeters to several meters. The approaches of developing the solutions by taking into account the skin thickness can be further divided into two categories. The first category assumes that the skin thickness is a parameter associated with the skin factor. The other treats the skin as another formation, and therefore one should handle a two-zone flow system for such an aquifer.

In an aquifer system subject to a pumping at a constant rate, Hawkins [162] took into account the skin thickness and developed a formula for the additional pressure drop to quantify its effect. Based on the pressure drop equation proposed by van Everdingen [163], Hawkins defined the formula of skin factor as

$$s_f = [(K/K_{skin}) - 1] \ln(r_{skin}/r_w) \qquad (14)$$

where r_{skin} denotes the skin radius, K and K_{skin} are the hydraulic conductivities of the formation and skin, respectively. The productivity ratio of a well defined by Hawkin is $\ln(r_e/r_w)/[\ln(r_e/r_w) + s_f]$ with drainage radius r_e. Thus the well productivity ratio becomes very small when K_{skin} tends zero and it depends only on drainage, wellbore and skin radius when K_{skin} tends to be a large value. Hawkin's skin factor has been adopted in CHTs, CRTs and slug tests by, for example, Faust and Mercer [58] and Chen and Chang [164].

Moench and Hsieh [165] considered an equation for describing the flow in skin zone with negligible storativity and provided a hydraulic head solution for the slug test. They defined the skin factor as

$$s_f = [(K/K_{skin})] \ln(r_{skin}/r_w) \qquad (15)$$

For CRTs, Moench [160] considered the flow toward a partially penetrating well in a slightly compressible water table aquifer and assumed that the storage capacity of the skin is negligible. He assumed that the flow rate through the skin zone is equal to that along the wellbore and gave another skin factor as

$$s_f = (Kd_s/K_{skin}r_w) \qquad (16)$$

where d_s is the skin thickness.

A finite skin region, however, has its storage capacity in nature. The newly introduced mathematical model is thus required for the aquifer system representing by two regions of radial flow and each of it with individual transmissivity, storativity, and thickness. A two-zone aquifer may be named as a patchy aquifer [166,167] and such an aquifer system may be referred to as a composite system [161] or a two-layered system [27]. By also considering the skin storativity, Moench and Hsieh [168] further extended their solution in Moehch and Hsieh [165] to solve the flow equations in both skin and formation zones and developed the head distributions for slug tests. Many other recent studies have modeled the two-zone aquifer system for CHTs [5,50,167], for CRTs [34,37,38, 161,169,170], and for slug tests [167,171]. In addition, Perina and Lee [153] further derived a generalized solution which is applicable to CHT, CRT, and slug test in the two-zone aquifer system.

3.3. Effect of well partial penetration

If the length of a well screen is less than the entire thickness of an aquifer, the well is called a partially penetrating well [172]. The well of partial penetration is often adapted for hydrogeological site investigation to characterize groundwater pollution problems. For example, the contaminant of light nonaqueous phase liquids generally forms a pool on the top of water table in the subsurface. Under this circumstance, the screen of the well is only open at the upper part of aquifer. On the other hand, the well is usually screened at the lower part of aquifer if the contaminant is dense nonaqueous phase liquid. The hydraulic head distribution in response to an aquifer test at a partially penetrating well will be influenced by the screen length and its location in the aquifer. The equation describing the groundwater flow should include the vertical flow component induced by pumping at a partially penetrating well. In this section, mathematical approaches for dealing with the effect of well partial penetration in both CHT and CRT are discussed.

3.3.1. Effect of partial penetration in constant rate test

The theory of a partially penetrating well pumped at a constant rate was first published in 1957 by Hantush for leaky aquifers [172]. In his solution, the discharge is assumed to be uniformly distributed over the well screen and the flow in the zone below the screen is neglected. The solution for leaky flow can be further simplified to the case in a non-leaky confined flow by making the hydraulic conductivity of the confining layer which overlays the confined aquifer equals zero. For the case that the screen does not install from the top of the aquifer, the solution can be found in Hantush [2]. The assumption of uniform discharge can also be applied to the case of unconfined aquifers. Neuman [173] developed a solution of drawdown distribution in an unconfined aquifer by considering the effect of well partial penetration on drawdown during CRT. Yang et al. [28] took into account the radius of the pumping well and derived an analytical solution of drawdown in a confined aquifer with a partially penetrating well for the CRT. More references on the issues of partially penetrating effect during CRT can be found in the well hydraulic literature [34,39,44].

3.3.2. Effect of partial penetration in constant head test

In the well hydraulic modeling, the well water lever is maintained constant at a CHT and the constant head is appropriate to specify at the screen section. If the aquifer is fully penetrated, the mathematical model describing the groundwater flow can generally be solved by the conventional integral transform techniques [2]. If the aquifer is partially penetrated and the effect of well skin is negligible, the constant head condition will be used for the screen part and the no flow (or zero flux) will be specified at the casing. The flow problem induced by the partially penetrating well therefore becomes a mixed boundary value problem [98]. Some analytical approaches had been used to deal with the mixed boundary value problems happen in CHTs in the well hydraulic literature. Those approaches may be classified into following five categories:

The first approach is to replace the constant head along the well screen by a uniform radial flux [151] which will transform the mixed boundary into a homogeneous Newman boundary and results in an approximate solution.

The second approach is the well screen discretization method which was adopted by Chang and Chen [174] and Perina and Lee [153]. Along the screen, the flux is assumed constant in the first approach and considered to be non-uniform in the second approach.

The third is the domain decomposition method which divides the flow domain into two or several sub-regions and each region has its own flow equation and associated initial and boundary conditions [5,52,53]. The flow equations are then solved simultaneously via the continuity conditions for the hydraulic head and flow rate at the interfaces of the sub-regions.

The fourth is to use the method of dual integral equation and directly solve the mixed boundary value problems (MBVPs) for aquifers of infinite vertical extent and the screen installed at the top of the aquifer [6,96]. For aquifers of finite vertical extent, Chang and Yeh [97] solved the MBVPs by the method of dual series equations in conjunction with perturbation method.

For cases of screen is not placed at the upper part of the aquifer, the previous approach becomes inapplicable. The use of triple inte-

gral equation method, i.e., the last approach, can solve the problems with a well screen arbitrarily located at any portion of the aquifer of finite vertical extent [98].

3.4. Unconfined flow problem

The modeling unconfined flow may be classified into five different approaches. The first approach is to use the confined flow equation to model the unconfined flow problems. The second approach neglects the vertical flow and uses the Boussinesq equation to represent the unconfined flow. The third is based on the radial confined flow equation embedded with a delayed yield term. The fourth employs Eq. (1) and adopts a free surface equation to represent the top boundary condition. The last is to solve the unconfined flow equation by accounting for unsaturated flow above the water table.

3.4.1. Confined flow equation

In some field or practical problems, the changes in water table due to pumping or recharge is very small and thus the confined flow model becomes applicable to simulate the water table or drawdown distribution [45].

3.4.2. Boussinesq equation

Based on the approach of mass balance and the Dupuit assumption of neglecting the vertical flow [122], the Boussinesq equation, which describes the 2-D unconfined flow, can be developed and written as

$$\frac{\partial}{\partial x}\left(K_x h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial h}{\partial y}\right) = S_y \frac{\partial h}{\partial t} - R \qquad (17)$$

where S_y is the specific yield. Eq. (17) is nonlinear because it contains the products of h and $\partial h/\partial x$. Therefore, the analytical solution for the Boussinesq equation is rather difficult to obtain. In the past, this equation has been employed in modeling and analyzing head distributions in many aspects of unconfined aquifer flow. Bear [122] mentioned two methods of linearization to facilitate the development for the solution of Eq. (17). The first method is to replace the variable thickness, h, in Eq. (17) with an average aquifer thickness, b, if the change of h due to pumping or recharge is very small compared with the saturated thickness. Then Eq. (17) becomes a linear equation with the same form as the confined flow equation when replacing $K_x h$ and $K_y h$ by T_x and T_y, respectively. The second method is to rewrite the first term on the right-hand side (RHS) of Eq. (17) as $(S_y/b)\partial(h^2/2)/\partial t$. Eq. (17) then becomes a linear equation in h^2. The published works associated with the first method are, for example, Verhoest and Troch [175], Bansal and Das [176], Parlange et al. [177], Li et al. [178] and with the second method, for example, Hantush [179], Yeh and Chang [180], Ilias et al. [181]. Different from those two methods given in Bear [122], Pulido-Velazquez et al. [182] also provided a linearization method to solve the Boussinesq equation based on the technique of change of variables.

In the past, some studies were also devoted to developing the analytical solutions from non-linear steady-state or transient Boussinesq equation without using the linearization method mentioned above [183–187]. The literature on steady-state solutions for the Boussinesq equation is, for example, given in the following. Basha and Maalouf [188] discussed two particular cases of the analytical solutions for surface and groundwater flows and compared these two nonlinear cases with the solutions derived from the linearized Boussinesq equation using the Green's function solution. Based on the Boussinesq equation, Batu and van Genuchten [130] utilized a singular perturbation method to solve the transient flow induced by a constant injection into a radial aquifer. Lockington

et al. [189] applied similarity transform to solve the problem of unconfined flow with a boundary condition in terms of a power of time. Semi-analytical approaches based on the Boltzmann transform have also been developed to solve problems involved the Boussinesq equation. For example, Guo [190] solved the Boussinesq equation for transient groundwater flow between a reservoir and an unconfined aquifer of semi-infinite extent using the Boltzmann transform and Newton–Raphson method. Wang and Zhan [99] also developed a solution based on the Boltzmann transform along with the Runge–Kutta method for transient confined–unconfined flow driven by a pumping well of infinitesimal radius with a constant rate of pumping. Their paper also presented a short literature review on the issue of confined–unconfined conversion due to pumping using analytical and numerical approaches.

The Boussinesq equation is often used to model the hydrological problems such as the flow due to surface recharge and/or evapotranspiration in horizontal aquifer [179,181] or in sloping aquifers [175,188], hillslope flow [191,192], coastal aquifer hydraulics [193,194], designing drainage systems [195], the flow caused by a flood event [196]; and the hydraulics of the stream-aquifer interaction [176,197]. In the review paper by Winter [198], he addressed the studies of the interaction of groundwater and surface water focusing on different landscapes from alpine to coastal.

Since Eq. (17) is nonlinear, numerical approaches such as finite-difference methods, finite-element methods, or hybrid numerical method are inevitable for solving the equation. Anderson and Woessner [199] mentioned that a few investigators in the 1970s had solved Eq. (17) with numerical techniques specifically designed to handle the problem of nonlinearity. Kim et al. [200] developed a transient, mixed analytical and finite difference models to simulate hydrological behavior for investigating the patterns of infiltration, evaporation, recharge and lateral flow across hillslope. Stagnitti et al. [201] developed an explicit finite difference scheme for the solution of the nonlinear Boussinesq equation to examine the profiles of the water table height in a shallow sloping aquifer.

Taigbenu [202] developed a simplified Galerkin's finite element model to solve the nonlinear Boussinesq equation with the advantages of computing efficiency and requiring less computer storage. Upadhyaya and Chauhan [203] presented a finite element solution of nonlinear Boussinesq equation to compare his analytical solution derived from the linearized equation for the problems of water table variation in a sloping aquifer caused by the sudden rise or fall of the water level in a nearby stream. By coupling the Boussinesq equation with the Richards equation, Hilberts et al. [192] developed a tetrahedral finite element model to investigate the role of unsaturated storage in the relationship between rainfall and recharge. Tang and Alshawabkeh [204] proposed a semi-analytical time integration approach for the simulations of 2-D transient unconfined flow described by the nonlinear Boussinesq equation.

3.4.3. Confined flow equation embedded with a delayed yield term

Boulton [30] extended the theory of transient confined flow to a pumped well and added the term, the rate of delayed yield, to account for slow drainage from the unsaturated zone. This term was expressed as $\mu S' \int_0^t e^{-\mu(t-t')}\partial s/\partial t' dt'$ where μ is an empirical constant; S' is the delayed yield per unit area, per unit drawdown; and s is the aquifer drawdown. He solved the drawdown equation by using Laplace transform method and the Faltung theorem. Later, he presented two approximated solutions based on his previous delayed-yield solution for unconfined aquifer flow [205]; one is for the case in which $\eta = (S + S')/S$ tends to infinity and the other is for the case that the pumping time is small. In addition, he also provided delayed yield type curves in the paper for the analyses of pumping test data. Notice that the mathematical model of using

Boussinesq equation and Boulton's model [30,205] do not consider vertical flow component in the flow equation and therefore cannot be used for the case of having partially penetration wells or for the determination of vertical hydraulic conductivity in pumping data analysis [32].

3.4.4. Flow equation with a free surface boundary

The free surface equation representing the top boundary condition of homogeneous and isotropic unconfined aquifers was first proposed by Boulton [206] using a substantial derivative method. The free surface boundary equation describing the head distribution of radial flow at the water table, in our notation, is

$$\frac{\partial h}{\partial t} = \frac{K}{S_y}\left[\left(\frac{\partial h}{\partial r}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 - \frac{\partial h}{\partial z}\right] \tag{18}$$

Later, Bear [207] gave a 3-D transient free surface equation with recharge for an anisotropic medium. Since Eq. (18) is nonlinear, the flow equation subject to this top boundary condition is therefore not readily solved analytically. Boulton [206] developed an analytical solution of a steady-state radial unconfined flow equation subject to the top boundary condition under following two assumptions: (1) the gradients of h in Eq. (18) are small and thus their squares are negligible and (2) the simplified free surface equation neglecting squared terms is then applicable to the case that the head change occurs at the initial water table position. Neuman [31] also presented an unconfined flow model similar to Boulton's approach [206] but considering a transient flow equation for homogeneous and anisotropic aquifers subject to a constant pumping at a fully penetration well. The time-domain solution of the model developed using Laplace and Hankel transforms can describe the delayed water table behavior in response to the pumping. Later, he extended his previous model to account for the effect of a partially penetrating well under an assumption of uniform flux along the well screen [173]. By a different approach in handling the water table boundary, Moench [32] presented an unconfined flow model with a well of infinitesimal radius and a simplified free surface equation in which the $S_y\partial h/\partial t$ term is replaced by the delayed yield term originally proposed by Boulton [30]. The top boundary equation with such a delayed yield term is hereinafter referred to as Moench's free surface equation. Later, he gave a Laplace domain solution for flow to a partially penetrating well with considering the effect of finite diameter [160] and mentioned that his free surface equation reduces to the simplified free surface equation for the case of $\mu \to \infty$ and becomes no-flow condition for the case of $\mu \to 0$. In order to examine the aquitard effect on the unconfined aquifer–aquitard system, Zlotnik and Zhan [208] gave a semi-analytical drawdown solution based on 3-D groundwater flow equation for the homogeneous and anisotropic unconfined aquifer and one-dimensional (1-D) vertical flow for the aquitard. Perina and Lee [153] developed a general well function for deriving a solution describing groundwater flow toward a pumping well of finite diameter with non-uniform flux along the screen and finite-thickness skin, partially penetrating an unconfined aquifer. This general well function is capable of describing the groundwater flow for CRT, CHT, or slug test. The top boundary condition that they used in modeling unconfined flow is the simplified free surface equation. Their solution is also applicable to the leaky aquifer case if the term $S_y\partial h/\partial t$ in the top boundary equation is replaced by a drawdown-dependent boundary flux (i.e., a leakage rate) and the confined aquifer case if S_y in the top boundary equation is set zero. Pasandi et al. [34] presented an analytical model for representing a CRT conducted in an aquifer having a partially penetrating well with a skin zone of finite thickness. The model has two flow equations along with two Moench's free surface equations to describe the flows in the skin zone and formation zone, separately. In a study on the dy-

namic response of tidal fluctuations in unconfined aquifers, Yeh and Kuo [18] examined the effect of neglecting the second-order terms in Eq. (18) on the accuracy of their analytical solution which is developed using the simplified free surface equation as the top boundary condition. They used Eq. (18) as the top boundary condition and developed an implicit finite difference solution to compare with the analytical solution. Their numerical results indicate that neglecting of the second order terms has no significant effect on the aquifer head distribution. Chang et al. [53] developed a new model for a constant-head pumping at a partially penetrating well in an unconfined aquifer without assuming an unknown flux along the screen for the constant-head boundary. They divided the flow domain into two regions and solved the model by separation of variable and Laplace transform techniques.

3.4.5. Unconfined flow equation accounting for unsaturated flow

The influence of the unsaturated zone on the drawdown due to pumping has been neglected for a long period of time [31,173]. Recently, the effect of drainage from the unsaturated zone on the unconfined flow has been mentioned in the groundwater literature [209]. To explore the importance of unsaturated zone characteristics in the analyses of unconfined aquifer test, analytical solutions for flow toward a well in an unconfined aquifer had been reported. In an early work, Kroszynski and Dagan [210] developed an approximate solution to describe transient flow toward a partially penetrating well pumped at a constant discharge in an unconfined aquifer coupled with a linearized unsaturated flow equation at the moving free surface. Mathias and Butler [211] extended the concept of Kroszynski and Dagan [210] by considering the finite thickness of unsaturated zones, aquifer compressibility, and the flexibility of having different moisture retention and relative permeability functions. In their work, a linear diffusion equation was used to describe the flow in the saturated zone, while Richards' equation was used to delineate the flow in the unsaturated zone. Because of the nonlinear property of Richards' equation and the presence of a moving interface (water table) between saturated and unsaturated zones, the system of equations becomes highly nonlinear. They selected a small parameter regarding to the pumping rate and expanded the dependent variable into a perturbation series of the small parameter. Ignoring the higher-order terms of the small parameter, the nonlinear boundary condition at the free surface can therefore be linearized. They further linearized Richards' equation by assuming that moisture content and hydraulic conductivity are in terms of exponential functions of pressure head and obtained a new drainage function for pumping test analysis in compressible aquifers Mathias and Butler's solution [211] is limited to 1-D vertical flow through the unsaturated zone and ignores the effect of well partially penetration. Tartakovsky and Neuman [33] used a similar approach to that of Mathias and Butler [211] and developed an analytical solution for characterizing flow to a partially penetrating well in an unconfined aquifer by accounting for unsaturated flow above the water table. Their solution considered both the vertical and horizontal flows in the unsaturated zone. Mishra and Neuman [212] improved the solution of Tartakovsky and Neuman [33] by introducing four-parameter representation of the hydraulic conductivity and water content functions and allowing the unsaturated zone to have a finite thickness. Mishra and Neuman [213] further considered the case of a finite diameter pumping well with storage and explored the effects of wellbore storage and delayed piezometer response on drawdowns in the unsaturated–saturated flow system.

The governing Richards' equation for describing the unsaturated flow is highly nonlinear and complicated; it is therefore difficult to develop analytical solutions in unsaturated–saturated flow system without relying on any assumption or simplification to the governing equations and their associated boundary conditions. The

numerical methods are regarded as alternative and flexible ways for solving the transient nature of the unsaturated–saturated flow problem with complex boundary conditions in heterogeneous geologic formations. Finite difference method and finite element method are two major approaches capable of solving the unsaturated–saturated flow problems.

In the past, the finite difference was a popular approach used to approximate the solution for flow in unsaturated–saturated formations [214]. Based on the finite difference method, Dogan and Motz [215] developed an unsaturated–saturated model for simulating the 3-D groundwater flow in response to the recharge and evapotranspiration. Since the conventional finite difference discretization assumes an orthogonal coordinate system and makes a finite difference model computationally less efficient than other numerical models that can treat nonorthogonal grids, An et al. [216] used a coordinate transformation method for handling the geometrically complex flow domain to avoid the disadvantage of using high-resolution orthogonal grids by conventional finite difference models. Their finite difference unsaturated–saturated flow model can fit a curvilinear flow domain. It is worthy of mention that TOUGH2, a famous computer program based on the integrated finite difference method, can be used to simulate the nonisothermal flows of multicomponent, multiphase fluids in porous and fractured media [217].

Based on the finite element method, the computer models such as FEFLOW, FEMWATER, HydroGeoSphere, and SUTRA can be employed to simulate the unsaturated–saturated flow. The FEFLOW applies finite element analysis to solve the saturated and unsaturated flow equations as well as the governing equations for mass and heat transports [218]. The 3-D finite element model called FEMWATER [219], originated from 3DFEMWATER and 3DLEWASTE models, can also be applied to simulate the unsaturated–saturated, density driven flow and transport. The HydroGeoSphere simulates coupled unsaturated–saturated flow and transport by using a 3-D control volume finite element method [220]. The SUTRA [221] can simulate the density-dependent saturated or unsaturated groundwater flow and transport of either energy or dissolved substances in a subsurface environment. It employs a 2-D hybrid finite element and integrated finite difference method to approximate the solutions of the flow and transport governing equations. For more detailed discussion and review on those and other computer codes, the readers may be referred to Zheng and Bennett [222] and Bear and Cheng [223].

3.5. Theory of image well

Most of existing analytical models dealing with the drawdown induced by pumping from a well assume that the aquifers are of infinitely lateral extent. However, the aquifer may have a physical barrier (such as fault or impervious formation) considered as an impermeable boundary (no-flow condition) or a surface water body (such as lake, reservoir, stream, or sea) providing a recharge boundary (constant head condition) near the well. The use of image-well method in well hydraulics problems allows one to remove the impermeable or recharge boundary and place image wells at judicious locations to take account for the effect of the boundary. The drawdown in a test well can then be superposed as the sum of the individual drawdown due to the real well and the image wells. The method of image well is applicable when the groundwater flow equation and its associated boundary conditions are linear in confined or leaky confined aquifers and the superposition of drawdown is therefore valid. The mathematical formulation for estimating the drawdown due to a discharging well in aquifers bounded by a nearby straight impermeable or recharge boundary can be found in most of groundwater books [224,225]. For wedge-shaped aquifers, the wedge angle can be assumed to be

an aliquot part of 360° and the number of the image wells, n, required in analyzing the flow toward the well is given by $n = 360°/\theta - 1$ where θ refers to the wedge angle [226]. However, in applying the image well theory, the angle must be an aliquot part of 90° for aquifers with boundaries that are either like or unlike; otherwise, it must be an aliquot part of 180° for aquifers with like boundaries [226]. This rule has its exception in the case where the wedge angle is an odd aliquot part of 360°, the test well is on the bisector of wedge angle, and the boundaries are both impermeable [226]. For aquifers with parallel boundaries, two situations are suitable to use image-well systems [122,226]; one is that the aquifer is in a shape of infinite strip, while the other of semi-infinite strip. In addition, the image-well method can also be applied to the case of triangle aquifers [16] and rectangle aquifers [226, 227]. Other applications of image-well method can be found, such as for flow to a drain near a leaky layer [228], pumping in sloping fault zone aquifers [229], tunnel water inflow or flow toward drains [230], pumping near a constant-head linear boundary [231], quantifying stream depletion in narrow alluvial aquifers [232], the drawdown in aquifers with irregularly shaped boundaries [233], and the drawdown for pumping at a finite-diameter well in a wedge-shaped aquifer [234].

3.6. Horizontal well

Horizontal wells were first installed in 1927 [235]; however, they were not widely adopted because of the lack of drilling techniques for horizontal wells. Until 1980s, the interest of horizontal wells was reignited with the significant advance of techniques in drilling horizontal wells [236]. Horizontal wells are commonly constructed in aquifers close to streams or lakes to produce large amount of water [122] or in a contaminated site for remediation of contaminated groundwater [237]. The use of horizontal wells offers the following four advantages over vertical wells [236]. First, horizontal wells can be installed in aquifers where ground surfaces may have obstructions such as buildings and roads. Second, the use of horizontal wells yields smaller drawdown near the well than vertical ones do under the same pumping rate and well length [84]. Third, a horizontal well can generally extract more water than a vertical well does in shallow aquifers because the screen of horizontal wells is completely placed in the aquifer. For example, Hoffman [238] demonstrated a conceptual design of horizontal wells to remediate a contaminated groundwater site in a shallow aquifer in northern Chicago. Instead of installing up to 40 vertical wells, only two horizontal wells were used via pump-and-treat approach to remediate the contaminated groundwater. Lastly, Joshi [236] mentioned that the operating cost of horizontal wells is about 42–44% lower than that of vertical wells and fewer horizontal wells are needed when extracting the same amount of oil as compared to vertical wells.

3.6.1. Analytical and semi-analytical methods

Many analytical and semi-analytical solutions for horizontal wells have been presented in the groundwater literature. Typically, two assumptions involved in describing horizontal well screens are made to simplify the problem. The first assumption is that the flux along the well screen is assumed to be uniform. For example, Zhan et al. [239] derived an analytical solution for describing the drawdown due to pumping from a horizontal well in an anisotropic confined aquifer and compared the difference between the horizontal-well type curves and vertical-well type curves. Zhan and Zlotnik [81] presented a 3-D semi-analytical solution to estimate the drawdown due to pumping at a horizontal well in an anisotropic unconfined aquifer. Huang et al. [85] provided a general analytical solution to investigate the drawdown and stream depletion rate induced by a single horizontal well in an unconfined aquifer

near a stream. Furthermore, they indicated that their solution can reduce to that of Zhan and Zlotnik [81] for the aquifer of infinitely horizontal extent. Most of studies in fact introduce this assumption to describe the flux to the horizontal well [84,117,240]. The second assumption for specifying the boundary along the horizontal well screens is the constant-head condition. For example, Samani et al. [241] modified the solution for the horizontal uniform-flux well developed by Zhan and Zlotnik [81] to obtain a Laplace-domain solution for horizontal drains in an anisotropic unconfined aquifer. They also provided the type curves and discharge to a horizontal drain computed by using the Stehfest method to invert the Laplace-domain solution. The analytical or semi-analytical solutions developed for flow in horizontal wells can also be grouped according to the type of aquifers, such as confined aquifers [239, 242,243], leaky aquifers [117,244,245], and unconfined aquifers [81,85,240,241]. Some studies are also concerned with a variety of problems associated with the flow to horizontal wells such as gas or vapor flow [246,247].

In reality, the boundary along the screen of horizontal well in the field problems is rarely subject to the constant flux or constant head condition. The water flow into well screen is often non-uniform and the hydraulic head along the well may vary due to head loss caused by the flow inside the well. Approximate methods and numerical methods are therefore needed in the simulations for complicated groundwater flow in horizontal wells.

3.6.2. Approximate methods

There are few studies in regard to the use of approximate methods in horizontal well problems. Langseth et al. [248], for example, presented an approach by which the established solutions of vertical wells with constant pumping rate can be applied to horizontal well pumping problems. Based on the principle of superposition, the horizontal well is represented by a series of partially penetrating vertical well with different inflow rates assigned along the axis of the well. Shedid and Zekri [249] provided six different steady-state equations of the horizontal wells for problems with a wide range of well lengths (30–900 m), formation thickness (15–150 m), and pressure drop (7–140 m) in the petroleum literature.

3.6.3. Numerical methods

There are two ways commonly found in the groundwater literature to solve the problems using the finite difference methods in regard to the flow toward a horizontal well. One way is to develop the finite difference code for some specific problems. For example, Kawecki and Al-Subaikhy [250] applied a finite difference model to verify and assess previously proposed equations for flow to a horizontal well in Kawecki [251]. The other way is to adopt the finite-difference model MODFLOW for the flow simulations. Mohamed and Rushton [252] presented a finite different model based on MODFLOW for describing groundwater response in a horizontal well system at a field test site in Malaysia with considering three flow regimes: flow within the aquifer, flow from aquifer to the well screen, and flow through the horizontal well. Haitjema et al. [253] used MODFLOW and the analytic element model, GFLOW, to simulate a Dupuit–Forchheimer flow underneath a river or in a confined aquifer with a Cauchy boundary along a horizontal well in shallow aquifers.

A series of studies for flow induced by a horizontal well using the analytic element method have been done by Steward and his coauthors. Steward [254] first built a 3-D steady-state groundwater flow model using the analytic element method to delineate a capture zone caused by a horizontal well in a contaminated site in an infinite aquifer. The model was then used to quantify the losing sections where water exits through the horizontal well [255], to model the drawdown and capture zone topology for nonvertical

wells [256], and to examine the impact of well design on the head distribution with a horizontal well [257].

3.7. Collector well

Practically, the collector well can be regarded as a special type of horizontal wells. The collector wells are commonly designed and installed adjacent to streams or other surface water bodies to take advantage of induced riverbank filtration for supplying the public-drinking water. A radial collector well generally comprises two main components: the central concrete caisson and lateral well screens or, simply named as laterals. Moore et al. [258] mentioned that the laterals may be constructed in many configurations and the use of the configuration influences well performance and filtering water quality. They also provided five lateral geometric design alternatives. The choice of the design may depend on the local setting and desired well yield.

3.7.1. Analytical methods

At the present time, there are few analytical solutions available for some simplified cases in the groundwater literature. Hantush and Papadopulos [259] was the first to develop analytical solutions via superposition principle for describing drawdown distribution in confined and unconfined aquifers with a uniform flow along line sinks reflecting the effects of the laterals. Strack [126] gave a solution of complex potential which can be used to describe the flow pattern or head distribution for a fully penetrating radial collector well in a steady and uniform flow field. Similar to the approach taken by Hantush and Papadopulos [259], Hunt [244] developed an analytical solution for the drawdown distribution in a homogeneous, anisotropic leaky confined aquifer due to pumping at a horizontal well or radial collector well.

3.7.2. Numerical methods

In predicting the head or drawdown distribution in aquifers induced by water extraction at collector wells, the numerical methods were commonly used because of their advantages in handling spatial variation of aquifer properties and accommodating complex geometry of aquifer boundaries or well configurations. In addition, numerical modeling can also be used for the design of collector wells or the determination of well field capacity. Several different numerical approaches were used for the simulation of flow induced by radial collector wells in the past; for example, finite difference method [260], finite element method [261,262], analytic element method [258,262,263], and boundary element method [264].

In regard to the use of finite difference method, Schafer [265] used MODFLOW to simulate the pumping test at a collector well adjacent to the Ohio River in Louisville, Kentucky to evaluate the aquifer/river hydraulics. Wang and Zhang [260] developed a 3-D finite difference model to simulate a complicated collector well system consisting of an interconnected caisson, galleries, chambers, and small-diameter radiating laterals. Su et al. [261] adopted the TOUGH2 to study the development of unsaturated region beneath a perennial river due to the pumping at two radial collector wells near the streams. Zhang et al. [262] also used the TOUGH2 to investigate crucial factors such as riverbed permeability, dam operation and river velocity influencing aquifer recharge, maximum pumping capacity, and development of an unsaturated zone beneath the riverbed. Xu et al. [266] developed a model based on Multi-Node Well package of MODFLOW to assess the effect of chemical clogging occurring around the caisson on the groundwater level.

Ophori and Farvolden [267] used a Galerkin's finite element model to simulate the drawdown distribution due to pumping at

a collector well for preventing groundwater contamination due to a nearby waste disposal site.

Bakker et al. [268] presented an analytic element method by treating the laterals as multi-aquifer line sinks to simulate groundwater flow to radial collector wells. For the hydraulic design of collector wells, Moore et al. [258] developed a model based on the analytic element method to discuss the challenges and comparisons associated with alternate designs. Similarly, Patel et al. [263] also developed a model based on the analytic element method to simulate the steady-state discharge–drawdown relation for a radial collector well in an unconfined riverbed aquifer. In addition, the model is used to investigate the effects such as different lateral configurations, hydraulic conductivity of aquifer, and conductance of laterals on the well discharge and resulting drawdown.

3.8. Flow in leaky and multilayered aquifers

The solution of radial flow due to the pumping in a single-layer aquifer bounded between two impervious layers is provided by Theis [1], where the hydraulic head was assumed not to be influenced by adjacent aquifers. His solution will not be valid if the bounding formations are semi-permeable, and there is flow exchange within the aquitards and aquifers. The flow exchange is generally known as "leakage". In this section, the "leaky aquifer" is referred to the aquifer system that has only one aquifer while the aquifer system has more than one aquifer denotes as a multilayered aquifer or simply multiaquifer. Intensive review on the studies of multilayered aquifers in last decade can be found in Cheng and Morohunfola [269] and Cheng [270].

Approaches for solving the leaky and multilayered aquifer systems may be divided into three categories. In the first category, the flow in aquifers is assumed completely horizontal and the flow in aquitards is ignored. The second one is to solve the horizontal flow equations in aquifers coupled with the vertical flow equations in aquitards. The third considers both horizontal and vertical flows in aquifers and aquitards. Here we introduce some typical solutions for the models belonging to these three categories.

3.8.1. Consider the horizontal flow in aquifer but ignore flow in aquitard

Without considering the flow in aquitard, the hydraulic gradient through the aquitard changes instantaneously in accordance with the change in hydraulic gradient in the confined aquifer. This change in hydraulic gradient causes a leakage flux from the aquitard and the leakage can be regarded as a source or sink term embedded in the governing equation [19]. The solution of transient drawdown due to a constant rate pumping at a well which fully penetrates a leaky aquifer is attributed to Hantush and Jacob [35]. The flow in Hantush and Jacob's model can be described by the following equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{S}{T}\frac{\partial s}{\partial t} \tag{19}$$

where S is the aquifer storativity, T is the transmissivity of the aquifer, $B = (Tb'/K')^{1/2}$ is the leaky factor and K' and b' are the hydraulic conductivity and the thickness of the aquitard, respectively. The zero drawdown condition at the remote boundary and wellbore boundary condition for the system of one aquifer and one aquitard are, respectively,

$$s = 0, \quad r \to \infty \tag{20}$$

and

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \tag{21}$$

Under the initial condition that drawdown is zero at $t = 0$, the solution is given as

$$s(r,t) = \frac{Q}{4\pi T}W\left(u, \frac{r}{B}\right) = \frac{Q}{4\pi T}\int_u^\infty \frac{1}{y}\exp\left(-y - \frac{r^2}{4B^2 y}\right)dy \tag{22}$$

When the $u > 2r/B$, the Hantush well function $W(u, 2r/B)$ can be approximated by $W(u)$ which is expressed as $-0.5772 - \ln u$.

The steady-state drawdown solution in the leaky aquifer introduced by Jacob [271] can be derived by eliminating the term $\partial s/\partial t$ on the right-hand side of Eq. (19) and it is in the form of

$$s = \frac{Q}{2\pi T}K_0\left(\frac{r}{B}\right) \tag{23}$$

In Eq. (23), $K_0(r/B)$ can be approximated by $-[0.5772 + \ln(r/2B)]$ and $\sqrt{\pi B/2r}(1 - B/8r)\exp(-r/B)$ for $r/B < 0.05$ and $r/B > 5$, respectively.

Related research on flow in leaky aquifers that treated the leakage from aquitard as a sink or source term contained in the governing flow equation of aquifer can be found in the groundwater literature [272,273].

Many studies were also devoted to the development of the analytical solutions for multilayered aquifers with considering only the horizontal flow in aquifers. For example, Hantush [274] developed an analytical expression for the two aquifer and one aquitard system that produces horizontal flow in two aquifers with an intervening aquitard. Hunt and Scott [140] further treated the top aquifer as unconfined with a specific yield S_y and obtained an approximate solution for the two aquifers and one aquitard system. Hunt [272] gave a semianalytical stream depletion solution for a two aquifer and one aquitard system in which a pumped unconfined aquifer is underlain by an aquitard and an unpumped aquifer.

3.8.2. Consider horizontal flow in aquifer and vertical flow in aquitard

In many cases, the drawdown in an aquitard has significant influence on the flow in an adjacent aquifer. Under this circumstance, the predicted drawdown from the solution based on the assumption of ignoring the flow in an aquitard will result in large errors. Hantush [138] investigated the problem in which the vertical flow in aquitards is taken into consideration. The leaky aquifer system is composed of an aquifer overlain and underlain by aquitards. Three cases with different boundary conditions in aquitards were studies by Hantush [138]. In the following content, symbols with subscripts 0, 1, and 2 denote the parameters or variables of the main aquifer, upper, and lower aquitards, respectively.

Without considering the horizontal flow in aquitard, the flow equations for the upper and lower aquitards are

$$\frac{\partial^2 s_i}{\partial z^2} = \frac{S_i}{K_i b_i}\frac{\partial s_i}{\partial t}, \quad i = 1,2 \tag{24}$$

In addition, considering only the horizontal flow in aquifer, the flow in the main aquifer can be described using the following equation:

$$\frac{\partial^2 s_0}{\partial r^2} + \frac{1}{r}\frac{\partial s_0}{\partial r} - \frac{K_1}{T_0}\frac{\partial s_1(r, b_2 + b_0, t)}{\partial z} - \frac{K_2}{T_0}\frac{\partial s_2(r, b_2, t)}{\partial z} = \frac{S_0}{T_0}\frac{\partial s_0}{\partial t} \tag{25}$$

The initial drawdowns in the main aquifer and aquitards are assumed to be zero. The hydraulic heads are distributed continuously at the interfaces between the upper aquitard and the main aquifer and between the lower aquitard and the main aquifer. In addition, the drawdown remains zero at infinite distance in the main aquifer and the flux along the wellbore can also be expressed as Eq. (21).

By sequentially applying the Laplace and Hankel transforms to the governing equation, initial and boundary conditions, the drawdown solution in Laplace domain for the main aquifer in case 1 can be found in Hantush [138]. Hantush also gave the approximate

solution for small and large values of time as shown in Hantush [138].

Recently, Moench and Barlow [275] addressed the issue of pumping in a system of one aquifer and one aquitard interacting with perennial stream. Wen et al. [55] presented a mathematical model for the CHT at a well with a finite-thickness skin in a system of one aquifer and two aquitard.

Some works modeled the flows in multilayered aquifer systems based on the second approach. For example, Neuman and Witherspoon [276] developed a theory for flow in a two aquifer and one aquitard system. Chen and Jiao [277] also used the second approach to investigate the flow behavior caused by pumping in multilayered aquifer system with non-Darcian vertical flow in the wellbore. Similarly, Butler et al. [278] utilized this approach to develop an analytical solution for flow in a multilayered aquifer system and explored the impact of groundwater pumping on a nearby stream.

3.8.3. Consider both horizontal and vertical flow in aquifers and aquitard

In nature, flow in aquifer and aquitard may occur in both vertical and horizontal directions. Hantush and Jacob [279] considered a pumping well in an aquifer overlain by an aquitard and solved a 3-D steady-state flow system without relying on the assumption that the permeability in aquifer must be much greater than that in aquitard made in Jacob [271]. The solution was obtained by simultaneously solving the flow equations of aquifer and aquitard. Coupling of flows in aquifer and aquitard was accomplished through acquiring the continuity conditions of drawdown and normal flux imposed at their interface.

Malama et al. [45] solved a transient problem that an unconfined aquifer bounded below by an aquitard is pumped continuously at a constant rate from a well of infinitesimal radius. They also treated the flows in aquifer and aquitard as both horizontal and vertical. By considering the horizontal and vertical flows in aquifers, Malama et al. [44] solved the flow problem in a two aquifer and one aquitard system in which the top unconfined aquifer is pumped at a partially penetrating well.

Different from those three approaches mentioned above, some studies can also be seen in the well hydraulic literature for flow in leaky aquifer systems [117,280,281] and for flow in multilayered aquifer systems [169,282,283]. Numerical solutions for flow in the leaky and multilayered aquifer systems have been accomplished by finite element methods [284,285], finite difference methods [285,286], and boundary element methods [269].

3.9. Capture zone delineation

An equation in terms of pumping rates, Darcy velocity, and aquifer thickness to stipple separating streamline was given by Muskat [287] which may be the earliest work associated with the capture zone delineation. A capture zone is defined as a portion of aquifer in which water is removed by one or more extraction wells under steady-state or transient flow condition. Generally, the shape of the capture zone depends on the groundwater flow velocity, the extraction rate, the aquifer boundaries and the hydraulic properties of the aquifer. The study of capture zone is very useful in solving problems involved in, for example, groundwater rights, wellhead protection program, groundwater remedial design, as well as study of surface water and groundwater interaction.

3.9.1. Analytical methods

Complex potential theory (also named as stream function methods or complex velocity potential) is usually applied to solve the problems of 2-D, planar, steady state flow in homogeneous formations [288]. It is the most widely used method to delineate the capture zone for one or more extraction wells in aquifers in an infinite domain [289–292].

By combing the velocity potential for flow to the well with that for the regional flow field, the potential theory can also be applied to determine 3-D capture zone for steady state flow toward a partially penetrating well in confined aquifers [293,294]. Additionally, some approaches were used to delineate 3-D capture zone under different flow conditions, such as flow in unconfined aquifers [295], the presence of multiple sinks and sources [296], or flow around horizontal drains and vertical wells in anisotropic aquifer in a uniform flow field [297]. Moreover, with the aid of image well theory, the potential theory can also be applied to determine the capture zone of a well either near a stream [298], located between two parallel streams [299], or in an infinite strip aquifer bounded by two constant head boundaries or by one constant head and one impermeable bound boundary [300].

Analytical approach to delineate capture areas induced by pumping involves the use of the Theis solutions for confined aquifers [301] and Hantush–Jacob equation for leaky confined aquifers [302]. Moreover, Huang and Goltz [303] developed a 3-D steady state model and solved it by using the Fourier cosine transform to calculate the interflow circulating between a pair of recirculating wells in a homogeneous, anisotropic aquifer. Moreover, the articles by Cunningham et al. [304] and Luo and Kitanidis [305] also deal with the recirculation zone around a pair of recirculating wells.

3.9.2. Numerical methods

Numerical simulations can take into account the effects of complex boundary conditions as well as heterogeneity, recharge/discharge, etc., and are therefore broadly used for studying vertical well capture zones [306,307]. Shafer [308] adopted a finite-difference groundwater flow model to simulate 2-D steady state head distribution and used a fourth-order Runge–Kutta scheme to determine the flow pathlines for a time-related capture zone around extraction wells. Townley and Davidson [309] used a boundary integral approach to define a capture zone for shallow water table lakes in a 2-D regional flow system. Taylor and Person [310] developed a finite element model to solve a coupled system of flow through the fresh water and salt water zones and used an approach of reverse particle tracking to generate fluid pathlines for determining the time-related capture zone. The analytic element method uses the superposition of appropriate potential functions and applied normally to the case of an aquifer of an infinite extent [126]. The analytic element method can be used to determine the shape of capture zone due to pumping for problems of 3-D flow induced by horizontal flow [255,256]. In capture zone delineation, this methods may be used with other approaches such as the locations of stagnation points in a flow field to describe the streamline [311] or the separation of variables and theory for multi-aquifer flow [312].

Most commonly used numerical approaches in the capture zone literature are to use MODFLOW model [313] along with a 3-D particle tracking program such as MODPATH [302,314] developed by Pollock [315] or PATH3D [316] developed by Zheng et al. [296], or with a 2-D particle tracking program such as GWPATH [317]. Few studies in the groundwater literature have addressed the topics of the capture zone produced by a horizontal well [242,243].

3.9.3. Optimization and uncertainty

The method of optimization can be employed in the capture zone analysis to find the best solution for determining the pumping rates and well locations [318,319], the minimum number of wells [320], or the number of wells and total pumping rates [307]. In addition, the optimization approaches can also be used

in the analysis of capture zone delineation to examine the effectiveness of various single-well pumping scheme [321] or the most cost-effective strategies for hydraulic control of groundwater contamination [322,323]. Some studies have devoted to the assessment of the uncertainty about the delineation of well capture zones in heterogeneous aquifers. The stochastic approach based on the Monte Carlo simulations is widely adopted for characterizing the uncertainty about the capture zone delineation [324–327]. Instead of Monte Carlo simulations, Esling et al. [328] attempted to reduce the capture zone uncertainty based on a systematic sensitivity analysis.

3.10. Non-Darcian flow

Darcy's law describes a linear relationship between the hydraulic gradient (i) and specific discharge (q). For flow through granular media, there are indications that the linear relationship is not valid in two situations. One is associated with flow through low permeability materials under very low gradients and the other is large flow through very high permeability media. Bear [122] mentioned that Darcy's law is applicable if the Reynolds number (N_R) based on average grain diameter falls in the range from 1 to 10. Within this range, the flow is classified as laminar and known as Darcian flow, and otherwise called as non-Darcian flow. In addition, the flow is turbulent when N_R is high (say, $N_R > 100$), while the flow between laminar and turbulent regimes is considered in the transition state. For flow near the extraction wells, the hydraulic gradients are usually very high and the flow velocities are enhanced due to the convergence of flow lines [329]. Consequently, the flow may become turbulent near the wells. Many formulae have been proposed to describe the relationship between i and q for non-Darcian flow [100,330]. Among them, the Forchheimer equation [331] and Izbash equation [332] (also called as power law function) are most commonly used. The Forchheimer equation can be expressed as $i = aq + bq^2$ where a and b are empirical coefficients whereas the Izbash equation may be written as $i = cq^m$ in which c is a constant and the parameter m ranges between one and two [333]. Notably, some studies indicated that the Izbash equation is better than the Forchheimer equation in describing the relationship between i and q under some circumstances [334]. In addition, Izbash equation is in continuity with Darcy's law which corresponds to the case $m = 1$ [333].

3.10.1. Forchheimer flow in porous media

Based on the Forchheimer equation, the analytical solutions for non-Darcian flow are provided by Bear [122] and Ewing et al. [335] for steady state groundwater flow toward a well. The analytical solutions for transient non-Darcian flow developed based on the Forchheimer equation and Boltzmann transform was first presented by Sen [336,337] for an infinitesimal well and later by Sen [101] for large-diameter wells in confined aquifers. However, Camacho-V. and Vasquez-C. [338] questioned the validity of using Boltzmann transform in solving the non-Darcian flow problems and considered those solutions as approximate solutions rather than analytical solutions. Based on the volumetric approach, Birpinar and Sen [339] developed an analytical solution for radial Forchheimer flow toward a fully penetrating and infinitesimally small diameter well in leaky aquifers. Mathias et al. [329] developed a large time solution for describing the Forchheimer flow toward a well using the method of matched asymptotic expansion and compared the predicted heads with those from the finite difference model. Moutsopoulos and Tsihrintzis [131] presented an approximate solution from the perturbation analysis of 1-D transient Forchheimer flow between two rivers.

Ewing et al. [335] developed a mixed numerical model for describing single phase Forchheimer flow in a hydrocarbon reservoir using the cell-centered finite differences, Galerkin's finite element, and mixed finite element techniques. Later, Ewing and Lin [340] extended the work of Ewing et al. [335] to simulate a steady state non-Darcian flow in porous media with the application of three different finite volume models. Mathias et al. [329] presented a finite difference model for describing the Forchheimer non-Darcian flow toward a pumping well in porous media and compared the predicted heads with those from three different approximate solutions. Wu [341] approximated the multiphase Forchheimer flow equations for a porous medium and fractured reservoir using an integral finite difference or control-volume finite element scheme.

3.10.2. Izbash flow in porous media

Sen [100] used the Boltzmann transform to derive a transient drawdown solution for Izbash flow toward a fully penetrating, infinitesimal well in a confined aquifer. Later, he used the principle of volumetric approach to develop a solution for investigating the transient Izbash flow behavior in leaky aquifers [342]. Some approximate solutions for describing the transient radial Izbash flow were proposed by Wen et al. [343] and Mathias et al. [329] in a confined aquifer using Laplace transform and the linearization approximation and also by Wen et al. [344] in an aquifer–aquitard system and Wen et al. [345] in a two-region flow system using linearization procedures.

The finite difference models were often used to simulate Izbash flow for various types of problems and the results were compared with those predicted by the approximate solution for validation purposes [329,345].

3.10.3. Forchheimer flow in fractured formations

Non-Darcian flow also occurs in fractures [346,347]. Some research has been devoted to the issue of non-Darcian flow in fractured formations with the use of the Forchheimer equation in hydrologic literature [341,348,349]. For example, Wu [349] developed a steady state solution for 1-D finite, radial Forchheimer flow toward an extraction well in a fracture system. Kohl et al. [346] developed a 3-D finite element model for the Forchheimer flow in a hot dry fractured rock to evaluate transient pressure responses for two flow tests at a wellbore. In addition, Kolditz [350] used the finite element method and Forchheimer equation to study non-Darcian flow behavior induced by the pumping tests in fractured rocks.

3.10.4. Izbash flow in fractured formations

Very few studies focused on the use of Izbash equation to develop non-Darcian flow in fractured media. Teh and Nie [351] developed a finite element model for describing non-Darcian flow based on the Izbash equation and coupled consolidation theory. Based on the Izbash equation, Wen et al. [352] presented a finite difference solution and two approximate solutions obtained by using the linearization method and the Boltzmann transform, respectively, for the non-Darcian flow toward an extended well.

3.11. Well function evaluation

The analytical solution arisen from solving various types of problems of aquifer tests in the well hydraulics is often given in terms of an improper integral. For example, Theis [1] presented a solution for a constant pumping from a fully penetrating well of infinitesimal diameter in a homogeneous, isotropic, and non-leaky confined aquifer of infinite extent. His solution describing the spatial and temporal distribution of aquifer drawdown due to pumping can be expressed in terms of an exponential integral which is also called the Theis well function. Hantush and Jacob [35] introduced a solution describing the drawdown in a leaky confined

aquifer due to the pumping at a fully penetration well. Their solution contains an improper integral called the Hantush well function, also called as Hantush–Jacob well function or leaky aquifer function. Later, Hantush [67] gave a solution for the unsteady drawdown due to a constant discharge at a partially penetrating well in a confined aquifer of uniform thickness and uniform hydraulic properties. This solution also includes an improper integral often called as Hantush M function. In addition, he provided a solution for describing the rise and decay of the water table in response to an areal uniform recharge in rectangular or circular type of shape [353] and another solution for estimating the stream depletion rate induced by a nearby pumping [179]. Both solutions contain different forms of improper integrals; the former is called as Hantush M function, while the latter is named as Hantush M^* function. In fact, the latter can be changed into a form with the variable of integration appearing in the denominator as demonstrated in Trefry [148]. Jaeger [354] presented an equation for the heat flow at the inner boundary of a solid. This equation is expressed in terms of an improper integral commonly called as Jaeger function or Jaeger integral. Interestingly, the Jaeger function also appears in a formula describing the flow rate across the wellbore obtained from solving the radial groundwater flow equation subject to the constant-head boundary condition at the wellbore [128]. The lower limit of above-mentioned functions is zero or a variable which may be very close to zero in some cases. In those functions, their variable of integration also appears in the denominator of the integrand. Such a problem causes the functions having singularity at or near the origin and therefore results in great difficulty to accurately compute those functions. This section thus reviews the numerical computations for those five functions mentioned above.

3.11.1. Theis well function

The Theis well function $W(u)$ is defined in the following drawdown equation as

$$s(r,t) = \frac{Q}{4\pi T} W(u) = \frac{Q}{4\pi T} \int_u^\infty \frac{e^{-y}}{y} dy \qquad (26)$$

where u is a dimensionless variable defined as $r^2 S/4Tt$ and y is a dummy variable. The integral in Eq. (26) is a function of the lower integration limit u and denoted as $W(u)$. This function tends to be infinity and therefore fails to be defined when u approaches zero. The integral in Eq. (26) can be expanded in a convergent series given in most groundwater textbooks [224,225]. Cody and Thacher [355] commented on the approximations to this integral in previous four articles which had the problems with insufficient accuracy or lack of efficiency in computing. He also mentioned that the series expression can be employed without loss of accuracy when $u \leqslant 1$. Furthermore, Tseng and Lee [146] pointed out that the series expression is suitable for practical applications when u is small (say, less than 4) and, however, has the problem of slow convergence when u is large.

Prabha and Yadav [356] developed a general polynomial expression based on rational approximations of the Bickley and Theis well functions. Interestingly, they also mentioned that both functions occur in the problem of reactor physics in computing first flight collision probability. Tseng and Lee [146] reviewed six approximation methods for calculating Theis well function. In addition, they also presented an algorithm based on the combination of a fast-converging series representation for small u and an easy-implementing Gauss–Laguerre quadrature formula when u becomes large. They mentioned that their algorithm will provide a required accuracy for any fixed argument of $0 < u < \infty$ if the number of quadrature points used in the computation is greater than 8 combined with the series expansion with less than 20 terms. Barry et al. [150] provided a single analytical approximation constructed

by interpolation between the small and large asymptotes of the Theis well function. The approximation has the maximum error less than 0.07% and the result is valid for $0 < u < \infty$. Prodanoff et al. [357] applied a smoothing procedure by decomposing a finite part of singular integrals, such as Theis and Hantush well functions, into two integrals, one of which is smooth and can be computed by standard integration quadrature, while the other can be easily integrated analytically. The advantage of using this approach is that it can directly deal with any related functions without further approximations and users can choose the desired accuracy.

3.11.2. Hantush well function

The Hantush well function $W(u,r/B)$ is defined in the drawdown equation, Eq. (22), for pumping in a leaky aquifer. When $u = 0$, $W(0,r/B)$ is equal to $2K_0(r/B)$. On the other hand, when $r/B = 0$ (i.e., $W(u,0)$ equals $W(u)$), the Hantush well function reduces to the Theis well function. Hunt [358] developed two infinite series to represent the Hantush well function. Those two series are absolutely convergent and can be used to estimate the Hantush well function for all possible values of the arguments. As mentioned above, Prodanoff et al.'s approach [357] can also be applied to evaluate Eq. (22). Nadarajah [359] commented on the work of Prodanoff et al. [357] and mentioned that both Theis and Hantush well functions can be expressed in terms of the modified Bessel function of the second kind of zero order and the Appell hypergeometric series of the first kind. Harris [360] gave a generalized formula to the Hantush well function as

$$K_v = \int_1^\infty \frac{dt}{t^{v+1}} e^{-xt-y/t} \qquad (27)$$

with v a real value but not necessarily an integral and $K_0(x,y)$ the original Hantush well function and identified as an incomplete Bessel function or a generalized incomplete gamma function. He mentioned that the function on the RHS of Eq. (27) appears in different areas such as hydrology, heat conduction, probability theory, and electronic structure in periodic systems. He also presented a number of new expansions which offer efficient computation over a broad range of all three parameters (v,x,y) in the incomplete Bessel function. Temme [361] also discussed the properties of K_v and suggested applying the trapezoidal rule to compute Eq. (27) for a wide range of the parameters. Veling and Mass [362] first reviewed the works of Hunt [358], Prodanoff et al. [357], and Nadarajah [359], then presented two analytic expressions based on a generalized hypergeometric function in two variables, and finally gave a very efficient approximate formula in terms of the exponential integral and the modified Bessel function K_0, which may be useful in programming. Moreover, Veling [363] expanded the generalized incomplete gamma function in terms of a sum of the function I_j and the modified Bessel function of the first kind of order j, and gave numerical techniques to evaluate the function.

3.11.3. Hantush M and M* functions

The Hantush M function was denoted as $M(\alpha, \beta)$ and written in terms of an infinite integral defined by the following [2,147,364]

$$M(\alpha, \beta) = \int_\alpha^\infty \frac{e^{-y}}{y} erf(\beta\sqrt{y}) dy \qquad (28)$$

where y is a dummy variable, α and β are parameters related to the physical properties of the aquifer, and $erf(x)$ is the error function. Tabular values of the $M(\alpha, \beta)$ function with four significant figures were given in Hantush [2,67,364] for α from 10^{-6} to 1 and β from 0.1 to 400. Later, Hantush [179,353] defined the $M^*(\alpha, \beta)$ function as

$$M^*(\alpha, \beta) = \frac{2}{\pi} \int_0^\alpha \frac{e^{-\beta(1+y^2)}}{1+y^2} dy \qquad (29)$$

Tabular values of the $M^*(\alpha, \beta)$ function were given in Hantush [179] for α from 0.1 to ∞ and β from 0.1 to 5.0 with five decimal figures.

Trefry [148] developed an exact algebraic expression for Eq. (29) in terms of a power series expansion with the incomplete gamma function. Furthermore, the expansion is also rearranged to two independent analytical partial summations. He demonstrated the numerical values computed by one of his equation [148] are correct to 9 decimal places for $M^*(1,1)$, 10 decimal places for $M^*(2,3/2)$, and 9 decimal places for $M^*(4/5, 7/10)$. In the comment on Trefry's article [148], Barry et al. [365] presented an alternative series for estimating $M^*(\alpha,\beta)$ for an extended range of arguments from that presented in Trefry [148]. Moreover, Barry et al. [366] gave some simple approximations to the $M^*(\alpha, \beta)$ function for all practical purposes. Mamedov and Ekenoglu [367] developed a general and simple analytical algorithm, which involves the incomplete Gamma function based on binomial expansion theorem, for calculating $M^*(\alpha, \beta)$ function with arbitrary accuracy. Yang and Yeh [368] gave a comment on Mamedov and Ekenoglu's paper [367] and provided a simple and efficient numerical approach using the Gaussian quadrature to calculate $M^*(\alpha, \beta)$ piece-wisely along the y-axis from $(0, \alpha)$ to $(-1, 1)$.

In a note on Hantush M function, Trefry [369] provided algebraic approximations to $M(\alpha, \beta)$ over the relevant (α, β) parameter space by using truncated Maclaurin and asymptotic series. Nadarajah [359] mentioned that both $M(\alpha, \beta)$ and $M^*(\alpha, \beta)$ functions can be reduced to the generalized incomplete exponential functions and demonstrated that the computed values for both functions accurate up to the first ten decimal places for selected values of α in the range of 0.01–13 and β in the range of 0.01−17.

3.11.4. Jaeger function

The Jaeger function is generally expressed as

$$I(0, 1; \tau) = \int_0^\infty \frac{e^{-\tau u^2}}{[J_0^2(u) + Y_0^2(u)]u} du \qquad (30)$$

where τ is dimensionless time, u is a dummy variable, and $J_0(\cdot)$ and $Y_0(\cdot)$ are the Bessel functions of the first and second kinds of order zero, respectively. This equation might be first given by Jaeger [354] for the heat flow problem. The values of $I(0, 1; \tau)$ with three decimal figures for τ from 0.01 to 1000 are given in a joint paper by Jaeger and Clarke [370]. Ingersoll et al. [371,372] adopted and interpolated the values of $I(0, 1; \tau)$ of Jaeger and Clarke [370] with three decimal places for τ from 0.01 to 25,000.

In an early paper, Smith [373] considered a problem in deep mining operations and developed solutions for the surface temperature and temperature flux across the wellbore. In his temperature flux solution, an integral defined as $G(\tau)$ in our notation can be written as [373]

$$G(\tau) = \frac{4\tau}{\pi} \int_0^\infty u e^{-\tau u^2} \left\{ \frac{\pi}{2} + \tan^{-1} \left(\frac{Y_0(u)}{J_0(u)} \right) \right\} du \qquad (31)$$

Later, Jacob and Lohman [48] studied a problem for the discharge at a well with constant drawdown in an extensive aquifer. They cited the solution given by Smith [373] and replaced the integral in Eq. (31) by a summation and developed a numerical integration approach similar to the trapezoidal rule to calculate the $G(\tau)$ function. They gave computed values of $G(\tau)$ in a table with three or four significant digits for τ from 10^{-4} to 10^{12}.

It is noteworthy that Eqs. (30) and (31) had been shown to be mathematically equivalent in the paper written by Peng et al. [128] which mainly focused on the numerical computation of solutions for groundwater flow subject to a constant-head pumping. They developed an approach including a singularity removal scheme, Newton's method, the Gaussian quadrature, and Shanks' method to evaluate the solutions of head distribution in aquifers

and the flow rate across the wellbore. The computed values of dimensionless flow rate to five decimal places for τ from 0.01 to 1000 are given in a table and compared with those of Jaeger and Clarke [370] and Jacob and Lohman [48] by multiplying a factor of $2/\pi$. Interestingly, Jaeger function also appears in the area of contemporary electrochemical techniques such as chronoamperometry and some efforts involved in the computations of this function have been made. For example, Aoki et al. [374] gave an approximation of $I(0, 1; \tau)$ with an error less than 1% for some ranges of τ, Szabo et al. [375] presented an approximation of $I(0, 2; \tau)$ within 1.3% for all values of τ, and Fang et al. [132] also provided a simple approximation with an accuracy of about 1% by extending from the steady-state approximation to the transient one. Moreover, Britz et al. [376] split the integration of Eq. (30) into a number of integrals and calculated each of the integrals using Romberg-integration to 6 significant figures for τ from 10^{-4} to 10^4. Recently, Bieniasz [377] decomposed the Jaeger function in Laplace domain into two terms; the first term, fully responsible for the singularity, can be easily computed after being inverted to the time domain, while the second term after taking the inverse Laplace transform is approximated by two finite series depending on the range of τ. Such an approximate procedure provides the results with at least 14–15 significant digits over the entire time domain. More recently, Philips and Mahon [378] presented a paper to desingularize Eq. (30) by subtracting and then adding a complimentary principal-valued integral, which can be evaluated analytically over the half-space. The integrand in the desingularized integral can then be calculated directly using Laguerre–Gauss quadrature. They provided the estimated results with 10 significant figures which agree the appropriate number of decimal places with those of Peng et al. [128] and Britz et al. [376].

4. Conclusion and future recommendations

In this review, we start with a concise introduction to the types of aquifers and mathematical formulations associated with the physical characteristics and configurations of aquifer hydraulics. Widely used aquifer tests and their related solution methodologies for solving the mathematical models are then addressed. Furthermore, a wealth of literature associated with the effects of finite well radius, wellbore storage, well partial penetration, and the presence of skin zone on the models has been intensively reviewed. Finally, recent advances on the subjects of capture zone delineation, non-Darcian flow, flows in horizontal well and collecting well, and various kinds of well function evaluation involved in the modeling of well hydraulics is also presented. Some related topics for future research are suggested as follows.

1. Most of researches on the investigation of the drainage, seepage, or recharge problems in unconfined aquifers have used the analytical approaches in dealing with the flow problems in sloping aquifers [175,188,192,379]. To the best of our knowledge, there are very few articles to deal with the flow induced by pumping in a sloping confined aquifer [229] or a confined aquifer with non-uniform thickness [259]. Antonio and Pacheco [229] developed an approximate approach applying the Cooper-Jacob equation and method of image well to describe the response of a sloping fault zone aquifer to the pumping at a fully penetration well. Their approximation, which neglects the effects of the vertical flow in the aquifer and the angle between the sloping aquifer and the horizontal axis, may result in poor predicted results. Based on a physical analogy, Hantush and Papadopulos [259] adopted the analytical solutions of the head distribution and flow rate for flow in a constant head pumping in infinite confined aquifers to describe the flow in

vertical wedge-shaped formations. Such an analogy may yield acceptable estimation of flow rate if the pumping well in the wedge-shaped formations is very close to a stream or lake. Yet, the head distribution predicted based on the solution of a constant head pumping in confined aquifers may be inaccurate, especially when the pumping well is located far away from the stream or lake. Very often, nature aquifer systems have dipping hydrostratigraphic formations. Thus, there may be a need to develop analytical models or numerical models to describe the flow in sloping aquifers in response to various types of aquifer tests.

2. During 1970s and 1980s, conceptual models for describing flow in naturally fractured media in response to a variety of hydraulic tests [380] had been made intensively in the areas of petroleum engineering and groundwater hydrology. Three models are commonly used to represent fractured formation systems such as equivalent porous medium, discrete fractures, and dual porosity [199,381]. Note that the subject of non-Darcian flow in discrete fractures of fractured formations has been covered in this review. Berkowitz [382] provided an intensive review on characterizing flow and transport in fractured media. He also raised many open questions at the end of some sections. One of the questions is that "How can we best quantify the non-linear relationship between volumetric fluid flow and hydraulic gradient ...?" It is of practical interest to evaluate the suitability or applicability of the use of the Forchheimer equation and Izbash equation developed in 1901 and 1931, respectively, in most field of fractured flow problems. If these two equations can not best quantify that non-linear relationship, then the development of a new model to simulate the non-Darcian flow in fracture media will be the task of future research.

3. In this review, we have addressed the issue of numerical calculations for five well functions encountered in the hydrology or other sciences. Among those functions, three functions including Theis well function, Hantush well function, and Hantush M function have been included in a list given by Hantush [2]. As a matter of fact, Hantush [2] gave a total of 20 functions often encountered in problems of groundwater flow modeling. Some of those functions are mathematical functions such as Bessel functions, error functions, and gamma functions. The numerical approximations for those functions are, in fact, available with high accuracy in Abramowitz and Stegun [104]. The first function $A(\tau,\rho)$ given in Hantush [2] is the dimensionless solution for groundwater flow subject to a constant-head pumping in a confined aquifer. The tabular values of $A(\tau,\rho)$ for very wide ranges of τ and ρ have three significant digits [2], while those of $A(\tau,\rho)$ estimated by Peng et al. [128] using their approach including a singularity removal scheme, Newton's method, the Gaussian quadrature, and Shanks' method have accuracy to five decimal places. The fourth function $G(\tau,\rho)$, fifth function $H(u,\beta)$, thirteen function $S(\tau,\rho)$, fifteen function $v(\tau,\rho)$, and nineteen function $Z(\tau,\rho,\beta)$ in Hantush's list [2] can also be evaluated using the approach of Peng et al. [128] or others approached suggested by Nadarajah [359], Philips and Mahon [378], or Veling [363] for high accuracy. Similarly, those above-mentioned approaches may also be used to evaluate complicated solutions such as the ones provided by Cooper et al. [57], Neuman and Witherspoon [276], Papadopulos and Cooper [75], or others obtained from well-hydraulic problems.

4. Very often, the analytical solutions developed for describing the flow in various groundwater hydrology problems are employed to generate the type curves which are generally plotted in figures with the curves of dimensionless hydraulic head (or dimensionless drawdown) versus dimensionless time (or distance). The hydraulic parameters are then determined by the graphical approach via curve-fitting procedure, i.e., to match the observed data to a type curve. In principle, the graphical approach is applicable only if the number of unknown parameters of aquifer properties is three or less. In reality, the number of hydraulic parameters may be up to four or more for the flow in leaky aquifers or unconfined aquifer systems. For example, the solution for hydraulic head (or drawdown) includes four parameters for flow in an aquifer system with one confined aquifer and one aquiclude [383] or in an unconfined aquifer [31]. The head solution has five parameters for flow in confined aquifers with a finite thickness skin subject to a pumping at a constant discharge [27] or constant head [167] if the skin thickness is considered as an unknown parameter. Moreover, the head solution contains six parameters for flow in an aquifer system with one confined aquifer and two confining layers [138] or with two confined aquifers and one confining layer [383]. In those cases that the parameter number is four or more, an alternative and feasible way we suggest for identifying the parameters is to develop a numerical approach by combining the analytical solution with the algorithm of extended Kalman filter [384] or with an optimization method such as simulated annealing [385,386] or the nonlinear least-squares [387].

5. In modeling groundwater flow to a partially penetrating well for a CRT, a uniform flux along the screen portion is commonly assumed as the boundary condition. Such an assumption makes the development of the analytical solutions workable, yet introduces some errors in the prediction of flow near the wellbore. A more appropriate treatment of the boundary condition along the screen portion is to specify the integrated flux rather than the uniform flux. Despite the widely application of analytical models based on the uniform flux assumption, the use of this assumption needs to be deliberated.

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