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Palindromic quadratization and structure-preserving algorithm for palindromic matrix polynomials of even degree

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Abstract In this paper, we propose a palindromic quadratization approach, transforming a palindromic matrix polynomial of even degree to a palindromic quadratic pencil. Based on the $(\mathcal{S} + \mathcal{S}^{-1})$-transform and Patel's algorithm, the structure-preserving algorithm can then be applied to solve the corresponding palindromic quadratic eigenvalue problem. Numerical experiments show that the relative residuals for eigenpairs of palindromic polynomial eigenvalue problems computed by palindromic quadratized eigenvalue problems are better than those via palindromic linearized eigenvalue problems or `polyeig` in MATLAB.

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1 Introduction

In this paper, we consider $(\star, \varepsilon)$-palindromic matrix polynomials of even degree $2d$

$$\mathcal{P}(\lambda) \equiv \sum_{k=0}^{d-1} \lambda^{2d-k} A_{d-k}^{\star} + \lambda^d A_0 + \varepsilon \sum_{k=1}^{d} \lambda^{d-k} A_k, \tag{1}$$

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where $d \geq 2$, $\varepsilon = \pm 1$ and $\star$ = H (Hermitian) or T (transpose), $A_k \in \mathbb{C}^{n\times n}$ ($k = 0, 1, \ldots, d$) and $A_0^\star = \varepsilon A_0$. The corresponding polynomial eigenvalue problem $\mathcal{P}(\lambda)x = 0$, with $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \backslash \{0\}$ being the eigenvalue and the associated eigenvector respectively, is called a $(\star, \varepsilon)$-palindromic polynomial eigenvalue problem ($(\star, \varepsilon)$-PPEP). It is also called a $\star$-PPEP if $\varepsilon = 1$ or a $\star$-anti-PPEP if $\varepsilon = -1$.

The underlying matrix polynomial $\mathcal{P}(\lambda)$ in (1) has the property that reversing the order of coefficients, followed by taking the (conjugate) transpose, leads to the original matrix polynomial (anti-)invariant, which satisfies

$$\mathcal{P}(\lambda) = \varepsilon\lambda^{2d}\mathcal{P}(1/\lambda)^\star \tag{2}$$

and explains the world "(anti-)palindromic" [20]. Consequently, taking the (conjugate) transpose of (1), we easily see that the eigenvalues of $\mathcal{P}(\lambda)$ satisfy a "reciprocal" property, that is, they appear in the pairs of the form $(\lambda, 1/\lambda^\star)$.

The $(\star, \varepsilon)$-PPEPs arise in solving higher order systems of ordinary or partial differential equations. A T-palindromic quadratic eigenvalue problem (T-PQEP) is first raised in the vibration analysis for fast trains in Germany [11,12] and then in the study of surface acoustic wave filters [28]. An H-palindromic quadratic eigenvalue problem (H-PQEP) arises in the computation of the Crawford number, for detecting definite Hermitian pairs or hyperbolic or elliptic quadratic eigenvalue problems [9]. A $\star$-PPEP of even degree is obtained when solving the linear quadratic discrete-time optimal control problem for higher order systems [2,27].

A standard approach for computing the eigenpairs of $\mathcal{P}(\lambda)$ in (1) is to linearize it to a $2dn \times 2dn$ linear matrix pencil by the companion linearization and compute its generalized Schur form [26]. However, the reciprocal property of eigenvalues of $\mathcal{P}(\lambda)$ is not preserved, generally, thus producing large numerical errors [6,13,15]. Recently, some pioneering works [20,21] propose some good linearizations that linearize $\mathcal{P}(\lambda)$ to palindromic linear pencils of the form $\lambda Z^\star + Z$ which preserves the reciprocity of eigenvalues. This does lead to a vast improvement over previous unstructured approaches, reflecting the palindromic structure in the original polynomial and enabling structure-preserving numerical methods to be designed. Later, a QR-like algorithm [24] and a hybrid method [19] which combines Jacobi-type method with the Laub's trick, a postprocessing step of the generalized Schur form, are proposed for solving T-palindromic linear eigenvalue problems efficiently. The QR-like algorithm typically requires $O(n^4)$ flops and the hybrid method requires $O(n^3 log(n))$ flops. Recently, a URV-decomposition based structured method of cubic complexity is developed in [25] to solve T-palindromic linear eigenvalue problems, producing eigenvalues which are reciprocally paired to working precision. A new structure-preserving doubling algorithm with cubic complexity for solving $\star$-palindromic linear eigenvalue problems is developed in [4]. On the other hand, for solving a $(\star, \varepsilon)$-PQEP, a structure-preserving doubling algorithm is developed in [5,6] via the computation of a solvent of a nonlinear matrix equation associated with the $(\star, \varepsilon)$-PQEP. Lately, a numerically stable structure-preserving algorithm (SPA), based on the $(\mathcal{S} + \mathcal{S}^{-1})$-transform [18] and Patel's algorithm [22], is proposed in [13] to solve the T-PQEP directly. The numerical results obtained by the SPA algorithm show much promise and the computational cost of SPA is about a half of that of the URV-based method.

The purpose of this paper is to develop a palindromic quadratization which transforms a $(\star, \varepsilon)$-palindromic matrix polynomial of even degree with $(\star, \varepsilon) \neq (T, -1)$ into a $(\star, \varepsilon)$-palindromic quadratic pencil. If $\star = T$ and $\varepsilon = 1$, then we can apply the SPA algorithm in [13] to solve the associated quadratized T-PQEP directly. If $\star = H$ and $\varepsilon = \pm 1$, we first transform the associated quadratized (H, ε)-PQEP to an H-skew-Hamiltonian pencil by the $(\mathcal{S} + \mathcal{S}^{-1})$-transform and enlarge the H-skew-Hamiltonian pencil to a real skew-Hamiltonian pencil, to which the SPA algorithm is applicable. Note that for the case $(\star, \varepsilon) = (T, -1)$, the T-anti-PPEP can then be solved by applying the URV-based method [23,25] to the linearized T-palindromic linear pencil.

Throughout this paper, $\mathbb{C}^{n\times m}$ is the set of all $n \times m$ complex matrices, $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ and $\mathbb{C} = \mathbb{C}^1$; I_n and $0_{n\times m}$ (or simply I and 0 if their dimensions are clear from the context) denote the $n \times n$ identity matrix and $n \times m$ zero matrix, respectively. The jth column of an identity matrix is denoted by e_j. The direct sum of two matrices is denoted by "$\oplus$".

This paper is organized as follows. In Sect. 2, we propose a palindromic quadratization for a $(\star, \varepsilon)$-palindromic matrix polynomial of even degree. In Sect. 3, we develop a structure-preserving algorithm for solving the H-PQEPs. We develop balancing techniques for PPEPs and PQEPs in Sect. 4. Comparisons of numerical results computed by the palindromic quadratization, the palindromic linearization and the standard companion linearization are presented in Sect. 5. Conclusions are given in Sect. 6.

2 P-Quadratization of $(\star, \varepsilon)$-PPEP

In [13], a structure-preserving algorithm is well-developed for solving the T-PQEP. A similar structure-preserving algorithm for solving the H-PQEP will be introduced in Sect. 3. As the H-anti-PQEP can be easily transformed to the H-PQEP, all $(\star, \varepsilon)$-PQEPs with $(\star, \varepsilon) \neq (T, -1)$ can be solved by structure-preserving algorithms. Furthermore when $(\star, \varepsilon) \neq (T, -1)$, we shall propose a new *palindromic* quadratization (P-quadratization) which can be utilized to transform a $(\star, \varepsilon)$-PPEP into a $(\star, \varepsilon)$-PQEP so that the structure-preserving algorithm in [13] is applicable.

Next we present definitions of quadratization and P-quadratization of a general matrix polynomial and a palindromic matrix polynomial, respectively.

Definition 1 (*Quadratization/P-Quadratization*)

(i) Let $\mathcal{P}(\lambda)$ be an arbitrary $\nu \times \nu$ matrix polynomial of degree $p \geq 2$ with $p\nu = 2q$. A $q \times q$ quadratic matrix polynomial (quadratic pencil) $\mathcal{Q}(\lambda)$ is a quadratization of $\mathcal{P}(\lambda)$ if there are matrix rational functions $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$ of size $q \times q$ with nonzero and constant determinants satisfying the two-sided factorization

$$\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \begin{bmatrix} \mathcal{P}(\lambda) & 0 \\ 0 & I_{q-\nu} \end{bmatrix}. \tag{3}$$

(ii) Let $\mathcal{P}(\lambda)$ be an arbitrary $\nu \times \nu$ $(\star, \varepsilon)$-palindromic matrix polynomial of degree $p \geq 2$ with $p\nu = 2q$ (i.e., $\mathcal{P}(\lambda) = \varepsilon\lambda^p\mathcal{P}(1/\lambda)^\star$ as in (2)). A quadratization $\mathcal{Q}(\lambda)$ of $\mathcal{P}(\lambda)$ having the $(\star, \varepsilon)$-palindromic structure is called a *P-quadratization* of $\mathcal{P}(\lambda)$.

Theorem 1 *Let $\mathcal{Q}(\lambda)$ be a $q \times q$ quadratization of a $\nu \times \nu$ matrix polynomial $\mathcal{P}(\lambda)$ of degree p with $p\nu = 2q$. Then*

(i) *$\lambda_0 \in \mathbb{C}$ is a finite eigenvalue of $\mathcal{Q}(\lambda)$ (i.e., $det(\mathcal{Q}(\lambda_0)) = 0$) if and only if λ_0 is a finite eigenvalue of $\mathcal{P}(\lambda)$ (i.e., $det(\mathcal{P}(\lambda_0)) = 0$).*
(ii) *∞ is an eigenvalue of $\mathcal{Q}(\lambda)$ (i.e., $det\big([\lambda^2\mathcal{Q}(\frac{1}{\lambda})]\big|_{\lambda=0}\big) = 0$) if and only if ∞ is an eigenvalue of $\mathcal{P}(\lambda)$ (i.e., $det\big(\big[\lambda^p\mathcal{P}\left(\frac{1}{\lambda}\right)\big]\big|_{\lambda=0}\big) = 0$).*

Proof (i) The factorization (3) implies that $\det(\mathcal{Q}(\lambda)) = c\ \det(\mathcal{P}(\lambda))$ for some nonzero constant c, so that $\mathcal{Q}(\lambda)$ and $\mathcal{P}(\lambda)$ are singular or nonsingular for precisely the same values of λ_0.
(ii) Since

$$\det\left[\lambda^2\mathcal{Q}\left(\frac{1}{\lambda}\right)\right] = \lambda^{2q}\det\left[\mathcal{Q}\left(\frac{1}{\lambda}\right)\right] = c\lambda^{2q}\det\left[\mathcal{P}\left(\frac{1}{\lambda}\right)\right]$$
$$= c\ \det\left[\lambda^p\mathcal{P}\left(\frac{1}{\lambda}\right)\right],$$

both $\mathcal{Q}(\lambda)$ and $\mathcal{P}(\lambda)$ have or have no infinite eigenvalues. □

Since both $\det(\mathcal{E}(\lambda))$ and $\det(\mathcal{F}(\lambda))$ are nonzero and constant, it is easily seen that the two-sided factorization (3) implies the existence of a more wide class of one-sided factorization

$$\mathcal{Q}(\lambda)F(\lambda) \equiv \mathcal{Q}(\lambda)\mathcal{F}(\lambda)\begin{bmatrix} I_n \\ 0 \end{bmatrix} = \mathcal{E}(\lambda)^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}\mathcal{P}(\lambda) \equiv G(\lambda)\mathcal{P}(\lambda), \tag{4}$$

where $F(\lambda)$ and $G(\lambda)$ are matrix rational functions of size $q \times \nu$. From the factorization (4) a close connection between eigenpairs of $\mathcal{P}(\lambda)$ and eigenpairs of $\mathcal{Q}(\lambda)$ has been shown in [8].

Theorem 2 [8] *Assume that (4) holds at $\lambda_0 \in \mathbb{C}$ with $F(\lambda_0)$ and $G(\lambda_0)$ being of full column rank. Then $F(\lambda_0)z_1$ is an eigenvector of $\mathcal{Q}(\lambda)$ if and only if z_1 is an eigenvector of $\mathcal{P}(\lambda)$, both corresponding to eigenvalue λ_0.*

In Definition 1(i), we give a new definition of quadratization for a general matrix polynomial. In Theorems 1 and 2, we show the connection between eigenpairs of a general matrix polynomial and its quadratization. We next present a P-quadratization for a palindromic matrix polynomial of even degree explicitly.

Theorem 3 *Let $\mathcal{P}(\lambda)$ be an $n \times n$ $(\star, \varepsilon)$-palindromic matrix polynomial of degree $2d$ as in (1) with $(\star, \varepsilon) \neq (T, -1)$. Then $\mathcal{P}(\lambda)$ can be P-quadratized into a $(\star, \varepsilon)$-palindromic quadratic pencil of the form*

$$\mathcal{Q}(\lambda) \equiv \lambda^2 \mathcal{A}_1^\star + \lambda \mathcal{A}_0 + \varepsilon \mathcal{A}_1 \tag{5}$$

with $\mathcal{A}_0^\star = \varepsilon \mathcal{A}_0$, where

(i) *(For $2d = 4m$) $\mathcal{A}_1$ and $\mathcal{A}_0$ are given by*

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & d_1^\star I \\ \varepsilon\sqrt{\varepsilon} d_1 A_2 & 0 \end{bmatrix}, \quad \mathcal{A}_0 = \begin{bmatrix} A_0 - \sqrt{\varepsilon} I - \sqrt{\varepsilon} A_2^\star A_2 & 0 \\ 0 & -\sqrt{\varepsilon} d_1 d_1^\star I \end{bmatrix} \tag{6}$$

if $m = 1$; otherwise,

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots & \cdots & \cdots & 0 & d_m^\star I \\ \varepsilon\sqrt{\varepsilon} d_1 A_{2m} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ A_{2m-1} & d_1^\star I & \ddots & & & & \vdots \\ 0 & & -\varepsilon d_2 I & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ A_3 & & & & d_{m-1}^\star I & 0 & \vdots \\ 0 & & & & & -\varepsilon d_m I & 0 \end{bmatrix}, \tag{7a}$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 - \sqrt{\varepsilon} I - \sqrt{\varepsilon} A_{2m}^\star A_{2m} & 0 & A_{2m-2}^\star & 0 & \cdots & A_2^\star & 0 \\ 0 & -\sqrt{\varepsilon} d_1 d_1^\star I & & & & & \vdots \\ \varepsilon A_{2m-2} & & 0 & & & & \vdots \\ 0 & & & 0 & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ \varepsilon A_2 & & & & & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}; \tag{7b}$$

(ii) *(For $2d = 4m + 2$) $\mathcal{A}_1$ and $\mathcal{A}_0$ are given by*

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & d_m^\star I \\ A_{2m+1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -\varepsilon d_1 I & \ddots & & & & & \vdots \\ A_{2m-1} & & d_1^\star I & \ddots & & & & \vdots \\ 0 & & & -\varepsilon d_2 I & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ A_3 & & & & & d_{m-1}^\star I & 0 & 0 \\ 0 & & & & & & -\varepsilon d_m I & 0 \end{bmatrix}, \quad (8a)$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 & A_{2m}^\star & 0 & A_{2m-2}^\star & 0 & \cdots & A_2^\star & 0 \\ \varepsilon A_{2m} & 0 & & & & & & \vdots \\ 0 & & 0 & & & & & \vdots \\ \varepsilon A_{2m-2} & & & 0 & & & & \vdots \\ 0 & & & & 0 & & & \vdots \\ \vdots & & & & & \ddots & & \vdots \\ \varepsilon A_2 & & & & & & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}; \quad (8b)$$

in which $d_1, \ldots, d_m$ are arbitrary nonzero constants.

Proof (i) (For $2d = 4m$) We define the $n \times n$ matrix rational functions for $j = 1, \ldots, m$:

$$F_{2j,1}(\lambda) = \left(\lambda^{2j-1} d_j^\star\right)^{-1} \left(\varepsilon\sqrt{\varepsilon}\lambda^{2m} I_n + \sum_{k=0}^{2j-2} \lambda^k A_{2m-k}\right), \quad (9a)$$

and for $j = 1, \ldots, m-1$:

$$F_{2j+1,1}(\lambda) = \lambda^{2m-2j} I_n. \quad (9b)$$

Let

$$F_1(\lambda) = \left[\begin{array}{c|c} I_n & F_{2,1}(\lambda)^T \; F_{3,1}(\lambda)^T \; \cdots \; F_{2m,1}(\lambda)^T \\ \hline 0 & I_{(2m-1)n} \end{array}\right]^T. \quad (10)$$

Routine but tedious calculation in terms of $F_1(\lambda)$ in (10), $\mathcal{Q}(\lambda)$ in (5) and $\mathcal{A}_0, \mathcal{A}_1$ in (6) or (7) leads to

$$\mathcal{Q}(\lambda) F_1(\lambda) \begin{bmatrix} I_n \\ 0_{(d-1)n,n} \end{bmatrix} = \begin{bmatrix} Q_1(\lambda) \\ 0_{(d-1)n,n} \end{bmatrix}, \quad (11)$$

where

$$Q_1(\lambda) = \lambda^2 A_1^\star + \lambda\left(A_0 - \sqrt{\varepsilon} I - \sqrt{\varepsilon} A_{2m}^\star A_{2m}\right) + \varepsilon A_1$$
$$+\lambda\sqrt{\varepsilon} A_{2m}^\star\left(\lambda^{2m}\varepsilon\sqrt{\varepsilon} I + A_{2m}\right) + \sum_{k=1}^{m-1}\lambda^{2m-2k}\left(\lambda^2 A_{2m-2k+1}^\star + \lambda A_{2m-2k}^\star\right)$$
$$+\varepsilon\lambda^{1-2m}\left(\varepsilon\sqrt{\varepsilon}\lambda^{2m} I + \sum_{k=0}^{2m-2}\lambda^k A_{2m-k}\right) = \lambda^{1-2m}\mathcal{P}(\lambda). \tag{12}$$

Next, we define for $j = 1, \ldots, m$:

$$E_{1,2j+1}(\lambda) = \lambda^{2j-2m} I_n, \quad \text{for } j = 1, \ldots, m-1,$$
$$E_{1,2j}(\lambda) = \left(\lambda^{2m-2j+1} d_j\right)^{-1}\left(\sum_{k=0}^{2j-2}\lambda^{2m-k} A_{2m-k}^\star + \sqrt{\varepsilon} I_n\right),$$

and let

$$E_1(\lambda) = \left[\begin{array}{c|c} I_n & E_{1,2}(\lambda)\;\; E_{1,3}(\lambda)\;\cdots\; E_{1,2m}(\lambda) \\ \hline 0 & I_{(2m-1)n}\end{array}\right]. \tag{13}$$

From (12) and the definition of $E_1(\lambda)$ in (13), we have

$$\left[\, I_n \quad 0_{n,(d-1)n} \,\right] E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda) = \left[\, \lambda^{1-2m}\mathcal{P}(\lambda)\; 0_{n,(d-1)n} \,\right].$$

Then from (10) and (13), it follows for $m = 1$ that

$$\left[\, 0 \quad I_n \,\right] E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda)\begin{bmatrix} 0 \\ I_n \end{bmatrix} = -\lambda\sqrt{\varepsilon} d_1 d_1^\star I_n;$$

or for $m \geq 2$:

$$\left[\, 0 \quad I_{(d-1)n} \,\right] E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda)\begin{bmatrix} 0 \\ I_{(d-1)n} \end{bmatrix}$$
$$= \begin{bmatrix} -\lambda\sqrt{\varepsilon} d_1 d_1^\star I_n & \lambda^2 d_1 I_n & 0 & \cdots & \cdots & 0 \\ \varepsilon d_1^\star I_n & 0 & -\lambda^2\varepsilon d_2^\star I_n & \ddots & & \vdots \\ 0 & -d_2 I_n & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 d_{m-1} I_n & 0 \\ \vdots & & \ddots & \varepsilon d_{m-1}^\star I_n & 0 & -\lambda^2\varepsilon d_m^\star I_n \\ 0 & \cdots & \cdots & 0 & -d_m I_n & 0 \end{bmatrix}.$$

Using the factorizations

$$\begin{bmatrix} I_n & 0 \\ \frac{\sqrt{\varepsilon}}{\lambda d_j} I_n & I_n \end{bmatrix} \begin{bmatrix} -\lambda\sqrt{\varepsilon} d_j d_j^\star I_n & \lambda^2 d_j I_n \\ \varepsilon d_j^\star I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & \frac{\lambda}{\sqrt{\varepsilon} d_j^\star} I_n \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} -\lambda\sqrt{\varepsilon} d_j d_j^\star I_n & 0 \\ 0 & \lambda\sqrt{\varepsilon} I_n \end{bmatrix}$$

and

$$\begin{bmatrix} I_n & 0 \\ \frac{d_{j+1}}{\sqrt{\varepsilon}\lambda} I_n & I_n \end{bmatrix} \begin{bmatrix} \lambda\sqrt{\varepsilon} I_n & -\lambda^2 \varepsilon d_{j+1}^\star I_n \\ -d_{j+1} I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & \lambda\sqrt{\varepsilon} d_{j+1}^\star I_n \\ 0 & I_n \end{bmatrix}$$
$$= \begin{bmatrix} \lambda\sqrt{\varepsilon} I_n & 0 \\ 0 & -\lambda\sqrt{\varepsilon} d_{j+1} d_{j+1}^\star I_n \end{bmatrix},$$

it holds that

$$E_{2m-1}(\lambda) \cdots E_2(\lambda) E_1(\lambda) \mathcal{Q}(\lambda) F_1(\lambda) F_2(\lambda) \cdots F_{2m-1}(\lambda)$$
$$= \operatorname{diag}\left\{\lambda^{1-2m}\mathcal{P}(\lambda), -\lambda\sqrt{\varepsilon} d_1 d_1^\star I_n, \lambda\sqrt{\varepsilon} I_n, \ldots, \lambda\sqrt{\varepsilon} I_n, -\lambda\sqrt{\varepsilon} d_m d_m^\star I_n\right\},$$

where

$$E_{2j}(\lambda) = I_{(2j-1)n} \oplus \begin{bmatrix} I_n & 0 \\ \frac{\sqrt{\varepsilon}}{\lambda d_j} I_n & I_n \end{bmatrix} \oplus I_{(2m-2j-1)n},$$
$$E_{2j+1}(\lambda) = I_{2jn} \oplus \begin{bmatrix} I_n & 0 \\ \frac{d_{j+1}}{\sqrt{\varepsilon}\lambda} I_n & I_n \end{bmatrix} \oplus I_{(2m-2j-2)n},$$

and

$$F_{2j}(\lambda) = I_{(2j-1)n} \oplus \begin{bmatrix} I_n & \frac{\lambda}{\sqrt{\varepsilon} d_j^\star} I_n \\ 0 & I_n \end{bmatrix} \oplus I_{(2m-2j-1)n},$$
$$F_{2j+1}(\lambda) = I_{2jn} \oplus \begin{bmatrix} I_n & \lambda\sqrt{\varepsilon} d_{j+1}^\star I_n \\ 0 & I_n \end{bmatrix} \oplus I_{(2m-2j-2)n},$$

for $j = 1, \ldots, m-1$. Finally, letting

$$E_{2m}(\lambda) := \operatorname{diag}\left\{\lambda^{2m-1} I_n, -(\lambda\sqrt{\varepsilon} d_1 d_1^\star)^{-1} I_n, (\lambda\sqrt{\varepsilon})^{-1} I_n, -(\lambda\sqrt{\varepsilon} d_2 d_2^\star)^{-1} I_n,\right.$$
$$\left. \ldots, (\lambda\sqrt{\varepsilon})^{-1} I_n, -(\lambda\sqrt{\varepsilon} d_m d_m^\star)^{-1} I_n\right\},$$

$\mathcal{E}(\lambda) := E_{2m}(\lambda) \cdots E_1(\lambda)$ and $\mathcal{F}(\lambda) := F_1(\lambda) \cdots F_{2m-1}(\lambda)$, one can easily verify that $\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \operatorname{diag}(\mathcal{P}(\lambda), I_{(2m-1)n})$. Furthermore, it holds that $\det(\mathcal{E}(\lambda)) = (-\varepsilon)^m / (\sqrt{\varepsilon} \prod_{j=1}^m d_j d_j^\star)$ and $\det(\mathcal{F}(\lambda)) = 1$.

(ii) (For $2d = 4m + 2$) Let

$$\Pi_{2j} = I_{(2j-1)n} \oplus \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \oplus I_{(2m-2j)n}, \quad \text{for } j = 1, \ldots, m.$$

Similar to part (i), we define $n \times n$ matrix rational functions $E_1(\lambda)$ and $F_1(\lambda)$ by

$$E_1(\lambda) = \Pi_2 \left[\begin{array}{c|c} I_n & E_{1,2}(\lambda) \;\; E_{1,3}(\lambda) \;\; \cdots \;\; E_{1,2m+1}(\lambda) \\ \hline 0 & I_{2mn} \end{array} \right],$$
$$F_1(\lambda) = \left[\begin{array}{c|c} I_n & F_{2,1}(\lambda)^T \;\; F_{3,1}(\lambda)^T \;\; \cdots \;\; F_{2m+1,1}(\lambda)^T \\ \hline 0 & I_{2mn} \end{array} \right]^T$$

with

$$\begin{aligned} E_{1,2j}(\lambda) &= \lambda^{2j-2m-2} I_n, \quad E_{1,2j+1}(\lambda) = d_j^{-1} \sum_{k=0}^{2j-1} \lambda^{k+1} A^\star_{2m+k-2j+2}, \\ F_{2j,1}(\lambda) &= \lambda^{2m-2j+2} I_n, \quad F_{2j+1,1}(\lambda) = \left(d_j^\star \lambda^{2j}\right)^{-1} \sum_{k=0}^{2j-1} \lambda^k A_{2m-k+1} \end{aligned} \tag{14}$$

for $j = 1, \ldots, m$. Via careful calculation we get

$$\mathcal{Q}(\lambda) F_1(\lambda) \begin{bmatrix} I_n \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda^{-2m} \mathcal{P}(\lambda) \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I_n & 0 \end{bmatrix} E_1(\lambda) \mathcal{Q}(\lambda) F_1(\lambda) = \begin{bmatrix} \lambda^{-2m} \mathcal{P}(\lambda) & 0 \end{bmatrix}.$$

Letting

$$E_j(\lambda) = \Pi_{2j} \left(I_{2(j-1)n} \oplus \begin{bmatrix} I_n & 0 \\ \lambda^{-2} I_n & I_n \end{bmatrix} \oplus I_{(2m-2j+1)n} \right),$$
$$F_j(\lambda) = I_{(2j-3)n} \oplus \begin{bmatrix} I_n & 0 & \lambda^2 I_n \\ 0 & I_n & I_n \\ 0 & 0 & I_n \end{bmatrix} \oplus I_{(2m-2j+1)n},$$

for $j = 2, \ldots, m$, and

$$\begin{aligned} &E_{m+1}(\lambda) \\ &\quad = \operatorname{diag}\left\{ \lambda^{2m} I_n, \frac{-1}{d_1} I_n, \frac{-1}{\lambda^2 \varepsilon d_1^\star} I_n, \frac{-1}{d_2} I_n, \frac{-1}{\lambda^2 \varepsilon d_2^\star} I_n, \ldots, \frac{-1}{d_m} I_n, \frac{-1}{\lambda^2 \varepsilon d_m^\star} I_n \right\}, \end{aligned}$$

one can also verify that $\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \text{diag}(\mathcal{P}(\lambda), I_{2mn})$, where $\mathcal{E}(\lambda) = E_{m+1}(\lambda) \cdots E_1(\lambda)$ and $\mathcal{F}(\lambda) = F_1(\lambda)\cdots F_m(\lambda)$. Furthermore, it holds that $\det(\mathcal{E}(\lambda)) = \varepsilon^{-m}/(\prod_{j=1}^{m}(d_j d_j^\star))$ and $\det(\mathcal{F}(\lambda)) = 1$. □

Note that the P-quadratization of a $(\star, \varepsilon)$-palindromic matrix polynomial of odd degree with even matrix dimension can be defined as in Definition 1. However, to the best of our knowledge, a P-quadratization of this type has not been found.

We now show the relationship between eigenpairs of $\mathcal{Q}(\lambda)$ in (5) and $\mathcal{P}(\lambda)$ in (1).

Theorem 4 *Let $\mathcal{Q}(\lambda)$ in (5) be a P-quadratization of $\mathcal{P}(\lambda)$ in (1) with $(\star, \varepsilon) \neq (T, -1)$. Denote $z = \begin{bmatrix} z_1^T & \cdots & z_d^T \end{bmatrix}^T$ with $z_j \in \mathbb{C}^n$ $(j = 1, \ldots, d)$. Then*

(i) *For $\lambda_0 \neq 0$, (λ_0, z_1) is an eigenpair of $\mathcal{P}(\lambda)$ if and only if (λ_0, z) is an eigenpair of $\mathcal{Q}(\lambda)$ with $z_j = F_{j,1}(\lambda_0)z_1$ $(j = 2, \ldots, d)$, where $\{F_{j,1}(\lambda)\}_{j=2}^{d}$ are given in (9) (for $d = 2m$) or (14) (for $d = 2m + 1$).*

(ii) *$(0, z_1)$ is an eigenpair of $\mathcal{P}(\lambda)$ if and only if $(0, z)$ is an eigenpair of $\mathcal{Q}(\lambda)$, where, for $j = 1, \ldots, m - 1$:*

$$(\text{for } d = 2m) \quad \begin{cases} z_{2m} = -(d_m^\star)^{-1}A_1 z_1, \; z_{2j+1} = 0, \\ z_{2j} = -(d_j^\star)^{-1}A_{2m-2j+1}z_1; \end{cases} \tag{15a}$$

$$(\text{or, for } d = 2m + 1) \quad \begin{cases} z_{2m+1} = -(d_m^\star)^{-1}A_1 z_1, \; z_{2m} = 0, \\ z_{2j} = 0, \; z_{2j+1} = -(d_j^\star)^{-1}A_{2m-2j+1}z_1. \end{cases} \tag{15b}$$

(iii) *(∞, z_2) is an eigenpair of $\mathcal{P}(\lambda)$ if and only if (∞, z) is an eigenpair of $\mathcal{Q}(\lambda)$, with $z_1 = z_3 = \cdots = z_d = 0$.*

Proof (i) From Theorem 3, there are $dn \times dn$ matrix rational functions $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$ with nonzero and constant determinants such that $\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \text{diag}(\mathcal{P}(\lambda), I_{(d-1)n})$. Since $\lambda = 0$ is the only pole of $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$, the matrices $F(\lambda_0)$ and $G(\lambda_0)$ defined in (4) are of full rank. The assertion in (i) follows immediately from Theorem 2 and the relation

$$F(\lambda_0)z_1 = \mathcal{F}(\lambda_0)\begin{bmatrix} z_1 \\ 0 \end{bmatrix} = F_1(\lambda_0)\begin{bmatrix} z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 \\ F_{2,1}(\lambda_0)z_1 \\ \vdots \\ F_{d,1}(\lambda_0)z_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{bmatrix}.$$

(ii) By the definition of $\mathcal{Q}(\lambda)$ in (5), we have $\mathcal{Q}(0)z = \varepsilon q \equiv \varepsilon \begin{bmatrix} q_1^T & \cdots & q_d^T \end{bmatrix}^T$, where for $d = 2m$:

$$\begin{cases} q_1 = A_1 z_1 + d_m^\star z_{2m}, \; q_2 = \varepsilon\sqrt{\varepsilon}d_1 A_{2m}z_1, \; q_3 = A_{2m-1}z_1 + d_1^\star z_2, \\ q_4 = -\varepsilon d_2 z_3, \ldots, q_{2m-1} = A_3 z_1 + d_{m-1}^\star z_{2m-2}, \; q_{2m} = -\varepsilon d_m z_{2m-1}; \end{cases} \tag{16a}$$

and for $d = 2m + 1$:

$$\begin{cases} q_1 = A_1 z_1 + d_m^\star z_{2m+1}, \; q_2 = A_{2m+1}z_1, \; q_3 = -\varepsilon d_1 z_2, \; q_4 = A_{2m-1}z_1 + d_1^\star z_3, \\ q_5 = -\varepsilon d_2 z_4, \ldots, q_{2m} = A_3 z_1 + d_{m-1}^\star z_{2m-1}, \; q_{2m+1} = -\varepsilon d_m z_{2m}. \end{cases} \tag{16b}$$

From (16), we see that $(0, z_1)$ is an eigenpair of $\mathcal{P}(\lambda)$; i.e., $A_d z_1 = 0$ if and only if $(0, z)$ is an eigenpair of $\mathcal{Q}(\lambda)$; i.e., $\mathcal{Q}(0)z = \varepsilon q = 0$, where $\{z_j\}_{j=1}^d$ satisfy (15).

(iii) By the definition of $\mathcal{Q}(\lambda)$ in (5) again, we have $\left[\lambda^2 \mathcal{Q}\left(\frac{1}{\lambda}\right)\right]_{\lambda=0} z = q \equiv \left[q_1^T \cdots q_d^T\right]^T$, where for $d = 2m$:

$$\begin{cases} q_1 = A_1^\star z_1 + \sqrt{\varepsilon} d_1^\star A_{2m}^\star z_2 + \sum_{k=1}^{m-1} A_{2m-2k+1}^\star z_{2k+1}, \ q_2 = d_1 z_3, \\ q_3 = -\varepsilon d_2^\star z_4, \ \ldots, \ q_{2m-2} = d_{m-1} z_{2m-1}, \ q_{2m-1} = -\varepsilon d_m^\star z_{2m}, \ q_{2m} = d_m z_1; \end{cases} \tag{17a}$$

and for $d = 2m + 1$:

$$\begin{cases} q_1 = A_1^\star z_1 + \sum_{k=1}^{m} A_{2m-2k+3}^\star z_{2k}, \ q_2 = -\varepsilon d_1 z_3, \ q_3 = d_1^\star z_4, \\ \ldots, q_{2m} = -\varepsilon d_m z_{2m+1}, \ q_{2m+1} = d_m^\star z_1. \end{cases} \tag{17b}$$

From (17), it follows that (∞, z_2) is an eigenpair of $\mathcal{P}(\lambda)$; i.e., $A_d^\star z_2 = 0$ if and only if (∞, z) is an eigenpair of $\mathcal{Q}(\lambda)$, or $\left[\lambda^2 \mathcal{Q}\left(\frac{1}{\lambda}\right)\right]_{\lambda=0} z = q = 0$, with $z_1 = z_3 = \cdots = z_d = 0$. □

Remark 1 (i) Theorem 3 cannot be applied to the case $(\star, \varepsilon) = (T, -1)$. In fact, up to now, there is no structure-preserving algorithm to solve the T-anti-PQEP directly. So it is pointless to transform a T-anti-PPEP to a T-anti-PQEP. Thus, for a $(T, -1)$-palindromic matrix polynomial, we can apply the palindromic linearization [20] to transform it into a T-palindromic linear pencil $\lambda Z^T + Z$, and then solve it by the QR-like algorithm [24], the hybrid method [19], the URV-based method [25] or the doubling algorithm [4].

(ii) On the other hand, if we rewrite $\lambda Z^T + Z$ to a T-palindromic quadratic pencil $\widehat{\mathcal{Q}}(\widehat{\lambda}) \equiv \widehat{\lambda}^2 Z^T + \widehat{\lambda} 0 + Z$ by letting $\widehat{\lambda}^2 = \lambda$, then the SPA algorithm [13] can also be used to solve its eigenpairs. It is shown in [13] that applying the SPA to solve $\widehat{\mathcal{Q}}(\widehat{\lambda})y = 0$ is mathematically equivalent to applying the URV-based method to solve $\lambda Z^T + Z$.

Applying the P-quadratization in Theorem 3, an H-anti-PPEP can be quadratized into an H-anti-PQEP whose eigenpairs can then be computed from an H-PQEP by the following relationship.

Proposition 1 *Given an H-anti-PQEP:* $(\lambda^2 A_1^H + \lambda A_0 - A_1)x = 0$, *with* $A_0^H = -A_0$. *Then* $(\imath\omega, x)$ *is an eigenpair of the H-anti-PQEP if and only if* (ω, x) *is an eigenpair of the H-PQEP:* $\left[\omega^2(-A_1)^H + \omega(\imath A_0) + (-A_1)\right]x = 0$.

Proof The result can be easily obtained by setting $\lambda = \imath\omega$ and using the fact $(\imath A_0)^H = \imath A_0$. □

We now consider the $\star$-even and $\star$-odd polynomial eigenvalue problems of even degree. Let $\mathcal{C}(\lambda) = \sum_{k=0}^{2d} \lambda^k C_k$, where $C_k \in \mathbb{C}^{n\times n}$ $(k = 0, 1, \ldots, 2d)$ and $C_{2d} \neq 0$. The polynomial eigenvalue problem $\mathcal{C}(\lambda)x = 0$ is called a $\star$-even polynomial eigenvalue problem, if $C_{2k}^\star = C_{2k}$ $(k = 0, 1, \ldots, d)$ and $C_{2k-1}^\star = -C_{2k-1}$ $(k = 1, \ldots, d)$; and it is called a $\star$-odd polynomial eigenvalue problem, if $C_{2k}^\star = -C_{2k}$ $(k = 0, 1, \ldots, d)$

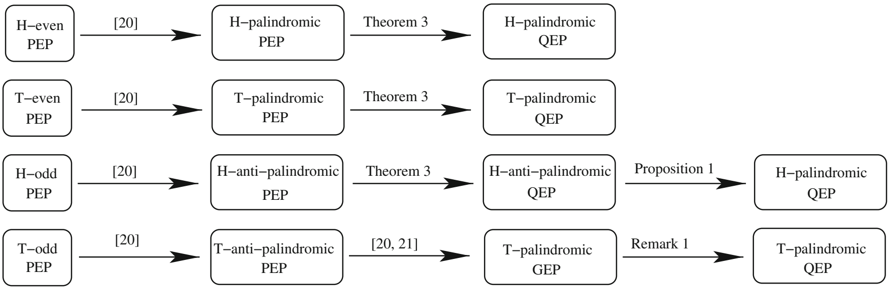

Fig. 1 Relations between various structured polynomial eigenvalue problems (PEPs)

and $C^{\star}_{2k-1} = C_{2k-1}$ $(k = 1, \dots, d)$. By the Cayley transformation, it was shown in [20] that a $\star$-even/odd polynomial eigenvalue problem can be transformed to a $(\star, \pm 1)$-PPEP, respectively.

We illustrate the relationship among various structured polynomial eigenvalue problems in Fig. 1. We see that T-even, T-odd, T-anti-palindromic and T-palindromic polynomial eigenvalue problems of even degree can be P-quadratized to T-PQEPs. Thus, the SPA algorithm in [13] can be applied to solve the associated T-PQEPs. On the other hand, we see that H-even, H-odd, H-anti-palindromic and H-palindromic polynomial eigenvalue problems of even degree can be P-quadratized to H-PQEPs. Therefore, we are motivated to develop a structure-preserving algorithm in the next section to solve the H-PQEP.

3 H-palindromic quadratic eigenvalue problems

Consider the H-PQEP

$$\mathcal{Q}(\lambda)x \equiv \left(\lambda^2 A_1^H + \lambda A_0 + A_1\right) x = 0, \tag{18}$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$ with $A_0^H = A_0$. The eigenvalues of $\mathcal{Q}(\lambda)$ clearly appear in the "reciprocal" pairs of the form $(\lambda, 1/\bar{\lambda})$.

Classical linearizations of (18) in a companion form, generally, do not preserve the symplectic structure. Fortunately, the special linearization

$$(\mathcal{M} - \lambda\mathcal{L})z \equiv \left(\begin{bmatrix} A_1 & 0 \\ -A_0 & -I \end{bmatrix} - \lambda \begin{bmatrix} 0 & I \\ A_1^H & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{19}$$

of (18) (see [5] or [13]) satisfies

$$\mathcal{M}\mathcal{J}\mathcal{M}^H = \mathcal{L}\mathcal{J}\mathcal{L}^H, \quad \mathcal{J} = \mathcal{J}_{2n} \equiv \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{20}$$

so that the matrix pair $(\mathcal{M}, \mathcal{L})$ has eigenvalues λ and $1/\bar{\lambda}$, preserving reciprocity. The pencil $\mathcal{M} - \lambda\mathcal{L}$ or the matrix pair $(\mathcal{M}, \mathcal{L})$ are called H-symplectic.

For any real symplectic matrix pair $(\mathcal{M}, \mathcal{L})$ satisfying (20), a structure-preserving $(\mathcal{S}+\mathcal{S}^{-1})$-transform for the computation of all eigenvalues was proposed by [18]. The $(\mathcal{S}+\mathcal{S}^{-1})$-transform $(\mathcal{M}_s, \mathcal{L}_s)$ of an H-symplectic matrix pair $(\mathcal{M}, \mathcal{L})$ is defined by

$$\mathcal{M}_s \equiv \mathcal{M}\mathcal{J}\mathcal{L}^H + \mathcal{L}\mathcal{J}\mathcal{M}^H \equiv \mathcal{K}\mathcal{J}, \qquad \mathcal{L}_s \equiv \mathcal{L}\mathcal{J}\mathcal{L}^H \equiv \mathcal{N}\mathcal{J}. \tag{21}$$

It is easily seen that $\mathcal{K}$ and $\mathcal{N}$ are both H-skew-Hamiltonian, i.e., $\mathcal{K}\mathcal{J} = \mathcal{J}\mathcal{K}^H$ and $\mathcal{N}\mathcal{J} = \mathcal{J}\mathcal{N}^H$. Hence, if μ is an eigenvalue of $(\mathcal{K}, \mathcal{N})$, so is $\bar{\mu}$.

The relationship between eigenpairs of an H-symplectic matrix pair and its $(\mathcal{S} + \mathcal{S}^{-1})$-transform has been given in [18].

Theorem 5 [18] *Let $(\mathcal{M}, \mathcal{L})$ be H-symplectic and $(\mathcal{M}_s, \mathcal{L}_s)$ be its $(\mathcal{S} + \mathcal{S}^{-1})$-transform as in (21). Suppose z_s is an eigenvector of $(\mathcal{M}_s, \mathcal{L}_s)$ corresponding to $\mu = \nu + \frac{1}{\nu}$. If $\left(\mathcal{L}^H - \frac{1}{\nu}\mathcal{M}^H\right) z_s \neq 0$ or $(\mathcal{L}^H - \nu\mathcal{M}^H)z_s \neq 0$, then $\left(\nu, \mathcal{J}\left(\mathcal{L}^H - \frac{1}{\nu}\mathcal{M}^H\right) z_s\right)$ or $(1/\nu, \mathcal{J}(\mathcal{L}^H - \nu\mathcal{M}^H)z_s)$ is an eigenpair of $(\mathcal{M}, \mathcal{L})$, respectively.*

Then the relationship between eigenpairs of $(\mathcal{M}_s, \mathcal{L}_s)$ and $\mathcal{Q}(\lambda)$ is given as follows.

Theorem 6 *Let $(\mathcal{M}, \mathcal{L})$ be H-symplectic of the form in (19) and $(\mathcal{M}_s, \mathcal{L}_s)$ be its $(\mathcal{S} + \mathcal{S}^{-1})$-transform. Suppose that z_s is an eigenvector of $(\mathcal{M}_s, \mathcal{L}_s)$ corresponding to the eigenvalue $\mu \neq \pm 2$, and denote $z_s \equiv [z_{s1}^T,\ z_{s2}^T]^T$ with $z_{s1}, z_{s2} \in \mathbb{C}^n$. Let ν be a root of the quadratic equation $\lambda + \frac{1}{\lambda} = \mu$. Then*

(i) *at least one of vectors $z_{s1} + \frac{1}{\nu}z_{s2}$ and $z_{s1} + \nu z_{s2}$ is nonzero;*
(ii) *if $z_{s1} + \frac{1}{\nu}z_{s2} \neq 0$, then $z_{s1} + \frac{1}{\nu}z_{s2}$ is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to ν;*
(iii) *if $z_{s1} + \frac{1}{\nu}z_{s2} = 0$, then z_{s2} is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to $\frac{1}{\nu}$.*

Proof (i) Suppose that $z_{s1} + \frac{1}{\nu}z_{s2} = 0$ and $z_{s1} + \nu z_{s2} = 0$. It implies that $\left(\nu - \frac{1}{\nu}\right) z_{s2} = 0$. If $z_{s2} = 0$, then $z_{s1} = 0$ and $z_s = 0$ which contradicts the fact that z_s is an eigenvector. Hence, $\nu = \pm 1$ and then $\mu = \pm 2$ which contradicts the assumption that $\mu \neq \pm 2$. Therefore, $z_{s1} + \frac{1}{\nu}z_{s2} \neq 0$ or $z_{s1} + \nu z_{s2} \neq 0$.

(ii) Since $\mathcal{M}\mathcal{J}\mathcal{M}^H = \mathcal{L}\mathcal{J}\mathcal{L}^H$, by (21) it holds that

$$0 = (\mathcal{M}_s - \mu\mathcal{L}_s)\, z_s = (\mathcal{M} - \nu\mathcal{L})\mathcal{J}\left(\mathcal{L}^H - \frac{1}{\nu}\mathcal{M}^H\right) z_s. \tag{22}$$

From (22), we obtain

$$(\mathcal{M} - \nu\mathcal{L}) \begin{bmatrix} z_{s1} + \frac{1}{\nu}z_{s2} \\ x_\nu \end{bmatrix} = 0, \tag{23}$$

where

$$x_\nu \equiv \frac{1}{\nu}A_1^H z_{s1} - \frac{1}{\nu}A_0 z_{s2} - A_1 z_{s2}. \tag{24}$$

Substituting $(\mathcal{M},\mathcal{L})$ of (19) into (23), we have

$$x_\nu = \frac{1}{\nu}A_1\left(z_{s1}+\frac{1}{\nu}z_{s2}\right) \tag{25}$$

and

$$A_0\left(z_{s1}+\frac{1}{\nu}z_{s2}\right)+x_\nu+\nu A_1^H\left(z_{s1}+\frac{1}{\nu}z_{s2}\right)=0. \tag{26}$$

Substituting x_ν of (25) into (26) and multiplying (26) by ν, we get $\mathcal{Q}(\nu)\left(z_{s1}+\frac{1}{\nu}z_{s2}\right)=0$.

(iii) Since $z_{s1}+\frac{1}{\nu}z_{s2}=0$, it follows that $z_{s1}=-\frac{1}{\nu}z_{s2}\neq 0$ and $x_\nu=0$ in (25). Substituting these results into (24), it holds that

$$0=x_\nu=-\left(\frac{1}{\nu}\right)^2 A_1^H z_{s2}-\frac{1}{\nu}A_0z_{s2}-A_1z_{s2}.$$

Therefore, z_{s2} is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to the eigenvalue $\frac{1}{\nu}$. □

In [13], a structure-preserving algorithm (SPA) based on Patel's algorithm [22] has been developed for solving T-PQEPs. In order to solve an H-PQEP, we apply the $(\mathcal{S}+\mathcal{S}^{-1})$-transform to the H-symplectic pair $(\mathcal{M},\mathcal{L})$ of the form (19) and get the generalized eigenvalue problem $\mathcal{K}z_s=\mu\mathcal{N}z_s$, where $\mathcal{K}$ and $\mathcal{N}$ are defined in (21). Substituting $(\mathcal{M},\mathcal{L})$ in (19) into (21), the H-skew-Hamiltonian $\mathcal{K}$ and $\mathcal{N}$ can be represented as

$$\mathcal{K}=\begin{bmatrix} A_0 & A_1^H-A_1\\ A_1-A_1^H & A_0\end{bmatrix},\quad \mathcal{N}=\begin{bmatrix}-A_1 & 0\\ 0 & -A_1^H\end{bmatrix}. \tag{27}$$

However, Patel's algorithm can only be applied to $(\mathcal{K},\mathcal{N})$ of (27) in the real case, but cannot be directly applied to $(\mathcal{K},\mathcal{N})$ in the complex conjugate case. In the following, we convert $(\mathcal{K},\mathcal{N})$ of (27) into an enlarged real T-skew-Hamiltonian pair so that Patel's algorithm can be applied. We extend $(\mathcal{K},\mathcal{N})$ in (27) to a real $4n\times 4n$ matrix pair $(\mathcal{K}_2,\mathcal{N}_2)$ by

$$\mathcal{K}_2=\begin{bmatrix}\mathcal{K}_R & -\mathcal{K}_I\\ \mathcal{K}_I & \mathcal{K}_R\end{bmatrix},\quad \mathcal{N}_2=\begin{bmatrix}\mathcal{N}_R & -\mathcal{N}_I\\ \mathcal{N}_I & \mathcal{N}_R\end{bmatrix}\in\mathbb{R}^{4n\times 4n}, \tag{28}$$

where $\mathcal{K}=\mathcal{K}_R+\imath\mathcal{K}_I$ and $\mathcal{N}=\mathcal{N}_R+\imath\mathcal{N}_I$. From (28), it is easily seen that if μ is an eigenvalue of $(\mathcal{K},\mathcal{N})$, then μ and $\bar{\mu}$ are eigenvalues of $(\mathcal{K}_2,\mathcal{N}_2)$.

Theorem 7 *The multiplicities of eigenvalues of* $(\mathcal{K}_2,\mathcal{N}_2)$ *are all even.*

Proof Define $\widetilde{\mathcal{K}}_2\equiv\Pi\mathcal{K}_2\Pi$ and $\widetilde{\mathcal{N}}_2\equiv\Pi\mathcal{N}_2\Pi$, where $\Pi=I_n\oplus\begin{bmatrix}0 & I_n\\ I_n & 0\end{bmatrix}\oplus I_n$. It is easy to check that $\widetilde{\mathcal{K}}_2$ and $\widetilde{\mathcal{N}}_2$ are real skew-Hamiltonian; i.e., $(\widetilde{\mathcal{K}}_2\mathcal{J}_{4n})^T=-\widetilde{\mathcal{K}}_2\mathcal{J}_{4n}$ and $(\widetilde{\mathcal{N}}_2\mathcal{J}_{4n})^T=-\widetilde{\mathcal{N}}_2\mathcal{J}_{4n}$. Therefore, from the result of [18], it follows that the multiplicities of eigenvalues of $(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{N}}_2)$ are all even. □

From (21), we see that μ is an eigenvalue of $(\mathcal{K}, \mathcal{N})$ if and only if $\bar{\mu}$ is also an eigenvalue. We now give the relationship between eigenpairs of $(\mathcal{K}, \mathcal{N})$ and $(\mathcal{K}_2, \mathcal{N}_2)$.

Theorem 8 (i) *If* $(\alpha + \imath\beta, x + \imath y)$ *is an eigenpair of* $(\mathcal{K}, \mathcal{N})$, *then* $\begin{bmatrix} x \\ y \end{bmatrix} \pm \imath \begin{bmatrix} y \\ -x \end{bmatrix}$ *are eigenvectors of* $(\mathcal{K}_2, \mathcal{N}_2)$ *corresponding to the eigenvalues* $\alpha \pm \imath\beta$.

(ii) *If* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \imath \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *is an eigenvector of* $(\mathcal{K}_2, \mathcal{N}_2)$ *corresponding to the eigenvalue* $\alpha + \imath\beta$, *then* $(x_1 - y_2) + \imath(x_2 + y_1)$ *is an eigenvector of* $(\mathcal{K}, \mathcal{N})$ *corresponding to* $\alpha + \imath\beta$.

Proof (i) Since $\mathcal{K}(x + \imath y) = (\alpha + \imath\beta)\mathcal{N}(x + \imath y)$, comparing the real and the imaginary parts of both sides leads to

$$\mathcal{K}_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \pm \imath \begin{bmatrix} y \\ -x \end{bmatrix} \right) = (\alpha \pm \imath\beta)\mathcal{N}_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \pm \imath \begin{bmatrix} y \\ -x \end{bmatrix} \right).$$

(ii) Since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \imath \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is an eigenvector of $(\mathcal{K}_2, \mathcal{N}_2)$ corresponding to the eigenvalue $\alpha + \imath\beta$, it holds that

$$\begin{aligned} \mathcal{K}_R x - \mathcal{K}_I y &= \alpha(\mathcal{N}_R x - \mathcal{N}_I y) - \beta(\mathcal{N}_I x + \mathcal{N}_R y), \\ \mathcal{K}_I x + \mathcal{K}_R y &= \beta(\mathcal{N}_R x - \mathcal{N}_I y) + \alpha(\mathcal{N}_I x + \mathcal{N}_R y), \end{aligned}$$

by setting $x = x_1 - y_2$ and $y = x_2 + y_1$. Thus, $(\alpha + \imath\beta, x + \imath y)$ is an eigenpair of $(\mathcal{K}, \mathcal{N})$. □

From Theorem 8, the eigenpairs of $(\mathcal{K}, \mathcal{N})$ can be computed from the eigenpairs of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$. Since $\widetilde{\mathcal{K}}_2$ and $\widetilde{\mathcal{N}}_2$ are both real skew-Hamiltonian, based on Patel's approach [13,22], the pair $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ can be reduced to block upper triangular forms

$$\widetilde{\mathcal{K}}_2 := Q^T \widetilde{\mathcal{K}}_2 Z = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{11}^T \end{bmatrix}, \qquad \widetilde{\mathcal{N}}_2 := Q^T \widetilde{\mathcal{N}}_2 Z = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}, \tag{29}$$

where $Q, Z \in \mathbb{R}^{4n \times 4n}$ are orthogonal satisfying $Q = \mathcal{J}_{4n}^T Z \mathcal{J}_{4n}$, and $K_{11}, N_{11} \in \mathbb{R}^{2n \times 2n}$ are upper Hessenberg and upper triangular, respectively.

From Theorem 7 and (29), we see that the pair (K_{11}, N_{11}) has the same spectrum as $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$. We then apply the QZ algorithm to (K_{11}, N_{11}) to compute all its eigenpairs $\{(\mu_j, \tilde{z}_j)\}_{j=1}^{2n}$. Consequently, $\left\{ \left(\mu_j, \Pi Z \begin{bmatrix} \tilde{z}_j \\ 0 \end{bmatrix} \right) \right\}_{j=1}^{2n}$ are the $2n$ eigenpairs of $(\mathcal{K}_2, \mathcal{N}_2)$. Let $\mu_j = \alpha_j + \imath\beta_j$ and $\Pi Z \left[\tilde{z}_j^T, 0^T \right]^T \equiv \left[x_{1j}^T, x_{2j}^T \right]^T + \imath \left[y_{1j}^T, y_{2j}^T \right]^T$ with $\alpha_j, \beta_j \in \mathbb{R}$ and $x_{1j}, x_{2j}, y_{1j}, y_{2j} \in \mathbb{R}^{2n}$. From Theorem 8, $\left\{ \left(\alpha_j + \imath\beta_j, \mathcal{J}^T (x_{1j} - y_{2j} + \imath(x_{2j} + y_{1j})) \right) \right\}_{j=1}^{2n}$ are eigenpairs of $(\mathcal{M}_s, \mathcal{L}_s)$. Finally, we compute all eigenvalues and the associated eigenvectors of $\mathcal{Q}(\lambda)$ by Theorem 6. We present the structure-preserving algorithm for solving H-PQEP in Algorithm 1.

Algorithm 1 Structure-Preserving Algorithm (SPA) for H-PQEP

Input: An H-palindromic quadratic pencil $\mathcal{Q}(\lambda) \equiv \lambda^2 A_1^H + \lambda A_0 + A_1$ with $A_0, A_1 \in \mathbb{C}^{n\times n}$ and $A_0^H = A_0$;
Output: All eigenvalues and eigenvectors of $\mathcal{Q}(\lambda)$.
1: Form the matrix pair $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2) = (\Pi\mathcal{K}_2\Pi, \Pi\mathcal{N}_2\Pi)$ as in (28);
2: Reduce $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ to block upper triangular forms as in (29);
3: Compute eigenpairs $\{(\mu_j, \tilde{z}_j)\}_{j=1}^{2n}$ of (K_{11}, N_{11}) defined in (29) by the QZ algorithm;
4: Compute $\Pi Z \begin{bmatrix} \tilde{z}_j \\ 0 \end{bmatrix} \equiv \begin{bmatrix} x_{1j} \\ x_{2j} \end{bmatrix} + \imath \begin{bmatrix} y_{1j} \\ y_{2j} \end{bmatrix}$, $j = 1, 2, \ldots, 2n$;
5: Compute the eigenpair (μ_j, z_j), for $j = 1, 2, \ldots, 2n$, of $(\mathcal{M}_s, \mathcal{L}_s)$ by

$$z_j = \mathcal{J}^T(x_{1j} - y_{2j} + \imath(x_{2j} + y_{1j})) \equiv \left[z_{j1}^T, z_{j2}^T\right]^T;$$

6: Compute ν_j and $\frac{1}{\nu_j}$ by solving $\nu^2 - \mu_j\nu + 1 = 0$; Compute $x_{j1} \equiv z_{j1} + \frac{1}{\nu_j} z_{j2}$ and $x_{j2} \equiv z_{j1} + \nu_j z_{j2}$ for $j = 1, 2, \ldots, 2n$;
7: If $x_{j1} \neq 0$, then it is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to ν_j; If $x_{j2} \neq 0$, then it is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to $\frac{1}{\nu_j}$;

4 Balancing of $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$

Scaling [1,3,7,16] is a commonly used technique for standard eigenvalue problems for the improvement of the sensitivity of eigenvalues. In this section, we first propose a diagonal scaling for $\mathcal{P}(\lambda)$ in (1). Then, we determine the free parameters $d_1, \ldots, d_m$ in (7) and (8) to improve the backward errors of eigenpairs for $\mathcal{P}(\lambda)$ as in [10,14,17].

In order to balance the entries of coefficient matrices in $\mathcal{P}(\lambda)$, we define a complex diagonal matrix

$$D \equiv \text{diag}\left(2^{\alpha_1}, 2^{\alpha_2+\imath\beta_2}, \ldots, 2^{\alpha_n+\imath\beta_n}\right)$$

with $\alpha_j, \beta_j \in \mathbb{R}$ so that the magnitudes of entries of coefficient matrices in the new $(\star, \varepsilon)$-palindromic matrix polynomial

$$D\left(\sum_{k=0}^{d-1} \lambda^{2d-k} A_{d-k}^{\star} + \lambda^d A_0 + \varepsilon \sum_{k=1}^{d} \lambda^{d-k} A_k\right) D^{\star}$$

are close to one as much as possible. That is, we determine $\alpha_1, \ldots, \alpha_n$ and $\beta_2, \ldots, \beta_n$ so that

$$2^{\alpha_j+\imath\beta_j} A_k(j,\ell) 2^{\alpha_\ell-\imath\beta_\ell} \approx 1, \tag{30}$$

for $j, \ell = 1, 2, \ldots, n$ and $k = 0, 1, \ldots, d$, where $A_k(j,\ell)$ is the (j,ℓ)-th entry of A_k. By taking logarithm of (30), the parameters, $\alpha_1, \ldots, \alpha_n$ and $\beta_2, \ldots, \beta_n$ can be determined by solving the least square problems

$$\alpha_j + \alpha_\ell = -\mathfrak{Re}(\log_2(A_k(j,\ell))), \quad \beta_j - \beta_\ell = -\mathfrak{Im}(\log_2(A_k(j,\ell))),$$

where $\mathfrak{Re}(c)$ and $\mathfrak{Im}(c)$ represent the real and imaginary parts of c, respectively. Then, the parameters $\alpha_1, \ldots, \alpha_n$ and $\beta_2, \ldots, \beta_n$ are determined by the associated normal equations

$$B^T B\,[\alpha_1, \ldots, \alpha_n]^T = B^T b, \quad C^T C\,[\beta_2, \ldots, \beta_n]^T = C^T c.$$

We now determine $d_1, \ldots, d_m$ in (7) or (8), to balance the magnitudes of entries of $\mathcal{A}_0$ and $\mathcal{A}_1$ in $\mathcal{Q}(\lambda)$. For convenience, we define

$$d_i = \begin{cases} d_i^{(1)}, & \text{if } 2d = 4m, \\ d_i^{(2)}, & \text{if } 2d = 4m + 2; \end{cases} \quad \text{for } i = 1, \ldots, m.$$

From the row balancing of $\mathcal{A}_1$ in (7a) or (8a), we first set

$$\eta_i^{(s)} = \max\{1,\ \max\{\|A_{2m-k+s-1}\|_1 : k = 0, 1, \ldots, 2i - 3 + s\}\}.$$

Then we take $\delta_i^{(s)}$ to be the geometric average of $\eta_i^{(s)}$ and the average of the absolute magnitudes of entries of $A_{2m-2i+1}$ in $\mathcal{A}_1$; i.e.,

$$\delta_i^{(s)} = \sqrt{\eta_i^{(s)} \left(\sum_{j=1}^{n} \sum_{\ell=1}^{n} |A_{2m-2i+1}(j, \ell)|/n^2 \right)}$$

for $i = 1, \ldots, m$ and $s = 1, 2$. Although the value of $d_i^{(s)}$ can be set to $\delta_i^{(s)}$ to balance the entries of $\mathcal{A}_1$, we also need to consider the balance of the entries of both $\mathcal{A}_0$ and $\mathcal{A}_1$ in (7) or (8). As a result, we take the values of $d_1^{(s)}, \ldots, d_m^{(s)}$ to be the geometric average of the maximal values of $\delta_1^{(s)}, \ldots, \delta_m^{(s)}$ and the maximal average for the absolute magnitudes of entries of A_k $(k = 0, \ldots, d)$; i.e., for $i = 1, \ldots, m$ and $s = 1, 2$, we set

$$d_i^{(s)} := \sqrt{\rho^{(s)} \max_{0 \le k \le d} \left\{ \sum_{j=1}^{n} \sum_{\ell=1}^{n} |A_k(j, \ell)|/n^2 \right\}}$$

with

$$\rho^{(s)} = \max\{\delta_i^{(s)}; i = 1, \ldots, m\}.$$

5 Numerical results

In [13], an SPA is proposed for solving T-PQEPs. Numerical experiments show that SPA performs well on the T-PQEP arising from a finite element model of high-speed trains and rails. In this section, we shall focus on the numerical comparison of the performance and accuracy for solving H-PPEP of even degree by using structure-preserving algorithms and companion linearization.

For solving an $n \times n$ H-PPEP of even degree $2d$, we apply the P-quadratization in Sect. 2 to transform it into a $dn \times dn$ H-PQEP. We then apply the SPA (Algorithm 1) in Sect. 3 to solve the H-PQEP. The combination of the P-quadratization and SPA is called the PQ_SPA algorithm. On the other hand, we can also use the "good" linearization [20,21] to transform the H-PPEP into a palindromic linear pencil $\lambda Z^H + Z$, and then utilize SPA to solve the H-PQEP: $(\hat{\lambda}^2 Z^H + \hat{\lambda} 0 + Z)x = 0$ with $\lambda = \hat{\lambda}^2$. The combination of the "good" linearization and SPA is called the PL_SPA algorithm. As mentioned in Remark 1 (ii), we see that applying the SPA to $\hat{\lambda}^2 Z^T + \hat{\lambda} 0 + Z$ is mathematically equivalent to applying the URV-based method [25] to $\lambda Z^T + Z$.

5.1 Computational cost

For making PQ_SPA more efficient, we reorder the submatrices of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ in step 1 of Algorithm 1 by the permutations

$$\Pi_1 = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{(d-1)n} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{(d-2)n} \\ 0 & 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & I_{(d-2)n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 & I_{(d-2)n} & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We substitute $\mathcal{A}_1$ in Theorem 3 into $\widetilde{\mathcal{N}}_2$ and get

$$\begin{aligned} \widetilde{\mathcal{N}}_2 &:= \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \widetilde{\mathcal{N}}_2 \begin{bmatrix} \Pi_2^T & 0 \\ 0 & \Pi_1^T \end{bmatrix} \\ &= \begin{bmatrix} D_1 & 0 & -V_1 & V_2 \\ 0 & D_1 & V_2 & V_1 \\ 0 & 0 & V_3 & -V_4 \\ 0 & 0 & V_4 & V_3 \end{bmatrix} \oplus \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \\ -V_1^T & V_2^T & V_3^T & V_4^T \\ V_2^T & V_1^T & -V_4^T & V_3^T \end{bmatrix}, \end{aligned}$$

where D_1 is a $(2d-2)n \times (2d-2)n$ diagonal matrix and $V_3, V_4 \in \mathbb{R}^{n \times n}$. Set

$$\widetilde{\mathcal{K}}_2 := \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \widetilde{\mathcal{K}}_2 \begin{bmatrix} \Pi_2^T & 0 \\ 0 & \Pi_1^T \end{bmatrix}.$$

Table 1 Computational flops of PQ_SPA and PL_SPA

	Eigenvalues	Eigenvectors	Total
PQ_SPA	$(408d^3 + 32d + 80/3)n^3$	$(408d^3 + 32d)n^3$	$(816d^3 + 64d + 80/3)n^3$
PL_SPA	$3605\frac{1}{3}d^3n^3$	$1920d^3n^3$	$5525\frac{1}{3}d^3n^3$

Now, we compare the computational costs for PQ_SPA and PL_SPA:

– (QR factorization and updating with real arithmetic operations) Compute Q_1 and Z_1 such that $Q_1^T\widetilde{\mathcal{N}}_2 Z_1 = \operatorname{diag}\left(N_{11}^{(1)}, \left(N_{11}^{(1)}\right)^T\right)$, where $N_{11}^{(1)}$ is upper triangular, and update $Q_1^T\widetilde{\mathcal{K}}_2 Z_1$. It requires $(80/3n^3 + 32dn^3)$ and $341\frac{1}{3}d^3n^3$ flops for PQ_SPA and PL_SPA, respectively.
– (Given's rotations and updating with real arithmetic operations) Reducing the new pair $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ produced by above step to block upper triangular forms of (29), it requires $232d^3n^3 - (296d^2 - 24d)n^2$ and $1856d^3n^3 - 800d^2n^2$ flops for PQ_SPA and PL_SPA, respectively.
– (Computing eigenvalues of (K_{11}, N_{11})) Computing eigenvalues of the real upper Hessenberg and triangular pair (K_{11}, N_{11}) by QZ algorithm, it requires $176d^3n^3$ and $1408d^3n^3$ flops for PQ_SPA and PL_SPA, respectively, to obtain the upper quasi-triangular and triangular pair.
– The eigenvectors of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ can be computed by an additional $(408d^3+32d)n^3 - (332d^2 - 16d)n^2$ and $1920d^3n^3 - 1088d^2n^2$ flops for PQ_SPA and PL_SPA, respectively.

We summarize the computational flops of PQ_SPA and PL_SPA in Table 1.

5.2 Numerical experiments

For an approximate eigenpair (λ, x) of the palindromic matrix polynomial $\mathcal{P}(\lambda)$, we define the associated relative residual by

$$\text{RRes} \equiv \text{RRes}(\lambda, x) := \frac{\|\mathcal{P}(\lambda)x\|_2}{\left[\sum_{k=1}^{d}\left(|\lambda|^{d+k} + |\lambda|^{d-k}\right)\|A_k\|_2 + |\lambda|^d\|A_0\|_2\right]\|x\|_2}.$$

We will show numerical results of RRes and the reciprocal property of eigenpair (λ, x) for the H-PPEPs, computed by PQ_SPA, PL_SPA and `polyeig` in MATLAB (applied directly to (1)).

As mentioned before, theoretically, the eigenvalues of H-PPEP appear in reciprocal pairs $(\lambda, 1/\bar{\lambda})$. So, if we sort the eigenvalues in ascending order by modulus, the product of the i-th eigenvalue and the conjugate of the $(2dn + 1 - i)$-th eigenvalue should be one. Therefore, we define the reciprocities of the computed eigenvalues by

$$r_i \equiv |\lambda_i\bar{\lambda}_{2dn+1-i} - 1| \quad (i = 1, \ldots, dn).$$

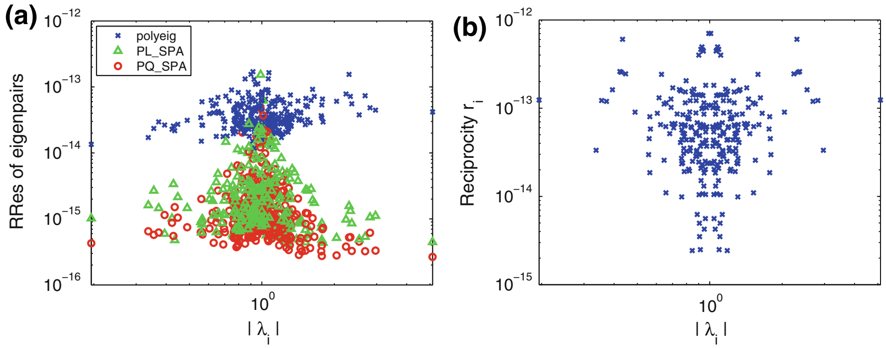

Fig. 2 Relative residuals of eigenpairs and the associated reciprocity for Example 1

All numerical experiments are carried out using MATLAB 2008b with machine precision *eps* $\approx 2.22 \times 10^{-16}$.

Let $\mathcal{C}_{n,b}$ denote the set of $n \times n$ complex matrices which real and imaginary parts are randomly generated by the normal distribution with zero mean and standard deviation b.

Example 1 Consider the H-PPEP with $d = 5$ and $A_k \in \mathcal{C}_{n,100}$ $(k = 0, \ldots, 5)$.

Example 2 Consider the H-PPEP with $d = 4$ and $A_1, A_3, A_4 \in \mathcal{C}_{n,100}$, and A_0 and A_2 being defined as

$$A_{k-1} = B_{1k} \cdot \operatorname{diag}\left\{\varphi_1^{(k)}, \ldots, \varphi_n^{(k)}\right\} \cdot B_{2k} \in \mathbb{C}^{n\times n} \quad (k = 1, 3)$$

where $B_{1k}, B_{2k} \in \mathcal{C}_{n,1}$, and

$$\begin{cases} \varphi_i^{(k)} = 4^{i+k-\ell}, \quad \varphi_{\ell+i}^{(k)} = 4^{i-k} & (i = 1, \ldots, \ell), \\ \varphi_n^{(k)} = 4^{n/2-k} & \text{if } n \text{ is odd;} \end{cases} \tag{31}$$

with $\ell = n/2$ (if n is even) or $\ell = (n-1)/2$ (otherwise).

Example 3 Consider the H-PPEP with $d = 4$ and $A_0, A_1, A_3, A_4 \in \mathcal{C}_{n,100}$, and A_2 being defined as

$$A_2 = B_{13} \cdot \operatorname{diag}\left\{\varphi_1^{(3)}, \ldots, \varphi_n^{(3)}\right\} \cdot B_{23} \in \mathbb{C}^{n\times n},$$

where $B_{13}, B_{23} \in \mathcal{C}_{n,1}$, and $\varphi_i^{(3)}$ is defined in (31) with $k = 3$.

We present the relative residuals (RRes) and the reciprocities of eigenpairs computed by the `polyeig`, PL_SPA and PQ_SPA for Examples 1–3, using the balancing technique in Sect. 4 with $n = 30$. Numerical results are shown in Figs. 2, 3 and 4. We indicate the results computed by `polyeig`, PL_SPA and PQ_SPA by "$\times$", "$\triangle$" and "o", respectively. For the PL_SPA and PQ_SPA, all reciprocities of eigenvalues

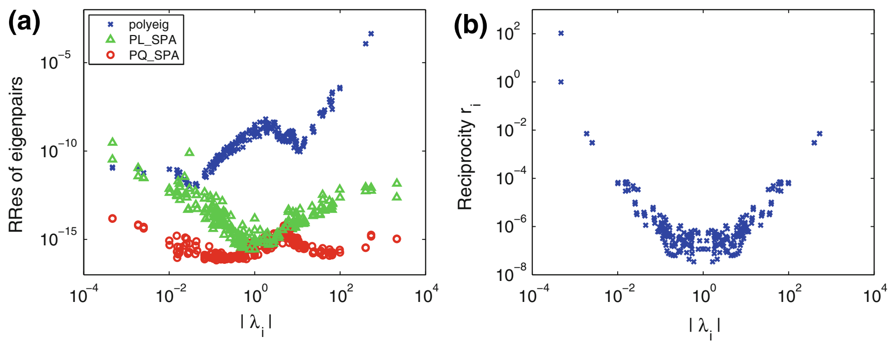

Fig. 3 Relative residuals of eigenpairs and the associated reciprocity for Example 2 with larger $\|A_0\|_2$ and $\|A_2\|_2$

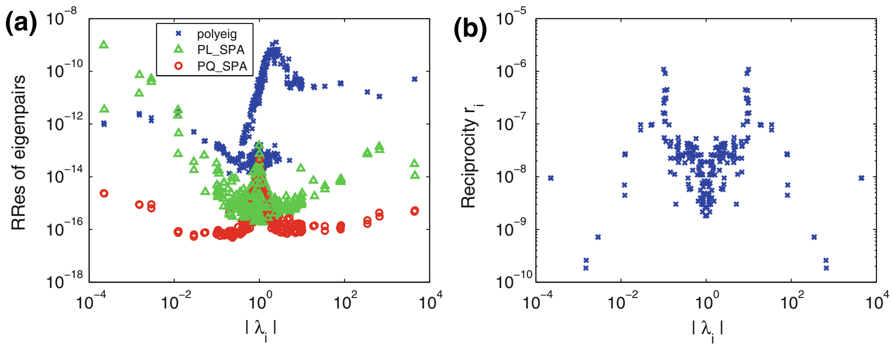

Fig. 4 Relative residuals of eigenpairs and the associated reciprocity for Example 3 with larger $\|A_2\|_2$

are preserved to machine accuracy, which are ignored in Figs. 2b–4b. From Figs. 2–4, we see that most of relative residuals of eigenpairs computed by the PQ_SPA are better than that computed by the PL_SPA, only a few exceptions. Overall, we conclude that applying P-quadratization and SPA (Algorithm 1) to solve PPEPs not only preserves the reciprocal property but also provides higher accuracy than that by PL_SPA and `polyeig` in MATLAB.

6 Conclusions

In this paper, we mainly propose a palindromic quadratization to transform the $(\star, \varepsilon)$-palindromic matrix polynomial of even degree with $(\star, \varepsilon) \neq (T, -1)$ to a $(\star, \varepsilon)$-palindromic quadratic pencil, instead of the orthodox palindromic linearization approach. The structure-preserving algorithm for solving palindromic quadratic eigenvalue problem based on $(\mathcal{S}+\mathcal{S}^{-1})$-transform and Patel's algorithm can then be applied. Numerical experiments show that relative residuals of approximate eigenpairs for the palindromic polynomial eigenvalue problem computed by the PQ_SPA are better than those by the PL_SPA and `polyeig` in MATLAB. Moreover, the computational cost for PQ_SPA is much cheaper than that for PL_SPA.

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