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# A Note on "Capability Assessment for Processes with Multiple Characteristics: A Generalization of the Popular Index $C_{pk}$"

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**The generalized yield index $C^{T}_{pk}$ establishes the relationship between the manufacturing specifications and the actual process performance, which provides a lower bound on process yield for two-sided processes with multiple characteristics. The results attended are very practical for industrial application. In this article, we extended the results in cases with one-sided specification and multiple characteristics. The generalized index $C^{T}_{PU}$ was considered, and the asymptotic distribution of the natural estimator $\hat{C}^{T}_{PU}$ was developed. Then, we derived the lower confidence bounds as well as the critical values of index $C^{T}_{PU}$. We not only provided some tables but also presented an application example. Copyright © 2012 John Wiley & Sons, Ltd.**

**Keywords:** critical values; lower confidence bounds; multiple characteristics; one-sided specification; process capability index

## 1. Introduction

Process yield has been the most basic and common criterion used in the manufacturing industry as a base for measuring process performance. Recently, Pan and Lee[2] developed two novel indices to evaluate the performance of multivariate manufacturing process. The effects of the estimation of process capability index (PCI) on the nonconforming units in parts per million (NCPPM) estimates are analyzed by Ozkaya and Testik.[3] Later, Lin and Pearn[4] used the yield index $S_{pk}$ to deal with process selection problem. Itay *et al.*[5] investigated an advanced multistage sampling plan based on $C_{pk}$ index.

Afterward, Yum and Kim[6] provided a bibliography of PCIs for 2000–2009. Spiring[7] then presented several method of process capability using Mathematica 7 software. Next, an applicable methodology to achieve the robustness of the multivariate process capability vector was proposed by Awad and Kovach.[8] Lately, more dissertations about PCI were published such as Pearn *et al.*,[9] Hsu *et al.*,[10] Yen and Pearn,[11] Pearn and Cheng,[12] and Goethals and Cho.[13] It can be observed that the new investigations of PCI mainly focus on processes with multivariate or multiple characteristics.

Pearn *et al.*[14] proposed a generalization of the popular index $C_{pk}$ for evaluating the yield of a gold bumping manufacturing process with multiple characteristics. The $C^{T}_{pk}$ index is defined for a process with multiple characteristics and two-sided specifications. However, the quality characteristics often have only one-sided specification. At this time, the overall capability index $C^{T}_{PU}$ proposed by Wu and Pearn[1] is considered for this purpose. The index $C^{T}_{PU}$ is defined as follows:

$$C^{T}_{PU} = \frac{1}{3}\Phi^{-1}\left\{\prod_{i=1}^{m}\Phi(3C_{PUi})\right\} \tag{1}$$

where $C_{PUi}$ denotes the $C_{PU}$ value of the $i$th characteristic for $i = 1, 2,\ldots, m$, and $m$ is the number of characteristics. $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution. A one-to-one correspondence relationship between $C^{T}_{PU}$ and the overall process NCPPM can be demonstrated as

$$\text{NCPPM} = 10^{6} \times \left[1 - \Phi\left(3C^{T}_{PU}\right)\right] \tag{2}$$

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## 2. Approximation distribution of the natural estimator

Consider the natural estimator of $C_{PU}^{T}$ as

$$\hat{C}_{PU}^{T}=\frac{1}{3}\Phi^{-1}\left\{\prod_{i=1}^{m}\Phi\left(3\hat{C}_{PUi}\right)\right\}=\frac{1}{3}\Phi^{-1}\left\{\prod_{i=1}^{m}\Phi\left(\frac{USL-\bar{X}_i}{S_i}\right)\right\},\ i=1,\ldots,m \tag{3}$$

where $\bar{X}_i$ and $S_i$ denote the sample mean and sample variance of $i$th characteristic. The exact distribution of $\hat{C}_{PU}^{T}$is mathematically intractable. By taking the first order of the Taylor expansion (see Appendix), the asymptotic distribution of $\hat{C}_{PU}^{T}$is

$$\hat{C}_{PU}^{T}\approx N\left(C_{PU}^{T},\ \frac{1}{9n\left[\phi\left(3C_{PU}^{T}\right)\right]^2}\sum_{i=1}^{m}\left(a_i^2+b_i^2\right)\right) \tag{4}$$

where

$$a_i=\left[\prod_{j=1,j\neq i}^{m}\Phi\left(3C_{PUj}\right)\right]\phi\left(3C_{PUj}\right)\ \text{ and }\ b_i=\frac{3C_{PUi}}{\sqrt{2}}a_i,\ i=1,\ldots,m$$

## 3. Estimation and testing on $C_{PU}^{T}$

From Equation (4), it can be seen that $\hat{C}_{PU}^{T}$ is an asymptotic unbiased estimator of $\hat{C}_{PU}^{T}$. The determination of the lower confidence bound on the actual process capability is essential for quality assurance. An approximate $100(1-\alpha)\%$ lower confidence bound for $C_{PU}^{T}$ can be expressed as

$$C_{PU}^{T\ LB}\approx\hat{C}_{PU}^{T}-Z_{\alpha}\left[\frac{1}{9n\left[\phi\left(3C_{PU}^{T}\right)\right]^2}\sum_{i=1}^{m}\left(a_i^2+b_i^2\right)\right]^{1/2} \tag{5}$$

It is noted that $a_i$ and $b_i$ parameters are unknown in Equation (5). To investigate the effects of $C_{PU_i}$ on $C_{PU}^{T\ LB}$, the cases with processes that have two independent characteristics are considered. Figure 1 displays the curves of $C_{PU}^{T\ LB}$ for various combinations of $C_{PU_1}$ and $C_{PU_2}$, with $C_{PU}^{T}$=1.0, 1.33, 1.5, 1.67. Then, we examined the results presented in Figure 1, which indicate that

(i) $C_{PU}^{T\ LB}$ obtains its absolute maximum as $C_{PU_1}=C_{PU_2}$.
(ii) The minimum $C_{PU}^{T\ LB}$ occurs when one of $C_{PU_i}$ approaches infinity, that is, another $C_{PU_i}$ equals $C_{PU}^{T}$. Under this condition, the minimum $C_{PU}^{T\ LB}$ is the most reliable lower confidence bound for a given $C_{PU}^{T}$.

By the discussion mentioned earlier, after doing some algebra, the asymptotic distribution of $\hat{C}_{PU}^{T}$ becomes

$$\hat{C}_{PU}^{T}\approx N\left(C_{PU}^{T},\ \frac{1}{9n}+\frac{1}{2n}{C_{PU}^{T}}^{2}\right) \tag{6}$$

Therefore, an approximate $100(1-\alpha)\%$ lower confidence bound for $C_{PU}^{T}$ can be expressed as

$$C_{PU}^{T\ LB}=\frac{2\hat{C}_{PU}^{T}-\sqrt{\frac{4z_{\alpha}^2}{9n}+\frac{2z_{\alpha}^2}{n}{\hat{C}_{PU}^{T}}^{2}-\frac{2z_{\alpha}^4}{9n^2}}}{2-z_{\alpha}^2/n} \tag{7}$$

Table I tabulates the 95% lower confidence bound $C_{PU}^{T\ LB}$ for $\hat{C}_{PU}^{T}$ =1.0(0.1)2.0, $n$ = 10(10)400. For the convenience of hypothesis testing, we also provide the critical value. From the asymptotic distribution listed in Equation (6), the critical value to hypothesis testing $H_0 : C_{PU}^{T}\leqslant C$ versus $H_a : C_{PU}^{T}>C$ is expressed as

$$c_0=C+z_{\alpha}\sqrt{\frac{1}{9n}+\frac{C^2}{2n}}. \tag{8}$$

Table II performs the critical values for type I error $\alpha$ = 0.05 with $C$=1.0(0.1)2.0, $n$ = 10(10)400. Next, an application example is presented.

## 4. A case study

We applied the methodology to a set of real data ($n$ = 100) presented in Wu and Pearn[1] for measuring manufacturing capability of a process making couplers and wavelength division multiplexers (WDM). Two quality characteristic including the polarization-dependent

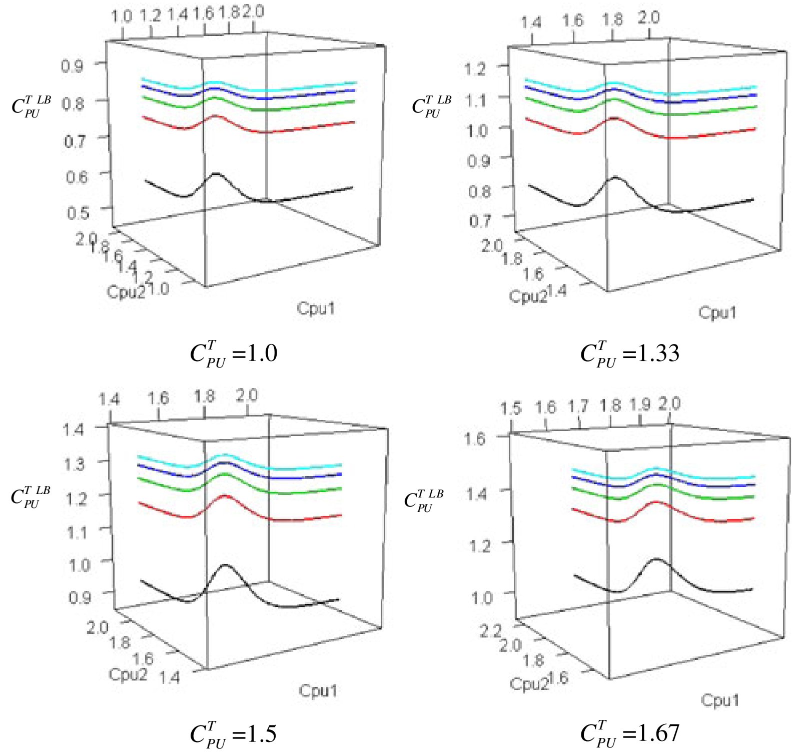


**Figure 1.** Curves of $C_{PU}^{T\ LB}$ with $\alpha = 0.05$, $n = 10(20)90$ (bottom to top in plot) and $C_{PU}^{T}$ =1.0, 1.33, 1.5, 1.67

**Table I.** 95% lower confidence bounds of $C_{PU}^{T}$ for $\hat{C}_{PU}^{T}$ =1.0(0.1)2.0, $n = 10(10)400$

| | $\hat{C}_{PU}^{T}$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 |
| 10 | 0.6920 | 0.7684 | 0.8443 | 0.9198 | 0.9950 | 1.0700 | 1.1447 | 1.2192 | 1.2936 | 1.3679 | 1.4420 |
| 20 | 0.7661 | 0.8477 | 0.9291 | 1.0101 | 1.0909 | 1.1716 | 1.2521 | 1.3324 | 1.4127 | 1.4929 | 1.5729 |
| 30 | 0.8024 | 0.8867 | 0.9708 | 1.0547 | 1.1384 | 1.2219 | 1.3053 | 1.3886 | 1.4718 | 1.5550 | 1.6380 |
| 40 | 0.8252 | 0.9113 | 0.9972 | 1.0828 | 1.1683 | 1.2537 | 1.3390 | 1.4241 | 1.5092 | 1.5943 | 1.6792 |
| 50 | 0.8414 | 0.9287 | 1.0158 | 1.1027 | 1.1895 | 1.2762 | 1.3628 | 1.4493 | 1.5358 | 1.6221 | 1.7085 |
| 60 | 0.8536 | 0.9419 | 1.0299 | 1.1178 | 1.2056 | 1.2933 | 1.3809 | 1.4684 | 1.5559 | 1.6433 | 1.7307 |
| 70 | 0.8633 | 0.9523 | 1.0411 | 1.1298 | 1.2184 | 1.3069 | 1.3953 | 1.4836 | 1.5719 | 1.6601 | 1.7483 |
| 80 | 0.8712 | 0.9608 | 1.0503 | 1.1396 | 1.2288 | 1.3180 | 1.4070 | 1.4960 | 1.5850 | 1.6739 | 1.7627 |
| 90 | 0.8778 | 0.9680 | 1.0580 | 1.1479 | 1.2376 | 1.3273 | 1.4169 | 1.5065 | 1.5960 | 1.6854 | 1.7749 |
| 100 | 0.8835 | 0.9741 | 1.0646 | 1.1549 | 1.2451 | 1.3353 | 1.4254 | 1.5154 | 1.6054 | 1.6953 | 1.7852 |
| 120 | 0.8928 | 0.9841 | 1.0753 | 1.1664 | 1.2574 | 1.3483 | 1.4392 | 1.5300 | 1.6208 | 1.7115 | 1.8022 |
| 150 | 0.9032 | 0.9954 | 1.0874 | 1.1794 | 1.2712 | 1.3630 | 1.4548 | 1.5465 | 1.6381 | 1.7297 | 1.8213 |
| 180 | 0.9111 | 1.0039 | 1.0965 | 1.1891 | 1.2816 | 1.3741 | 1.4665 | 1.5588 | 1.6511 | 1.7434 | 1.8357 |
| 200 | 0.9153 | 1.0084 | 1.1015 | 1.1944 | 1.2873 | 1.3801 | 1.4728 | 1.5655 | 1.6582 | 1.7509 | 1.8435 |
| 250 | 0.9237 | 1.0175 | 1.1112 | 1.2048 | 1.2984 | 1.3919 | 1.4854 | 1.5788 | 1.6722 | 1.7656 | 1.8589 |
| 300 | 0.9300 | 1.0243 | 1.1185 | 1.2126 | 1.3067 | 1.4008 | 1.4948 | 1.5887 | 1.6827 | 1.7766 | 1.8705 |
| 350 | 0.9349 | 1.0296 | 1.1242 | 1.2188 | 1.3133 | 1.4077 | 1.5021 | 1.5965 | 1.6909 | 1.7852 | 1.8795 |
| 400 | 0.9389 | 1.0339 | 1.1289 | 1.2237 | 1.3186 | 1.4134 | 1.5081 | 1.6028 | 1.6975 | 1.7922 | 1.8869 |

loss and the insertion loss, which are critical in fiber-optic transmission quality, are considered. Table III (cited from Wu and Pearn[1]) displays the manufacturing capabilities and the corresponding NCPPM for coupler and WDM process using $\hat{C}_{PU}^{T}$ values and $C_{PU}^{T\ LB}$.

In the coupler case, if the quality requirement is $C_{PU}^{T} \geqslant$; 1.3, some statistical inferences can be made. First, because the 95% lower confidence bound $C_{PU}^{T\ LB} = 1.35588 > 1.3$, we say that the process satisfies the requirement. Moreover, $C_{PU}^{T\ LB} = 1.35588$ means that there are no more than 23.7 NCPPM, or, from Table III, the critical value 1.460835 ($n = 100$, $C = 1.3$) is less than the observation value $\hat{C}_{PU}^{T} = 1.5261$. The two results are agreed. In WDM case, $\hat{C}_{PU}^{T} = 0.7352 < 1$ is obviously inadequate and incapable for high-tech product manufacturing.

**Table II.** Critical values $c_0$ for $C$ =1.0(0.1)2.0, $n$ = 10(10)400

| | $C$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 |
| 10 | 1.4066 | 1.5401 | 1.6741 | 1.8086 | 1.9433 | 2.0783 | 2.2134 | 2.3488 | 2.4843 | 2.6200 | 2.7557 |
| 20 | 1.2875 | 1.4112 | 1.5353 | 1.6596 | 1.7841 | 1.9089 | 2.0338 | 2.1588 | 2.2839 | 2.4091 | 2.5344 |
| 30 | 1.2347 | 1.3541 | 1.4737 | 1.5936 | 1.7136 | 1.8338 | 1.9541 | 2.0746 | 2.1951 | 2.3156 | 2.4363 |
| 40 | 1.2033 | 1.3200 | 1.4370 | 1.5543 | 1.6716 | 1.7891 | 1.9067 | 2.0244 | 2.1421 | 2.2600 | 2.3778 |
| 50 | 1.1818 | 1.2968 | 1.4120 | 1.5274 | 1.6429 | 1.7586 | 1.8743 | 1.9901 | 2.1060 | 2.2219 | 2.3379 |
| 60 | 1.1660 | 1.2796 | 1.3935 | 1.5076 | 1.6218 | 1.7360 | 1.8504 | 1.9648 | 2.0793 | 2.1939 | 2.3085 |
| 70 | 1.1536 | 1.2663 | 1.3792 | 1.4922 | 1.6053 | 1.7185 | 1.8318 | 1.9452 | 2.0586 | 2.1721 | 2.2856 |
| 80 | 1.1437 | 1.2556 | 1.3676 | 1.4798 | 1.5920 | 1.7044 | 1.8169 | 1.9294 | 2.0419 | 2.1545 | 2.2672 |
| 90 | 1.1355 | 1.2467 | 1.3580 | 1.4695 | 1.5811 | 1.6927 | 1.8044 | 1.9162 | 2.0281 | 2.1400 | 2.2519 |
| 100 | 1.1285 | 1.2391 | 1.3499 | 1.4608 | 1.5718 | 1.6828 | 1.7940 | 1.9051 | 2.0164 | 2.1276 | 2.2389 |
| 120 | 1.1173 | 1.2270 | 1.3368 | 1.4468 | 1.5568 | 1.6669 | 1.7771 | 1.8873 | 1.9975 | 2.1078 | 2.2181 |
| 150 | 1.1049 | 1.2136 | 1.3224 | 1.4313 | 1.5402 | 1.6493 | 1.7584 | 1.8675 | 1.9767 | 2.0859 | 2.1951 |
| 180 | 1.0958 | 1.2037 | 1.3117 | 1.4198 | 1.5280 | 1.6363 | 1.7446 | 1.8529 | 1.9613 | 2.0697 | 2.1781 |
| 200 | 1.0909 | 1.1984 | 1.3060 | 1.4137 | 1.5214 | 1.6293 | 1.7371 | 1.8450 | 1.9530 | 2.0609 | 2.1689 |
| 250 | 1.0813 | 1.1880 | 1.2948 | 1.4017 | 1.5086 | 1.6156 | 1.7226 | 1.8297 | 1.9368 | 2.0440 | 2.1511 |
| 300 | 1.0742 | 1.1803 | 1.2865 | 1.3928 | 1.4991 | 1.6055 | 1.7120 | 1.8184 | 1.9249 | 2.0314 | 2.1379 |
| 350 | 1.0687 | 1.1744 | 1.2801 | 1.3859 | 1.4918 | 1.5977 | 1.7036 | 1.8096 | 1.9156 | 2.0217 | 2.1277 |

**Table III.** Calculations for process capability of the coupler and WDMS

| Characteristic | $\hat{C}^{T}_{PU}$ | NCPPM | $\hat{C}^{T}_{PU}$ LB | NCPPM |
|---|---|---|---|---|
| Coupler | 1.5261 | 2.3439 | 1.3588 | 22.86916 |
| WDM | 0.7352 | 13706.01 | 0.6425 | 26958.67 |

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