PAPER

The Number of Isolated Nodes in a Wireless Network with a Generic Probabilistic Channel Model

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SUMMARY A wireless node is called isolated if it has no links to other nodes. The number of isolated nodes in a wireless network is an important connectivity index. However, most previous works on analytically determining the number of isolated nodes were not based on practical channel models. In this work, we study this problem using a generic probabilistic channel model that can capture the behaviors of the most widely used channel models, including the disk graph model, the Bernoulli link model, the Gaussian white noise model, the Rayleigh fading model, and the Nakagami fading model. We derive the expected number of isolated nodes and further prove that their distribution asymptotically follows a Poisson distribution. We also conjecture that the nonexistence of isolated nodes asymptotically implies the connectivity of the network, and that the probability of connectivity follows the Gumbel function.

key words: *connectivity, isolated nodes, multihop wireless networks, wireless channel models*

1. Introduction

Wireless ad hoc and sensor networks dispense with the need for fixed infrastructures, such as cables or base stations, which means they can be flexibly deployed with minimal cost for various tasks. However, without the aid of infrastructures, connectivity is a major concern [1]–[4]. In the past, many theoretical studies on network connectivity were based on disk graph models, in which two nodes have a link if and only if the distance between them is no more than their transmission radii [5]; even so, the disk graph models were criticized for oversimplifying wireless channels. In this work, using a generic probabilistic channel model, we study the connectivity problem by investigating the number of isolated nodes in a randomly deployed wireless network.

In many applications of multihop wireless networks, e.g., wireless sensor networks and mobile ad hoc networks, wireless nodes are distributed in a random manner, so it is natural to represent wireless nodes by a set of random points. A good tutorial on modeling wireless networks and techniques is presented [6]. Nodes are called isolated if they do not have links to other nodes. The nonexistence of isolated nodes is a prerequisite for connectivity. In [7], it was proved that if n nodes with transmission radius $\sqrt{(\ln n + \xi)/\pi n}$, in which ξ is a tunable parameter that remains constant during the analysis, are independently distributed over a unit-area square, then the network asymptotically has no isolated nodes with probability $\exp\left(-e^{-\xi}\right)$. Furthermore, in [8], it was proved that the network is asymptotically connected if $\xi \to \infty$ and asymptotically disconnected if $\xi \to -\infty$. However, neither node failures nor link failures were considered in these works. In [9], assuming that each node has the same probability to be active independently, p_1, and that each link has the same probability to be up independently, p_2, the authors proved that the total number of isolated active nodes asymptotically follows a Poisson distribution with mean $p_1 e^{-\xi}$, if the transmission radius is given by $\sqrt{(\ln n + \xi)/\pi p_1 p_2 n}$. In [10]–[12], the impact on connectivity due to link failures caused by shadowing or fading was investigated. An analytical procedure for the computation of the node isolation probability and coverage in an ad hoc network in the presence of channel randomness was presented [13]. In [14], the node isolation probability under a specific channel model, the Nakagami fading model, was derived.

However, the majority of previous analytical works were not based on practical channel models. In this paper, our study is based on a generic probabilistic model that can capture the behaviors of the most widely used models, including the disk graph model, the Bernoulli link model, the Gaussian white noise model, the Rayleigh fading model, and the Nakagami fading model. In this generic probabilistic model, we let $f(r)$ be the probability of the event that two nodes have a link if they are separated by a distance of r. It is realistic to assume that $f(r)$ is a decreasing function and that there are two constants $0 \leq R_1 \leq R_2 < \infty$ such that $f(r) = 1$ for $0 \leq r \leq R_1$ and $f(r) = 0$ for $r \geq R_2$. The values of R_1, R_2, and $f(r)$ are set to define the generic model.

The nonexistence of isolated nodes is a prerequisite for network connectivity and is also a good indicator for it. We will first derive the expected number of isolated nodes. In addition, we will further prove that the probability distribution of the number of isolated nodes asymptotically follows a Poisson distribution. The equations given in this work will allow users to control the expected number of isolated nodes by tuning the node density or even the transmission power. Thus, the desired level of connectivity can be achieved. These results can be used to obtain the connectivity features of the network. In addition, they are of practical value for researchers or designers in this field. We run exten-

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sive simulations to verify our probabilistic analysis. Moreover, we conjecture that the nonexistence of isolated nodes asymptotically implies connectivity. This conjecture is also verified through simulations.

A similar work focused on the connectivity of wireless networks with arbitrary wireless channel models [15]. This was built upon the wireless model, in which a total of n nodes are randomly, independently, and uniformly distributed in a unit square, and each node has uniform transmission power. The authors proved that the probability of having a connected network and the probability of having no isolated nodes asymptotically converges to the same value as n tends to infinity. In contrast, our work uses a different network model, which is not scaling. Furthermore, besides the node isolation probability, we also investigate the distribution of the number of isolated nodes.

The rest of this paper is organized as follows. In Sect. 2, we give our main results. In Sect. 3, lemmas needed to prove the main results are given. In Sect. 4, we derive the expected number of isolated nodes. In Sect. 5, we give the asymptotic distribution of the number of isolated nodes. In Sect. 6, the channel models used in the simulations are introduced, and simulation results are given to verify our theoretical theorems. We summarize the paper in Sect. 7.

2. Main Results

The nonexistence of isolated nodes is a precondition for network connectivity. The probability of nonexistence of these isolated nodes is considered as a tight upper bound for the probability of connectivity [16]. It was proved that if there are no isolated nodes in a random geometric graph, then the graph is asymptotically almost surely connected.

In this paper, we assume that wireless nodes are represented by a Poisson point process over a deployment region $\mathbb{D} = [0, l]^2$ with density $\lambda(l)$. For convenience, we will suppress the parameter l in $\lambda(l)$ from this point on. To avoid tedious arguments on boundary effects and to simplify the calculations, we will apply the torus convention described, for example, in [17]–[19]. Hence, instead of the Euclidean distance, the toroidal distance [20], [21] is applied. The toroidal distance between nodes u and v is denoted by $d(u, v)$. In addition, scaling disk models, like that used in [8] and [9], are not adopted. Instead, the transmission radius of the nodes is independent of the number of nodes, and the node density is a function of the deployment region.

As stated before, $f(r)$ is the probability of the event that two nodes have a link if they are separated by r. Let $a = \int_0^\infty f(r)\, 2\pi r dr$, $\lambda = \left(\ln l^2 + \ln\ln l^2 + \xi\right)/a$, and $\omega = \xi + \ln a$. Here, ξ is a tunable parameter that remains constant during the analysis and $\lambda a = \ln l^2 + \ln\ln l^2 + \xi$ is the expected number of nodes with which a node has links. Our first result gives the expected total number of isolated nodes in the network.

Theorem 1: The expected total number of isolated nodes in the network is $e^{-\omega}$.

Based on Theorem 1, the expected number of isolated nodes can be controlled by tuning the parameter ω that depends only on ξ and a. The ratio of a to πR_2^2 is the conditional probability that two nodes have a link between them if they are within the distance R_2. Specifically, if $f(r) = 1$ for $r \in [0, R_2]$, the ratio is equal to 1. In other words, any two nodes within a distance R_2 always have a link, and this is the traditional r-disk graph. Different channel models have different settings for R_1, R_2, and $f(r)$. Accordingly, $a = \int_0^\infty f(r)\, 2\pi r dr$ also differs. This shows variation in the expected number of isolated nodes. The greater the value of a, the lesser the expected number of isolated nodes. In some applications, it is tolerable to have a certain percentage of nodes isolated. With the knowledge of the expected number of isolated nodes, this can be achieved by choosing a proper ξ. We further give the probability distribution of the number of isolated nodes in the following theorem.

Theorem 2: The total number of isolated nodes is asymptotically Poisson with mean $e^{-\omega}$.

According to Theorem 2, the probability of the event that there are k isolated nodes in the network is asymptotically equal to $\left(e^{-k\omega}/k!\right)\exp\left(-e^{-\omega}\right)$. Specifically, the event of the nonexistence of isolated nodes is asymptotic with probability $\exp\left(-e^{-\omega}\right)$, and is called the Gumbel function. This probability is an upper bound for the probability of connectivity. In addition, we conjecture that, even considering link failures, a network without isolated nodes is almost surely connected. If this conjecture is true, then

$$\Pr(\text{the network is connected}) \sim \exp\left(-e^{-\omega}\right). \tag{1}$$

This conjecture will be verified through simulations in Sect. 6.

3. Preliminaries

In this section, lemmas for proving the above theorems are given. The proofs of these lemmas are given in the Appendix. Let $G_r(x_1, \cdots, x_k)$ denote the r-disk graph over $x_1, \cdots, x_k$ in which there is an edge between two nodes if and only if their distance is at most r. For any positive integers k and m with $1 \le m \le k$, let C_{km} denote the set of $(x_1, \cdots, x_k) \in \mathbb{D}^k$ satisfying the condition that $G_{2R_2}(x_1, \cdots, x_k)$ has exactly m connected components. The space $\mathbb{D}^k$ can be partitioned into $C_{k1}, C_{k2} \cdots, C_{kk}$.

We use $X_1, X_2, \cdots$ and $X_1', X_2', \cdots$ to denote independently and uniformly distributed random points in $\mathbb{D}$. Let $\mathcal{P}_\lambda = \{X_1, X_2, \cdots, X_{\mathrm{Po}(\lambda|\mathbb{D}|)}\}$ denote a Poisson point process with density λ over $\mathbb{D}$ where $\mathrm{Po}(\cdot)$ is the Poisson random variable, and $\mathcal{X}_k = \left\{X_1', X_2', \cdots, X_k'\right\}$ denotes a random k-point process over $\mathbb{D}$. Specifically, $\mathcal{X}_k$ is independent of $\mathcal{P}_\lambda$. Let B_i for $i = 1, 2, \cdots$ denote the event that the node X_i is isolated in the network over $\mathcal{P}_\lambda$, and let B_i' for $i = 1, \cdots, k$ denote the event that X_i' is isolated in the network over $\mathcal{X}_k \cup \mathcal{P}_\lambda$. Recall that ξ is a constant, $a = \int_0^\infty f(r)\, 2\pi r dr$, $\lambda = \left(\ln l^2 + \ln\ln l^2 + \xi\right)/a$, and $\omega = \xi + \ln a$. Thus, we have the following lemmas.

Lemma 3: For any integer $k \geq 2$ and $(x_1, x_2, \cdots, x_k) \in C_{kk}$,

$$\Pr\left(\bigwedge_{i=1}^{k} B_i' \left| \begin{array}{c} X_1' = x_1 \\ \vdots \\ X_k' = x_k \end{array} \right.\right) = e^{-k\lambda a}. \tag{2}$$

Lemma 4: For any positive integer k,

$$\lambda^k \int_{(x_1,\cdots,x_k)\in C_{kk}} e^{-k\lambda a} \left(\prod_{i=1}^{k} dx_i\right) \sim \left(e^{-\omega}\right)^k. \tag{3}$$

Lemma 5: For any two integers $k \geq 2$ and $1 \leq m \leq k-1$, there is a positive constant c such that, for any $(x_1, \cdots, x_k) \in C_{km}$,

$$\Pr\left(\bigwedge_{i=1}^{k} B_i' \left| \begin{array}{c} X_1' = x_1 \\ \vdots \\ X_k' = x_k \end{array} \right.\right) \leq e^{-(m+c)\lambda a}. \tag{4}$$

Lemma 6: For any two integers $k \geq 2$ and $1 \leq m \leq k-1$,

$$\lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{km}} e^{-(m+c)\lambda a} \left(\prod_{i=1}^{k} dx_i\right) = o(1). \tag{5}$$

We write $g(\lambda) = o(h(\lambda))$ if $\lim_{\lambda\to\infty} g(\lambda)/h(\lambda) = 0$. Generally speaking, Lemmas 3 and 4 are for estimating the probability of the existence of k widely spaced isolated nodes, and Lemmas 5 and 6 are for estimating the probability of the existence of k isolated nodes among which some are close to others. The density λ and a together will affect the probability of the isolated nodes.

4. The Expected Number of Isolated Nodes

Briefly, the expected number of isolated nodes equals the total number of nodes in the network multiplied by the probability of a typical node being isolated. To derive the expected number of isolated nodes, we use Palm theory [22]. Let X' be a point randomly located in $\mathbb{D}$ and independent of $\mathcal{P}_\lambda$, and let $B(x, r)$ denote a disk with center x and radius r. Let $h(\mathcal{Y}, \mathcal{X})$ for $\mathcal{Y} \subseteq \mathcal{X}$ be the indicator function such that $h(\mathcal{Y}, \mathcal{X}) = 1$ if $\#(\mathcal{Y}) = 1$, where $\#(\cdot)$ is the cardinality function, and the node in $\mathcal{Y}$ is isolated in $\mathcal{X}$; otherwise, $h(\mathcal{Y}, \mathcal{X}) = 0$. Then, we have

$$\text{The expected number of isolated nodes in } \mathcal{P}_\lambda \tag{6}$$

$$= \sum_{\{X\}\subseteq\mathcal{P}_\lambda} \Pr(X \text{ is isolated}) \tag{7}$$

$$= \sum_{\{X\}\subseteq\mathcal{P}_\lambda} \mathbf{E}\left[h(\{X\}, \mathcal{P}_\lambda)\right] \tag{8}$$

$$= \mathbf{E}\left[\sum_{\{X\}\subseteq\mathcal{P}_\lambda} h(\{X\}, \mathcal{P}_\lambda)\right] \tag{9}$$

$$= \left(\lambda l^2\right) \mathbf{E}\left[h(\{X'\}, \{X'\} \cup \mathcal{P}_\lambda)\right] \tag{10}$$

$$= \lambda l^2 \Pr(X' \text{ is isolated in } \{X'\} \cup \mathcal{P}_\lambda). \tag{11}$$

The introduction of the random point X' in Eq. (10) is to ensure that there is at least one node in $\mathbb{D}$. The equality from Eqs. (9) to (10) is based on Palm theory, and λl^2 is the expected total number of nodes in the network. The probability $\Pr(X' \text{ is isolated in } \{X'\} \cup \mathcal{P}_\lambda)$ can be given by

$$\begin{aligned} &\Pr(X' \text{ is isolated in } \{X'\} \cup \mathcal{P}_\lambda) \\ &= \int_{x\in\mathbb{D}} \Pr(X' \text{ is isolated} \mid X' = x) \Pr(X' = x)\, dx \\ &= \frac{1}{l^2} \int_{x\in\mathbb{D}} \Pr(X' \text{ is isolated} \mid X' = x)\, dx. \end{aligned} \tag{12}$$

To calculate $\Pr(X' \text{ is isolated} \mid X' = x)$, we apply the partition technique of the Riemann integral. The disk $B(x, R_2)$ is divided into k annuli by k concentric circles with centers at x and radii $r_1 < r_2 < \cdots < r_k = R_2$, respectively. For convenience, let $r_0 = 0$. For $1 \leq i \leq k$, the annulus with radii r_{i-1} and r_i is called the ith annulus. Let $\triangle r_i = r_i - r_{i-1}$. The area of the ith annulus can be approximated by $2\pi r_i \triangle r_i$. If a node is in the ith annulus, the event that X' has a link to that node is approximately the probability $f(r_i)$. Let N_i denote the number of nodes in the ith annulus. Then,

$$\begin{aligned} &\Pr\left(\begin{array}{c} X' \text{ does not have links with} \\ \text{nodes in the } i\text{th annulus} \end{array} \middle| X' = x\right) \\ &= \sum_{j=0}^{\infty} \Pr\left(\begin{array}{c} \text{All links between } X' \text{ and} \\ \text{nodes in the } i\text{th annulus fail} \end{array} \middle| N_i = j\right) \\ &\cdot \Pr(N_i = j \mid X' = x) \\ &= \sum_{j=0}^{\infty} (1 - f(r_i))^j \left(\frac{(\lambda 2\pi r_i \triangle r_i)^j}{j!} e^{-\lambda 2\pi r_i \triangle r_i}\right) \\ &= e^{-f(r_i)\lambda 2\pi r_i \triangle r_i}. \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} &\Pr\left(X' \text{ is isolated} \middle| X' = x\right) \\ &= \Pr\left(\begin{array}{c} \text{For all } 1 \leq i \leq k, \\ X' \text{ does not have links with} \\ \text{nodes in the } i\text{th annulus} \end{array} \middle| X' = x\right) \\ &= \lim_{k\to\infty} \prod_{i=1}^{k} \Pr\left(\begin{array}{c} X' \text{ does not have links with} \\ \text{nodes in the } i\text{th annulus} \end{array} \middle| X' = x\right) \\ &= \lim_{k\to\infty} \prod_{i=1}^{k} e^{-f(r_i)\lambda 2\pi r_i \triangle r_i} = \lim_{k\to\infty} e^{-\lambda \sum_{i=1}^{k} f(r_i) 2\pi r_i \triangle r_i} \\ &= e^{-\lambda \int_0^{R_2} f(r) 2\pi r dr} = e^{-\lambda a}. \end{aligned} \tag{14}$$

Putting Eqs. (11), (12), and (14) together, we have

$$\begin{aligned} &\text{The expected number of isolated nodes in } \mathcal{P}_\lambda \\ &= \lambda \int_{x\in\mathbb{D}} e^{-\lambda a} dx \sim e^{-\omega}. \end{aligned} \tag{15}$$

The last equality holds due to Lemma 4. Thus, Theorem 1 is proved.

5. Asymptotic Distribution of the Number of Isolated Nodes

To prove Theorem 2, we apply Brun's sieve in the form, for example, used in [23]. This is used to derive the asymptotic distribution of the number of isolated nodes. For completeness, we give Brun's sieve here.

Theorem 7 (Brun's Sieve): Assume $m(n)$ is a non-negative random integer variable. Let $B_1, \cdots, B_{m(n)}$ be events and let Y be the number of B_i that hold, and let

$$S^{(j)} = \sum_{\{i_1,\cdots,i_j\}\subseteq\{1,\cdots,m(n)\}} \Pr\left(B_{i_1} \wedge \cdots \wedge B_{i_j}\right). \quad (16)$$

Suppose there is a constant μ such that for every fixed j,

$$\mathbf{E}\left[S^{(j)}\right] \sim \frac{1}{j!}\mu^j. \quad (17)$$

Then Y is also asymptotically Poisson with mean μ.

Let Y be the number of B_i events that hold. In other words, Y is the total number of isolated nodes. To prove Theorem 2 by applying Brun's sieve, we need to show that for every fixed k,

$$\mathbf{E}\left[\sum_{\{i_1,\cdots,i_k\}\subseteq\{1,\cdots,\mathrm{Po}(\lambda l^2)\}} \Pr\left(B_{i_1} \wedge \cdots \wedge B_{i_k}\right)\right] \sim \frac{1}{k!}\left(e^{-\omega}\right)^k. \quad (18)$$

Again, by applying Palm theory to prove Eq. (18), we have

$$\mathbf{E}\left[\sum_{\{i_1,\cdots,i_k\}\subseteq\{1,\cdots,\mathrm{Po}(\lambda l^2)\}} \Pr\left(B_{i_1} \wedge \cdots \wedge B_{i_k}\right)\right] = \frac{\left(\lambda l^2\right)^k}{k!}\Pr\left(B'_1 \wedge \cdots \wedge B'_k\right). \quad (19)$$

Comparing Eqs. (19) with (18), we can see that if $\left(\lambda l^2\right)^k \Pr\left(B'_1 \wedge \cdots \wedge B'_k\right) \sim \left(e^{-\omega}\right)^k$ for all $k = 1, 2, \cdots$ then the proof is complete. The case of $k = 1$ was proved in the previous section. Thus, here we only need to prove that, for any integer $k \geq 2$,

$$\left(\lambda l^2\right)^k \Pr\left(B'_1 \wedge \cdots \wedge B'_k\right) \sim \left(e^{-\omega}\right)^k. \quad (20)$$

As described in Sect. 3, for any positive integers k and m with $1 \leq m \leq k$, C_{km} denotes the set of $(x_1, \cdots, x_k) \in \mathbb{D}^k$ satisfying the condition that $G_{2R_2}(x_1, \cdots, x_k)$ has exactly m connected components and the space $\mathbb{D}^k$ can be partitioned into $C_{k1}, C_{k2} \cdots, C_{kk}$. Therefore, we use a divide and conquer strategy to prove this statement. We have

$$\begin{aligned}
&\left(\lambda l^2\right)^k \Pr\left(B'_1 \wedge \cdots \wedge B'_k\right)\\
&= \left(\lambda l^2\right)^k \int_{(x_1,x_2,\cdots,x_k)\in\mathbb{D}^k} \Pr\left(\bigwedge_{i=1}^k B'_i \,\middle|\, \begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right) \cdot \Pr\left(\begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right)\left(\prod_{i=1}^k dx_i\right)\\
&= \lambda^k \int_{(x_1,x_2,\cdots,x_k)\in\mathbb{D}^k} \Pr\left(\bigwedge_{i=1}^k B'_i \,\middle|\, \begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right)\left(\prod_{i=1}^k dx_i\right)\\
&= \sum_{i=1}^k \lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{ki}} \Pr\left(\bigwedge_{i=1}^k B'_i \,\middle|\, \begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right) \cdot \left(\prod_{i=1}^k dx_i\right).
\end{aligned} \quad (21)$$

For the integral over C_{km} with $1 \leq m \leq k-1$, according to Lemmas 5 and 6, we have

$$\begin{aligned}
&\lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{km}} \Pr\left(\bigwedge_{i=1}^k B'_i \,\middle|\, \begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right)\left(\prod_{i=1}^k dx_i\right)\\
&\leq \lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{km}} e^{-(m+c)\lambda a}\left(\prod_{i=1}^k dx_i\right) = o(1).
\end{aligned} \quad (22)$$

For the integral over C_{kk}, according to Lemmas 3 and 4, we have

$$\begin{aligned}
&\lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{kk}} \Pr\left(\bigwedge_{i=1}^k B'_i \,\middle|\, \begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right)\left(\prod_{i=1}^k dx_i\right)\\
&= \lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{kk}} e^{-k\lambda a}\left(\prod_{i=1}^k dx_i\right) \sim \left(e^{-\omega}\right)^k.
\end{aligned} \quad (23)$$

If we combine Eqs. (21), (22), and (23) we have

$$\left(\lambda l^2\right)^k \Pr\left(B'_1 \wedge \cdots \wedge B'_k\right) \sim \left(e^{-\omega}\right)^k. \quad (24)$$

6. Simulation Results

The generic probabilistic channel model used in this work is a generalization of many widely used channel models. Channel models can be described by setting R_1, R_2, and $f(r)$. In this section, we begin with a brief introduction to several well-known channel models: the log-distance path loss model, the Bernoulli link model, the Gaussian white noise model, the Rayleigh fading model, and the Nakagami fading model. Extensive simulation results are given to verify our theoretical results. For the sake of succinctness, similar figures for different channel models will not be included.

6.1 Channel Models and Simulation Parameters

Let S_{ref} (dBm) be the signal strength measured at a reference distance d_0 from the transmitter. Based on the log-distance path loss model [24], the signal strength received at a distance d from the transmitter will be $S_{\text{ref}} - 10\alpha \log(d/d_0)$ (dBm) where α is the path loss exponent, whose value is between 2 and 6 depending on the environment. Let S_{thr} (dBm) be the minimum received signal strength for decoding a signal. Hence, the transmission radius denoted as R can be derived from the equation

$$S_{\text{thr}} = S_{\text{ref}} - 10\alpha \log \frac{R}{d_0}. \tag{25}$$

In the simulation, we assume that $d_0 = 1$, $S_{\text{ref}} = 35$ dBm, $S_{\text{thr}} = 15$ dBm, and $\alpha = 2$. Thus, for the log-distance path loss model, we have $R_1 = R_2 = 10$. We should note that $f(r)$ is not needed here.

In a realistic system, links may be down due to the environment or barriers between nodes. To characterize the uncertainty of the existence of links, the Bernoulli link model assumes that two nodes within each other's transmission range may have a link with probability p, depending on the specific propagation environment. In other words, $R_1 = 0$ and $f(r) = p$. Let R_B denote the R_2 in the Bernoulli link model. To achieve a fair comparison, the average node degree should be kept the same. The degree of a node, or 'node degree', is the number of connections it has to other nodes. A node of degree 0 is isolated. Thus, we have $\pi R^2 = p\pi R_B^2$. In the simulation, we set $p = 0.8$, which can be set according to the environment, and therefore, $R_1 = 0$, $R_2 = 11.2$, and $f(r) = 0.8$.

To model background noises, the Gaussian white noise model assumes that the signal strength received at a distance r from the transmitter is given by $S_{\text{ref}} - 10\alpha \log(r/d_0) - \mathcal{N}$, where $\mathcal{N}$ is a log-normal random variable with mean $\mu = 0$ and standard deviation σ. Therefore,

$$\begin{aligned} f(r) &= \Pr\left(\mathcal{N} \le S_{\text{ref}} - S_{\text{thr}} - 10\alpha \log \frac{r}{d_0}\right) \\ &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{S_{\text{ref}} - S_{\text{thr}} - 10\alpha \log \frac{r}{d_0}}{\sigma\sqrt{2}}\right)\right), \end{aligned} \tag{26}$$

where $\operatorname{erf}(\cdot)$ is the error function. Let R_G denote the R_2 for the Gaussian white noise model. To have the same mean node degree, R_G is given by

$$\pi R^2 = \int_0^{R_G} \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{S_{\text{ref}} - S_{\text{thr}} - 10\alpha \log \frac{r}{d_0}}{\sigma\sqrt{2}}\right)\right) 2\pi r dr. \tag{27}$$

In the simulation, we set $\sigma = 8$, and therefore, $f(r) = \frac{1}{2}\left(1 + \operatorname{erf}\left((5 - 5\log r)/\left(2\sqrt{2}\right)\right)\right)$, $R_1 = 0$, and $R_2 = 13.186$.

In the Rayleigh fading model [25], the amplitude of a signal will vary according to the Rayleigh distribution. The cumulative distribution function (CDF) of the Rayleigh distribution with standard deviation σ is

$$R(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{28}$$

where $2\sigma^2 = 10^{(S_{\text{ref}} - 10\alpha \log(r/d_0))/10}$ is the average received signal power, and for successful reception, the received signal power x^2 must be at least $10^{S_{\text{thr}}/10}$. Therefore,

$$\begin{aligned} f(r) &= \Pr\left(x^2 \ge 10^{\frac{S_{\text{thr}}}{10}}\right) = \exp\left(-10^{-\frac{S_{\text{ref}} - S_{\text{thr}} - 10\alpha \log\left(\frac{r}{d_0}\right)}{10}}\right) \\ &= \exp\left(-\left(\frac{r}{d_0}\right)^{\alpha} 10^{-\frac{S_{\text{ref}} - S_{\text{thr}}}{10}}\right). \end{aligned} \tag{29}$$

Let R_R denote the R_2 for the Rayleigh fading model. To have the same mean node degree, R_R is given by

$$\pi R^2 = \int_0^{R_R} \exp\left(-\left(\frac{r}{d_0}\right)^{\alpha} 10^{-\frac{S_{\text{ref}} - S_{\text{thr}}}{10}}\right) 2\pi r dr. \tag{30}$$

In the simulation, $f(r) = \exp\left(-r^2/100\right)$, $R_1 = 0$, and $R_2 = 38.1$ for the Rayleigh fading model.

The Nakagami fading model is described by two parameters: μ is the shape parameter denoting the severity of fading, and ω is the scale parameter equal to the average received power. We have

$$\begin{aligned} f(r) &= \Pr\left(\text{received power} \ge 10^{\frac{S_{\text{thr}}}{10}}\right) \\ &= 1 - G\left(10^{\frac{S_{\text{thr}}}{10}}; \mu, \frac{\omega}{\mu}\right), \end{aligned} \tag{31}$$

where G is the CDF of the gamma distribution and $\omega = 10^{(S_{\text{ref}} - 10\alpha \log(r/d_0))/10}$. Let R_N denote the R_2 for the Nakagami fading model. To have the same mean node degree, R_N must satisfy

$$\pi R^2 = \int_0^{R_N} \left(1 - G\left(10^{\frac{S_{\text{thr}}}{10}}; \mu, \frac{1}{\mu} 10^{\frac{S_{\text{ref}} - 10\alpha \log\left(\frac{r}{d_0}\right)}{10}}\right)\right) 2\pi r dr. \tag{32}$$

In the simulation, we set $\mu = 2$, and therefore, $f(r) = \left(r^2/50 + 1\right) e^{-r^2/50}$, $R_1 = 0$, and $R_2 = 28.3$.

6.2 Network Snapshots

A schematic of a wireless network with the Gaussian white noise model is depicted in Fig. 1. In the network, nodes are generated by a Poisson point process with mean density $\lambda = 0.02$ in a square region $\mathbb{D} = [0, 100]^2$. The solid lines represent the links between the nodes under the Euclidean metric; and the dotted lines denote additional links between nodes under the toroidal metric that connects pairs of nodes lying on opposite sides.

6.3 Expected Number of Isolated Nodes

To verify Theorem 1, i.e., the theorem of the expected total number of isolated nodes, we depict the average number of isolated nodes w.r.t. ξ in networks with $l = 500$. We are interested in the condition where the network is close to being

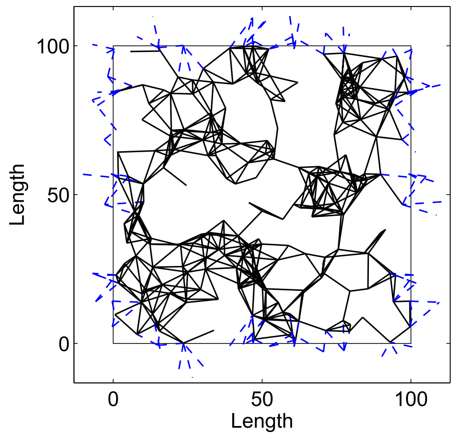

Fig. 1 A schematic of a wireless network with the Gaussian white noise model.

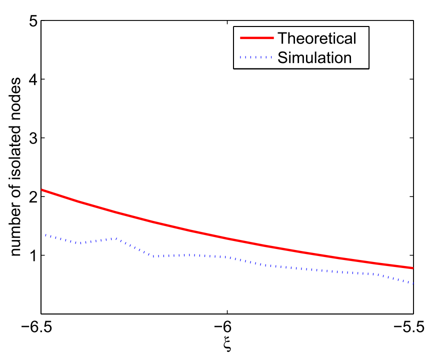

Fig. 2 The number of isolated nodes in a network with the Gaussian white noise model.

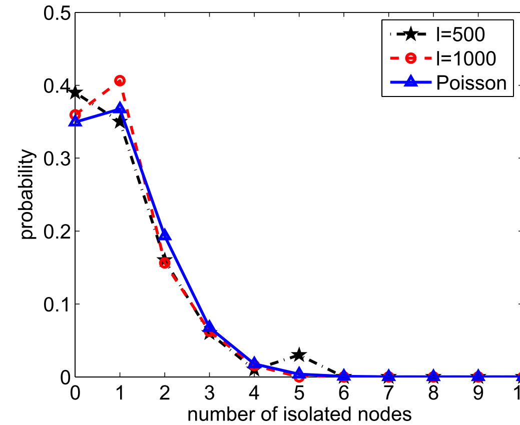

Fig. 3 The probability distribution functions of the total number of isolated nodes.

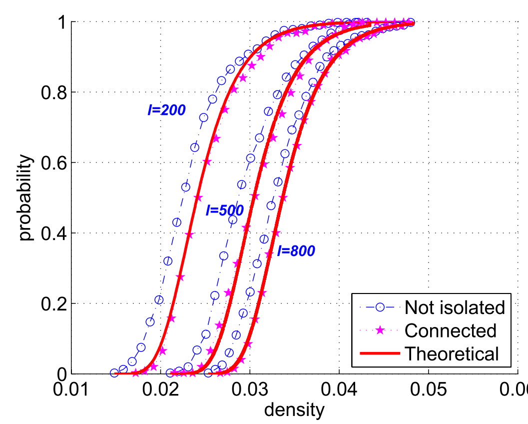

Fig. 4 The CDFs of $D_{\rm iso}$, $D_{\rm con}$, and $D_{\rm th}$ for the log-distance path loss model.

connected, i.e., there are few isolated nodes in the network. Thus, we choose the corresponding domain and range for the parameter ξ. Figure 2 illustrates the outcome for the Gaussian white noise model. The red solid line represents the expected number of isolated nodes given by Theorem 1, and the blue dotted line represents the average number of isolated nodes given by the simulations. We can see that there is only a small gap between the two curves. The rounding in the mathematical derivation for the theorem results in the gap between the expected number and the average number of isolated nodes. Even so, Theorem 1 accurately approximates the number of isolated nodes in a network.

6.4 Distribution of the Number of Isolated Nodes

To verify Theorem 2, i.e., the theorem of the probability distribution of the total number of isolated nodes, network instances are generated with $\xi = -5.8$ and $l = 500, 1000$. As Fig. 2 shows, there is about one isolated node on average in the network when $\xi = -5.8$. This helps us understand the features of the network in such circumstances. The node densities corresponding to $l = 500$ and $l = 1000$ are 0.029 and 0.339, respectively. The probability distribution functions of the total number of isolated nodes are depicted in Fig. 3 for the Gaussian white noise model. The blue solid line marked by triangles denotes the asymptotic probability distribution, i.e., the Poisson distribution with parameter $e^{-\omega}$. The black dash-dot line marked by stars and the red dash line marked by circles denote the experimental probability distribution corresponding to $l = 500$ and $l = 1000$, respectively. The results show that Theorem 2 can accurately capture network behavior.

6.5 Network Connectivity

Lastly, we investigate the conjecture that a random network without isolated nodes is almost surely connected. In the simulation, nodes are added into the network one by one. After each node is added, isolated nodes are counted and network connectivity is verified. Let $D_{\rm iso}$ denote the node density the first time that the network has no isolated nodes in the simulation, let $D_{\rm con}$ be the node density the first time that the network becomes connected in the simulation, and let $D_{\rm th}$ denote the theoretical node density for nonexistence of isolated nodes.

Figure 4 depicts the CDFs of $D_{\rm iso}$, $D_{\rm con}$, and $D_{\rm th}$ based on 400 network instances over the log-distance path loss model for $l = 200$, $l = 500$, and $l = 800$. In addition, Figs. 5,

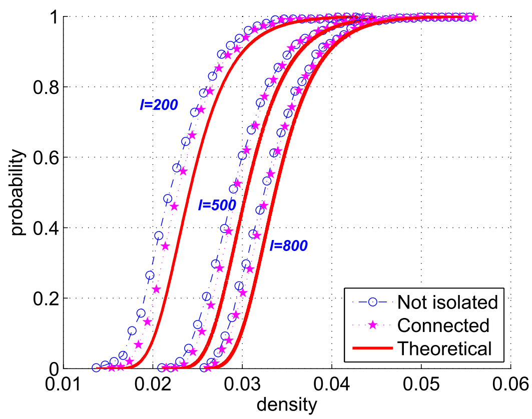

Fig. 5 The CDFs of $D_{\rm iso}$, $D_{\rm con}$, and $D_{\rm th}$ for the Bernoulli link model.

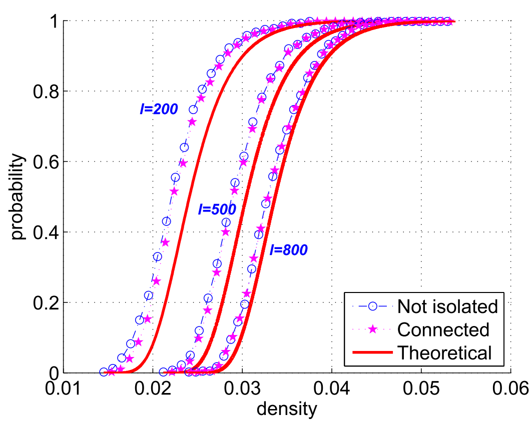

Fig. 6 The CDFs of $D_{\rm iso}$, $D_{\rm con}$, and $D_{\rm th}$ for the Gaussian white noise model.

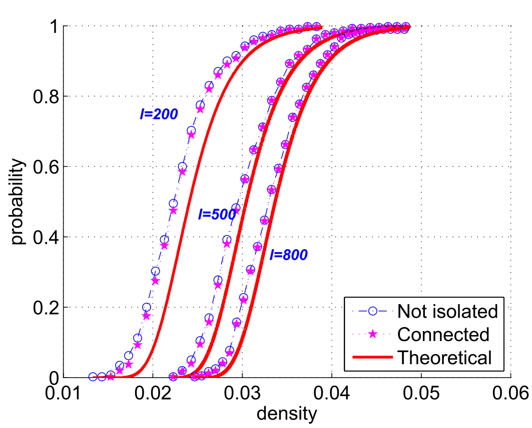

Fig. 7 The CDFs of $D_{\rm iso}$, $D_{\rm con}$, and $D_{\rm th}$ for the Rayleigh fading model.

6, and 7 depict the CDFs for the Bernoulli link model, the Gaussian white noise model and the Rayleigh fading model, respectively, using the same value of l. The simulation results support our conjecture. It is worth noting that, under the same average node degree condition, there are no significant differences between the behaviors of the different models. In fact, the node degree is one of the fundamental indicators to measure the connectivity of a wireless network. The simulation results show that the node degree and the connectivity are closely related. It can infer that a simple model, e.g., the disk graph model, can capture the behaviors of network connectivity, and that theoretical analysis can provide insights for designing a real network.

7. Conclusions

In this work, we derive the expected total number of isolated nodes and the asymptotic probability distribution of the total number of isolated nodes in a wireless network, based on a generic probabilistic wireless channel model. The probabilistic model studied in this work was a generalization of many widely used channel models, including the log-distance path loss model, the Bernoulli link model, the Gaussian white noise model, the Rayleigh fading model, and the Nakagami fading model. We presented the exact equation for the expected number of isolated nodes, and proved that the distribution of the number of isolated nodes asymptotically follows a Poisson distribution. In future work, we would like to prove the conjecture that a random network with a generic channel model without isolated nodes is almost surely connected.

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Appendix A: Proof of Lemma 3

For any $(x_1, \cdots, x_k) \in C_{kk}$, since $B(x_1, R_2), \cdots, B(x_k, R_2)$ are pairwise disjoint, we have

$$\Pr\left(\bigwedge_{i=1}^{k} B'_i \;\middle|\; \begin{matrix} X'_1 = x_1 \\ \vdots \\ X'_k = x_k \end{matrix}\right) = \left(e^{-\lambda a}\right)^k = e^{-k\lambda a}. \quad (\text{A}\cdot 1)$$

□

Appendix B: Proof of Lemma 4

This can be proved by straightforward calculation.

$$\begin{aligned}
&\lambda^k \int_{(x_1,\cdots,x_k)\in C_{kk}} e^{-k\lambda a}\left(\prod_{i=1}^{k} dx_i\right) \\
&= \left(\frac{1}{a}\left(\ln l^2 + \ln\ln l^2 + \xi\right)\right)^k e^{-k\left(\ln l^2 + \ln\ln l^2 + \xi\right)} \\
&\cdot \left(\prod_{i=1}^{k}\left(l^2 - (i-1)\pi R_2^2\right)\right) \\
&\sim \left(\frac{1}{a}e^{-\xi}\right)^k = \left(e^{-\omega}\right)^k. \quad (\text{A}\cdot 2)
\end{aligned}$$

□

Appendix C: Proof of Lemma 5

First, we prove the inequality for $k = 2$ and $m = 1$. Consider the case in which $d(x_1, x_2) \geq R_2/2$. Let $\mathbf{B}_2$ be the event that X'_2 does not have links to nodes in $B(x_2, R_2) - B(x_1, R_2)$. Then,

$$\begin{aligned}
&\Pr\left(B'_1 \wedge B'_2 \;\middle|\; \begin{matrix} X'_1 = x_1 \\ X'_2 = x_2 \end{matrix}\right) \\
&\leq \Pr\left(B'_1 \,\middle|\, X'_1 = x_1\right) \Pr\left(\mathbf{B}_2 \;\middle|\; \begin{matrix} X'_1 = x_1 \\ X'_2 = x_2 \end{matrix}\right). \quad (\text{A}\cdot 3)
\end{aligned}$$

Since it is known from Eq. (14) that $\Pr\left(B'_1 \,\middle|\, X'_1 = x_1\right) = e^{-\lambda a}$, we only need to show that there is a positive constant c_1 such that

$$\Pr\left(\mathbf{B}_2 \;\middle|\; \begin{matrix} X'_1 = x_1 \\ X'_2 = x_2 \end{matrix}\right) \leq e^{-c_1\lambda a}. \quad (\text{A}\cdot 4)$$

Let $\rho = d(x_1, x_2)$. For any $\rho \in [R_2/2, R_2]$ and $r \in [0, R_2]$, let $\theta(\rho, r)$ denote the angle of the arc of $\partial B(x_2, r)$ not contained in $B(x_1, R_2)$. See Fig. A·1. Since $\theta(\rho, r)$ is increasing w.r.t. ρ and $f(r) \geq 0$ for $r \in [0, R_2]$, we have

$$\begin{aligned}
&\int_0^{R_2} f(r)\,\theta(\rho, r)\, r dr \\
&\geq \int_{\frac{1}{2}R_2}^{R_2} f(r)\,\theta\left(\frac{1}{2}R_2, r\right) r dr. \quad (\text{A}\cdot 5)
\end{aligned}$$

Let

$$c_1 = \frac{\int_{\frac{1}{2}R_2}^{R_2} f(r)\,\theta\left(\frac{1}{2}R_2, r\right) r dr}{\int_0^{R_2} f(r)\,\theta(\rho, r)\, r dr}.$$

Applying the same approach to deriving the probability

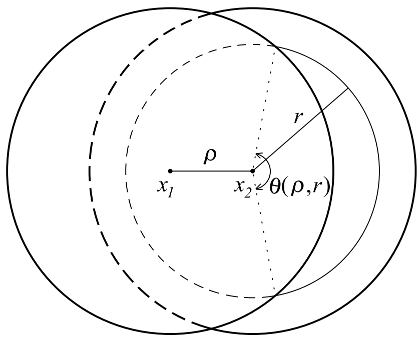

Fig. A·1 $\theta(\rho, r)$ is the angle of the arc of $\partial B(x_2, r)$ not contained in $B(x_1, R_2)$.

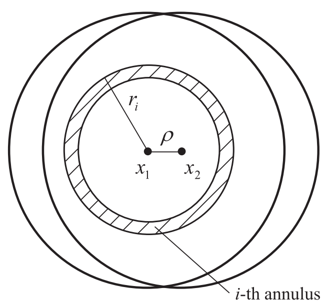

Fig. A·2 An annulus with center at x_1.

$\Pr(X \text{ is isolated}|\, X = x)$ in Sect. 4, we have

$$\Pr\left(\mathbf{B}_2 \,\middle|\, \begin{array}{l} X_1' = x_1 \\ X_2' = x_2 \end{array}\right) = e^{-\lambda \int_0^{R_2} f(r)\theta(\rho,r) r dr}$$
$$\leq e^{-\lambda \int_{\frac{1}{2}R_2}^{R_2} f(r)\theta(R_2/2,r) r dr}$$
$$= e^{-c_1 \lambda \int_0^{R_2} f(r) 2\pi r dr} = e^{-c_1 \lambda a}. \quad \text{(A·6)}$$

Therefore, if $R_2/2 \leq d(x_1, x_2) \leq R_2$, we have $\Pr\left(B_1' \wedge B_2' \,\middle|\, \begin{array}{l} X_1' = x_1 \\ X_2' = x_2 \end{array}\right) \leq e^{-(1+c_1)\lambda a}$.

Now, consider the case in which $0 \leq d(x_1, x_2) \leq R_2/2$. For this case, we only consider nodes in $B(x_1, R_2)$ and divide $B(x_1, R_2)$ by h concentric circles with center at x_1 and radii $r_1 < r_2 < \cdots < r_h = R_2$ as illustrated in Fig. A·2. Since $f(r)$ is a decreasing function, we have

$$\Pr\left(\begin{array}{l} X_1' \text{ and } X_2' \text{ do not have links} \\ \text{with nodes in the } i\text{th annulus} \end{array} \,\middle|\, \begin{array}{l} X_1' = x_1 \\ X_2' = x_2 \end{array}\right)$$
$$\leq \sum_{j=0}^{\infty} \left(\frac{(\lambda 2\pi r_i \triangle r_i)^j}{j!} e^{-\lambda 2\pi r_i \triangle r_i} \right)$$
$$\cdot (1 - f(r_i))^j (1 - f(r_i + \rho))^j$$
$$= e^{-(f(r_i)+f(r_i+\rho)-f(r_i)f(r_i+\rho))\lambda 2\pi r_i \triangle r_i}. \quad \text{(A·7)}$$

We should note that the inequality still holds for annuli not fully contained in $B(x_2, R_2)$. Thus,

$$\Pr\left(B_1' \wedge B_2' \,\middle|\, \begin{array}{l} X_1' = x_1 \\ X_2' = x_2 \end{array}\right)$$
$$\leq \lim_{k\to\infty} \prod_{i=1}^{k} e^{-(f(r_i)+f(r_i+\rho)-f(r_i)f(r_i+\rho))\lambda 2\pi r_i \triangle r_i}$$
$$= e^{-\lambda \int_0^{R_2} (f(r)+f(r+\rho)-f(r)f(r+\rho)) 2\pi r dr}$$
$$= e^{-\lambda \int_0^{R_2} f(r) 2\pi r dr - \lambda \int_0^{R_2} (f(r+\rho)-f(r)f(r+\rho)) 2\pi r dr}. \quad \text{(A·8)}$$

Since $\int_0^{R_2} f(r) 2\pi r dr = a$, if we can prove that there is a positive constant c_2 such that

$$\int_0^{R_2} (f(r+\rho) - f(r) f(r+\rho)) 2\pi r dr \geq c_2 a, \quad \text{(A·9)}$$

then this case is also proved. For any $r \in [R_2/8, R_2/4]$, we have $f(r+\rho) \geq f(3R_2/4)$ and $1 - f(r) \geq 1 - f(R_2/8)$. Let

$$c_2 = \frac{\int_{\frac{1}{8}R_2}^{\frac{1}{4}R_2} f\left(\frac{3}{4}R_2\right)\left(1 - f\left(\frac{1}{8}R_2\right)\right) 2\pi r dr}{\int_0^{R_2} f(r) 2\pi r dr}. \quad \text{(A·10)}$$

Then,

$$\int_0^{R_2} f(r+\rho)(1 - f(r)) 2\pi r dr$$
$$\geq \int_{\frac{1}{8}R_2}^{\frac{1}{4}R_2} f(r+\rho)(1 - f(r)) 2\pi r dr$$
$$\geq \int_{\frac{1}{8}R_2}^{\frac{1}{4}R_2} f\left(\frac{3}{4}R_2\right)\left(1 - f\left(\frac{1}{8}R_2\right)\right) 2\pi r dr$$
$$= c_2 a. \quad \text{(A·11)}$$

Thus, if $0 \leq d(x_1, x_2) \leq R_2/2$, we have

$$\Pr\left(B_1' \wedge B_2' \,\middle|\, \begin{array}{l} X_1 = x_1 \\ X_2 = x_2 \end{array}\right) \leq e^{-(1+c_2)\lambda a}. \quad \text{(A·12)}$$

If we choose $c = \min(c_1, c_2)$, the lemma for $k = 2$ is proved.

For any $k \geq 3$ and $m = 1$, since there are always two overlapping disks in the component $\bigcup_{i=1,\cdots,k} B(x_i, R_2)$, it is trivial to see that the inequality is still correct.

For any $k \geq 3$ and $2 \leq m \leq k-1$, if $(x_1, \cdots, x_k) \in C_{km}$, then $\{x_1, \cdots, x_k\}$ is partitioned into m sets $K_1, K_2, \cdots, K_m$ such that for each $j = 1, \cdots, m$, $\bigcup_{x \in K_j} B(x, R_2)$ is a maximal component. Let $n_j = |K_j|$ be the number of elements in K_j. Suppose $K_j = \{x_{j1}, \cdots, x_{jn_j}\}$. Then,

$$\Pr\left(\bigwedge_{x_i \in K_j} B_i' \,\middle|\, \begin{array}{c} X_{j1}' = x_{j1} \\ \vdots \\ X_{jn_j}' = x_{jn_j} \end{array}\right) = e^{-\lambda a} \text{ if } n_j = 1 \quad \text{(A·13)}$$

and

$$\Pr\left(\bigwedge_{x_i \in K_j} B_i' \,\middle|\, \begin{array}{c} X_{j1}' = x_{j1} \\ \vdots \\ X_{jn_j}' = x_{jn_j} \end{array}\right) \leq e^{-(1+c)\lambda a} \text{ if } n_j > 1.$$
$$\text{(A·14)}$$

Since there is at least one component that contains more than one node, we have

$$\Pr\left(\bigwedge_{i=1}^{k} B_i' \,\middle|\, \begin{array}{c} X_1' = x_1 \\ \vdots \\ X_k' = x_k \end{array}\right) = \prod_{j=1}^{m} \Pr\left(\bigwedge_{x_i\in K_j} B_i' \,\middle|\, \begin{array}{c} X_{j1}' = x_{j1} \\ \vdots \\ X_{jn_j}' = x_{jn_j} \end{array}\right) \le e^{-(m+c)\lambda a}. \quad \text{(A· 15)}$$

Thus, the lemma is proved. □

Appendix D: Proof of Lemma 6

First, consider $m = 1$. This can be validated by straightforward calculation.

$$\begin{aligned} &\lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{k1}} e^{-(1+c)\lambda a}\left(\prod_{i=1}^{k} dx_i\right) \\ &\le \lambda^k l^2 \left(\prod_{i=2}^{k}\left(\pi\left(2\left(i-1\right)R_2\right)^2\right)\right) e^{-(1+c)\lambda a} \\ &= O\left(1\right)\left(\ln l^2 + \ln\ln l^2 + \xi\right)^k l^2 e^{-(1+c)\left(\ln l^2 + \ln\ln l^2 + \xi\right)} \\ &= o\left(1\right). \end{aligned} \quad \text{(A· 16)}$$

We write $g(\lambda) = O(h(\lambda))$ if there are λ_0, c_0 such that $|g(\lambda)/h(\lambda)| \le c_0$ for all $\lambda \ge \lambda_0$.

Next, consider $2 \le m \le k-1$. If $(x_1,\cdots,x_k) \in C_{km}$, then $\{x_1,\cdots,x_k\}$ can be partitioned into m sets $K_1, K_2, \cdots, K_m$ such that for each $j = 1,\cdots,m$, $\bigcup_{x\in K_j} B(x,R_2)$ is a maximal connected component. Let $n_j = |K_j|$ be the number of elements in K_j, and suppose $K_j = \{x_{j1},\cdots,x_{jn_j}\}$. For fixed k and m, the number of m-partitions of $\{x_1,\cdots,x_k\}$ are constant. Then,

$$\begin{aligned} &\lambda^k \int_{(x_1,x_2,\cdots,x_k)\in C_{km}} e^{-(m+c)\lambda a}\left(\prod_{i=1}^{k} dx_i\right) \\ &= O\left(1\right) e^{-c\lambda a} \prod_{j=1}^{m}\left(\lambda^{n_j} \int_{\left(x_{j1},\cdots,x_{jn_j}\right)\in C_{n_j 1}} e^{-\lambda a}\left(\prod_{j=1}^{n_j} dx_i\right)\right) \\ &= o\left(1\right). \end{aligned} \quad \text{(A· 17)}$$

The last equality holds because of at least one $n_j > 1$. Thus, the lemma is proved. □

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