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# Spectral radius and degree sequence of a graph

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ABSTRACT

Let $G$ be a simple connected graph of order $n$ with degree sequence $d_1, d_2, \ldots, d_n$ in non-increasing order. The *spectral radius* $\rho(G)$ of $G$ is the largest eigenvalue of its adjacency matrix. For each positive integer $\ell$ at most $n$, we give a sharp upper bound for $\rho(G)$ by a function of $d_1, d_2, \ldots, d_\ell$, which generalizes a series of previous results.



## 1. Introduction

Let $G$ be a simple connected graph of $n$ vertices and $m$ edges with degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$. The *adjacency matrix* $A = (a_{ij})$ of $G$ is a binary square matrix of order $n$ with rows and columns indexed by the vertex set $VG$ of $G$ such that for any $i, j \in VG$, $a_{ij} = 1$ if $i, j$ are adjacent in $G$. The *spectral radius* $\rho(G)$ of $G$ is the largest eigenvalue of its adjacency matrix, which has been studied by many authors.

The following theorem is well-known [6, Chapter 2].

**Theorem 1.1.** *If $A$ is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(A)$ and row-sums $r_1, r_2, \ldots, r_n$, then*

$$\rho(A) \leqslant \max_{1 \leqslant i \leqslant n} r_i$$

*with equality if and only if the row-sums of $A$ are all equal.*

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In 1985 [1, Corollary 2.3], Brauldi and Hoffman showed the following result.

**Theorem 1.2.** *If* $m \leqslant k(k-1)/2$, *then*

$$\rho(G) \leqslant k-1$$

*with equality if and only if G is isomorphic to the complete graph* $K_n$ *of order* $n$.

In 1987 [8], Stanley improved Theorem 1.2 and showed the following result.

**Theorem 1.3.**

$$\rho(G) \leqslant \frac{-1+\sqrt{1+8m}}{2}$$

*with equality if and only if G is isomorphic to the complete graph* $K_n$ *of order* $n$.

In 1998 [3, Theorem 2], Yuan Hong improved Theorem 1.3 and showed the following result.

**Theorem 1.4.**

$$\rho(G) \leqslant \sqrt{2m-n+1}$$

*with equality if and only if G is isomorphic to the star* $K_{1,n-1}$ *or to the complete graph* $K_n$.

In 2001 [4, Theorem 2.3], Hong et al. improved Theorem 1.4 and showed the following result.

**Theorem 1.5.**

$$\rho(G) \leqslant \frac{d_n-1+\sqrt{(d_n+1)^2+4(2m-nd_n)}}{2}$$

*with equality if and only if G is regular or there exists* $2 \leqslant t \leqslant n$ *such that* $d_1 = d_{t-1} = n-1$ *and* $d_t = d_n$.

In 2004 [7, Theorem 2.2], Jinlong Shu and Yarong Wu improved Theorem 1.1 in the case that $A$ is the adjacency matrix of $G$ by showing the following result.

**Theorem 1.6.** *For* $1 \leqslant \ell \leqslant n$,

$$\rho(G) \leqslant \frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4(\ell-1)(d_1-d_\ell)}}{2}$$

*with equality if and only if G is regular or there exists* $2 \leqslant t \leqslant \ell$ *such that* $d_1 = d_{t-1} = n-1$ *and* $d_t = d_n$.

Moreover, they also showed in [7, Theorem 2.5] that if $p+q \geqslant d_1+1$ then Theorem 1.6 improves Theorem 1.5 where $p$ is the number of vertices with the largest degree $d_1$ and $q$ is the number of vertices with the second largest degree. The special case $\ell = 2$ of Theorem 1.6 is reproved [2].

In this research, we present a sharp upper bound of $\rho(G)$ in terms of the degree sequence of $G$, which improves Theorem 1.2 to Theorem 1.6.

**Theorem 1.7.** *For* $1 \leqslant \ell \leqslant n$,

$$\rho(G) \leqslant \phi_\ell := \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4\sum_{i=1}^{\ell-1}(d_i - d_\ell)}}{2},$$

*with equality if and only if G is regular or there exists* $2 \leqslant t \leqslant \ell$ *such that* $d_1 = d_{t-1} = n - 1$ *and* $d_t = d_n$.

This result improves Theorem 1.5 and Theorem 1.6 since $\phi_n$ is exactly the upper bounds in Theorem 1.5 and is at most the upper bound appearing in Theorem 1.6. Additionally, generalized from this research, a similar upper bound of the spectral radius in terms of the average 2-degree sequence of a graph will be presented in [5].

Notice that the number $\phi_\ell$ defined in Theorem 1.7 is at least $d_\ell$. The sequence $\phi_1, \phi_2, \ldots, \phi_n$ is not necessary to be non-increasing. We show that this sequence is first non-increasing and then non-decreasing, and determine its lowest value in Section 3.

## 2. Proof of Theorem 1.7

**Proof.** Let the vertices be labeled by $1, 2, \ldots, n$ with degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$, respectively. For each $1 \leqslant i \leqslant \ell - 1$, let $x_i \geqslant 1$ be a variable to be determined later. Let $U = diag(x_1, x_2, \ldots, x_{\ell-1}, 1, 1, \ldots, 1)$ be a diagonal matrix of size $n \times n$. Then $U^{-1} = diag(x_1^{-1}, x_2^{-1}, \ldots, x_{\ell-1}^{-1}, 1, 1, \ldots, 1)$.

Let $B = U^{-1}AU$. Notice that $A$ and $B$ have the same eigenvalues.

Let $r_1, r_2, \ldots, r_n$ be the row-sums of $B$. Then for $1 \leqslant i \leqslant \ell - 1$ we have

$$\begin{aligned} r_i &= \sum_{k=1}^{\ell-1} \frac{x_k}{x_i} a_{ik} + \sum_{k=\ell}^{n} \frac{1}{x_i} a_{ik} = \frac{1}{x_i}\sum_{k=1}^{n} a_{ik} + \frac{1}{x_i}\sum_{k=1}^{\ell-1}(x_k - 1)a_{ik} \\ &\leqslant \frac{1}{x_i} d_i + \frac{1}{x_i}\left(\sum_{k=1, k\neq i}^{\ell-1} x_k - (\ell - 2)\right), \end{aligned} \tag{2.1}$$

and for $\ell \leqslant j \leqslant n$ we have

$$\begin{aligned} r_j &= \sum_{k=1}^{\ell-1} x_k a_{jk} + \sum_{k=\ell}^{n} a_{jk} = \sum_{k=1}^{n} a_{jk} + \sum_{k=1}^{\ell-1}(x_k - 1)a_{jk} \\ &\leqslant d_\ell + \left(\sum_{k=1}^{\ell-1} x_k - (\ell - 1)\right). \end{aligned} \tag{2.2}$$

For $1 \leqslant i \leqslant \ell - 1$ let

$$x_i = 1 + \frac{d_i - d_\ell}{\phi_\ell + 1} \geqslant 1, \tag{2.3}$$

where $\phi_\ell$ is defined in Theorem 1.7. Then for $1 \leqslant i \leqslant \ell - 1$ we have

$$r_i \leqslant \frac{1}{x_i} d_i + \frac{1}{x_i}\left(\sum_{k=1, k\neq i}^{\ell-1} x_k - (\ell - 2)\right) = \phi_\ell,$$

and for $\ell \leqslant j \leqslant n$ we have

$$r_j \leqslant d_\ell + \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) = \phi_\ell.$$

Hence by Theorem 1.1,

$$\rho(G) = \rho(B) \leqslant \max_{1 \leqslant i \leqslant n} \{r_i\} \leqslant \phi_\ell. \tag{2.4}$$

The first part of Theorem 1.7 follows.

The sufficient condition of $\phi_\ell = \rho(G)$ follows from the fact that

$$\phi_\ell \leqslant \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}$$

and applying the second part in Theorem 1.6.

To prove the necessary condition of $\phi_\ell = \rho(G)$, suppose $\phi_\ell = \rho(G)$. Then the equalities in (2.1) and (2.2) all holds. If $d_1 = d_\ell$, then $d_1 = \phi_1 = \phi_\ell = \rho(G)$, and $G$ is regular by the second part of Theorem 1.1. Suppose $2 \leqslant t \leqslant \ell$ such that $d_{t-1} > d_t = d_\ell$. Then $x_i > 1$ for $1 \leqslant i \leqslant t-1$ by (2.3). For each $1 \leqslant i \leqslant \ell - 1$, the equality in (2.1) implies that $a_{ik} = 1$ for $1 \leqslant k \leqslant t-1$, $k \neq i$. For each $\ell \leqslant j \leqslant n$, the equality in (2.2) implies that $a_{jk} = 1$ for $1 \leqslant k \leqslant t-1$ and $d_j = d_\ell$. Hence $n - 1 = d_1 = d_{t-1} > d_t = d_\ell = d_n$.

We complete the proof. □

## 3. The sequence $\phi_1, \phi_2, \ldots, \phi_n$

The sequence $\phi_1, \phi_2, \ldots, \phi_n$ is not necessarily non-increasing. For example, the path $P_n$ of $n$ vertices has $2 = d_1 = d_{n-2} > d_{n-1} = d_n = 1$, and it is immediate to check that if $n \geqslant 6$ then $\phi_1 = \phi_2 = 2 < \sqrt{n-1} = \phi_{n-1} = \phi_n$.

Clearly that for all $1 \leqslant s < t \leqslant n$, $d_s = d_t$ implies that $\phi_s = \phi_t$. However, $\phi_s = \phi_t$ dose not imply $d_s = d_t$. For example, in the graph with degree sequence $(4, 3, 3, 2, 1, 1)$, one can check that $\phi_4 = \phi_5 = 3$ but $d_4 > d_5$.

Recall that $d_s = d_{s+1}$ implies $\phi_s = \phi_{s+1}$ for $1 \leqslant s \leqslant n-1$. The following proposition describes the shape of the sequence $\phi_1, \phi_2, \ldots, \phi_n$.

**Proposition 3.1.** *Suppose $d_s > d_{s+1}$ for $1 \leqslant s \leqslant n-1$, and let $\succeq \ \in \{>, =\}$. Then*

$$\phi_s \succeq \phi_{s+1} \ \textit{iff} \ \sum_{i=1}^{s} d_i \succeq s(s-1).$$

**Proof.** Recall that

$$\phi_s = \frac{d_s - 1 + \sqrt{(d_s + 1)^2 + 4\sum_{i=1}^{s-1}(d_i - d_s)}}{2}.$$

The proposition follows from the following equivalent relations step by step:

$$\begin{aligned} & \phi_s \succeq \phi_{s+1} \\ \Leftrightarrow \quad & d_s - d_{s+1} + \sqrt{(d_s + 1)^2 + 4\sum_{i=1}^{s-1}(d_i - d_s)} \end{aligned}$$

$$
\begin{aligned}
& \succeq \sqrt{(d_{s+1}+1)^2 + 4\sum_{i=1}^{s}(d_i - d_{s+1})} \\
\Leftrightarrow \quad & \sqrt{(d_s+1)^2 + 4\sum_{i=1}^{s-1}(d_i - d_s)} \succeq 2s - (d_s+1) \\
\Leftrightarrow \quad & (d_s+1)^2 + 4\sum_{i=1}^{s}(d_i - d_s) \succeq 4s^2 - 4s(d_s+1) + (d_s+1)^2 \\
\Leftrightarrow \quad & \sum_{i=1}^{s} d_i \succeq s(s-1),
\end{aligned}
$$

where the relation in (3.1) is obtained from the second by taking square on both sides, simplifying it, and deleting the common term $d_s - d_{s+1}$. Notice that if $2s - (d_s+1) < 0$ in (3.1) then in the case that $\succeq$ is $=$, all statements fails, and in the case that $\succeq$ is $>$ the left hand side of (3.1) is at least $d_s + 1$, which is greater than $|2s - (d_s+1)|$, so the equivalent relation in the next step holds. □

**Corollary 3.2.** *Let $3 \leqslant \ell \leqslant n$ be the smallest integer such that $\sum_{i=1}^{\ell} d_i < \ell(\ell-1)$. Then for $1 \leqslant j \leqslant n$ we have*

$$\phi_j = \min\{\phi_k \mid 1 \leqslant k \leqslant n\}$$

*if and only if $d_j = d_\ell$, or $d_j = d_{\ell-1}$ with $\sum_{i=1}^{\ell-1} d_i = (\ell-1)(\ell-2)$.*

**Proof.** From Proposition 3.1, $\sum_{i=1}^{\ell-1} d_i = (\ell-1)(\ell-2)$ implies $\phi_{\ell-1} = \phi_\ell$. Also, clearly that $d_j = d_\ell$ implies $\phi_j = \phi_\ell$. We show that $\phi_\ell = \min\{\phi_k \mid 1 \leqslant k \leqslant n\}$ in the following.

For $1 \leqslant s \leqslant \ell - 1$, from Proposition 3.1 we have $\phi_s \geqslant \phi_{s+1}$ since $\sum_{i=1}^{s} d_i \geqslant s(s-1)$. For $\ell \leqslant t \leqslant n-1$, notice that $\sum_{i=1}^{t} d_i < t(t-1)$ implies $d_t < t-1$, and hence $\sum_{i=1}^{t+1} d_i < t(t-1) + (t-1) < t(t+1)$. From Proposition 3.1 we have $\phi_\ell \leqslant \phi_{\ell+1} \leqslant \cdots \leqslant \phi_n$ since $\sum_{i=1}^{\ell} d_i < \ell(\ell-1)$. The result follows. □

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