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Measuring PPM Non-conformities for Processes with Asymmetric Tolerances

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Process yield has been the most basic and common criterion used in the manufacturing industry for evaluating process capability. The C_{pk} index has been used widely in the manufacturing industry. In this note, we considered a generalization of C_{pk} index which handles processes involving a target T with asymmetric tolerances. Particularly, we established a formula for measuring the PPM non-conformities for given ratios of the two-side tolerances. We proved the validity of the established formula and tabulated the upper bounds on PPM non-conformities for various given C_{pk} index values and ratios of the two-side tolerances. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

In manufacturing industry, the process yield is one of the major criteria for interpreting the process capability. C_{pk} index is a yield-based index, which provides an upper bound on the non-conforming units in parts per million (NCPPM) for a normally distributed process. The C_{pk} index was defined as

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}. \quad (1)$$

where μ is the process mean, σ is the standard deviation, $M=(USL+LSL)/2$ is the midpoint of the specification interval, and $d=(USL-LSL)/2$. For normal processes distributed as $N(\mu,\sigma^2)$, the bounds on the NCPPM for processes with a specific value of C_{pk} can be represented as

$$\left[2-2\Phi\left(3C_{pk}\right)\right]\times 10^6 \geqslant \text{NCPPM} \geqslant \left[1-\Phi\left(3C_{pk}\right)\right]\times 10^6. \quad (2)$$

where function Φ is the cumulative probability function of the standard normal distribution. Table I displays some C_{pk} index values with the upper bounds of NCPPM for a normally distributed process.

2. The generalization C''_{pk} index

The formula presented in equation (2) only applies to processes with symmetric tolerances. For processes with asymmetric tolerances, Pearn and Chen[1] proposed a generalization of C_{pk} index which was referred to as C''_{pk}. The generalization C''_{pk} index is superior to other existing generalizations of C_{pk} index for processes with asymmetric tolerances. The C''_{pk} index was defined as (see Pearn and Chen[1]):

$$C''_{pk} = \frac{d^* - \max\{d^*(\mu-T)/D_U, d^*(T-\mu)/D_L\}}{3\sigma} \quad (3)$$

where $d^*=\min\{D_L, D_U\}$, $D_U=USL-T$, $D_L=T-LSL$ and T is the target value. Obviously, for processes with symmetric tolerances ($D_U=D_L$), C''_{pk} reduces to the C_{pk} index which mentioned earlier in equation (1).

Pearn and Lin[2] investigated the statistical estimation of C''_{pk} index. Comparisons among several process capability indices (PCI) for processes with asymmetric tolerances were proposed by Chen and Pearn.[3] Lin and Pearn[4] analyzed the large sample properties of the natural estimator of C''_{pk} under general condition and provided an approximate confidence interval using the limiting distribution.

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Table I. Some specific values of C_{pk} and the upper bounds on the NCPPM

C_{pk}	NCPPM
1.00	2699.796
1.25	176.835
1.33	66.073
1.45	13.614
1.50	6.795
1.60	1.587
1.67	0.544
2.00	0.002

The explicit forms of the cumulative distribution function and the probability density function for the natural estimator $\hat{C}''_{pk}$ under the assumption of normality were derived by Pearn *et al.*[5] Various applications of PCI in manufacturing industry were considered by Pearn *et al.*[6] and Chen *et al.*[7] More recent researches on PCI include Pearn *et al.*,[8] Awad and Kovach,[9] Wu *et al.*,[10] and Yum and Kim.[11]

The measurement of process PPM non-conformities for normally distributed processes with asymmetric tolerance using C''_{pk} index have never been investigated. In this note, we obtain an upper bound formula on the NCPPM for normally distributed processes with asymmetric tolerances and given ratios of the two-side tolerances.

3. Non-conformity measures based on C''_{pk}

To prove the formula, we first rewrite C''_{pk} as follows:

$$C''_{pk} = \frac{d^* - d^* \max\{R_U, R_L\}}{3\sigma} = \frac{d^*(1 - \max\{R_U, R_L\})}{3\sigma} \tag{4}$$

where the symbols $R_U = (\mu - T)/D_U$ and $R_L = (T - \mu)/D_L$ reflect the corresponding departure ratios on right and left tolerances, respectively. For a process with characteristic X distributed as $N(\mu, \sigma^2)$, the process yield can be represented as

$$\begin{aligned}
\text{yield} &= P(LSL < X < USL) = P[(LSL - \mu)/\sigma < Z < (USL - \mu)/\sigma] \\
&= \Phi\left[\frac{(USL - \mu)}{\sigma}\right] - \Phi\left[\frac{(LSL - \mu)}{\sigma}\right] = \Phi\left[\frac{(USL - \mu)}{\sigma}\right] + \Phi\left[\frac{(\mu - LSL)}{\sigma}\right] - 1 \\
&= \Phi\left[\frac{(USL - T)}{\sigma} - \frac{(\mu - T)}{\sigma}\right] + \Phi\left[\frac{(T - LSL)}{\sigma} - \frac{(T - \mu)}{\sigma}\right] - 1 \\
&= \Phi\left[\frac{D_U}{\sigma}(1 - R_U)\right] + \Phi\left[\frac{D_L}{\sigma}(1 - R_L)\right] - 1
\end{aligned} \tag{5}$$

Let $\kappa = \max\{D_U/D_L, D_L/D_U\}$ represents the larger one of the ratios of two-side tolerances. Four cases are discussed in the following:

Case 1

$d^* = D_U$, $R_U > R_L$, $\kappa = D_L/D_U$ and $C''_{pk} = \frac{D_U(1-R_U)}{3\sigma}$

$$\begin{aligned}
\text{yield} &= \Phi\left[\frac{D_U}{\sigma}(1 - R_U)\right] + \Phi\left[\frac{D_L}{\sigma}(1 - R_L)\right] - 1 = \Phi[3C''_{pk}] + \Phi\left[\frac{D_L}{D_U}\frac{D_U}{\sigma}(1 - R_L)\right] - 1 \\
&\geqslant \Phi[3C''_{pk}] + \Phi\left[\frac{D_L}{D_U}\frac{D_U}{\sigma}(1 - R_U)\right] - 1 = \Phi[3C''_{pk}] + \Phi[3\kappa C''_{pk}] - 1,
\end{aligned}$$

and

$$\begin{aligned}
\text{yield} &= \Phi\left[\frac{D_U}{\sigma}(1 - R_U)\right] + \Phi\left[\frac{D_L}{\sigma}(1 - R_L)\right] - 1 \\
&\leqslant \Phi\left[\frac{D_L}{\sigma}(1 - R_U)\right] + \Phi\left[\frac{D_L}{\sigma}(1 - R_L)\right] - 1 \\
&= \Phi\left[\frac{D_L}{\sigma}(1 - R_U)\right] + \Phi[3\kappa C''_{pk}] - 1 < \Phi[3\kappa C''_{pk}].
\end{aligned}$$

Case 2

$d^* = D_U$, $R_L > R_U$, $\kappa = D_L/D_U$ and $C''_{pk} = \frac{D_U(1-R_L)}{3\sigma}$

$$\begin{aligned}\text{yield} &= \Phi\left[\frac{D_U}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1\\ &> \Phi\left[\frac{D_U}{\sigma}(1-R_L)\right]+\Phi\left[\frac{D_L}{D_U}3C''_{pk}\right]-1=\Phi\left[3C''_{pk}\right]+\Phi\left[3\kappa C''_{pk}\right]-1,\end{aligned}$$

and

$$\text{yield}=\Phi\left[\frac{D_U}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1<\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]=\Phi\left[3\kappa C''_{pk}\right].$$

Case 3

$d^*=D_L$, $R_U>R_L$, $\kappa=D_U/D_L$ and $C''_{pk}=\frac{D_L(1-R_U)}{3\sigma}$

$$\begin{aligned}\text{yield} &= \Phi\left[\frac{D_U}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1\\ &> \Phi\left[\frac{D_U}{D_L}\frac{D_L}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_U)\right]-1\\ &= \Phi\left[3\kappa C''_{pk}\right]+\Phi\left[3C''_{pk}\right]-1,\end{aligned}$$

and

$$\text{yield}=\Phi\left[\frac{D_U}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1<\Phi\left[\frac{D_U}{D_L}\frac{D_L}{\sigma}(1-R_U)\right]=\Phi\left[3\kappa C''_{pk}\right].$$

Case 4

$d^*=D_L$, $R_L>R_U$, $\kappa=D_U/D_L$ and $C''_{pk}=\frac{D_L(1-R_L)}{3\sigma}$

$$\begin{aligned}\text{yield} &= \Phi\left[\frac{D_U}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1\\ &> \Phi\left[\frac{D_U}{\sigma}(1-R_L)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1=\Phi\left[3\kappa C''_{pk}\right]+\Phi\left[3C''_{pk}\right]-1,\end{aligned}$$

and

$$\begin{aligned}\text{yield} &= \Phi\left[\frac{D_U}{\sigma}(1-R_U)\right]+\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]-1<\Phi\left[\frac{D_L}{\sigma}(1-R_L)\right]\\ &< \Phi\left[\frac{D_U}{\sigma}(1-R_L)\right]<\Phi\left[\frac{D_U}{D_L}\frac{D_L}{\sigma}(1-R_L)\right]=\Phi\left[3\kappa C''_{pk}\right].\end{aligned}$$

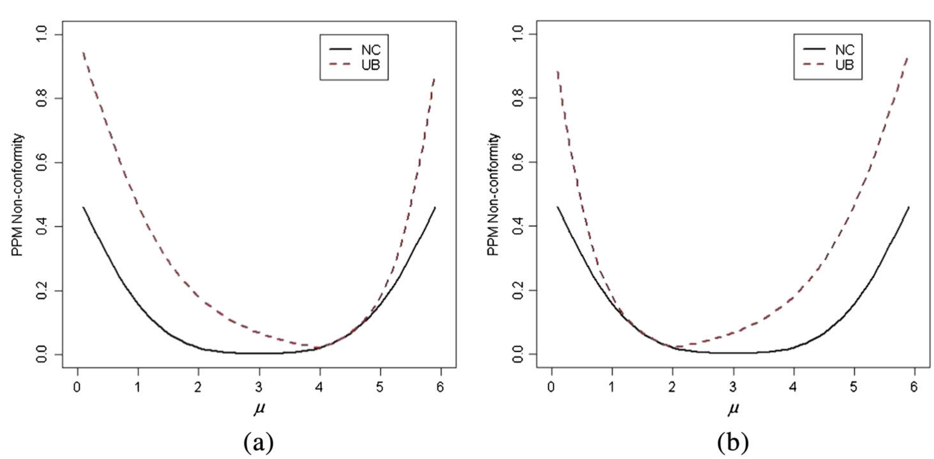

Figure 1. (a) Upper bounds on NCPPM and true NCPPM for fixed target $T=4$ with $0<\mu<6$. (b) Upper bounds on NCPPM and true NCPPM for fixed target $T=2$ with $0<\mu<6$.

From cases 1–4, we establish the bounds on process yield based on the C''_{pk} index. Consequently, a two-sided bound on NCPPM for normally distributed processes with asymmetric tolerances can be represented as follows:

$$\left[2-\Phi\left(3\kappa C''_{pk}\right)-\Phi\left(3C''_{pk}\right)\right]\times 10^6 \geqslant \text{NCPPM} \geqslant \left[1-\Phi\left(3\kappa C''_{pk}\right)\right]\times 10^6. \tag{6}$$

For various values of process mean μ, Figs. 1(a)–1(b) display the upper bound on NCPPM and true NCPPM for a normally distributed process with specifications $LSL=0$, $USL=6$ and variance $\sigma^2=1$. In Figs. 1(a)–1(b), the dotted red line and the black line represent the NCPPM upper bounds and the true NCPPM of the process, respectively. We note that the true NCPPM is minimized by $\mu=M=3$ for a given ratios of two-side specifications. The NCPPM upper bounds and the true NCPPM are plotted in Figure 2(a) 2(b) as a function of the target value T.

4. Non-conformity bounds calculation

Table II displays the upper bounds on the NCPPM for various values of $C''_{pk}=1.00(0.05)2.00$ and the larger one of the ratios of two-side tolerances $\kappa=\max\{D_U/D_L, D_L/D_U\}$, $\kappa=1.00(0.05)1.50$. For instance, for a normally distributed process with asymmetric

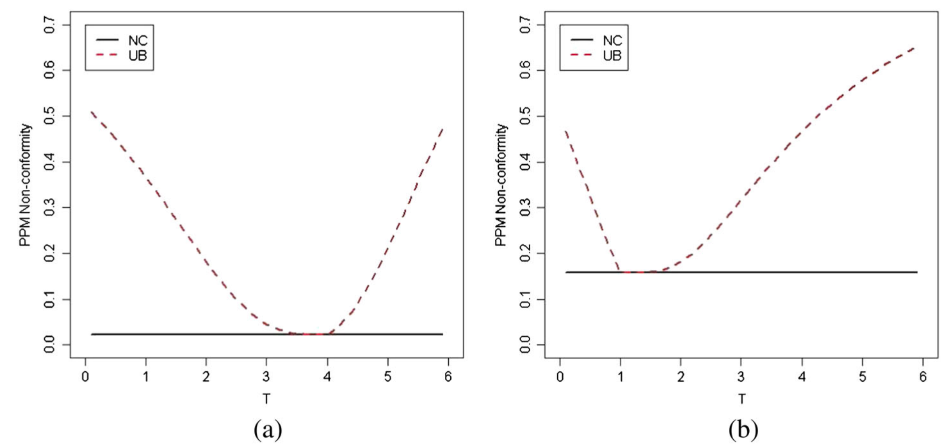

Figure 2. (a) Upper bounds on NCPPM and true NCPPM for fixed process mean $\mu=4$ with $0<T<6$. (b) Upper bounds on NCPPM and true NCPPM for fixed process mean $\mu=1$ with $0<T<6$.

Table II. The upper bounds on NCPPM for various values of C''_{pk} and κ

	κ										
C''_{pk}	1.000	1.050	1.100	1.150	1.200	1.250	1.300	1.350	1.400	1.450	1.500
1.00	2699.8	2166.3	1833.3	1630.2	1509.0	1438.3	1398.0	1375.5	1363.2	1356.7	1353.3
1.05	1632.7	1287.0	1081.5	962.24	894.77	857.52	837.46	826.92	821.52	818.82	817.50
1.10	966.85	748.54	625.14	557.25	520.90	501.96	492.36	487.62	485.34	484.28	483.80
1.15	560.59	426.18	354.12	316.61	297.67	288.36	283.94	281.89	280.98	280.58	280.41
1.20	318.22	237.52	196.58	176.47	166.91	162.51	160.54	159.70	159.34	159.20	159.14
1.25	176.84	129.59	106.95	96.488	91.815	89.800	88.961	88.624	88.493	88.444	88.427
1.30	96.193	69.205	57.030	51.742	49.531	48.640	48.295	48.166	48.120	48.104	48.099
1.35	51.218	36.179	29.803	27.209	26.196	25.816	25.679	25.632	25.616	25.611	25.609
1.40	26.691	18.514	15.264	14.028	13.579	13.422	13.370	13.353	13.348	13.346	13.346
1.45	13.614	9.275	7.662	7.090	6.896	6.834	6.815	6.809	6.807	6.807	6.807
1.50	6.795	4.548	3.769	3.512	3.431	3.407	3.400	3.398	3.398	3.398	3.398
1.55	3.319	2.183	1.817	1.704	1.672	1.663	1.660	1.660	1.660	1.660	1.660
1.60	1.587	1.026	0.858	0.810	0.798	0.794	0.794	0.793	0.793	0.793	0.793
1.65	0.742	0.472	0.397	0.377	0.372	0.371	0.371	0.371	0.371	0.371	0.371
1.70	0.340	0.213	0.180	0.172	0.170	0.170	0.170	0.170	0.170	0.170	0.170
1.75	0.152	0.094	0.080	0.077	0.076	0.076	0.076	0.076	0.076	0.076	0.076
1.80	0.067	0.040	0.035	0.034	0.033	0.033	0.033	0.033	0.033	0.033	0.033
1.85	0.029	0.017	0.015	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
1.90	0.012	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006
1.95	0.005	0.003	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
2.00	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001

tolerances which satisfies $C''_{pk} = 1.40$ and $\kappa = 1.30$, the product's fractions of defectives is at most 13.37 ppm. From Table II, for case of $\kappa = \max\{D_U/D_L, D_L/D_U\} = 1$, the upper bounds on NCPPM are the same with the results which mentioned earlier in equation (2). Obviously, if $D_U = D_L$ (symmetric tolerance), then C''_{pk} defined in equation (3) reduces to C_{pk} defined in equation (1), and formula we established in equation (6) reduced to equation (2).

For fixed C''_{pk} value, when κ increases, the upper bounds on NCPPM decrease and the bounds are closer to the true NCPPM. It is evident since the larger the value κ, the smaller the value $d^*(C''_{pk})$. That is, when the tolerances become more asymmetric, it requires the process to have a lower variance for keeping the C''_{pk} value remains the same. For example, two on-target processes A and B with same value of C''_{pk} index and identical specifications $(LSL, USL) = (0, 50)$ are considered. Since processes A and B have identical C''_{pk} index value, the expected proportions non-conforming are the same for both processes. On the cases that $\mu_A = T_A = 25$ ($\kappa_A = 1.00$) and $\mu_B = T_B = 30$ ($\kappa_B = 1.50$), because $C''_{pk} = 25/(3\sigma_A) = 20/(3\sigma_B) = C''_{pkB}$ implies that process B has smaller variance ($\sigma_B < \sigma_A$), process B is better than process A.

5. Conclusions

In this note, the generalization C''_{pk} index purposed by Pearn and Chen[1] was considered. Based on C''_{pk} index, we established a formula for measuring the PPM non-conformities for given ratios of the two-side tolerances. The validity of the established formula was also proved. The upper bounds on NCPPM and true NCPPM for various values of process mean and target were presented graphically. For practice and convenience, we tabulated the upper bounds on NCPPM for various C''_{pk} index values and given ratios of the two-side tolerances.

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