ELSEVIER

0965-9978(95)00058-5

Advances in Engineering Software **24** (1995) 57–63
Elsevier Science Limited
Printed in Great Britain.

The Trefftz method as an integral equation

S. C. Huang

National Chiao Tung University, Hsinchu, Taiwan

&

R. P. Shaw

S.U.N.Y. at Buffalo, Buffalo, New York, USA

The Trefftz method may be described in terms of an integral equation formulation based on eigenfunction expansions rather than the usual element (BEM) solution method. Using an 'Embedding Integral Equation' approach, the Trefftz method may not only be seen to be related to integral equation methods but it is also seen to be amenable to partitioning, allowing more rapid convergence.

INTRODUCTION

The Trefftz method, e.g. Herrera,[1] is well known as an eigenfunction expansion approach to the solution of partial differential equations. In a sense, it may be considered as the generalization of separation of variables, e.g. Wylie & Barrett,[2] just as the boundary integral equation approach, e.g. Cruse & Rizzo,[3] may be considered as the generalization of the classical Green's function approach, e.g. Morse & Feshbach.[4] In the separation of variables, the basic set of eigenfunctions must be particular not only to a given equation but also to a specific geometry and type of boundary condition. The Trefftz approach relaxes this to a basic set of eigenfunctions which merely satisfies the governing equation, but at the price of leading to a coupled system for the unknown coefficients when the particular boundary conditions given are satisfied in some manner on the given geometry. Similarly, 'THE' Green's function as used in the classical Green's theorem approach is also specific to a given equation, geometry and type of boundary condition. The boundary integral equation approach is based on the use of 'A' Green's function which satisfies the governing equation but is unconnected from the geometry and type of boundary condition, but at the price of leading to an integral equation rather than a quadrature for the solution. While element expansions are the usual method for solving BIE's, there is a substantial literature which is based on eigenfunction expansion methods, primarily on wave scattering problems, e.g. Varadan & Varadan.[5] The purpose of this paper is to review some integral approaches, as discussed in some previous work, and show their connection to the Trefftz method as well as showing how the Trefftz method may be improved from using a 'different point of view'.

THE TREFFTZ FORMULATION

Consider a problem governed by a linear second order elliptic partial differential equation in some domain, $V(\mathbf{r})$, bounded by a surface, $S(\mathbf{r})$, with no exterior forcing term as an example. A separation of variables solution to such problems for a specific geometry and boundary condition would depend on whether or not the governing equation separated in a coordinate system whose surfaces of constant independent variable matched the boundaries of that particular geometry, $S(\mathbf{r})$, on whether the boundary conditions were appropriate, e.g. homogeneous in one direction, and whether the equation itself separated in this coordinate system in the domain $V(\mathbf{r})$. For arbitrary geometries, this will generally not be the case. Nevertheless an eigenfunction expansion approach can be used. If the dependent variable is assumed to have an expansion in terms of a basic set of eigenfunctions, each of which satisfies the governing Laplace equation, i.e.

$$U(\mathbf{r}) = \sum_{j=0}^{\infty} \mathbf{a}_j \Phi_j(\mathbf{r}); \qquad \text{in } V(\mathbf{r}) \tag{1}$$

the coefficients, $\mathbf{a}_j$, may be found for an approximating finite sum going from $j = 0$ to $j = N$ by the solution of

an $N \times N$ set of simultaneous algebraic equations obtained by forcing eqn (1) to satisfy the given boundary conditions on the given boundary, $S(\mathbf{r})$, e.g. by collocation at N specific points or by a Galerkin method using, for example, the $\Phi_j(\mathbf{r})$ as weights. Neumann conditions require the derivative of eqn (1) but this present no difficulty since this series is to hold for all of the original domain. The main change from the separation of variables approach is that these equations are now completely coupled and a change in the upper limit N changes all of the coefficients whereas the separation of variables approach determined the coefficients one by one independently of the upper limit N, but only for specific geometries. For example, a Dirichlet problem with $U(\mathbf{r}_b) = F(\mathbf{r}_b)$ with $\mathbf{r}_b$ on the boundary, would have, in the collocation approach,

$$F(\mathbf{r}_b)_i = \sum_{j=0}^{N} \mathbf{a}_j \Phi(\mathbf{r}_b)_i; \qquad i = 0, \ldots, N; \qquad \text{on } S(\mathbf{r}) \tag{2}$$

as the system of linear algebraic equations to be solved.

THE EMBEDDING INTEGRAL EQUATION FORMULATION

Consider the original volume V extended to include a new volume V' which terminates at some convenient surface, S_e, which is called the embedding surface, e.g. Fig. 1. The original boundary surface, S, is now 'embedded' in this extended volume and is therefore called the embedded surface. A distribution of sources and/or doublets will be placed on the embedding surface which will be chosen for convenience, e.g. to be a sphere in three-dimensional problems or a circle in two-dimensional problems, and the strength of this distribution will be determined by satisfying the original boundary conditions on the embedded surface. Consider as an example, a distribution of sources of strength $\sigma(\mathbf{r}_e)$ on S_e which will be taken as a circle for a two-dimensional illustration. The three-dimensional form follows the same procedures but with somewhat more arithmetic. This provides a solution, $U(\mathbf{r})$, within $V \cup V'$ as

$$U(\mathbf{r}) = \int_{S_e} \sigma(\mathbf{r}_s)\, G(\mathbf{r}, \mathbf{r}_s)\, \mathrm{d}S_e \tag{3}$$

with an outward normal derivative,

$$\partial U(\mathbf{r})/\partial n = q(\mathbf{r}) = \int_{S_e} \sigma(\mathbf{r}_e)\, \partial G(\mathbf{r}, \mathbf{r}_e)/\partial n \, \mathrm{d}S_e \tag{4}$$

EMBEDDING EIGENFUNCTION EXPANSION SOLUTION — POTENTIAL EXAMPLE

The eigenfunction expansion technique assumes that all of the variables in this problem may be expanded in terms of a convenient set of basic eigenfunctions, e.g. Shaw & Huang.[6] While it would be nice to have the 'correct' set of eigenfunctions, i.e. that corresponding to this differential equation for this specific geometry and boundary conditions, the classical Trefftz approach

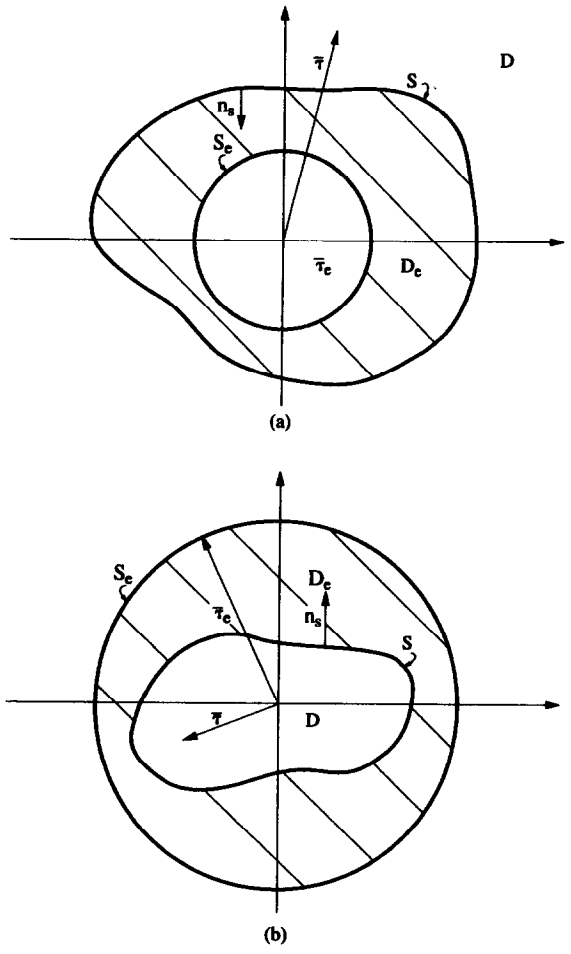

Fig. 1. Embedding surface S_e as a circle of radius $\mathbf{r}_e = b$ 'around' original domain D with added domain D_e; (a) exterior problem and (b) interior problem.

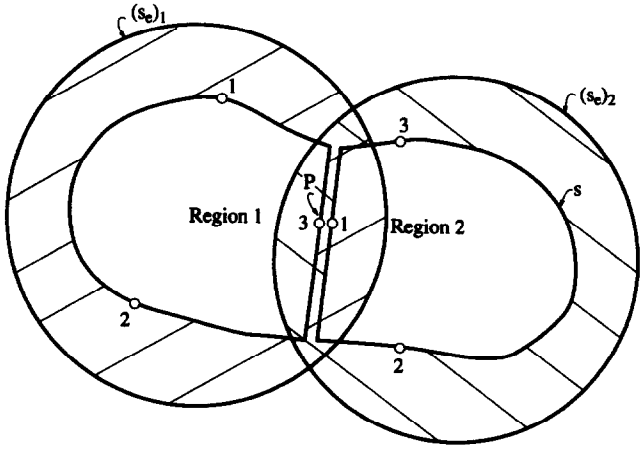

Fig. 2. Partitioned interior domain with interface point P and two embedding surfaces.

described above suggests that any set satisfying just the differential equation will do, e.g. those corresponding to the geometry chosen for the embedding surface, with the restriction that no singularities are introduced in the added volume as seen from the embedding viewpoint. Then, as an illustration, for a two-dimensional interior potential problem of relatively square or box-like shape, those eigenfunctions for a circumscribing circle could be chosen, Fig. 1. For long slender geometries, either those eigenfunctions corresponding to an ellipse of a similar aspect ratio or a partitioning into box-like shapes with several 'embedding' circles could be used, e.g. Fig. 2. For a box-like interior potential problem as discussed first, a circle of radius b will be used as the embedding surface as shown in Fig. 1(b). The eigenfunctions for a potential or Laplace equation are

$$\Phi_p(r,\theta) = r^p\cos(p\theta), r^p\sin(p\theta); \qquad p = 0,1,2,\ldots, \tag{5}$$

$$\sigma(\mathbf{r}_e) = \sum_{m=0}^{\infty} A_m\cos(m\theta_e) + B_m\sin(m\theta_e) \qquad \text{on } r_e = b \tag{6}$$

with the Green's function for $r < r_e = b$ (the radius of the embedding circle, S_e, around the original surface, S) given as, e.g. Morse & Feshbach,[4]

$$G(r,\theta;b,\theta_e) = (1/2\pi) \times \left\{-\ln(b) + \sum_{p=1}^{\infty}(1/p)(r/b)^p\cos(p\,|\,\theta-\theta_e\,|)\right\} \tag{7}$$

Then the solution is given by

$$\begin{aligned} U(r,\theta) &= (1/2\pi)\int_{\theta_e=0}^{2\pi}\sum_{m=0}^{\infty}\{A_m\cos(m\theta_e) + B_m\sin(m\theta_e)\} \\ &\quad\times\left\{-\ln(b) + \sum_{p=1}^{\infty}(1/p)(r/b)^p(\cos(p\theta)\cos(p\theta_e) \right. \\ &\quad\left. + \sin(p\theta)\sin(p\theta_e))\,b\,\mathrm{d}\theta_e\right\} \\ &= \alpha_0 + \sum_{m=1}^{\infty}(1/m)\{\alpha_m r^m\cos(m\theta) + \beta_m r^m\sin(m\theta)\} \end{aligned} \tag{8}$$

where

$$\begin{aligned} \alpha_0 &= -b\ln(b)A_0 \\ \alpha_m &= (1/2)A_m/b^{m-1}; \qquad m = 0,1,2,\ldots \\ \beta_m &= (1/2)B_m/b^{m-1}; \qquad m = 1,2,\ldots \end{aligned} \tag{9}$$

A similar expression is obtained for the normal derivative of U which will be required for Neumann boundary conditions on S,

$$\begin{aligned} \partial U(r,\theta)/\partial n &= (1/2\pi)\int_{\theta_e=0}^{2P}\sum_{m=0}^{\infty}\{A_m\cos(m\theta_e) \\ &\quad + B_m\sin(m\theta_e)\}\partial\{-\ln(b) + \sum_{p=1}^{\infty}(1/p)(r/b)^p \\ &\quad\times(\cos(p\theta)\cos(p\theta_e) \\ &\quad + \sin(p\theta)\sin(p\theta_e))\,b\,\mathrm{d}\theta_e\}/\partial n = (1/2) \\ &\quad\times\sum_{m=1}^{\infty}\{A_m[(r/b)^{m-1}\cos(m\theta)\,\partial r/\partial n - (r^m/b^{m-1}) \\ &\quad\times\sin(m\theta)\,\partial\theta/\partial n]\} + B_m[(r/b)^{m-1}\sin(m\theta)\,\partial r/\partial n \\ &\quad + (r^m/b^{m-1})\cos(m\theta)\,\partial\theta/\partial n] = \sum_{m=1}^{i}\{\alpha_m r^{m-1} \\ &\quad\times\cos(\lambda+m\theta) + \beta_m r^{m-1}\sin(\lambda+m\theta)\} \end{aligned} \tag{10}$$

where λ is the angle between the r direction and the unit normal vector to the surface with counterclockwise as positive, i.e.

$$\cos(\lambda) = \partial r/\partial n; \qquad \sin(\lambda) = (1/r)\,\partial\theta/\partial n \tag{11}$$

Details of this derivation and the corresponding exterior problem derivation are given in the thesis by Huang.[7] Note that the embedding surface radius b does not appear explicitly in the coefficients α and β as they are used in eqns (8) and (10) to determine U and $\partial U/\partial n$, i.e. the actual size of the embedding surface is not relevant to the solution procedure. It does appear in the expression for σ through the coefficients A_m and B_m but this is not used directly. In fact, this shows that the present procedure is simply an alternative derivation of the Trefftz method. However, this approach is still of conceptual value especially when partitioning is discussed.

NUMERICAL PROCEDURE FOR POTENTIAL EXAMPLE

One of the simplest methods for the solution of the above formulation is that of collocation. Select the first K coefficients of α_m and the first $(N-K)$ coefficients of β_m where $K = N/2$ or $(N+1)/2$ for even and odd N respectively. This implies an approximate solution for U to K terms in the eigenfunction expansion. Equations (8) and (10) will be forced to satisfy the given boundary conditions at N collocation points on the original surface, S. In general,

$$\begin{aligned} U(\mathbf{r}_i) &= U_i = \epsilon_j G_{ij}; \qquad i,j = 1,2,\ldots,N \\ \partial U(\mathbf{r}_i)/\partial n &= q_i = \epsilon_j H_{ij}; \qquad i,j = 1,2,\ldots,N \end{aligned} \tag{12}$$

where α_j is represented by the first K terms of ϵ_j and β_j is represented by the last $(N-K)$ terms. The coefficient

matrices G_{ij} and H_{ij} are given by

$$G_{ij} = 2; \quad j = 1$$
$$\times \cos([j-1]\theta)(r_i)^{j-1}(j-1);$$
$$j = 2,3,\ldots,K$$
$$\times \sin([j-K]\theta)(r_i)^{j-k}(j-K);$$
$$j = K+1,\ldots,N$$
$$H_{ij} = 0; \quad j = 1$$
$$\times \cos(\lambda + [j-1]\theta)(r_i)^{j-2}; \quad j = 2,3,\ldots,K$$
$$\times \sin(\lambda + [j-K]\theta)(r_i)^{j-k-1};$$
$$j = K+1,\ldots,N \qquad (13)$$

unless the problem has pure Neumann boundary conditions in which case α_0 will not be present and U is determined only to within an arbitrary reference level. The simplest way to avoid this difficulty is to pin U down to a specific value by replacing the Neumann boundary condition at one of the collocation points by a reference level for U, thereby making the problem one with mixed boundary conditions. Nothing is lost in this procedure since this particular value of the normal derivative of U is completely specified by the condition that the net total boundary integral, or sum of collocation values times segment lengths, around the entire original surface must be zero.

Using B_i as the known boundary values of either U_i or q_i and C_{ij} as either the corresponding G or H, the equation on ϵ_j is simply

$$C_{ij}\epsilon_j = B_i; \quad i = 1,2,\ldots,N \qquad (14)$$

Once ϵ_j is known, i.e. all of the α's and β's within this level of approximation, the remainder of the field solution is found directly from eqns (8) and (10).

EXTENSION TO PARTITIONED GEOMETRIES

When the original geometry is irregular or has a large aspect ratio such that the eigenfunctions for an embedding circle might appear inappropriate or distances from the embedding elements to the collocation points may be too large, the possibility of partitioning the original geometry into subdomains that are better suited geometrically to this embedding geometry arises. So of course does the possibility of choosing another embedding geometry such as an ellipse, etc., but this choice loses the simplicity of the present approach. In the partitioning approach, the original domain is divided into subdomains that are closer to the circular embedding geometry, e.g. box-like sections. This introduces a small number of interior points which now become boundary points on the partitioned geometries, e.g. Fig. 2. At these 'new' points, rather than one boundary condition there will be two continuity conditions, one on the temperature and one on the flux or normal derivative, which still provides a complete system of algebraic equations on the eigenfunction expansion coefficients. Again details are given by Huang[7] for both the eigenfunction expansion and the element approaches.

NUMERICAL POTENTIAL PROBLEM EXAMPLES

Several numerical examples were given in Shaw, Huang & Zhao[8] based on the dissertation of Huang,[7] e.g. an L shaped region of equal legs made up of three squares. Boundary values corresponding to simple specified nonsingular temperature fields, e.g. $y+3$, $xy+2x+3y$ and x^2-y^2, were used with boundary conditions of the Dirichlet, Neumann and mixed types, chosen from these analytical solutions. Forty collocation points were spaced equally around the perimeter of the original surface. Numerical results were checked against the exact solution. The L shaped region, shown in Fig. 3 with one embedding surface and 32 collocation points, gave results accurate to which errors of the order of 10^{-8}% or better for boundary points and points at the centers of the three small squares that made up this domain. When three subregions were used as shown in Fig. 4, increasing the number of collocation points to 40 through the introduction of interface points but keeping the same spacing around the boundary, this error was reduced to less than 10^{-12}%, an improvement by several orders of magnitude. This was of course not a stringent test since the reentrant corner did not have any singular behavior, except in geometry.

The next test considered a discontinuous prescribed

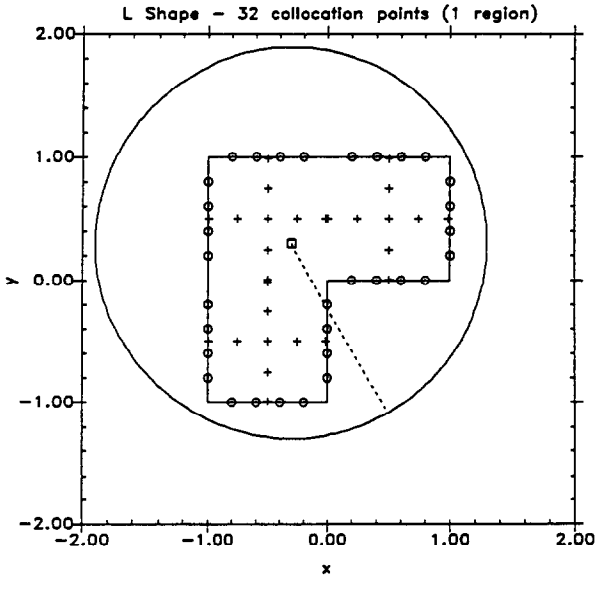

Fig. 3. L shaped domain with circular embedding surface and 32 collocation points.

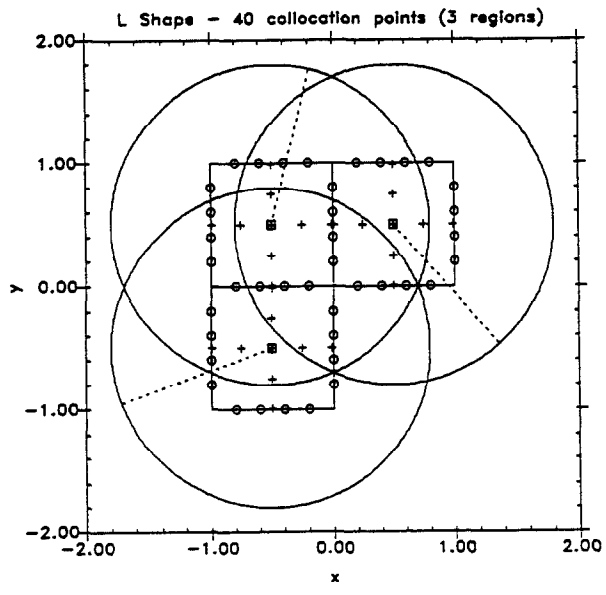

Fig. 4. L shaped domain with three embedding circles and 40 collocation points.

temperature field which has a singularity in flux for a problem with a known separation of variables solution. Figure 5 shows results for a unit circle with a temperature discontinuity, $T(1,\theta) = 1$ for $0 < \theta < \pi$ and $= 0$ for $\pi < \theta < 2\pi$. Numerical results were found with 18, 30 and 60 collocation points for $r = 0{\cdot}999$ and a range of values of θ. While the singularity in radial flux is reasonably well represented, the solution was not as accurate as those above. This might be improved by using a matching method involving the entire boundary such as a Galerkin method. Of course, as the solution point was moved further from the boundary, the accuracy of the solution improved considerably.

Finally a test problem widely used as a benchmark

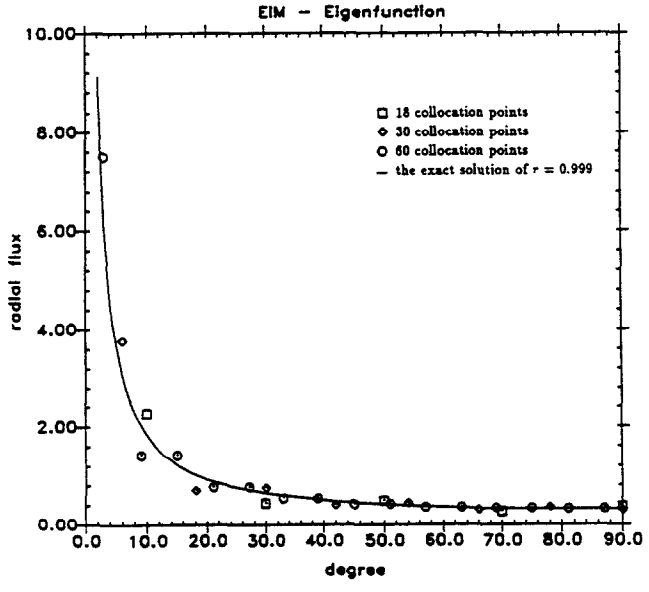

Fig. 5. Radial flux at $r = 0{\cdot}999$ for unit circle with discontinuous prescribed boundary temperature of 1 for $0 < \theta < \pi$ and 0 for $\pi < \theta < 2\pi$ by an embedded eigenfunction expansion.

Table 1

X	Y	Symm	Eigenfunction	BEM
0·5	0·5	0·2881	0·2863	0·2889
−0·5	0·5	0·8018	0·8020	0·7986
−0·5	−0·5	0·9001	0·9117	0·9119

was considered. This is an L shaped region $T = 0$ on the right face, $T = 1$ on the left face and $q = 0$ on all other faces including the two sides of the reentrant corner. Calculations using a single embedding circle gave very poor results, indicative of the difficulty that the Trefftz method has with singular behavior. The reentrant corner in this case does have a steep temperature gradient in the benchmark numerical solution to which these results were compared. When a three region partition was introduced, it gave results which were in very good agreement with a numerical solution by Symm,[9] e.g. within about 2%, even at the reentrant corner. Table 1 compares these numerical results at three interior field points corresponding to the centers of the three squares which make up the L shaped domain with 48 collocation points and three embedding circles of radius 1·5 of the subregion square sides to results from a standard 64 element BEM program using 'constant' shape functions, i.e. mean values, and show excellent agreement. Table 2 gives the solutions, using three regions, for the temperature along the two sides of the reentrant corner where a large temperature gradient is found; again the agreement with a 64 element BEM program is very good, as it was for the other boundary points.

EMBEDDING EIGENFUNCTION EXPANSION SOLUTION — HELMHOLTZ EXAMPLE

The major difference between the potential example and a Helmholtz problem lies in the form of the basic set of eigenfunctions used. These, in 2D, differ only in the radial dependence with the angular dependence still essentially a Fourier series. For an exterior problem, i.e. excluding $r = 0$, as occurs in wave scattering, the roles of r and r_e are reversed in that r would be larger than r_e, e.g. Shaw & Huang.[10] Furthermore, the dependent variable must vanish as r approaches infinity, i.e. the scattered or radiated field must be used, to avoid contributions from a closing circle at infinity in the

Table 2

X	Y	Emb. Eig.	BEM
0	−0·8	0·8954	0·9000
0	−0·6	0·8812	0·8822
0	−0·4	0·8531	0·8502
0	−0·2	0·8047	0·7977
0·2	0	0·4784	0·4768
0·4	0	0·3534	0·3507
0·6	0	0·2340	0·2311
0·8	0	0·1170	0·1141

original Green's theorem. Then the basic set would be

$$\Phi_p(r,\theta) = J_p(kr_e)H_p^{(1)}(kr)\cos(p\theta), \times J_p(kr_e)H_p^{(1)}(kr)\sin(p\theta); \quad p = 0,1,2,\ldots, \tag{15}$$

where $r > r_e = a$. Thus the form for the source strength on the embedding boundary of constant radius is unchanged from the potential problem, i.e.

$$\sigma(\mathbf{r}_e) = \sum_{m=0}^{\infty} A_m \cos(m\theta_e) + B_m \sin(m\theta_e) \qquad \text{on } r_e = a \tag{16}$$

with the Green's function for $r > r_e = a$ (the radius of the embedding circle, S_e, 'inside' the original surface, S, given as, e.g. Morse & Feshbach[4])

$$G(r,\theta;a,\theta_e) = (i/4)\sum_{p=0}^{\infty} \epsilon_p \cos(p\,|\,\theta-\theta_e\,|)J_p(ka)H_p^{(1)}(kr) \tag{17}$$

Here ϵ_p is the Neumann function and is 1 if $p = 0$ and 2 otherwise. Then the solution is given by

$$U(r,\theta) = (i/4)\int_{\theta_e=0}^{2\pi}\sum_{m=0}^{\infty}\{A_m\cos(m\theta_e) + B_m\sin(m\theta_e)\} \times \left\{\sum_{p=0}^{\infty}\epsilon_p\cos(p\,|\,\theta-\theta_e\,|)J_p(ka)H_p^{(1)}(kr)\,b\,\mathrm{d}\theta_e\right\} = (i\pi/2)\sum_{m=0}^{\infty}\{\alpha_m\cos(m\theta) + \beta_m\sin(m\theta)\}H_p^{(1)}(kr) \tag{18}$$

where α_m and β_m are combinations of A_m, B_m and $J_m(ka)$. The bothersome 'fictitious interior eigenvalue' problem, which arises at wavenumbers such that $J_m(ka) = 0$, has been circumvented by this approach. This is not surprising since the Trefftz method, which is directly related to this approach, does not suffer from this difficulty. An example of a plane wave scattering by a circular cylinder was given by Shaw & Huang,[10] where no breakdown at these eigenvalues occurred, as occurred in the BEM method and the T Matrix method. This approach worked equally well on interior problems, e.g. obtaining eigenfrequencies for a closed domain, e.g. Shaw & Huang.[11]

CONCLUSION

The embedding integral method is seen to lead to a form of the Trefftz method when eigenfunction expansions are used for the approximate solution of this formulation. One of the major drawbacks to the Trefftz method in general (and its solution by collocation in particular) in its application to irregularly shaped geometries lies in the fact that a large number of terms in the expansion are required for accuracy in the geometrical representation leading then to a possible large accumulation of errors and 'high frequency oscillations' between collocation points, if the procedure converges at all. When this is viewed as an embedding integral however, the logical resolution of this difficulty lies in partitioning which provides much better results at a modest increase in the number of solution points, i.e. due to the added interior points, through a better match of the partitioned geometry to the original eigenfunction geometry. Such partitioning is particularly well suited to parallel processing since the coefficients for each partitioned domain may be solved simultaneously down separate pipelines. This partitioning has the added advantage of reducing the added 'embedding' volume which reduces the possibility of singularities occurring in the added region. The question of added singularities occurring outside of the original domain but within the extended domain has been examined to some extent by considering a potential problem on a square for boundary conditions arising from a singularity near to but outside of the original domain. The resulting coefficients in the eigenfunction expansion showed rapid oscillation at higher modes which served as a warning that the method was inadequate. In principle, the eigenfunction expansion (Trefftz) method should still converge since the actual boundary conditions were bounded, but the embedding approach indicated where the difficulty lay, i.e. the singularity was included in the embedding domain. Attempts to remove this singularity by subtracting out a known solution caused by a similar nearby singularity, i.e. a 'guess' as to where the original lay, met with only limited success.

REFERENCES

1. Herrera, I. Trefftz Method. In *Topics in Boundary Element Research*. ed. C. A. Brebbia. Chap. 10. Springer/CMP, New York, 1984.
2. Wylie, C. R. & Barrett, L. C. *Advanced Engineering Mathematics*. 5th edn, McGraw-Hill, New York, 1982.
3. Cruse, T. A. & Rizzo, F. J. (eds), *Boundary-Integral Equation Method: Computational Applications in Applied Mechanics*. AMD, Vol. 11, A.S.M.E., New York, 1975.
4. Morse, P. & Feshbach, H. *Methods of Theoretical Physics*. Vols. 1 and 2, McGraw Hill, New York, 1953.
5. Varadan, V. K. & Varadan, V. V. (eds), *Acoustic Electromagnetic and Elastic Wave Scattering — Focus on the T Matrix Approach*. Pergamon Press, New York, 1980.
6. Shaw, R. P. & Huang, S. C. *Element and Eigenfunction Forms for an Embedding Integral Equation Approach to Acoustics*. BEM 11, ed. C. A. Brebbia & J. Connor. Springer/CMP, New York, pp. 319–40, 1989.
7. Huang, S.-C. The embedding integral equation method for potential and acoustic problems. PhD. Thesis, Dept. of Civil Eng., S.U.N.Y. at Buffalo, 1989.
8. Shaw, R. P., Huang, S. C. & Zhao, C. X. The embedding

integral and the Trefftz method for potential problems. *Engng Anal.*, 1992, **9**, 83–90.
9. Symm, G. T. Treatment of singularities in the solution of Laplace's equation by an integral equation method. NPL Report NAC31, National Physical Laboratory, Teddington, U.K., 1973. Also see Benchmark Test 1 in *Boundary Elements* — Abstracts and Newsletter, Vol. 1, No. 2, Int. Soc. for Boundary Elements, Mar. 1990.
10. Shaw, R. P. & Huang, S. C. The embedding integral equation method with application to acoustic problems. *2nd. IMACS Int. Symp. on Computational Acoustics*, Princeton, New Jersey. ed. D. Lee, A. Cakmak & R. Vichnevetsky. Vol. 1, pp. 83–99. North Holland/Elsevier, New York, 1990.
11. Shaw, R. P. & Huang, S. C. Helmholtz eigenvalues by embedding methods. *Wave Motion*, 1989, **11**, 367–70.