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A note on on-shell recursion relation of string amplitudes

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Abstract: In the application of on-shell recursion relation to string amplitudes, one challenge is the sum over infinite intermediate on-shell string states. In this note, we show how to sum these infinite states explicitly by including unphysical states to make complete Fock space.

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1 Introduction

Whilst its application requires merely the knowledge of analytic structure of the scattering amplitude of interest, the on-shell recursion relation (BCFW) [1, 2] has achieved tremendous success in calculations of scattering amplitudes, a task would very often seem practically impossible using conventional methods even when there are only a few of external particles involving gluons or gravitons.[1] In contrast to perturbative off-shell formulation, the on-shell recursion relation uses fewer-point *physical* amplitude as building blocks,

$$A(123\ldots n) = \sum_{\text{poles}} A_L(\hat{1}2\ldots,\hat{P}^h)\,\frac{1}{P^2}\,A_R(-\hat{P}^{-h},\ldots n), \tag{1.1}$$

thereby avoiding large amount of unnecessary cancelation in intermediate step of computations. An important point of eq. (1.1) is the sum over all possible physical poles and allowed helicity configurations. Generalization of on-shell relation to string amplitudes was pioneered in [4, 5] and [6] and further elaborated in [7–9]. Recent applications at 4-point and to eikonal Regge limit can be found in [10] and [11] respectively. The validity of on-shell recursion relation in string theory context was argued both from the better convergent UV behavior generically observed in string amplitudes and from analyzing explicit expressions of string amplitudes.

However, when applying on-shell recursion relation to string amplitudes, we are facing the problem of summing over infinite number of physical states in (1.1). Although it could be done in principle, there is no efficient algorithm doing so. For scattering amplitudes of tachyons, based on known analytic expressions, it has been conjectured in [7] that amplitudes can be effectively reduced to factorization of two lower-point tachyon-like sub-amplitudes.

In this paper, we provide an algorithm to do the sum over infinity number of physical states in (1.1). Applying our algorithm to tachyon amplitudes, we see that the sum over physical states at each mass level predicted by open string theory does produce the conjectured scalar-behaved residue observed in [6]. In contrast with the experiences with amplitude calculations in field theory, the key of our algorithm is to enlarge the sum over intermediate physical states to over intermediate complete Fock space states. The zero contributions of extra states are guaranteed by no-ghost theorem (i.e., the Ward-like identity in string theory).[2]

The structure of this paper is organized as the following: in section 2, we present a very brief review of BCFW on-shell recursion relation of generic field theory amplitudes. In section 3 we start with the familiar 4-point Veneziano amplitude as an example and demonstrate how the tachyonic recursion relation can be understood from carrying out sum directly. Section 4 consists of analysis on 5-point string amplitudes, in which case the pole structure becomes much more complicated. A discussion on pole structure of generic n-point amplitude is presented in section 5. In section 6 we consider higher-spin scatterings and demonstrate that generically the mathematical connection between BCFW and tachyonic recursion descriptions can be found in the generating function for Stirling number of the first kind outlined in appendix A, while the relation between on-shell condition and decoupling of unphysical states is discussed in appendix B.

[1] A review of the principles of BCFW on-shell recursion relation as well as its some applications can be found in [3].

[2] We have summarized the no-ghost theorem in appendix B for reference.

2 A brief review of BCFW on-shell recursion relation

In this section we provide a short review of on-shell recursion relation [1, 2]. Derivation of BCFW on-shell recursion relation starts from taking analytic continuation of amplitudes. An amplitude can be regarded as function of complex momenta defined by standard Feynman rules. When the momenta of a pair of particle lines manually chosen are shifted in a complex q-direction,

$$\widehat{k}_1(z) = k_1 + zq, \qquad \widehat{k}_n(z) = k_n - zq, \tag{2.1}$$

with $q^2 = q \cdot k_a = q \cdot k_n = 0$, the shifted amplitude $A(z)$ defines a complex function. While the explicit analytic structure of amplitude is determined by individual theory and does not concern us here, $A(z)$ thus defined will contain simple poles produced by propagators, which is the consequence of local interaction and the null condition of q. From Cauchy's Theorem, integrating over a contour large enough to enclose all finite poles yields

$$\oint dz \, \frac{A(z)}{z} = A(0) + \sum_{\text{poles } \alpha} Res_{z=z_\alpha}, \tag{2.2}$$

where an unshifted amplitude $A(0)$ contributes as residue at $z = 0$ and residues from other finite poles assume the form as cut-amplitudes, $Res_{z_\alpha} = -A(z_\alpha)\frac{1}{P^2}A_R(z_\alpha)$. In various theories shifted amplitudes posses convergent large-z asymptotic behavior and the integral (2.2) vanish, we are then entitled to write down the BCFW recursion relation[3]

$$A_n = \sum_{\text{poles}} \sum_{\text{physical states}} A_L(\ldots, P(z_\alpha)) \frac{2}{P^2 + M^2} A_R(-P(z_\alpha), \ldots), \tag{2.3}$$

where the first sum is over all finite simple poles z_α of z, and the second sum is over all physical states at the given simple pole z_a.

3 Example I: BCFW of 4-tachyon amplitude in bosonic open string theory

As was demonstrated in the previous section, a key feature making BCFW on-shell recursion relation possible is that in perturbative field theory, at tree-level amplitude can often be determined entirely from its poles and related residues. The locations of poles are determined by propagators while the residues, by factorization properties. Same analytic structure holds for string theory, with one complication: there is an infinite number of poles and related residues. As an consequence, there are several expressions for amplitudes, for example, the Veneziano formula assumes the form of a worldsheet integral, making the pole structure obscured. In [6] through binomial expansions of these integral formulas, the pole structure can be made manifest. In this section, we will use four-point tachyon amplitude as an example to demonstrate our idea and method.

[3]We have assumed the boundary contribution to be zero. If it is no zero, we need to modify recursion relation, see [12].

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3.1 Pole structure extraction

Consider the four tachyon scattering amplitude in bosonic open string theory, given by Koba-Nielson formula as

$$A(1234) = \int_0^1 dz_2\, (1-z_2)^{k_3\cdot k_2}\, z_2^{k_2\cdot k_1}, \tag{3.1}$$

where we have used the conformal symmetry to fix $z_1 = 0$, $z_3 = 1$ and $z_4 = +\infty$. For arbitrary complex power w we have following binomial expansion

$$(x-y)^w = \sum_{a=0}^{\infty} \begin{pmatrix} w \\ a \end{pmatrix} x^{w-a} y^a \tag{3.2}$$

where coefficient $\begin{pmatrix} w \\ a \end{pmatrix}$ is defined as

$$\begin{pmatrix} w \\ a \end{pmatrix} = \frac{w(w-1)(w-2)\dots(w-a+1)}{a!} \tag{3.3}$$

Applying (3.2) to $(1-z_2)^{k_3\cdot k_2}$ and collecting relative terms we have

$$A(1234) = \sum_{a=0}^{\infty} \begin{pmatrix} k_3\cdot k_2 \\ a \end{pmatrix} (-)^a \int dz_2\, z_2^{\frac{1}{2}(k_1+k_2)^2+a-2} \tag{3.4}$$

where we have used the mass-shell condition for tachyon that $k_1^2 = k_2^2 = -M^2 = +2$.[4] The worldsheet integration can be explicitly carried out, producing an s-channel propagator.[5] Inserting it back, we obtain

$$A(1234) = \sum_{a=0}^{\infty} \begin{pmatrix} k_3\cdot k_2 \\ a \end{pmatrix} (-)^a \frac{2}{(k_1+k_2)^2+2(a-1)} \tag{3.5}$$

3.2 Interpreting pole expansion formula from BCFW perspective

Having derived an explicit analytic expression (3.5) for tree-level four tachyon scattering amplitude, it is then interesting to see if the result can be understood in the language of BCFW on-shell recursion relation. We choose the shifted pair to be $(1,4)$ to be consistent with the manifest s-channel expansion. Assuming there is no boundary contribution for on-shell recursion relation, equation (3.5) should be given by on-shell recursion relation (2.3):

$$A_n = \sum_{\text{poles}} \sum_{\text{physical}} A_L(\dots, P(z_\alpha)) \frac{2}{P^2+M^2} A_R(-P(z_\alpha), \dots) \tag{3.6}$$

[4] We have used the convention $\alpha' = 1/2$, so the mass of bosonic open string state is $M^2 = -2 + 2\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n$

[5] In this expansion, only s-channel is manifest. However, by string duality, t-channel is also contained.

In denominator we see infinitely many single poles occurs at

$$z_a = \frac{(k_1+k_2)^2 + 2(a-1)}{-2q \cdot (k_1+k_2)}, \qquad a = 0, 1, \dots \tag{3.7}$$

where $P = k_1 + k_2$ and the mass square $M_a^2 = 2(a-1)$ for every integer a is precisely the mass spectrum prescribed by bosonic open string theory. In addition, matching residues of (3.5) with (3.6) indicates that, at each level a, there should be a number of physical states, collectively yielding

$$\sum_{\text{states} \quad h} A_L(1, 2, P_a^h(z_a)) A_R(-P_a^{\widetilde{h}}(z_a), 3, 4) = (-1)^a \begin{pmatrix} k_3 \cdot k_2 \\ a \end{pmatrix}. \tag{3.8}$$

Thus to understand (3.5) from BCFW recursion relation (2.3), we need to be able to interpret the scalar-behaved residue (3.8) as sum over physical states at each fixed level a.

3.3 Summing over physical states

Before undertaking a state-by-state calculation of residues over bosonic string spectrum, let us make a slight detour and consider how the analytic structure featuring intermediate states fits into the picture of BCFW on-shell recursion relation in quantum field theory. Although in Feynman rules scalar, fermion and gauge boson each are assigned with a propagator in distinct representations, we note that the propagator appearing in BCFW recursion relation (3.6) is always scalar-like. The reason is following. For example, if the intermediate particles are massless fermions, BCFW recursion relation reads

$$A \sim \sum_{h=\pm} A_L(\sigma_L, P^h) A_R(-P^{-h}, \sigma_R). \tag{3.9}$$

We can rewrite the on-shell sub-amplitude $A_L(\sigma_L, P^h) = \sum_{a=1,2} \widetilde{A}_L(\sigma_L, P^h)^a u^h(P)_a$, i.e., we have decomposed the on-shell amplitude into two parts: wave function for external on-shell particle P and the rest. Similar decomposition can be done for $A_R(-P^{-h}, \sigma_R)$. Thus the sum over physical states becomes

$$A \sim \widetilde{A}_L(\sigma_L, P^h) \left(\sum_h u^s(P) \overline{u}^s(p) \right) \widetilde{A}_R(-P^{-h}, \sigma_R) \sim \widetilde{A}_L(\sigma_L, P^h) \, (\gamma \cdot P) \, \widetilde{A}_R(-P^{-h}, \sigma_R) \tag{3.10}$$

where in the middle, $\gamma \cdot P$ is exactly the factor needed to translate scalar propagator into the familiar fermion propagator.

A similar mechanism supports the translation from scalar propagator into gauge boson propagator when summed over physical states, but with some subtleties. The sum over two transverse physical states for gauge boson is $(\epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+)$ while the familiar Feynman gauge uses $g_{\mu\nu}$. In fact, in 4-dimensions we need four polarization vectors, and

$$g_{\mu\nu} = \epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+ + \epsilon_\mu^L \epsilon_\nu^T + \epsilon_\mu^T \epsilon_\nu^L \tag{3.11}$$

where ϵ_μ^L and ϵ_μ^T are longitude and time-like polarization vector [13]. The reason that these two sums (Namely a summation over two physical states and another over all four states) give same answer depends crucially on Ward Identity of gauge theory, i.e., if all $(n-1)$ particles are physical polarized while the n-th particle is longitude (i.e., proportional to k_μ), the amplitude is zero. Thus we have

$$
\begin{aligned}
\sum_{\text{all states}} A_L(\sigma_L, P^h) A_R(-P^{\widetilde{h}}, \sigma_R) &\sim \widetilde{A}_L^\mu(\sigma_L, P) g_{\mu\nu} \widetilde{A}_R^\nu(-P, \sigma_R) \\
&\sim \widetilde{A}_L^\mu(\sigma_L, P) \left(\epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+ + \epsilon_\mu^L \epsilon_\nu^T + \epsilon_\mu^T \epsilon_\nu^L\right) \widetilde{A}_R^\nu(-P, \sigma_R) \\
&\sim \widetilde{A}_L^\mu(\sigma_L, P) \left(\epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+\right) \widetilde{A}_R^\nu(-P, \sigma_R) \\
&= \sum_{\text{physical states}} A_L(\sigma_L, P^h) A_R(-P^{-h}, \sigma_R)
\end{aligned} \tag{3.12}
$$

Having understood the effect of summing over physical states from quantum field theory, let us return to the problem of interpreting scalar-behaved residue (3.5) as sum over physical states. In old covariant quantization framework, the Fock space in bosonic open string theory is constructed by linear combinations of states obtained from acting creation modes successively on ground state

$$
\alpha_{-n_1}^{\mu_1} \alpha_{-n_2}^{\mu_2} \dots \alpha_{-n_n}^{\mu_n} \left|0; k\right\rangle . \tag{3.13}
$$

Generically, a Fock state can carry $N_{\mu,1}$-multiple of α_{-1}^μ mode operators[6] and $N_{\mu,2}$-multiple of α_{-2}^μ mode and so on. In the following discussions we use the set of numbers $\{N_{\mu,n}\}$ as label of normalized Fock state

$$
\left|\{N_{\mu,n}\},\, k\right\rangle = \left[\prod_{\mu=0}^{D-1} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^\mu)^{N_{\mu,n}}}{\sqrt{n^{N_{\mu,n}} N_{\mu,n}!}}\right] \left|0,\, k\right\rangle . \tag{3.14}
$$

Physical states however, in addition must satisfy Virasoro constraints $(L_0 - 1)\left|\phi\right\rangle = 0$, $L_{m>0}\left|\phi\right\rangle = 0$ and constitute only a subset in Fock space. An immediate consequence is that physical states are automatically on the mass-shell, $-k^2 = M^2 = 2(N-1)$, where N is the level

$$
N = \sum_{\mu=0}^{D-1} \sum_{n=1}^{\infty} n N_{\mu,n} . \tag{3.15}
$$

Note however, for a generic Fock state its center-of-mass momentum k^μ and modes $\{N_{\mu,n}\}$ are considered as independent degrees of freedom and does not *a priori* satisfy mass-shell condition, and yet in a BCFW on-shell recursion relation, Fock states that happen to be the on mass-shell are picked out because as we have seen from (3.7) that only these states contribute to residues.

Now we come to our central point. The prescription given by BCFW on-shell recursion relation is to sum over physical states satisfying on-shell condition plus remaining Virasoro constraints $L_{m>0}\left|\phi\right\rangle = 0$. However, a rather technical difficulty carrying out above prescription in string theory is that it requires the knowledge of physical polarization tensor at arbitrarily high mass level N, which is very hard to write down explicitly. To bypass the

[6]It should be emphasized that α_{-1}^μ and α_{-1}^ν should be considered as different operators when $\mu \neq \nu$.

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problem, inspired by the observation given in [13] for gauge theory (3.11), we can enlarge the sum over physical states to all states in Fock space satisfying on-shell condition. The fact that these two sums are same is guaranteed by the famous "No-Ghost Theorem".[7] With this understanding, we can write

$$\begin{aligned} A_n &= \sum_{\text{poles}}\sum_{\text{physical}} A_L(\ldots,P(z_\alpha))\frac{2}{P^2+M^2}A_R(-P(z_\alpha),\ldots) \\ &= \sum_{\text{poles}}\sum_{\text{Fock}} A_L(\ldots,P(z_\alpha))\frac{2}{P^2+M^2}A_R(-P(z_\alpha),\ldots) \\ &= \sum_{\text{poles}}\sum_{\text{Fock}} (\widetilde{A}_L(\ldots,P(z_\alpha))\cdot\xi_P\frac{2}{P^2+M^2}\widetilde{A}_R(-P(z_\alpha),\ldots)\cdot\xi^*(P) \end{aligned} \tag{3.16}$$

where at the last step we have stripped away the polarization tensor of intermediate state P from on-shell amplitude. Since the sum is taken over whole Fock space, we are free to choose any convenient basis, for example, the one given in (3.14), to perform the sum. Thus if we take pair $(1,n)$ to conduct BCFW-deformation and sum over the polarization tensor of intermediate state, BCFW on-shell relation of a string amplitude reads

$$\begin{aligned} A_n = &\sum_{i=2}^{n-2}\sum_{N=0}^{+\infty}\sum_{\{N_{\mu,n}\}}\left\langle\phi_1(\widehat{k}_1)|V_2(k_2)\ldots V_i(k_i)|\{N_{\mu,n}\},\widehat{P}\right\rangle\frac{2\mathcal{T}_{\{N_{\mu,n}\}}}{(\sum_{t=1}^{i}k_i)^2+2(N-1)} \\ &\left\langle\{N_{\mu,n}\},\widehat{P}|V_{i+1}(k_{i+1})\ldots V_{n-1}(k_{n-1})|\phi_n(\widehat{k}_n)\right\rangle \end{aligned} \tag{3.17}$$

In this formula, the first sum is over the splitting of particles into left and right handed sides while the second sum is over poles fixed by the mass level N. The third sum is over all allowed choice of the set $\{N_{\mu,n}\}$ as long as they satisfy (3.15). The tensor structure $\mathcal{T}_{\{N_{\mu,n}\}}$ is determined by the set $\{N_{\mu,n}\}$. To demonstrate the rule for the tensor structure, we list the tensor structure for first three levels:

- **Level $N=0$:** For the first level, all $N_{\mu,n}=0$ so we have $\mathcal{T}=1$.

- **Level $N=1$:** The choice is $N_{\mu,1}=1$ for $\mu=0,1,\ldots,D-1$, thus we have $\mathcal{T}=g_{\mu\nu}$, i.e., we have

$$\left\langle\phi_1|\ldots V_i\ \alpha^\mu_{-1}|0;P\right\rangle\frac{2g_{\mu\nu}}{P^2+2(N-1)}\left\langle 0;P|\alpha^\nu_{+1}\ V_{i+1}\ldots|\phi_n\right\rangle \tag{3.18}$$

where when we conjugate $\left|\alpha^\mu_{-1}|0;P\right\rangle$ we get $\left\langle 0;P|\alpha^\nu_{+1}\right|$

[7]We have collected some facts of "No-Ghost Theorem" in appendix B.

- **Level $N = 2$:** There are several choices and the structure is given by

$$\sum_{\mu,\nu=0}^{D-1} \left\langle \phi_1 | \dots V_i \, \frac{\alpha^{\mu}_{-2}}{\sqrt{2}} |0;P \right\rangle \frac{2g_{\mu\nu}}{P^2+2(N-1)} \left\langle 0;P| \frac{\alpha^{\nu}_{+2}}{\sqrt{2}} \, V_{i+1} \dots |\phi_n \right\rangle$$
$$+ \sum_{0\le\mu_1<\mu_2\le D-1} \sum_{0\le\nu_1<\nu_2\le D-1} \langle \phi_1 | \dots V_i \, \alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1} |0;P\rangle \frac{2g_{\mu_1\nu_1}g_{\mu_2\nu_2}}{P^2+2(N-1)}$$
$$\langle 0;P|\alpha^{\nu_2}_{+1}\alpha^{\nu_1}_{+1} \, V_{i+1}\dots|\phi_n\rangle$$
$$+ \sum_{\mu,\nu=0}^{D-1} \left\langle \phi_1 | \dots V_i \, \frac{(\alpha^{\mu}_{-1})^2}{\sqrt{2}} |0;P \right\rangle \frac{2(g_{\mu\nu})^2}{P^2+2(N-1)} \left\langle 0;P| \frac{(\alpha^{\nu}_{+1})^2}{\sqrt{2}} \, V_{i+1} \dots |\phi_n \right\rangle \quad (3.19)$$

  where at the second line, to avoid repetition, we must have the ordering $0 \le \mu_1 < \mu_2 \le D-1$.

- **Level $N = 3$:** There are several choices which are given respectively by

$$T_1 = \sum_{\mu,\nu=0}^{D-1} \left\langle \phi_1 | \dots V_i \, \frac{\alpha^{\mu}_{-3}}{\sqrt{3}} |0;P \right\rangle \frac{2g_{\mu\nu}}{P^2+2(N-1)} \left\langle 0;P| \frac{\alpha^{\nu}_{+3}}{\sqrt{3}} \, V_{i+1} \dots |\phi_n \right\rangle$$
$$T_2 = \sum_{\mu_1,\mu_2,\nu_1,\nu_2=0}^{D-1} \left\langle \phi_1 | \dots V_i \, \frac{\alpha^{\mu_1}_{-2}}{\sqrt{2}} \alpha^{\mu_2}_{-1} |0;P \right\rangle \frac{2g_{\mu_1\nu_1}g_{\mu_2\nu_2}}{P^2+2(N-1)}$$
$$\left\langle 0;P|\alpha^{\nu_2}_{+1} \frac{\alpha^{\nu_1}_{+2}}{\sqrt{2}} \, V_{i+1}\dots|\phi_n \right\rangle$$
$$T_3 = \sum_{0\le\mu_1<\mu_2<\mu_3\le D-1} \sum_{0\le\nu_1<\nu_2<\nu_3\le D-1} \langle \phi_1 | \dots V_i \, \alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}\alpha^{\mu_3}_{-1} |0;P\rangle$$
$$\frac{2g_{\mu_1\nu_1}g_{\mu_2\nu_2}g_{\mu_3\nu_3}}{P^2+2(N-1)} \langle 0;P|\alpha^{\nu_3}_{+1}\alpha^{\nu_2}_{+1}\alpha^{\nu_1}_{+1} \, V_{i+1}\dots|\phi_n\rangle$$
$$T_4 = \sum_{\mu_1,\mu_2,\nu_1,\nu_2=0}^{D-1} \left\langle \phi_1 | \dots V_i \, \frac{(\alpha^{\mu_1}_{-1})^2}{\sqrt{2}} (a_{-1})^{\mu_2} |0;P \right\rangle \frac{2(g_{\mu_1\nu_1})^2 g_{\mu_2\nu_2}}{P^2+2(N-1)}$$
$$\left\langle 0;P|(\alpha_{+1})^{\nu_2} \frac{(\alpha^{\nu_1}_{+1})^2}{\sqrt{2}} \, V_{i+1}\dots|\phi_n \right\rangle$$
$$T_5 = \sum_{\mu,\nu=0}^{D-1} \left\langle \phi_1 | \dots V_i \, \frac{(\alpha^{\mu}_{-1})^3}{\sqrt{3!}} |0;P \right\rangle \frac{2(g_{\mu\nu})^3}{P^2+2(N-1)} \left\langle 0;P| \frac{(\alpha^{\nu}_{+1})^3}{\sqrt{3!}} \, V_{i+1} \dots |\phi_n \right\rangle$$

  So we have

$$N = 3 : \quad T_1 + T_2 + T_3 + T_4 + T_5 \quad (3.20)$$

These examples demonstrate the general pattern of tensor structures. However, because when we have several oscillators with same n, there are freedoms with the choice of μ, we need to distinguish if these μ are same or different from each other. This makes the tensor structure a little bit of complicated. This complication can be simplified further.

For example, at the level $N = 2$, we have

$$\begin{aligned}
&\sum_{0\le\mu_1\le\mu_2\le D-1}\ \sum_{0\le\nu_1\le\nu_2\le D-1} \alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}g_{\mu_1\nu_1}g_{\mu_2\nu_2}\alpha^{\nu_2}_{+1}\alpha^{\nu_1}_{+1} \\
&= \frac{1}{2}\sum_{\mu_1\neq\mu_2=0}^{D-1}\ \sum_{\nu_1\neq\nu_2=0}^{D-1} \alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}g_{\mu_1\nu_1}g_{\mu_2\nu_2}\alpha^{\nu_2}_{+1}\alpha^{\nu_1}_{+1} \\
&= \sum_{\mu_1\neq\mu_2=0}^{D-1}\ \sum_{\nu_1\neq\nu_2=0}^{D-1} \frac{\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}}{\sqrt{2}}g_{\mu_1\nu_1}g_{\mu_2\nu_2}\frac{\alpha^{\nu_2}_{+1}\alpha^{\nu_1}_{+1}}{\sqrt{2}}
\end{aligned} \tag{3.21}$$

With this rewriting, the second and third line of (3.19) can be combined to

$$\sum_{\mu_1,\mu_2,\nu_1,\nu_2=0}^{D-1} \left\langle \phi_1|\dots V_i\ \frac{\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}}{\sqrt{2}}|0;P\right\rangle \frac{2g_{\mu_1\nu_1}g_{\mu_2\nu_2}}{P^2+2(N-1)}\left\langle 0;P|\frac{\alpha^{\nu_1}_{+1}\alpha^{\nu_2}_{+1}}{\sqrt{2}}\ V_{i+1}\dots|\phi_n\right\rangle \tag{3.22}$$

Similar argument can show that the sum T_3, T_4, T_5 of (3.20) gives

$$\begin{aligned}
T_3+T_4+T_5 = \sum_{\mu_i,\nu_i=0}^{D-1}&\left\langle \phi_1|\dots V_i\ \frac{\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}\alpha^{\mu_3}_{-1}}{\sqrt{3!}}|0;P\right\rangle \frac{2g_{\mu_1\nu_1}g_{\mu_2\nu_2}g_{\mu_3\nu_3}}{P^2+2(N-1)} \\
&\left\langle 0;P|\frac{\alpha^{\nu_3}_{+1}\alpha^{\nu_2}_{+1}\alpha^{\nu_1}_{+1}}{\sqrt{3!}}\ V_{i+1}\dots|\phi_n\right\rangle
\end{aligned}$$

It is easy to see that when multiple operators of the same mode n are present in the Fock state, each may or may not be carrying the same Lorentz index 0 , or 1 , or … , or $D-1$, the general pattern is given by the expansion $(a_0+a_1+\dots+a_{D-1})^{N_n}/N_n!$ where $a_i = \alpha^i_{-n}g_{ii}\alpha^i_{+n}$. The coefficient of term $(\alpha^0_{-n})^{n_0}(\alpha^1_{-n})^{n_1}\dots(\alpha^{D-1}_{-n})^{n_{D-1}}$ in the Fock state is given by the coefficient of term $a_0^{n_0}a_1^{n_1}\dots a_{D-1}^{n_{D-1}}$ with $N_n = \sum_{i=0}^{D-1} n_i$ in the expansion, which reads

$$\frac{1}{N!}C^N_{n_0}C^{N-n_0}_{n_1}C^{N-n_0-n_1}_{n_2}\dots C^{n_{D-1}}_{n_{D-1}} = \frac{1}{N!}\frac{N!}{\prod_{i=0}^{D-1}(n_i)!} \tag{3.23}$$

thus we can drop the $\mu_1 < \mu_2 < \dots$ arrangement and rewrite the sum in (3.17) as

$$\begin{aligned}
&\sum_{\{N_{\mu,n}\}}\left|\{N_{\mu,n}\};\widehat{P}\right\rangle \mathcal{T}_{\{N_{\mu,n}\}}\left\langle\{N_{\mu,n}\};\widehat{P}\right| \\
&= \sum_{\sum_n nN_n=N}\left\{\prod_{n=1}^{\infty}\frac{(\alpha^{\mu_{N_n,1}}_{-n}\alpha^{\mu_{N_n,2}}_{-n}\dots\alpha^{\mu_{N_n,N_n}}_{-n})}{\sqrt{N_n!n^{N_n}}}\right\}\left|0;\widehat{P}\right\rangle \\
&\prod_{n=1}^{\infty}(g_{\mu_{N_n,1}\nu_{N_n,1}}g_{\mu_{N_n,2}\nu_{N_n,2}}\dots g_{\mu_{N_n,N_n}\nu_{N_n,N_n}}) \\
&\left\langle 0;\widehat{P}\right|\left\{\prod_{n=1}^{\infty}\frac{(\alpha^{\nu_{N_n,1}}_{+n}\alpha^{\nu_{N_n,2}}_{+n}\dots\alpha^{\nu_{N_n,N_n}}_{+n})}{\sqrt{N_n!n^{N_n}}}\right\}
\end{aligned} \tag{3.24}$$

Having the simplified version (3.24), we can give following explicit calculations.

3.3.1 Explicit calculation

Recalling the vertex of tachyon

$$V_0(k,z) =: e^{ik\cdot X(z)} := Z_0 W_0\,, \tag{3.25}$$

where

$$Z_0 = e^{ik\cdot x + k\cdot p \ln z} = e^{ikx} z^{k\cdot p+1} = z^{k\cdot p-1} e^{ik\cdot x} \tag{3.26}$$

and

$$W_0 = e^{\sum_{n=1}^{\infty} \frac{z^n}{n} k\cdot\alpha_{-n}} e^{-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} k\cdot\alpha_n}\,, \tag{3.27}$$

it is easy to calculate the left three-point amplitude

$$\langle 0; -k_1 | V_0(k_2, z) | \{N_{\mu,n}\}; P\rangle = \delta(k_1 + k_2 + P) \prod_{\mu=0}^{D-1} \prod_{m=1}^{\infty} \frac{(-k_2^\mu)^{N_{\mu,m}}}{\sqrt{m^{N_{\mu,m}} N_{\mu,m}!}} \tag{3.28}$$

where N is the level defined in (3.15) and the right three-point amplitude

$$\langle \{N_{\mu,n}\}; P | V_0(k_3, z) | 0; k_4\rangle = \delta(P - k_3 - k_4) \prod_{\mu=0}^{D-1} \prod_{m=1}^{\infty} \frac{(k_3^\mu)^{N_{\mu,m}}}{\sqrt{N_{\mu,m}! m^{N_{\mu,m}}}} \tag{3.29}$$

Using (3.28) and (3.29) it is easy to calculate first few mass levels. In fact, the same calculation has been done in our simplification leading to the simplified tensor structure (3.24). Thus we have when $N = 0$, it is 1, while when $N = 1$ it is $(-k_2 \cdot k_3)$. Finally when $N = 2$ it is $\frac{(k_2\cdot k_3)(k_2\cdot k_3 - 1)}{2}$. They do satisfy (3.8) for $N = 0, 1, 2$.

For general level N, from (3.24), (3.28) and (3.29) we find

$$I_N = \sum_{\sum nN_n = N} \prod \frac{(-k_2 \cdot k_3)^{N_n}}{N_n! n^{N_n}} \tag{3.30}$$

Let us define

$$N = \sum_{n=1}^{\infty} nN_n, \qquad J = \sum_{n=1}^{\infty} N_n \tag{3.31}$$

with obviously that $J \leq a$, then using the definition (A.2) of Stirling number of the first kind, I_N can be rewritten as

$$I_N = (-)^N \sum_{J=1}^{N} \frac{S(N,J)}{N!} (k_2\cdot k_3)^J = (-)^N \begin{pmatrix} k_2 \cdot k_3 \\ N \end{pmatrix} \tag{3.32}$$

where we have used the formula (A.1).[8] This is exactly the result (3.8) we try to prove.

[8]Since $S(N, 0) = 0$ when $N > 0$, we can extend the sum over J from region $[1, N]$ to region $[0, N]$.

4 Example II: BCFW of 5-tachyon amplitude in bosonic open string theory

Having shown that a 4-point Veneziano amplitude can be indeed described by BCFW on-shell recursion relation, let us consider the 5-tachyon scattering amplitude, which contains slightly richer analytic structure because unlike 4-point amplitude with only pole s_{12}, there are two types of poles from s_{12}, s_{123} for deformation (1,5). Multiple pole structure is seen for general amplitudes, we need to study this simplest nontrivial example.

4.1 Pole expansion

The Koba-Nielson formula for 5-point tachyon amplitude is given by

$$A(12345) = \int_0^1 dz_3 \int_0^{z_3} dz_2\, (1-z_3)^{k_4\cdot k_3}(1-z_2)^{k_4\cdot k_2}(z_3-z_2)^{k_3\cdot k_2} z_2^{k_2\cdot k_1} z_3^{k_3\cdot k_1}. \tag{4.1}$$

where we have fixed $z_1 = 0, z_4 = 1, z_5 = \infty$. Unlike in quantum field theory, where analytic behavior of an amplitude is transparent from Feynman rules, kinematic dependence in Koba-Nielson's formulation were implicitly introduced through exponents of worldsheet integration variables, making it less easier to locate poles. However as we have seen in the previous section, worldsheet integrals can be explicitly carried out after binomial expansions. Expanding $(z_3 - z_2)^{k_3\cdot k_2}$ with respect to z_2, which is the variable that assumes smaller value (than z_3), and expand similarly $(1 - z_2)^{k_4\cdot k_2}$ and $(1 - z_3)^{k_4\cdot k_3}$ we have

$$\begin{aligned}
(1-z_2)^{k_4\cdot k_2} &= \sum_{a=0}^{\infty} \binom{k_4\cdot k_2}{a} (-)^a z_2^a, \\
(z_3-z_2)^{k_3\cdot k_2} &= \sum_{b=0}^{\infty} \binom{k_3\cdot k_2}{b} (-)^b z_3^{k_s\cdot k_2 - b} z_2^b, \\
(1-z_3)^{k_4\cdot k_3} &= \sum_{c=0}^{\infty} \binom{k_4\cdot k_3}{c} (-)^c z_3^c,
\end{aligned} \tag{4.2}$$

Grouping z_2 and z_3 dependence in equation (4.1) together we arrive

$$\begin{aligned}
A(12345) = &\sum_{a,b,c=0}^{\infty} \binom{k_4\cdot k_2}{a}\binom{k_3\cdot k_2}{b}\binom{k_4\cdot k_3}{c} (-)^{a+b+c} \\
&\times \int_0^1 dz_3 \int_0^{z_3} dz_2 z_3^{k_3\cdot(k_1+k_2)-b+c} z_2^{k_1\cdot k_2+a+b}
\end{aligned} \tag{4.3}$$

Carrying out the integration in order, i.e., $\int dz_2$ first and then $\int dz_3$ we obtain

$$\begin{aligned}
A(12345) = &\sum_{a,b,c=0}^{\infty} \binom{k_4\cdot k_2}{a}\binom{k_3\cdot k_2}{b}\binom{k_4\cdot k_3}{c} (-)^{a+b+c} \\
&\times \frac{2}{s_{12}+2(a+b-1)} \frac{2}{s_{123}+2(a+c-1)},
\end{aligned} \tag{4.4}$$

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where we have used $s_{12} = (k_1+k_2)^2$, $s_{123} = (k_1+k_2+k_3)^2$, and the mass-shell conditions for tachyons, $k_1^2 = k_2^2 = k_3^2 = 2$.

Now we consider the pole structure under the deformation (2.1) with pair $(1,5)$. For s_{12}, the poles are located at

$$z_N = \frac{(k_1+k_2)^2 + 2(N-1)}{-q\cdot(k_1+k_2)}, \qquad N = a+b = 0,1,\ldots \tag{4.5}$$

while for s_{123} the poles are located at

$$w_M = \frac{(k_1+k_2+k_3)^2 + 2(M-1)}{-q\cdot(k_1+k_2+k_3)}, \qquad M = a+c = 0,1,2,\ldots \tag{4.6}$$

Using the BCFW recursion relation, we have

$$A(1,2,3,4,5) = \sum_{z_N} \frac{2}{s_{12}+2(N-1)}\mathcal{R}_N + \sum_{w_M} \frac{2}{s_{123}+2(M-1)}\mathcal{S}_M \tag{4.7}$$

where $\mathcal{R}_N$ and $\mathcal{S}_M$ are corresponding residues of poles.

Residue $\boldsymbol{\mathcal{R}_N}$: from (4.4) we can read out the residue $\mathcal{R}_N$ as

$$\mathcal{R}_N = \sum_{\substack{a,b=0\\a+b=N}}^{\infty} \sum_{c=0}^{\infty} \binom{k_4\cdot k_2}{a}\binom{k_3\cdot k_2}{b}\binom{k_4\cdot k_3}{c}(-)^{a+b+c}\left[\frac{2}{\widehat{s}_{123}(z_N)+2(a+c-1)}\right] \tag{4.8}$$

Noticing that

$$\widehat{s}_{12}(z_N) + k_3^2 + 2k_3\cdot\widehat{k}_{12}(z_N) + 2(a+c-1) = 2k_3\cdot\widehat{k}_{12}(z_N) + 2(c-b+1)$$

we can rewrite

$$\begin{aligned}\binom{k_4\cdot k_3}{c}(-)^c\left[\frac{2}{\widehat{s}_{123}(z_N)+2(a+c-1)}\right] &= \binom{k_4\cdot k_3}{c}(-)^c\left[\frac{1}{k_3\cdot\widehat{k}_{12}(z_N)+(c-b+1)}\right]\\ &= \sum_{c=0}^{\infty}\int_0^1 dz_3\, z_3^{k_3\cdot(\hat{k}_1+k_2)-b+c}\binom{k_4\cdot k_3}{c}(-)^c\\ &= \int_0^1 dz_3\, z^{k_3\cdot(\hat{k}_1+k_2)-b}\,(1-z_3)^{k_4\cdot k_3},\end{aligned} \tag{4.9}$$

The reason we write the sum over c as the integration is clear: the subamplitude at the right handed side should be $A(\widehat{P},3,4,\widehat{5})$. With this rewriting we have

$$\mathcal{R}_N = \sum_{\substack{a,b=0\\a+b=N}}^{\infty} \binom{k_4\cdot k_2}{a}\binom{k_3\cdot k_2}{b}(-)^N\int_0^1 dz_3\, z^{k_3\cdot(\hat{k}_1+k_2)-b}\,(1-z_3)^{k_4\cdot k_3} \tag{4.10}$$

Residue $\mathcal{S}_M$: from (4.4) we can read out the residue $\mathcal{S}_M$ as

$$\mathcal{S}_M = \sum_{\substack{a,\, c=0 \\ a+c=M}}^{\infty} \sum_{b=0}^{\infty} \binom{k_4\cdot k_2}{a}\binom{k_3\cdot k_2}{b}\binom{k_4\cdot k_3}{c}(-)^{a+b+c}\left[\frac{2}{\widehat{s}_{12}(w_N)+2(a+b-1)}\right] \tag{4.11}$$

Using

$$\begin{aligned}\sum_{b=0}^{\infty}\left.\frac{\binom{k_3\cdot k_2}{b}(-)^b}{\hat{k}_1\cdot k_2+(a+b)+1}\right|_{z=w_M} &= \sum_{b=0}^{\infty}\int_0^1 dz_2\, z_2^{\hat{k}_1\cdot k_2+a+b}\binom{k_3\cdot k_2}{b}(-)^b \\ &= \int_0^1 dz_2\, z_2^{\hat{k}_1\cdot k_2+a}(1-z_2)^{k_3\cdot k_2},\end{aligned} \tag{4.12}$$

which remind us the subamplitude $A(\widehat{1},2,3,\widehat{P})$, we get another form

$$\mathcal{S}_M = \sum_{\substack{a,\, c=0 \\ a+c=M}}^{\infty} \binom{k_4\cdot k_2}{a}\binom{k_4\cdot k_3}{c}(-)^M\int_0^1 dz_2\, z_2^{\hat{k}_1\cdot k_2+a}(1-z_2)^{k_3\cdot k_2} \tag{4.13}$$

4.2 Four point scattering amplitude

Now we try to reproduce the same residue from the BCFW recursion relation. To do this, we need to calculate the three point and four point amplitudes with one general Fock state. The three point case has been given in section 3. Now we give the four point result.

First let us consider a simple example

$$\langle 0, k_4|V_0(k_3,z_3)\, V_0(k_2,z_2)\alpha^{\mu}_{-m}|0,k_1\rangle \tag{4.14}$$

where $V_0(k,z)$ stands for tachyon vertex operator (B.1) inserted at z, and the initial state $\alpha^{\mu}_{-m}|0,k_1\rangle$ is raised from the ground state by a $-m$ mode operator. Following the standard treatment moving this mode operator to the left until it finally annihilate the final state we obtain

$$(-k_2^{\mu}z_2^m - k_3^{\mu}z_3^m)\,\langle 0,k_4|V_0(k_3)\,V_0(k_2)|0,k_1\rangle\,. \tag{4.15}$$

In addition to all-tachyon amplitude we receive factors $(-k_2^{\mu}z_2^m - k_3^{\mu}z_3^m)$ picked up from the commutator

$$[: e^{ik\cdot X(z)}:,\alpha^{\mu}_{-m}] = -k^{\mu}z^m\left(: e^{ik\cdot X(z)}:\right)\,. \tag{4.16}$$

For a generic normalized Fock state (3.14) we repeat the same manipulation, moving mode operators α^{μ}_{-m} one by one to the left, picking up a factor $(-k^{\mu}z^m)$ when passing a tachyon vertex $V(k,z)$. Putting all together we finally have

$$\langle 0,p|V_0(k_3)\,V_0(k_2)|\left\{N_{\mu,m}\right\},k_1\rangle=\left\langle 0,p|V_0(k_3)\,V_0(k_2)\prod_{\mu=0}^{D-1}\prod_{m=1}^{\infty}\frac{(\alpha_{-m}^{\mu})^{N_{\mu,m}}}{\sqrt{N_{\mu,m}!m^{N_{\mu,m}}}}|0,k_1\right\rangle$$
$$=\prod_{\mu=0}^{D-1}\prod_{m=1}^{\infty}\frac{(-k_2^{\mu}z_2^m-k_3^{\mu}z_3^m)^{N_{\mu,m}}}{\sqrt{N_{\mu,m}!m^{N_{\mu,m}}}}\langle 0,p|V_0(k_3)\,V_0(k_2)|0,k_1\rangle \tag{4.17}$$

where $\langle 0,p|V_0(k_3)\,V_0(k_2)|0,k_1\rangle$ is known.

Similarly, if the Fock state defines the final state instead of the initial state of an amplitude we move mode operator α_m^{μ} to the right hand side, yielding

$$\langle\left\{N_{\mu,m}\right\},k_5|V_0(k_4)\,V_0(k_3)|0,p\rangle=\left\langle 0,k_5|\prod_{\mu=0}^{D-1}\prod_{m=1}^{\infty}\frac{(\alpha_{m}^{\mu})^{N_{\mu,m}}}{\sqrt{N_{\mu,m}!m^{N_{\mu,m}}}}V_0(k_4)\,V_0(k_3)|0,p\right\rangle.$$
$$=\prod_{\mu=0}^{D-1}\prod_{m=1}^{\infty}\frac{(k_4^{\mu}z_4^m+k_3^{\mu}z_3^m)^{N_{\mu,m}}}{\sqrt{N_{\mu,m}!m^{N_{\mu,m}}}}\langle 0,k_5|V_0(k_4)\,V_0(k_3)|0,p\rangle\,. \tag{4.18}$$

It is worth to notice that the factors picked up by modes have different signs from (4.17) due to the fact that opposite signs were assigned to positive and negative modes in a tachyon vertex operator,

$$W_0=e^{\sum_{n=1}^{\infty}\frac{z^n}{n}k\cdot\alpha_{-n}}e^{-\sum_{n=1}^{\infty}\frac{z^n}{n}k\cdot\alpha_n} \tag{4.19}$$

so that

$$[\alpha_m^{\mu},:e^{ik\cdot X(z)}:]=k^{\mu}z^m\left(:e^{ik\cdot X(z)}:\right) \tag{4.20}$$

4.3 Calculation of residue $\mathcal{S}_M$

Having above preparation, we can calculate residue by summing over immediate Fock states at given mass level M. In other words, at level M, we should have

$$\mathcal{S}_M=\int dz_2\sum_{\{N_{\mu,m}\}}\left\langle 0,\hat{k}_5|V_0(k_4)|\{N_{\mu,m}\},\hat{p}\right\rangle\left\langle\{N_{\mu,m}\},\hat{p}|V_0(k_3)\,V_0(k_2)|0,\hat{k}_1\right\rangle\Big|_{z_4=z_3=1}, \tag{4.21}$$

where the summation is over modes$\{N_{\mu,m}\}$ at fixed mass level $N=\sum_{\mu,m}(m\times N_{\mu,m})$, so $\widehat{p},\widehat{k}_5,\widehat{k}_1$ are all fixed by M. Before giving the general discussion, let us see a few examples:

- **Level** $N=0$: at $N=0$, $N_{\mu,m}$ must be all zero, so that equation (4.21) simply yields

$$\mathcal{S}_0=\int dz_2\left\langle 0,\hat{k}_5|V_0(k_4)|0,\hat{p}\right\rangle\left\langle 0,\hat{p}|V_0(k_3)\,V_0(k_2)|0,\hat{k}_1\right\rangle\Big|_{z_4=z_3=1}$$
$$=1\times\int_0^1 dz_2\,z^{k_2\cdot\hat{k}_1}(1-z_2)^{k_3\cdot k_2}, \tag{4.22}$$

  and we have an agreement with (4.13) at $a=c=0$.

- **Level** $N=1$: the $N=1$ state can only arise from states having a single $N_{\mu,m}=1$ for $\mu=0,\dots,D-1$, while powers of other modes remain zero

$$
\begin{aligned}
\mathcal{S}_1 &= \sum_{\mu,\nu}\int dz_2\,\left\langle 0,\hat{k}_5|V_0(k_4)|N_{\mu,1},\hat{p}\right\rangle g^{\mu\nu}\left\langle N_{\nu,1},\hat{p}|V_0(k_3)\,V_0(k_2)|0,\hat{k}_1\right\rangle\Big|_{z_4=z_3=1}\\
&= \int_0^1 dz_2(-k_4)\cdot(k_3z_3+k_2z_2)\;z^{k_2\cdot\hat{k}_1}(1-z_2)^{k_3\cdot k_2}\Big|_{z_3=1}
\end{aligned}
\tag{4.23}
$$

In addition to the usual tachyonic Koba-Nielson formula we obtain a factor $-(k_4\cdot k_3)\,z_3-(k_4\cdot k_2)\,z_2|_{z_3=1}$. These two terms correspond to $(a,c)=(0,1)$ and $(1,0)$ respectively.

- **Level** $N=2$: the first non-trivial case happens at $N=2$. As in the previous mass level we receive an additional term to the tachyonic formula. For $N_{\mu,2}$ states this factor is $\frac{-1}{2}k_4\cdot\left(k_3z_3^2+k_2z_2^2\right)$, while for states with $N_{\mu_1,1}=N_{\mu_2,1}=1$ and $0\le\mu_1<\mu_2\le D-1$ the factor is $\frac{1}{2}\left[k_4\cdot(k_3z_3+k_2z_2)\right]^2-\frac{1}{2}\sum_\mu\left[k_4^\mu(k_3z_3+k_2z_2)^\mu\right]^2$, and for states with $N_{\mu,1}=2$ we obtain $\sum_\mu\left[k_4^\mu(k_3z_3+k_2z_2)\right]^2$. Adding all these contribution gives

$$
\begin{aligned}
&\frac{-1}{2}k_4\cdot\left(k_3z_3^2+k_2z_2^2\right)+\frac{1}{2}\left[k_4\cdot(k_3z_3+k_2z_2)\right]^2\\
&=\frac{(k_4\cdot k_3)(k_4\cdot k_3-1)}{2}z_3^2+\frac{(k_4\cdot k_2)(k_4\cdot k_2-1)}{2}z_2^2+(k_4\cdot k_3)(k_4\cdot k_2)z_3z_2
\end{aligned}
\tag{4.24}
$$

Explicit expansion into series shows again agreement with $\binom{k_4\cdot k_2}{a}\binom{k_4\cdot k_3}{c}(-)^{a+c}z_2^a$, with the first, second, third terms corresponding to $(a,c)=(0,2)$, $(2,0)$ and $(1,1)$ respectively.

For general level $N=\sum_{n=1}^{\infty}n\,N_n$ in addition to the all-tachyon formula we have[9]

$$
\begin{aligned}
&\sum_{\substack{\text{partitions of }N\\ \text{into }\{N_n\}}}\prod_{n=1}^{\infty}\frac{\left[-k_4\cdot(k_3z_3^n+k_2z_2^n)\right]^{N_n}}{N_n!\,n^{N_n}}\\
&=\sum_{\substack{\text{partitions of }N\\ \text{into }\{N_n\}}}\prod_{n}\sum_{N_n^{(2)}=0}^{\infty}\frac{\binom{N_n}{N_n^{(2)}}}{N_n!\,n^{N_n}}(k_4\cdot k_3)^{N_n-N_n^{(2)}}z_3^{n\,(N_n-N_n^{(2)})}(k_4\cdot k_3)^{N_n^{(2)}}z_2^{n\,N_n^{(2)}}.
\end{aligned}
\tag{4.25}
$$

where in the second line above we expanded the numerator with respect to power of z_2, which we denote as $N_n^{(2)}$. Introducing the notation $N_n^{(3)}=N_n-N_n^{(2)}$, the combinatorial

[9] Note that at every step these factors are produced in the same pattern observed in the 4-point case, as was discussed in appendix B, except with k_3 now replaced by $k_3z_3^n+k_2z_2^n$.

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factor can be written as

$$\binom{N_n}{N_n^{(2)}} \frac{1}{N_n!\, n^{N_n}} = \frac{1}{N_n^{(2)}!\, (N_n - N_n^{(2)})!\, n^{N_n}} = \frac{1}{N_n^{(2)}!\, N^{(3)}!\, n^{N_n^{(2)}} n^{N_n^{(3)}}}$$

Now we notice that in equation (4.25), summing over partitions of fixed N_n into $N_n^{(2)}$ and $N_n^{(3)}$ first and then summing over partitions of N into $\{N_n\}$ secondly can be replaced by summing over partitions of N directly into $\{N_n^{(2)}\}$ and $\{N_n^{(3)}\}$, so (4.25) can be written as

$$\sum_{\text{partitions into } N_n^{(2)}, N_n^{(3)}} \prod_n \frac{1}{N_n^{(2)}!\, N^{(3)}!\, n^{N_n^{(2)}} n^{N_n^{(3)}}} (k_4 \cdot k_2)^{N_n^{(3)}} (k_4 \cdot k_3)^{N_n^{(2)}} z_2^{n\, N_n^{(2)}} z_3^{n\, N_n^{(3)}} . \tag{4.26}$$

Defining

$$K = \sum_n N_n^{(2)}, \quad J \equiv \sum_n N_n^{(3)}, \quad a = \sum_n n\, N_n^{(2)}, \quad c = \sum_n n\, N_n^{(3)}, \tag{4.27}$$

sum in equation (4.26) can be divided into summations over partitions of $\{N_n^{(2)}\}$ and $\{N_n^{(3)}\}$ with fixed J, K, a, c at first, and then summing over J, K, and a,[10] i.e., equation (4.26) is equal to

$$\sum_a \sum_{J,K} \frac{S(c,J)}{c!} \frac{S(a,K)}{a!} (k_4 \cdot k_2)^J (k_4 \cdot k_3)^K\ z_2^a z_3^c, \tag{4.28}$$

where Striling numbers of the first kind are given by

$$S(a,K) = \sum_{\text{partitions } N_n^{(2)}} \frac{a!}{N_n^{(2)}!\, n^{N_n^{(2)}}}, \qquad S(c,J) = \sum_{\text{partitions } N_n^{(3)}} \frac{c!}{N_n^{(3)}!\, n^{N_n^{(3)}}}, \tag{4.29}$$

Now we are almost done. Summing equation (4.28) over J and K yields

$$\binom{k_4 \cdot k_2}{a} \binom{k_4 \cdot k_3}{c} (-)^{a+c} z_2^a\, z_3^c. \tag{4.30}$$

Inserting the result back into (4.21) we see that

$$\begin{aligned}
&\mathcal{S}_M \int dz_2 \left\langle 0, \hat{k}_5 | V_0(k_4) | \{N_{\mu,m}\}, \hat{p} \right\rangle \left\langle \{N_{\mu,m}\}, \hat{p} | V_0(k_3)\, V_0(k_2) | 0, \hat{k}_1 \right\rangle \Big|_{z_4=z_3=1} \\
&= \sum_a \int dz_2 \left\langle 0, \hat{k}_5 | V_0(k_4) | 0, \hat{p} \right\rangle \left\langle 0, \hat{p} | V_0(k_3)\, V_0(k_2) | 0, \hat{k}_1 \right\rangle \\
&\quad \times \binom{k_4 \cdot k_2}{a} \binom{k_4 \cdot k_3}{c} (-)^{a+c} z_2^a\, z_3^c \Big|_{z_4=z_3=1} \\
&= \sum_{a=0, a+c=M}^{M} \binom{k_4 \cdot k_2}{a} \binom{k_4 \cdot k_3}{c} (-)^{a+c} \int_0^1 dz_2\, z_2^{\hat{k}_1 \cdot k_2 + a} (1 - z_2)^{k_3 \cdot k_2},
\end{aligned} \tag{4.31}$$

which is the form (4.13) we want to prove.

The other residue $\mathcal{R}_N$ can be derived from BCFW prescription following similar procedures.

[10]However note that c should not be summed over here because the mass level $(a + c) = \sum_n n(N_n^{(2)} + N_n^{(3)}) = N$ is understood as a fixed number at every pole.

5 The general proof

Having done above two examples, we would like to have a general understanding. The method we will use in this section will be a little different although it is easy to translate languages between these two approaches.

5.1 String theory calculation

In open string theory, the ordered tree-level amplitude is given by

$$A_M = g^{M-2} \int \delta(y_A - y_A^0)\delta(y_B - y_B^0)\delta(y_c - y_c^0)(y_A - y_B)(y_A - y_C)(y_B - y_C) \prod_{i=2}^{M} \theta(y_{i-1} - y_i) \prod_{j=1}^{M} dy_j \left\langle 0;0 \left| \frac{V(k_1, y_1)}{y_1} \ldots \frac{V(k_M, y_M)}{y_M} \right| 0;0 \right\rangle \tag{5.1}$$

Using three delta-function, we can take $y_M = 0, y_2 = 1, y_1 = \infty$, so the amplitude can be written as

$$A_M = g^{M-2} \int_0^1 dy_3 \int_0^{y_3} dy_4 \ldots \int_0^{y_{M-2}} dy_{M-1} \left\langle \phi_1(k_1) \left| V(k_2, 1) \frac{V(k_3, y_3)}{y_3} \ldots \frac{V(k_{M-1}, y_{M-1})}{y_{M-1}} \right| \phi_M(k_M) \right\rangle \tag{5.2}$$

where we have used the definition of initial state and final state

$$|\Lambda; k\rangle = \lim_{y \to 0} \frac{V_\Lambda(k, y)}{y} |0;0\rangle, \qquad \langle \Lambda; k| = \lim_{y \to \infty} y V_\Lambda(k, y) |0;0\rangle \tag{5.3}$$

Next we define $y_i = z_3 z_4 \ldots z_i$ with $i = 3, \ldots, M-1$, from which we can solve

$$z_3 = y_3, \qquad z_i = \frac{y_i}{y_{i-1}}, \qquad i = 4, \ldots, M-1 \tag{5.4}$$

Now let us fix all y_i except transform $y_{M-1} = z_{M-1} y_{M-2}$, then using

$$V_\Lambda(k, z) = z^{L_0} V_\Lambda(k, z=1) z^{-L_0} \tag{5.5}$$

we get

$$\begin{aligned} &\ldots \int_0^1 dz_{M-1} y_{M-2} y_{M-1}^{L_0-2} V(k_{M-1}, 1) y_{M-1}^{-L_0+1} |\phi_M(k_M)\rangle \\ &= \ldots \left[\int_0^1 dz_{M-1} y_{M-2}^{L_0-1} z_{M-1}^{L_0-2} \right] V(k_{M-1}, 1) |\phi_M(k_M)\rangle \end{aligned}$$

where we have used the physical condition $(L_0 - 1)|\phi_M\rangle = 0$. Now we change $y_{M-2} = z_{M-2} y_{M-3}$, then we have

$$\begin{aligned} &\ldots \int_0^1 dz_{M-2} y_{M-3} y_{M-2}^{L_0} \frac{V(k_{M-2}, 1)}{y_{M-2}} y_{M-2}^{-L_0} \int_0^1 dz_{M-1} y_{M-2}^{L_0-1} z_{M-1}^{L_0-2} V(k_{M-1}, 1) |\phi_M(k_M)\rangle \\ &= \ldots \int_0^1 dz_{M-2} y_{M-3} y_{M-2}^{L_0-2} V(k_{M-2}, 1) \int_0^1 dz_{M-1} z_{M-1}^{L_0-2} V(k_{M-1}, 1) |\phi_M(k_M)\rangle \\ &= \ldots \left[\int_0^1 dz_{M-2} y_{M-3}^{L_0-1} z_{M-2}^{L_0-2} \right] V(k_{M-2}, 1) \frac{1}{L_0 - 1} V(k_{M-1}, 1) |\phi_M(k_M)\rangle \end{aligned}$$

where we have used $\int_0^1 dz z^{L_0-2} = \frac{1}{L_0-1}$ is the string propagator.

Comparing expressions from last two steps, we see that we can iterate this procedure to

$$A_M = g^{M-2}\left\langle \phi_1 \left| V_2(k_2)\frac{1}{L_0-1}V_3(k_3)\dots\frac{1}{L_0-1}V_{M-1}(k_{M-1})\right| \phi_M\right\rangle \tag{5.6}$$

Form (5.6) is the convenient one to compare with BCFW recursion relation, because locations of poles are clearly indicated by propagator $\frac{1}{L_0-1}$. For example, for $\frac{1}{L_0-1}$ between vertex operators V_i and V_{i+1}, pole locations are given by

$$\frac{1}{2}(k_1+\dots+k_i)^2+N-1=0, \qquad N=0,1,2,\dots \tag{5.7}$$

Now let us consider the $(1,M)$-deformation given in (2.1) and use z_{iN} to indicate the solution obtained from equation (5.7) with $k_1\to k_1+zq$. Because it has been proved that boundary contribution is zero under the deformation at least for some kinematic region, we have immediately

$$A_M = g^{M-2}\sum_{i=2}^{M-2}\sum_{N=0}^{\infty}\frac{2\mathcal{R}_{i,N}}{(k_1+k_2+\dots+k_i)^2+2(N-1)} \tag{5.8}$$

where

$$\begin{aligned}
\mathcal{R}_{i,N} &= \langle\Phi_{i,N}|\Psi_{i,N}\rangle\\
\langle\Phi_{i,N}| &= \left\langle \phi_1(k_1+z_{i,N}q)\right| \left| V_2(k_2)\frac{1}{L_0-1}V_3(k_3)\dots\frac{1}{L_0-1}V_i(k_i)\right|\\
|\Psi_{i,N}\rangle &= \left| V_{i+1}(k_{i+1})\frac{1}{L_0-1}\dots V_{M-1}(k_{M-1})\right|\left.\phi_M(k_M-z_{i,N}q)\right\rangle
\end{aligned} \tag{5.9}$$

What we want to prove is that residue $\mathcal{R}_{i,N}$ can be obtained from summing over intermediate physical states prescribed by BCFW on-shell recursion relation.

5.2 The proof

Now we give our proof. First, we notice that both states $\langle\Phi_{i,N}|, |\Psi_{i,N}\rangle$ are physical states,[11] thus in the frame work of DDF-state construction, both physical states can be written as $|s_{phy}\rangle+|f\rangle$, where $|f\rangle$ is the DDF-state while $|s_{phy}\rangle$ is physical spurious states. Using the property of spurious state, we have

$$\langle\Phi_{i,N}|\Psi_{i,N}\rangle = \left\langle s^L_{i,N}+f^L_{i,N}|s^R_{i,N}+f^R_{i,N}\right\rangle = \left\langle f^L_{i,N}|f^R_{i,N}\right\rangle \tag{5.10}$$

Having established (5.10) we insert identity operator in the Fock space with given momentum $P_{i,N}=k_1+z_{i,N}q+k_2+\dots+k_i$ and annihilated by (L_0-1), so

$$\left\langle f^L_{i,N}|f^R_{i,N}\right\rangle = \sum_i\left\langle f^L_{i,N}|\psi_i^\dagger(P_{i,N})\right\rangle\langle\psi_i(P_{i,N})|f^R_{i,N}\rangle \tag{5.11}$$

where set $\{|\psi_i(P_{i,N})\rangle\}$ can be any normalized orthogonal basis. In DDF-frame work, a general state can be written as the linear combination of $|k\rangle, |s\rangle, |f\rangle$, i.e., a choice of the

[11] The proof can be found in a standard text, for example in *Superstring Theory* by Green, Schwarz and Witten [16] (chapter 7, vol. 1.).

basis is $|k\rangle, |s\rangle, |f\rangle$. Using the definition of states, we see immediately that $\langle s|f\rangle = 0$ and $\langle k|f\rangle = 0$, thus

$$\begin{aligned}\left\langle f_{i,N}^L|f_{i,N}^R\right\rangle &= \sum_i \left\langle f_{i,N}^L|f_i^\dagger(P_{i,N})\right\rangle \left\langle f_i(P_{i,N})|f_{i,N}^R\right\rangle \\ &= \sum_i \left\langle s_{i,N}^L + f_{i,N}^L|f_i^\dagger(P_{i,N})\right\rangle \left\langle f_i(P_{i,N})|s_{i,N}^R + f_{i,N}^R\right\rangle \\ &= \sum_i \left\langle \Phi_{i,N}|f_i^\dagger(P_{i,N})\right\rangle \left\langle f_i(P_{i,N})|\Psi_{i,N}\right\rangle \end{aligned} \tag{5.12}$$

Using (5.10) and (5.12) we see immediately

$$\mathcal{R}_{i,N} = \sum_i \left\langle \Phi_{i,N}|f_i^\dagger(P_{i,N})\right\rangle \left\langle f_i(P_{i,N})|\Psi_{i,N}\right\rangle \tag{5.13}$$

which is the prescription given by BCFW recursion relation. Thus we have given our proof.

5.3 Practical method for summing over physical states

Having shown that BCFW recursion relation gives the right string amplitude, we need to explain how to sum over physical states. The difficulty of the sum is that the physical state is hard to describe in general, i.e., we do not know how to write down polarization vector for a given physical state. However, from the equivalent between (5.11) and (5.12) we see that we can replace the sum over all physical states to the sum over whole Fock space with given momentum and annihilated by $(L_0 - 1)$. For the Fock space, there is a freedom with the choice of basis and the one convenient for real calculation is oscillation basis defined in (3.14). Thus the residue can be calculated by

$$\begin{aligned}\mathcal{R}_{i,N} &= \sum_{\{N_{\mu,n}\}} \left\langle \Phi_{i,N}|\{N_{\mu,n}\};\widehat{P}\right\rangle \mathcal{T}_{\{N_{\mu,n}\}} \left\langle \{N_{\mu,n}\};\widehat{P}|\Psi_{i,N}\right\rangle \\ &= \sum_{\sum_n nN_n=N} \left\langle \Phi_{i,N}\left|\left\{\prod_{n=1}^{\infty}\frac{(\alpha_{-n}^{\mu_{N_n,1}}\alpha_{-n}^{\mu_{N_n,2}}\ldots\alpha_{-n}^{\mu_{N_n,N_n}})}{\sqrt{N_n!n^{N_n}}}\right\}\right|0;\widehat{P}\right\rangle \\ &\quad \prod_{n=1}^{\infty}(g_{\mu_{N_n,1}\nu_{N_n,1}}g_{\mu_{N_n,2}\nu_{N_n,2}}\cdots g_{\mu_{N_n,N_n}\nu_{N_n,N_n}}) \\ &\quad \left\langle 0;\widehat{P}\left|\left\{\prod_{n=1}^{\infty}\frac{(\alpha_{+n}^{\nu_{N_n,1}}\alpha_{+n}^{\nu_{N_n,2}}\ldots\alpha_{+n}^{\nu_{N_n,N_n}})}{\sqrt{N_n!n^{N_n}}}\right\}\right|\Psi_{i.N}\right\rangle\end{aligned} \tag{5.14}$$

6 Scattering with higher spin particles

Having established the general method given in (5.14), let us consider scatterings when higher spin particles are present. However, before doing this, let us recall some results coming from scattering amplitudes of pure tachyons. By checking with (3.32) and (4.28), we see that residues are given as series of Lorentz invariants $k_i \cdot k_j$ with coefficients given by Stirling number of the first kind $s(N,J) = \sum_{\{N_n\}} \prod_{n=1}^{\infty} \frac{1}{N_n! n^{N_n}}$. Summing over powers of $k_i \cdot k_j$ reproduces the residue in combinatorial form observed in [6]. This relation is established by writing generating function of Stirling number into two different forms

$$\begin{aligned} e^{X\,ln(1-z)} &= e^{-X\,(z+\frac{z^2}{2}+\frac{z^3}{3}+\dots)} = e^{-X\,z}e^{-X\,\frac{z^2}{2}}e^{-X\,\frac{z^3}{3}}\dots \\ &= \left(1+(-)Xz+\frac{(-)^2}{2!}X^2z^2+\dots\right)\left(1+(-)X\frac{z^2}{2}+(-)^2X\left(\frac{z^2}{2}\right)^2+\dots\right)\dots \end{aligned} \tag{6.1}$$

and

$$(1-z)^X = \sum_{a=1}^{\infty}(-)^a\frac{s(a,J)}{a!}X^Jz^a = \sum_a(-)^a\binom{X}{a}z^a \tag{6.2}$$

by matching power of z and setting $X = k_2 \cdot k_3$. In fact, it is straightforward to see that residues in an arbitrary n-point pure tachyon scattering amplitude can be read off from products of generating functions

$$e^{X_{23}\,ln(1-z_{23})}e^{X_{24}\,ln(1-z_{24})}\dots e^{X_{n-2,n-1}\,ln(1-z_{n-2,n-1})} \tag{6.3}$$

with $X_{ij} = k_i \cdot k_j$, $z_{ij} = z_j/z_i$, and residues in tachyonic recursion relation can be found through binomial expansion of

$$(1-z_{23})^{X_{23}}\,(1-z_{24})^{X_{24}}\dots(1-z_{n-2,n-1})^{X_{n-2,n-1}}. \tag{6.4}$$

Having recalled the experience from tachyon amplitude, now we discuss the scattering amplitude of 3-tachyon and 1-vector, which is given by

$$A(1\bar{2}34) = \int_0^1\frac{dz_2}{z_2}\left\langle 0,k_1\right|\left(\epsilon_2\cdot\dot{X}\;:e^{ik_2\cdot X(z_2)}:\right)\left(:e^{ik_3\cdot X(z_3)}:\right)\left|0,k_4\right\rangle\Big|_{z_3=1} \tag{6.5}$$

$$= \int_0^1 dz_2\,\left(-\epsilon_2\cdot k_1(1-z_2)^{k_3\cdot k_2}\,z_2^{k_1\cdot k_2-1}+\epsilon_2\cdot k_3\,(1-z_2)^{k_2\cdot k_3-1}\,z_2^{k_1\cdot k_2}\right) \tag{6.6}$$

where $\bar{2}$ means that the second particle is a vector. As in the case of pure tachyon scattering we binomially expanding $(1-z_2)^{k_3\cdot k_2}$ in (6.5) and integrating over z_2, yielding

$$\begin{aligned} A(1\bar{2}34) = &-\sum_{a=0}^{\infty}(-)^a\epsilon_2\cdot k_1\binom{k_3\cdot k_2}{a}\frac{2}{(k_2+k_1)^2+2(a-1)} \\ &+\sum_{a=1}^{\infty}(-)^{a-1}\epsilon_2\cdot k_3\binom{k_3\cdot k_2-1}{a-1}\frac{2}{(k_2+k_1)^2+2(a-1)}. \end{aligned} \tag{6.7}$$

We are interested in relating residue in (6.7) with residue given by BCFW prescription

$$\left\langle 0,k_1\right|\epsilon_2\cdot\dot{X}\;:e^{ik_2\cdot X(z_2)}:|\{N_{\mu,m}\},p\rangle\,\mathcal{T}_{\{N_{\mu,m}\}}\,\langle\{N_{\mu,m}\},p|:e^{ik_3\cdot X(z_3)}:\left|0,k_4\right\rangle\Big|_{z_2=z_3=1}. \tag{6.8}$$

It is straightforward to see at the first few levels, residues in (6.7) agree with those prescribed by (6.8) table 1.

	intermediate state $\|\{N_{\mu,m}\}\rangle\, \mathcal{T}_{\{N_{\mu,m}\}}\, \langle\{N_{\mu,m}\}\|$	contribution $\sim \epsilon_2\cdot k_3$
$N=0$	$\|0\rangle\,\langle 0\|$	absent
$N=1$	$\frac{\alpha^{\mu}_{-1}}{\sqrt{1}}\,\|0\rangle\,\eta_{\mu\nu}\,\langle 0\|\,\frac{\alpha^{\nu}_{1}}{\sqrt{1}}$	$(-)\,(\epsilon_2\cdot k_3)$
$N=2$	$\frac{\alpha^{\mu}_{-2}}{\sqrt{2}}\,\|0\rangle\,\eta_{\mu\nu}\,\langle 0\|\,\frac{\alpha^{\nu}_{2}}{\sqrt{2}}$ $\sum_{\mu_1<\mu_2}\frac{\alpha^{\mu_1}_{-1}}{\sqrt{1}}\frac{\alpha^{\mu_2}_{-1}}{\sqrt{1}}\,\|0\rangle\,\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\,\langle 0\|\,\frac{\alpha^{\nu_1}_{1}}{\sqrt{1}}\frac{\alpha^{\nu_2}_{1}}{\sqrt{1}}$ $\frac{1}{\sqrt{2!}}\frac{\alpha^{\mu}_{-1}}{\sqrt{1}}\frac{\alpha^{\mu}_{-1}}{\sqrt{1}}\,\|0\rangle\,(\eta_{\mu\nu})^2\,\langle 0\|\,\frac{1}{\sqrt{2!}}\frac{\alpha^{\nu}_{1}}{\sqrt{1}}\frac{\alpha^{\nu}_{1}}{\sqrt{1}}$	$(-)\,(\epsilon_2\cdot k_3)$ $(\epsilon_2\cdot k_3)\,(k_3\cdot k_2)$

Table 1. Residues of 3-tachyon, 1-vector scattering for first three levels

Note that algebraically, the first term proportional to $\epsilon_2\cdot k_1$ in (6.7) was obtained from moving an operator $\epsilon_2\cdot\alpha_0$ in $\epsilon_2\cdot\dot{X}(z_2)=\epsilon_2\cdot(\alpha_{-1}z_2^1+\dots+\alpha_0 z_2^0+\alpha_1 z_2^{-1}+\dots)$ to the left, acting upon final state $|0,k_1\rangle$ in the standard process of normal ordering, which simply reproduces the pure tachyon residue since rest of its kinematic dependence was contributed from $\langle 0,k_1\big|:e^{ik_2\cdot X(z_2)}::e^{ik_3\cdot X(z_3)}:\big|0,k_4\rangle$. It is therefore straightforward to show that, following the same expansion as in the case of pure tachyon scattering, at each mass level residue contributed from this term is connected to BCFW prescription by generating function for Stirling number of the first kind. New structure however, is found in the second term proportional to $\epsilon_2\cdot k_3$ in (6.7), which was produced by moving positive mode operators $\alpha_1 z_2^{-1}+\alpha_2 z_2^{-2}+\dots$ in $\epsilon_2\cdot\dot{X}(z_2)=\epsilon_2\cdot(\alpha_{-1}z_2^1+\dots+\alpha_0 z_2^0+\alpha_1 z_2^{-1}+\dots)$ to the right and contracting with intermediate states. For example when we have a Fock state $\frac{\alpha^{\mu_1}_{-q}\alpha^{\mu_2}_{-r}}{\sqrt{q}\sqrt{r}}\frac{1}{\sqrt{2!}}|0,p\rangle$ as intermediate state, equation (6.8) reads

$$\begin{aligned}&\left\langle 0,k_1\left|(\epsilon_2\cdot\sum_{n=1}^{\infty}\alpha_n\,z_2^{-n})e^{-\frac{1}{n}k_2\cdot\alpha_n z_2^{-n}}\frac{\alpha^{\mu_1}_{-q}\alpha^{\mu_2}_{-r}}{\sqrt{q}\sqrt{r}}\frac{1}{\sqrt{2!}}\right|0,p\right\rangle\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\\&\left\langle 0,p\left|\frac{\alpha^{\nu_1}_{q}\alpha^{\nu_2}_{r}}{\sqrt{q}\sqrt{r}}\frac{1}{\sqrt{2!}}e^{-\frac{1}{n}k_3\cdot\alpha_n z_3^{n}}\right|0,k_4\right\rangle\Bigg|_{z_2=z_3=1}.\end{aligned}\tag{6.9}$$

Contribution proportional to $\epsilon_2\cdot k_3$ is produced by contracting an α_q or α_r in $\epsilon_2\cdot\dot{X}(z_2)$ with Fock state, yielding

$$\frac{(\epsilon_2\cdot k_3)\times q}{q}\left(\frac{z_3}{z_2}\right)^q\frac{(k_3\cdot k_2)}{r}\left(\frac{z_3}{z_2}\right)^r+\frac{(k_3\cdot k_2)}{q}\left(\frac{z_3}{z_2}\right)^q\frac{(\epsilon_2\cdot k_3)\times r}{r}\left(\frac{z_3}{z_2}\right)^r\Bigg|_{z_2=z_3=1}.\tag{6.10}$$

Therefore generically residue (6.8) proportional to $\epsilon_2\cdot k_3$ at level $N=a$ is given by z^a term expansion coefficient of the derivative of generating function

$$\begin{aligned}&\frac{(\epsilon_2\cdot k_3)}{(k_2\cdot k_3)}\,z\frac{d}{dz}e^{(k_2\cdot k_3)\ln(1-z)}\\&=\frac{(\epsilon_2\cdot k_3)}{(k_2\cdot k_3)}\,z\frac{d}{dz}\left(e^{-X\,z}e^{-X\,\frac{z^2}{2}}e^{-X\,\frac{z^3}{3}}\dots\right).\end{aligned}\tag{6.11}$$

Note that we may as well express the generating function (6.11) above as

$$(\epsilon_2\cdot k_3)\,z\frac{d}{dz}\left[\ln(1-z)\right]e^{(k_2\cdot k_3)\ln(1-z)}\,,\tag{6.12}$$

from which it is obvious that BCFW prescription yields the same residue as tachyonic recursion relation of 1-vector 3-tachyon amplitude, since the tachyonic recursion relation was derived from binomial expansion of standard worldsheet integral formula that takes the same form as (6.12).

Explicit recursion relation. Here we present an explicit calculation of the term proportional to $\epsilon_2 \cdot k_3$ in eq. (6.7). By using eq. (6.8), the term proportional to $\epsilon_2 \cdot k_3$ with mass level N can be calculated by gluing two 3-point functions

$$I_N = \sum_{\{\sum mN_m = N\}} \Big\langle\, k_1; 0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n \Big) V_0(k_2) \Big| \{N_m\}\,; P \Big\rangle \mathcal{T}_{\{N_m\}} \\ \Big\langle\, \{N_m\}\,; P \Big| V_0(k_3) \Big| k_4; 0 \,\Big\rangle \Big|_{z_2=1} .$$

For convenience, let us denote the two 3-point functions as

$$A_L = A_L(k_1, k_2, P) = \Big\langle\, k_1; 0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n \Big) V_0(k_2) \Big| \{N_m\}\,; P \Big\rangle \Big|_{z_2=1}, \tag{6.13}$$

$$A_R = A_R(P, k_3, k_4) = \Big\langle\, \{N_m\}\,; P \Big| V_0(k_3) \Big| k_4; 0 \,\Big\rangle \Big|_{z_2=1}. \tag{6.14}$$

The term A_R was obtained in eq. (3.29) previously, while A_L can be calculated to be (we ignore the momentum dependent part)

$$A_L = \sum_{n=1}^{\infty} \Big\langle 0 \Big| \Big(\epsilon_2 \cdot \alpha_n \Big) \prod_{m=1}^{\infty} e^{-\frac{k_2 \cdot \alpha_m}{m}} \frac{\big(\alpha_{-m}^{\mu}\big)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \Big| 0 \Big\rangle \tag{6.15}$$

$$= \sum_{n=1}^{\infty} \Big\langle 0 \Big| \Big(\epsilon_2 \cdot \alpha_n \Big) \left[e^{-\frac{k_2 \cdot \alpha_n}{n}} \frac{\big(\alpha_{-n}^{\mu}\big)^{N_n}}{\sqrt{n^{N_n}\, N_m!}} \right] \prod_{m=1, m\neq n}^{\infty} e^{-\frac{k_2 \cdot \alpha_m}{m}} \frac{\big(\alpha_{-m}^{\mu}\big)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \Big| 0 \Big\rangle. \tag{6.16}$$

In the presence of $\epsilon_2 \cdot \alpha_n$ term, one notes that only term of order $(N_n - 1)$ in the Taylor expansion of $\exp\big[-k_2 \cdot \alpha_n / n \,\big]$ inside the square bracket will contribute. By using $\big[\, \alpha_m^{\mu}\,,\, \alpha_n^{\nu} \,\big] = m\delta_{m+n}\eta_{\mu\nu}$, we get

$$A_L = \sum_{n=1}^{\infty} \left\{ \left[\frac{(-)^{N_n - 1}\, n\, N_n\, \epsilon_2^{\mu}\, (k_2^{\mu})^{N_n - 1}}{\sqrt{n^{N_n}\, N_n!}} \right] \prod_{m=1, m\neq n}^{\infty} \frac{\big(-k_2^{\mu}\big)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \right\}. \tag{6.17}$$

Combining A_R and A_L and summing over all states with $\sum_m mN_m = N$ yields

$$I_N = \sum_{\{N = \sum_m mN_m\}} (-)\, N\, \frac{\epsilon_2 \cdot k_3}{k_2 \cdot k_3} \prod_{m=1}^{\infty} \frac{\big(-k_2 \cdot k_3\big)^{N_m}}{m^{N_m}\, N_m!}. \tag{6.18}$$

We can now use the definition of Stirling number of the first kind to get

$$I_N = \epsilon_2 \cdot k_3 \sum_{J=1}^{N} \frac{s(N, J)}{N!} (-)^{N-1}\, N\, (k_2 \cdot k_3)^{J-1}. \tag{6.19}$$

Finally the expression can be further reduced to

$$I_N = \epsilon_2 \cdot k_3 \; (-)^{N-1} \binom{k_2 \cdot k_3 - 1}{N-1}. \tag{6.20}$$

In the following, instead of the operator method adopted previously, we will use path-integral approach [17] to calculate the generating function for the rank-two tensor, three tachyons amplitude. As a warm up exercise, we first use this method to rederive eq. (6.12) for the vector, three tachyons amplitude. We first note that the amplitude can be written as

$$\mathcal{A} = \int \prod_{i=1}^{1} dz_i < e^{ik_1 X(z_1)} \epsilon_2 \cdot \partial X(z_2) e^{ik_2 X(z_3)} e^{ik_3 X(z_3)} e^{ik_4 X(z_4)} > \tag{6.21}$$

$$= \int \prod_{i=1}^{4} dz_i < e^{ik_1 X(z_1)} e^{ik_2 X(z_2) + i\epsilon_2 \cdot \partial X(z_2)} e^{ik_3 X(z_3)} e^{ik_4 X(z_4)} >|_{\text{linear in } \epsilon_2} \tag{6.22}$$

$$= \int \prod_{i=1}^{4} dz_i \exp\left[-\sum_{l<j} k_{l\mu} k_{j\nu} < X^\mu(z_l) X^\nu(z_j) > - \sum_{j \neq 2} \epsilon_{2\mu} k_{j\nu} < \partial X^\mu(z_l) X^\nu(z_j) > \right] |_{\text{linear in } \epsilon_2} \tag{6.23}$$

$$= \int_0^1 dz (1-z)^{k_2 \cdot k_3} z^{k_1 \cdot k_2} \left[\frac{\epsilon_2 \cdot k_1}{z} - \frac{\epsilon_2 \cdot k_3}{1-z} \right]. \tag{6.24}$$

In the last equality, we have used the worldsheet $\mathrm{SL}(2,R)$ to set the positions of the four vertex at $0, z, 1$ and ∞, and the propagator $< X^\mu(z_l) X^\nu(z_j) >= -\eta^{\mu\nu} \ln(z_l - z_j)$. Note that the term proportional to $\epsilon_2 \cdot k_1$ has been considered previously for the calculation of four tachyons amplitude. One can now see from eq. (6.23) that the generating function for amplitude proportional to the term $\epsilon_2 \cdot k_3$ is

$$G_1 = \exp^{\{-k_3 \cdot k_2 [-\ln(1-z)]\}} \exp^{\left\{-\epsilon_2 \cdot k_3 z \frac{d}{dz}[-\ln(1-z)]\right\}} |_{\text{linear in } \epsilon_2} \tag{6.25}$$

$$= (\epsilon_2 \cdot k_3) z \frac{d}{dz} [\ln(1-z)] \exp^{\{k_3 \cdot k_2 [\ln(1-z)]\}} \tag{6.26}$$

which is the same with eq. (6.12). Therefore the derivative of generating function in eq. (6.11) can be traced back to the derivative part ∂X^μ of the vector vertex. We now generalize the calculation to the higher spin cases. For example, for the spin two case

$$\mathcal{A} = \int \prod_{i=1}^{4} dz_i < e^{ik_1 X(z_1)} \epsilon_{2\mu\nu} \cdot \partial X^\mu(z_2) \partial X^\nu(z_2) e^{ik_2 X(z_2)} e^{ik_3 X(z_3)} e^{ik_4 X(z_4)} > \tag{6.27}$$

$$= \int \prod_{i=1}^{4} dz_i < e^{ik_1 X(z_1)} e^{ik_2 X(z_2) + i\epsilon_2^{(1)} \cdot \partial X(z_2) + i\epsilon_2^{(2)} \cdot \partial X(z_2)} e^{ik_3 X(z_3)} e^{ik_4 X(z_4)} >|_{\text{multilinear in } \epsilon_2^{(1)}, \epsilon_2^{(2)}} \tag{6.28}$$

$$= \int_0^1 dz (1-z)^{k_2 \cdot k_3} z^{k_1 \cdot k_2} \left[\frac{\epsilon_2^{(1)} \cdot k_1}{z} - \frac{\epsilon_2^{(1)} \cdot k_3}{1-z} \right] \left[\frac{\epsilon_2^{(2)} \cdot k_1}{z} - \frac{\epsilon_2^{(3)} \cdot k_3}{1-z} \right] \tag{6.29}$$

where $\epsilon_{3\mu}^{(l)} \epsilon_{3\nu}^{(j)}$ is to be identified with $\epsilon_{3\mu\nu}$. Note that the terms proportional to $k_1^\mu k_1^\nu$ and $k_1^\mu k_3^\nu$ have been considered previously for the calculation of four tachyons and one vector, three tachyons amplitudes respectively. The only new term is the one proportional to $k_3^\mu k_3^\nu$,

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which can be expressed as

$$\mathcal{A}_4 = \sum_{a=2}^{\infty} \binom{k_2 \cdot k_3 - 2}{a-2} (-1)^{a-2} \frac{2}{(k_1+k_2)^2 + 2(a-1)} \epsilon_{2\mu\nu} k_3^\mu k_3^\nu. \tag{6.30}$$

The generating function for this term can be seen from eq. (6.28) as

$$G_2 = \exp^{\{-k_3 \cdot k_2[-\ln(1-z)]\}} \exp^{\left\{-\epsilon_2^{(1)} \cdot k_3 z \frac{d}{dz}[-\ln(1-z)]\right\}} \exp^{\left\{-\epsilon_2^{(2)} \cdot k_3 z \frac{d}{dz}[-\ln(1-z)]\right\}} |_{\text{multilinear in } \epsilon_2^{(1)}, \epsilon_2^{(2)}} \tag{6.31}$$

$$= \left(\epsilon_2^{(1)} \cdot k_3\right) z \frac{d}{dz}[\ln(1-z)] \exp^{\left\{\frac{k_3 \cdot k_2}{2}[\ln(1-z)]\right\}} \left(\epsilon_2^{(1)} \cdot k_3\right) z \frac{d}{dz}[\ln(1-z)] \exp^{\left\{\frac{k_3 \cdot k_2}{2}[\ln(1-z)]\right\}} \tag{6.32}$$

$$= \sum_{a=2}^{\infty} \binom{k_2 \cdot k_3 - 2}{a-2} (-1)^{a-2} \epsilon_{2\mu\nu} k_3^\mu k_3^\nu z^a. \tag{6.33}$$

Eq. (6.32) contains product of two derivative terms which again can be traced back to $\partial X^\mu \partial X^\nu$ part of the spin two vertex. After setting $z = 1$ in eq. (6.33) above, one can match with the correct result in eq. (6.30).

The calculation above can be generalized to arbitrary higher spin vertex. We thus conclude that generically generating function for Stirling number of the first kind connects BCFW precription with scalar-like recursion relation to arbitrary high spin level scattering, provided that the corresponding derivatives in its worldsheet integral expression are included.

7 Conclusions

Starting from the familiar 4-point Veneziano formula we have demonstrated that the scalar-like recursion relation observed by Cheung, O'Connell and Wecht in [6] and by Fotopoulos in [7] can indeed be understood from BCFW on-shell recursion relation of string amplitudes. We showed that explanation to the absence of higher-spin modes was very much like a similar mechanism observed in BCFW on-shell recursion relation of gauge theory amplitudes: while in gauge theory Ward identity guarantees that two unphysical degrees of freedom necessary to make up for the completeness relation [13]

$$g_{\mu\nu} = \epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+ + \epsilon_\mu^L \epsilon_\nu^T + \epsilon_\mu^T \epsilon_\nu^L \tag{7.1}$$

decouple, in bosonic string amplitude the No-Ghost Theorem does the same thing to decouple necessary unphysical degrees of freedom that make up for the whole Fock space completeness relation, which makes the translation between covariant and scalar-behaved on-shell relations of string amplitudes. The freedom to translate on-shell recursion relation between Fock state and physical state is especially of practical interests since writing down polarization tensors for generic physical high-spin modes can be quite complicated in string theory context.

Although our method can be used to calculate string scattering amplitudes using the on-shell recursion relation, it may be not the best way to do so. However, it could provide another point of view to discuss some analytic properties of string theory along, for example, the work of Benincasa and Cachazo [18], and the work of Fotopoulous and Tsulaia [9], based on consistency using different BCFW-deformations to calculate amplitudes. It can also be used to discuss possible loop amplitudes using unitarity cut method [14, 15].

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A Mathematical identity

Stirling Number of the first kind: the Stirling numbers of the first kind is defined from the generation function

$$(x)_n \equiv x(x-1)\dots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k \tag{A.1}$$

where $(x)_n$ is the *Pochhammer symbol* for the falling factorial and when $n=0$, $(x)_0 \equiv 1$. Using this, we can see that $s(0,0)=1$ but $s(n,0)=0$ if $n \neq 0$.

The signed Stirling numbers of the first kind are defined such that the number of permutations of n elements which contain exactly m permutation cycles is the nonnegative number

$$|s(n,m)| = (-)^{n-m}s(n,m) = n! \sum_{\{N_t\}} \prod_{t=1}^{\infty} \frac{1}{N_t! t^{N_t}}, \qquad \sum tN_t = n, \quad m = \sum N_t \tag{A.2}$$

There are other ways to see above identities. Considering following Taylor expansion

$$I_1 = (1-z)^X = \sum_{a=0}^{\infty} \begin{pmatrix} X \\ a \end{pmatrix} (-)^a z^a \tag{A.3}$$

which can be expanded by following alternative way

$$\begin{aligned} \exp\left[\, X \ln(1-z)\right] &= \exp\left[(-X) \left(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + \frac{1}{n}z^n + \cdots \right) \right] \\ &= \sum_{N=0}^{\infty} \sum_{J=0}^{N} \frac{\left|s(N,J)\right|}{N!} (-X)^J z^N \\ &= \sum_{N=0}^{\infty} \sum_{J=0}^{N} \frac{s(N,J)}{N!} (-)^N X^J z^N \end{aligned} \tag{A.4}$$

Comparing these two expansions we can refer (A.1) and (A.2).

B Decoupling of Ghosts in string amplitude

The content in this section can be found in [16]. In bosonic string theory, physical states are required to satisfy Virasoro constraints $(L_0 - 1)\,|\phi\rangle = 0$ and $L_{m>0}\,|\phi\rangle = 0$. As we have seen in section 3.2, the first of these two types of constraints was implemented as on-shell condition (3.7) so that it is satisfied by intermediate states that appear in BCFW recursion relation. In this appendix we prove that ghosts decouples from BCFW recursion relation. As a consequence we are allowed to introduce freely the physical states, for which the remaining Virasoro constraint $L_{m>0}\,|\phi\rangle = 0$ applies, or generic Fock states as intermediate states in the recursion relation. For the purpose of argument needed in this proof we first divide Fock space into three subspaces according to DDF construction.

B.1 DDF states

A standard DDF state is defined by acting a string of transverse A^i_{-n} operators on tachyonic vacuum

$$|f\rangle = A^{i_1}_{-n_1} A^{i_2}_{-n_2} \dots A^{i_m}_{-n_m} \,|0; p_0\rangle \ , \tag{B.1}$$

where DDF operator A^i_n is prescribed as the Fourier zero mode of vector vertex operator $V_j(nk_0, \tau) = \dot{X}^j(\tau) e^{inX^+(\tau)}$,

$$A^i_n = \frac{1}{2\pi} \int_0^{2\pi} \dot{X}^i(\tau) e^{inX^+(\tau)} d\tau, \quad i = 1, \dots, D-2\,, \tag{B.2}$$

and $p_0 = (p_0^+, p_0^-, p_0^i) = (1, -1, 0)$. It is easy to show that $L_{m>0}\,|f\rangle = 0$ since L_m commutates with all A^i_{-n} while $L_{m>0}\,|0; p_0\rangle = 0$. For L_0, using that $L_0\,\left|0; p_0^2\right\rangle = \alpha' p_0^2 = 1$ we get $(L_0 - 1)\,|0; p_0\rangle = 0$. The DDF states thus defined are positive definite, as can be easily checked using the commutation relation $[A^i_m, A^j_n] = m\delta_{ij}\delta_{m+n}$. We shall denote in the following a generic DDF state as $|f\rangle$. Note however, that in the standard construction these DDF states are automatically on the N-mass-shell,

$$\widehat{p}\,|f\rangle = \left(p_0 + k_0 \sum n_i\right) |f\rangle \tag{B.3}$$

so that $(p_0 + Nk_0)^2 = p_0^2 + 2N = 2 + 2N$, where we introduced $k_0 = (k_0^+, k_0^-, k_0^i) = (0, -1, 0)$, and here $N = \sum n_i$. In order to describe Fock states in DDF language, where center-of-mass momentum k^μ and mode number N are considered independent, let us define generalized off-shell DDF-like state, starting again from tachyonic vacuum but with momentum $q + N\,k_0$,

$$|f\rangle_{\mathrm{off-shell}} = A^{i_1}_{-n_1} A^{i_2}_{-n_2} \dots A^{i_m}_{-n_m} \,|0; q + N\,k_0\rangle\,. \tag{B.4}$$

Note that we shift ground state momentum by equal and opposite of the amount that is going to be shifted by DDF operators so that subsequent operations produces an off-shell state with arbitrary momentum q and mode eigenvalue $N = \sum_i n_i$. In addition to DDF operators we introduce operators K_m, defined as

$$K_m = k_0 \cdot \alpha_m = -\alpha^+_m \tag{B.5}$$

and consider states constructed by operating a string of Virasoro generator L_{-n} and K_{-m} on DDF-like state $|f\rangle_{\text{off-shell}}$ carrying off-shell momentum q in the following order

$$|\{\lambda,\mu\},f\rangle = L_{-1}^{\lambda_1}L_{-2}^{\lambda_2}\dots L_{-n}^{\lambda_n}K_{-1}^{\mu_1}\dots K_{-m}^{\mu_m}\,|f\rangle_{\text{off-shell}}\;. \tag{B.6}$$

The set of states $|\{\lambda,\mu\},f\rangle$ with $\sum r\lambda_r+\sum s\mu_s+\sum n_i=N$ are linearly independent and constitutes a basis that spans level-N subspace at fixed momentum q. In the following discussions for convenience we drop the lower script that distinguishes DDF state $|f\rangle$ and DDF-like state $|f\rangle_{\text{off-shell}}$, while it is understood that the center-of-mass momentum is considered as a independent degree of freedom, on-shell or not, whenever a DDF basis is referred to.

B.2 Decoupling of ghosts in string amplitude

States (B.6) can be divided into two types. The first type is with L_{-n} in front, so it is spurious state $|s\rangle$. The second one is without L_{-n} and we denote it as $|k\rangle$. Thus any state in the Fock space can be uniquely decomposed as

$$|\phi\rangle = |s\rangle + |k\rangle \tag{B.7}$$

where $|s\rangle$ is the *spurious* state and $|k\rangle$ is the form in (B.6) without any L_{-n} in front of the expression. Since $|s\rangle\,,|k\rangle$ are linear independently, if $|\phi\rangle$ is the eigenstate of L_0, so are $|s\rangle\,,|k\rangle$. This means that if

$$(L_0-1)\,|\phi\rangle=0,\qquad \Longrightarrow (L_0-1)\,|s\rangle=(L_0-1)\,|k\rangle=0 \tag{B.8}$$

Next we show that if the state $|\phi\rangle$ is physical state, the decomposed states $|s\rangle$ and $|k\rangle$ are also physical states.

Because $|s\rangle$ is spurious and physical when $|\phi\rangle$ is physical, we have $\langle s|s\rangle=\langle s|k\rangle=0$, so $\langle\phi|\phi\rangle=\langle k|k\rangle$. We can decompose $|k\rangle=|f\rangle+\left|\widetilde{k}\right\rangle$ where $|f\rangle$ is DDF state and $|k\rangle$ is the form of (B.6) without string of L but at least one of K_{-m}. By the property of K_{-m}, it is easy to shown that $\left\langle\widetilde{k}|\widetilde{k}\right\rangle=\left\langle\widetilde{k}|f\right\rangle=0$, so finally we have $\langle\phi|\phi\rangle=\langle k|k\rangle=\langle f|f\rangle$. This is the familiar result known as the "No-ghost Theorem" for string *amplitude*, which can also be characterized as the absence of negative norm among general physical state $|\phi\rangle$.

In fact, there is a stronger statement. Using $[L_m,K_n]-nK_{m+n}$ and $L_{m>0}\,|f\rangle=0$, it can show that if $|k\rangle$ is physical, then $\left|\widetilde{k}\right\rangle=0$ in the expansion of $|k\rangle=|f\rangle+\left|\widetilde{k}\right\rangle$. Thus we see that the general physical state $|\phi\rangle$ can be written

$$|\phi\rangle=|f\rangle+|s\rangle \tag{B.9}$$

where $|f\rangle$ is a DDF state and $|s\rangle$ is a spurious physical state. The appearance of spurious physical state $|s\rangle$, i.e., the transformation $|f\rangle\to|f\rangle+|s\rangle$ is the string-theoretic analog of a gauge transformation.

B.3 Decoupling of ghosts in BCFW on-shell recursion relation

In section 5 we saw that pole structure in a bosonic string amplitude is manifest when expressed in algebraic form

$$A_M = \langle\phi_1| V_2\Delta V_3 \dots V_i\Delta V_{i+1} \dots V_{M-1}\Delta |\phi_M\rangle\,. \tag{B.10}$$

Residue at the $(i-1)$-th pole at mass level N is therefore given by the sum of products

$$\sum_{\substack{\text{level-N}\\ \text{states}}} \langle\phi_1(k_1+z_{i,N}\,q)| V_2\Delta V_3 \dots V_i |\{N_{\mu,m}\},\hat{p}\rangle\langle\{N_{\mu,m}\},\hat{p}| V_{i+1} \dots \Delta V_{M-1} |\phi_M(k_M-z_{i,N}\,q)\rangle\,, \tag{B.11}$$

where the above sum is taken only over intermediate Fock states that happen to be on the level-N mass-shell. Note that in BCFW recursion relation the mode eigenvalues $\{N_{\mu,m}\}$ and center-of-mass momentum $\hat{p}$ of intermediate states were originally considered as independent. *It is because $\{N_{\mu,m}\}$ and $\hat{p}$ assume the values $\sum N_{\mu,m} = N$ and $\frac{1}{2}\hat{p}^2(z)+N-1=0$ that a pole was created at $z=z_{i,N}$ in the first place*, so that at pole the mass-shell condition is automatically satisfied.

Consider the state

$$|\phi_R\rangle = V_{i+1}\Delta V_{i+2} \dots \Delta V_{M-1} |\phi_M\rangle \tag{B.12}$$

that appears on the right side of equation (B.11). Since we are only interested in its product with on-shell states, let us operate on it a projection operator P_1. For the purpose of proving decoupling of ghosts, first we would like to show that

$$L_{m>0}P_1 |\phi_R\rangle = 0, \tag{B.13}$$

where we defined P_k as a projection operator which projects states to subspace with $L_0 = k$. Using $[L_0, L_m] = -mL_m$, we find $L_0 L_m P_1 |\alpha\rangle = (1-m)L_m P_1 |\alpha\rangle$, so $L_m P_1 = P_{1-m}L_m P_1 = P_{1-m}L_m$, thus we need to prove

$$P_{1-m}L_m |\phi_R\rangle = 0, \qquad m > 0 \tag{B.14}$$

Using $P_{1-m}(-L_0 - m + 1) = 0$, we get

$$P_{1-m}(L_m - L_0 - m + 1) |\phi_R\rangle = 0, \qquad m > 0 \tag{B.15}$$

Finally we arrive at the identity

$$(L_m - L_0 - m + 1)V_N\Delta V_{N+1} \dots \Delta V_{M-1} |\phi_M\rangle = 0, \qquad m > 0 \tag{B.16}$$

Note that a vertex V has conformal dimension one, therefore satisfies

$$[L_m, V(k,z)] = \left(z^{m+1}\frac{d}{dz} + mz^m\right) V(k,z). \tag{B.17}$$

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Now using the (B.17) and set $z=1$ (since we have $\tau=0$ which is crucial) we have

$$[L_m - L_0, V] = mV, \quad or \quad (L_m - L_0 - m + 1)V = V(L_m - L_0 + 1) \tag{B.18}$$

where $\frac{d}{dz}$ has been canceled. Using Virasoro algebra it is straightforward to show that

$$(L_m - L_0 + 1)\frac{1}{L_0 - 1} = \frac{1}{L_0 + m - 1}(L_m - L_0 - m + 1) \tag{B.19}$$

Thus (B.18) and (B.19) give

$$(L_m - L_0 - m + 1)V\frac{1}{L_0 - 1} = V\frac{1}{L_0 + m - 1}(L_m - L_0 - m + 1) \tag{B.20}$$

so $(L_m - L_0 - m + 1)$ can be pushed step by step all the way to the right until it meets $|\phi_M\rangle$, and we obtain $(L_m - L_0 + 1)|\phi_M\rangle = 0$ because $|\phi_M\rangle$ is physical. From the argument above we see that when on-shell, $|\phi_M\rangle$ satisfies Virasoro constraints and is therefore a physical state. It is straightforward to see that the same argument applies to state $|\phi_L\rangle = V_i \Delta V_{i-1} \ldots \Delta V_2 |\phi_1\rangle$.

Proof: having done all the preparations we are now finally ready to derive our proof. We note that in the algebraic expression (B.11) for residue at mass level N, the summation of outer products of Fock states $|\{N_{\mu,m}\}, \hat{p}\rangle \mathcal{T}_{\{N_{\mu,m}\}} \langle \{N_{\mu,m}\}, \hat{p}|$ over level-N subspace works as a projection operator that maps $|\phi_R\rangle$ and $|\phi_L\rangle$ into the level-N subspace, so that if we decompose in this sector $|\phi_R\rangle$ and $|\phi_L\rangle$ according to DDF basis into $|s\rangle + \left|\tilde{k}\right\rangle + |f\rangle$, the residue (B.11) reads

$$\begin{aligned}\sum_{\substack{\text{level-N}\\ \text{states}}} \langle \phi_L | \{N_{\mu,m}\}, \hat{p}\rangle \mathcal{T}_{\{N_{\mu,m}\}} \langle \{N_{\mu,m}\}, \hat{p} | \phi_R\rangle &= \left\langle s_L + \tilde{k}_L + f_L | s_R + \tilde{k}_R + f_R \right\rangle \\ &= \langle f_L | f_R \rangle\,. \end{aligned} \tag{B.21}$$

As argued in the decoupling of ghosts in amplitudes, spurious state $|s\rangle$ drop out from (B.21) because both $|\phi_R\rangle$ and $|\phi_L\rangle$ are physical, and we remove subsequently $\left|\tilde{k}\right\rangle$ states since $\left\langle \tilde{k}|\tilde{k} \right\rangle = \left\langle \tilde{k}|f \right\rangle = 0$.

Inserting complete states again, but this time in DDF basis, into the product $\langle f_L|f_R\rangle$,

$$\begin{aligned}\langle f_L | f_R \rangle &= \sum_i \langle f_L | s_i + k_i + f_i \rangle \langle s_i + k_i + f_i | f_R \rangle \\ &= \sum_i \langle f_L | f_i \rangle \langle f_i | f_R \rangle = \sum_i \langle f_L + s_L | f_i \rangle \langle f_i | f_R + s_R \rangle \\ &= \sum_i \langle \phi_L | f_i \rangle \langle f_i | \phi_R \rangle \end{aligned} \tag{B.22}$$

and we see that spurious and $\left|\tilde{k}\right\rangle$ intermediate states drop out for the same reason, thus summing over the whole intermediate Fock space is equivalent to summing over the physical subspace.

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