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Numerical ranges of weighted shift matrices

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ARTICLE INFO

Article history:

Received 1 August 2010

Accepted 18 January 2011

Available online 15 February 2011

Submitted by M. Tsatsomeros

In memory of Ky Fan (1914–2010), a great mathematician.

AMS classification:

15A60

Keywords:

Numerical range

Weighted shift matrix

Nilpotent matrix

ABSTRACT

We show that if A is an n -by- n ($n \geq 3$) matrix of the form

$$\begin{bmatrix} 0 & a_1 & & & \\ & 0 & \ddots & & \\ & & \ddots & a_{n-1} & \\ a_n & & & & 0 \end{bmatrix},$$

then the boundary of its numerical range contains a line segment if and only if the a_j 's are nonzero and the numerical ranges of the $(n-1)$ -by- $(n-1)$ principal submatrices of A are all equal. For $n = 3$, this is the case if and only if $|a_1| = |a_2| = |a_3| \neq 0$, in which case $W(A)$, the numerical range of A , is the equilateral triangular region with vertices the three cubic roots of $a_1 a_2 a_3$. For $n = 4$, the condition becomes $|a_1| = |a_3| \neq 0$ and $|a_2| = |a_4| \neq 0$, in which case $W(A)$ is the convex hull of two (degenerate or otherwise) ellipses.

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An n -by- n ($n \geq 2$) weighted shift matrix A is one of the form

$$\begin{bmatrix} 0 & a_1 & & & \\ & 0 & \ddots & & \\ & & \ddots & a_{n-1} & \\ a_n & & & & 0 \end{bmatrix},$$

where the a_j 's, called the *weights* of A , are complex numbers. The purpose of this paper is to study the numerical ranges of such matrices.

Recall that for any n -by- n complex matrix A , its numerical range $W(A)$ is by definition the subset $\{ \langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1 \}$ of the plane, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard inner product and its associated norm in \mathbb{C}^n . It is known that $W(A)$ is a nonempty compact convex subset of \mathbb{C} . For its other properties, the reader may consult [7, Chapter 1] or [5].

The numerical ranges of certain weighted shift matrices are easier to determine. For example, if any of the weights of an n -by- n weighted shift matrix A is zero, then its numerical range is a circular disc centered at the origin. On the other hand, if all the weights of A have equal (nonzero) moduli, then $W(A)$ is a polygonal region with its boundary a regular n -gon. The main theorem below gives necessary and sufficient conditions for the boundary of $W(A)$ to have a line segment. More specifically, it is shown that this is the case if and only if the weights are all nonzero and all its $(n - 1)$ -by- $(n - 1)$ principal submatrices have identical numerical ranges (which are necessarily circular discs centered at the origin). This is then used to give a characterization of such a matrix A of size 4 with line segments on $\partial W(A)$ purely in terms of its weights, namely, for

$$A = \begin{bmatrix} 0 & a_1 & & \\ & 0 & a_2 & \\ & & 0 & a_3 \\ a_4 & & & 0 \end{bmatrix},$$

the boundary of $W(A)$ has a line segment if and only if $|a_1| = |a_3| \neq 0$ and $|a_2| = |a_4| \neq 0$. Along the way, we also prove various properties of the numerical ranges of such matrices.

In the literature, there are works on the numerical ranges and numerical radii of weighted shift matrices and operators. For example, [12, Lemma 2] gives a method to compute the numerical radius of a weighted shift matrix with at least one zero weight. The authors in Refs. [11, 12, 1] discuss properties of the numerical ranges and numerical radii of weighted shifts on l^2 with periodic or geometric weights.

For an n -by- n matrix A , let A^T denote its transpose, A^* its adjoint and $\text{Re } A$ its real part $(A + A^*)/2$. For $1 \leq i_1 < \dots < i_m \leq n$, let $A[i_1, \dots, i_m]$ denote the $(n - m)$ -by- $(n - m)$ principal submatrix of A obtained by deleting its rows and columns indexed by i_1, \dots, i_m . The *numerical radius* $w(A)$ and *generalized Crawford number* $w_0(A)$ of A are, by definition, $\max \{ |z| : z \in W(A) \}$ and $\min \{ |z| : z \in \partial W(A) \}$, respectively. A diagonal matrix with diagonals a_1, \dots, a_n is denoted by $\text{diag}(a_1, \dots, a_n)$. Our basic reference for properties of matrices is [6].

For any nonzero complex number $z = x + iy$ (x and y real), $\arg z$ is the angle θ , $0 \leq \theta < 2\pi$, from the positive x -axis to the vector (x, y) . If $z = 0$, then $\arg z$ can be an arbitrary real number. In the following, let $B(0; r) = \{ z \in \mathbb{C} : |z| \leq r \}$ for $r > 0$ and $\omega_n = e^{2\pi i/n}$ for $n \geq 1$. For any subset Δ of \mathbb{C} , Δ^\wedge denotes its convex hull.

The main result of this paper is the following.

Theorem 1. *Let A be an n -by- n ($n \geq 3$) weighted shift matrix with weights a_1, \dots, a_n . Then $\partial W(A)$ has a line segment if and only if the a_j 's are nonzero and $W(A[1]) = \dots = W(A[n])$. In this case, $W(A[j])$ is the circular disc centered at the origin with radius $w_0(A)$, the line segment lies on one of the lines $x \cos \theta_k + y \sin \theta_k = w_0(A)$, where $\theta_k = ((2k + 1)\pi + \sum_{j=1}^n \arg a_j)/n$, $0 \leq k < n$, and there are exactly n line segments on $\partial W(A)$.*

The necessity of this theorem is easier to establish. It follows from the next two results.

Lemma 2. Let A and B be n -by- n ($n \geq 2$) weighted shift matrices with weights a_1, \dots, a_n and b_1, \dots, b_n , respectively.

- (1) If, for some fixed k , $1 \leq k \leq n$, $b_j = a_{k+j} (a_{n+j} \equiv a_j)$ for all j , then A is unitarily equivalent to B .
- (2) If $|a_j| = |b_j|$ for all j , then A is unitarily equivalent to $e^{i\psi_k} B$, where $\psi_k = (2k\pi + \sum_{j=1}^n (\arg a_j - \arg b_j))/n$ for $0 \leq k < n$.

Proof

- (1) If U is the n -by- n weighted shift matrix with weights $1, \dots, 1$, then U is unitary and $AU^{n-k} = U^{n-k}B$. This proves the unitary equivalence of A and B .
- (2) If $U = \text{diag}(e^{i\phi_1}, \dots, e^{i\phi_n})$, where $\phi_1 = 0$ and $\phi_j = \phi_{j-1} + (\arg b_{j-1} - \arg a_{j-1}) + \psi_k$ for $2 \leq j \leq n$, then U is unitary and $AU = U(e^{i\psi_k} B)$. \square

Using this lemma, we can deduce many properties of the numerical range of a weighted shift matrix. These are gathered together in the following. Note that some of them have already been obtained in [8], the Ph.D. dissertation of Issos on irreducible nonnegative matrices. For example, (1) below is a special case of [8, Theorem 6] and (4) follows from [8, Theorem 7].

Proposition 3. Let A be an n -by- n ($n \geq 3$) weighted shift matrix with weights a_1, \dots, a_n , and let $\theta = (\sum_{j=1}^n \arg a_j)/n$.

- (1) $W(A)$ satisfies $W(A) = \omega_n W(A)$.
- (2) $W(A)$ is symmetric with respect to the lines $y = x \tan \gamma_k$, where $\gamma_k = (k\pi/n) + \theta$ for $0 \leq k < n$.
- (3) The following conditions are equivalent:
 - (a) $a_j = 0$ for some j ,
 - (b) A is unitarily equivalent to $e^{i\theta} A$ for all real θ , and
 - (c) $W(A)$ is a circular disc centered at the origin.
- (4) If $a_j \neq 0$ for all j , then $\partial W(A)$ intersects $\partial B(0; w(A))$ (resp., $\partial B(0; w_0(A))$) at exactly the n points $w(A)e^{i\alpha_k}$ (resp., $w_0(A)e^{i\beta_k}$), where $\alpha_k = (2k\pi/n) + \theta$ (resp., $\beta_k = ((2k - 1)\pi/n) + \theta$) for $0 \leq k < n$.
- (5) $w(A) \leq w_0(A) \sec(\pi/n)$ and

$$B(0; w_0(A)) \subseteq W(A) \subseteq w_0(A) \left(\sec \frac{\pi}{n} \right) e^{i\theta} \{1, \omega_n, \dots, \omega_n^{n-1}\}^\wedge.$$

- (6) If $\partial W(A)$ has a line segment L , then $\text{dist}(0, L) = w_0(A)$, L lies on one of the lines $x \cos \beta_k + y \sin \beta_k = w_0(A)$, where the β_k 's are as in (4), and there are exactly n line segments on $\partial W(A)$.

Proof. (1) Letting $B = A$ and $k = 1$ in Lemma 2 (2) yields the unitary equivalence of A and $\omega_n A$. The assertion $W(A) = \omega_n W(A)$ then follows.

(2) By Lemma 2 (2), A is unitarily equivalent to $e^{i\alpha_k} B$ (resp., $e^{i\beta_k} C$), where B (resp., C) is the n -by- n weighted shift matrix with weights $|a_1|, \dots, |a_n|$ (resp., $|a_1|, \dots, |a_{n-1}|, -|a_n|$) and $\alpha_k = (2k\pi/n) + \theta$ (resp., $\beta_k = ((2k - 1)\pi/n) + \theta$) for $0 \leq k < n$. Since the numerical ranges of the real matrices B and C are symmetric with respect to the x -axis, the unitary equivalences above yield the symmetry of $W(A)$ with respect to the lines $l_k : y = x \tan \alpha_k$ (resp., $l_k : y = x \tan \beta_k$), $0 \leq k < n$. Note that for odd n , l_k and $l_{(k+(n+1)/2) \pmod n}$ coincide while for even n , l_k and $l_{(k+(n/2)) \pmod n}$ (resp., l_k and $l_{(k+(n/2)) \pmod n}$) coincide. Thus $W(A)$ is symmetric with respect to the n distinct lines among them, namely, the lines $y = x \tan \gamma_k$, $0 \leq k < n$.

(3) If (a) holds, then the ψ_k 's in Lemma 2 (2) can be arbitrary. Letting $B = A$ in there, we obtain (b). The implication (b) \Rightarrow (c) is trivial. To prove (c) \Rightarrow (a), note that 0 , the center of the circular disc $W(A)$,

is an eigenvalue of A (cf. [10, Theorem 4.2]). Hence $\det A = (-1)^{n+1} a_1, \dots, a_n = 0$, which shows that $a_j = 0$ for some j .

(4) We first prove the assertion for $w(A)$. As shown in (2) above, A is unitarily equivalent to $e^{i\alpha_k} B$ for $0 \leq k < n$. Since B is a nonnegative matrix, $w(B)$ belongs to $W(B)$ (cf. [9, Proposition 3.3]). Hence $w(A)$ is in $e^{-i\alpha_k} W(A)$ and thus $w(A)e^{i\alpha_k}$ is in $W(A)$ for $0 \leq k < n$. If there are more than n points in $\partial W(A) \cap \partial B(0; w(A))$, then Anderson’s theorem (cf. [13, Lemma 6] or [10, Theorem 4.12]) implies that $W(A)$ and $B(0; w(A))$ coincide. This would contradict the assertions in (3) since $a_j \neq 0$ for all j . Hence $\partial W(A) \cap \partial B(0; w(A))$ consists of exactly the n points $w(A)e^{i\alpha_k}, 0 \leq k < n$.

For the assertion on $w_0(A)$, as in (2) above, A is unitarily equivalent to $e^{i\beta_k} C$ for $0 \leq k < n$. Let $w_0(C)e^{i\beta}$ (β real) be any point in $\partial W(C) \cap \partial B(0; w_0(C))$. Then $w_0(A)e^{i(\beta+\beta_k)}$ is in $\partial W(A) \cap \partial B(0; w_0(A))$ for $0 \leq k < n$. In particular, $W(A)$ contains the regular n -polygonal region with these points as vertices. From the convexity of $W(A)$ and the minimality of $w_0(A)$, we infer that $B(0; w_0(A)) \subseteq W(A)$. If there are more than n points in $\partial W(A) \cap \partial B(0; w_0(A))$, then [3, Theorem 2.5 (b)] implies that $\partial W(A)$ contains at least one arc of $\partial B(0; w_0(A))$. The n -symmetry of $W(A)$ from (1) then yields that either $\partial W(A) = \partial B(0; w_0(A))$ or $\partial W(A)$ contains n arcs of $\partial B(0; w_0(A))$. The former implies that $W(A) = B(0; w_0(A))$, which contradicts (3), while the latter would contradict [3, Theorem 2.5 (a)]. Thus $\partial W(A) \cap \partial B(0; w_0(A))$ consists of exactly n points. These points must be on the lines $y = x \tan \gamma_k$ in (2) for otherwise by symmetry there will be at least $2n$ points in $\partial W(A) \cap \partial B(0; w_0(A))$ contradicting what we have proved above. We conclude that the points in $\partial W(A) \cap \partial B(0; w_0(A))$ are exactly $w_0(A)e^{i\beta_k}, 0 \leq k < n$.

(5) Since the points $w(A)e^{i\alpha_k}, 0 \leq k < n$, are in $W(A)$ by (4), the regular n -polygonal region R whose vertices are these points is contained in $W(A)$. Hence

$$w_0(A) \geq \text{dist}(0, R) = w(A) \frac{1}{2} |e^{i\alpha_0} + e^{i\alpha_1}| = w(A) \cos \frac{\pi}{n}.$$

This proves that $w(A) \leq w_0(A) \sec(\pi/n)$.

The containment $B(0; w_0(A)) \subseteq W(A)$ was already noted in the proof of (4). For the other direction, note that if u is any point of $W(A)$ which is in a different half-plane, determined by the line L connecting $w_0(A) \sec(\pi/n)e^{i\alpha_0}$ and $w_0(A) \sec(\pi/n)e^{i\alpha_1}$, from the origin, then, by (2), its symmetric point u' with respect to the line connecting 0 and $w_0(A)e^{i\beta_1}$ is also in $W(A)$. Thus $(u + u')/2$ is in $W(A)$, which would contradict the fact that $w_0(A)e^{i\beta_1}$ is on the boundary of $W(A)$. This shows that $W(A)$ is contained in the same half-plane of L as the origin. The n -symmetry of $W(A)$ from (1) then yields that

$$\begin{aligned} W(A) &\subseteq w_0(A) \left(\sec \frac{\pi}{n} \right) \left\{ e^{i\alpha_k} : 0 \leq k < n \right\}^\wedge \\ &= w_0(A) \left(\sec \frac{\pi}{n} \right) e^{i\theta} \left\{ \omega_n^k : 0 \leq k < n \right\}^\wedge. \end{aligned}$$

(6) If L is a line segment on $\partial W(A)$, then L intersects $\partial W(A[j])$ for every $j, 1 \leq j \leq n$ (cf. [2, Lemma 5]). Since $W(A[j]) \subseteq W(A)$ and $W(A[j])$ is a circular disc centered at the origin, we obtain $\text{dist}(0, L) = w_0(A[j]) \leq w_0(A)$ for every j . But $\text{dist}(0, L) \geq w_0(A)$ is obviously true. This shows that $\text{dist}(0, L) = w_0(A[1]) = \dots = w_0(A[n]) = w_0(A)$. It follows from (4) that L must lie on one of the lines $x \cos \beta_k + y \sin \beta_k = w_0(A), 0 \leq k < n$. Together with (1), this implies that there are exactly n line segments on $\partial W(A)$. \square

The necessity of Theorem 1 then follows easily from Proposition 3 (3) and (6).

As a side result, the next proposition gives conditions for a weighted shift matrix to have a regular polygonal region as its numerical range. The equivalence of some conditions below can also be derived from [8, Theorem 13].

Proposition 4. *Let A be a nonzero n -by- n ($n \geq 3$) weighted shift matrix with weights a_1, \dots, a_n . Then the following conditions are equivalent:*

- (1) A is normal,
- (2) $|a_1| = \dots = |a_n|$,

- (3) A is unitarily equivalent to $\text{diag}(\lambda, \lambda\omega_n, \dots, \lambda\omega_n^{n-1})$, where $\lambda = (a_1, \dots, a_n)^{1/n}$,
- (4) $W(A)$ is a regular n -polygonal region with center at the origin and the distance from the center to its vertices equal to $|a_1, \dots, a_n|^{1/n}$,
- (5) $\partial W(A)$ has a nondifferentiable point, and
- (6) $w(A) = w_0(A) \sec(\pi/n)$.

Proof. That (1) \Rightarrow (2), (3) \Rightarrow (4) and (4) \Rightarrow (5) are trivial. To prove (2) \Rightarrow (3), note that, under (2), A is unitarily equivalent to $|a_1|e^{i\theta}B$, where B is the n -by- n weighted shift matrix with weights $1, \dots, 1$ by Lemma 2 (2). It is easily seen that B is unitarily equivalent to $\text{diag}(1, \omega_n, \dots, \omega_n^{n-1})$ and $|a_1|e^{i\theta} = |a_1|e^{i(\sum_{j=1}^n \arg a_j)/n} = (a_1, \dots, a_n)^{1/n} = \lambda$. Hence (3) follows. For (5) \Rightarrow (1), if λ is a nondifferentiable point of $\partial W(A)$, then so are $\lambda\omega_n^k, 0 \leq k < n$, by Proposition 3 (1). Since each of such points is a reducing eigenvalue of A , we obtain that A is unitarily equivalent to $\text{diag}(\lambda, \lambda\omega_n, \dots, \lambda\omega_n^{n-1})$. In particular, A is normal, that is, (1) holds. Finally, if (4) holds, then (2) is true and hence

$$w(A) = |a_1, \dots, a_n|^{1/n} = |a_1| = w_0(A) \sec \frac{\pi}{n},$$

that is, (6) holds. Conversely, if (6) is true, then Proposition 3 (5) says that $W(A) \subseteq w(A)e^{i\theta}\{1, \omega_n, \dots, \omega_n^{n-1}\}^\wedge$. But the vertices of this latter regular n -polygonal region, namely, $w(A)e^{i\theta}\omega_n^k, 0 \leq k < n$, are in $W(A)$ by Proposition 3 (4). Hence we must have $W(A) = w(A)e^{i\theta}\{1, \omega_n, \dots, \omega_n^{n-1}\}^\wedge$. Hence $\partial W(A)$ has nondifferentiable points, that is, (5) holds. \square

We now proceed to prepare ourselves for the proof of the sufficiency of Theorem 1. This will be done in a series of lemmas and propositions. We start with the following.

Lemma 5. *Let A and B be the n -by- n ($n \geq 2$) weighted shift matrices with weights $a_1, \dots, a_{n-1}, 0$ and $b_1, \dots, b_{n-1}, 0$, respectively.*

- (1) If $|a_j| \leq |b_j|$ for all j , then $W(A) \subseteq W(B)$.
- (2) If the b_j 's are nonzero, $|a_j| \leq |b_j|$ for all j and $|a_k| < |b_k|$ for some k , then $W(A) \subsetneq W(B)$.
- (3) If the a_j 's are nonzero, then $W(A[n]) \subsetneq W(A)$.

Proof. In view of Lemma 2 (2), we may assume that the a_j 's and b_j 's are all nonnegative. Since $W(A)$ and $W(B)$ are circular discs centered at the origin by Proposition 3 (3), the assertions in (1) and (2) are equivalent to $w(A) \leq w(B)$ and $w(A) < w(B)$, respectively. These in turn follow from [9, Corollary 3.6]. To prove (3), let $C = A[n] \oplus [0]$. Then $W(A[n]) = W(C) \subsetneq W(A)$ by (2). \square

The next lemma is needed for the proof of Proposition 7.

Lemma 6. *If A and B are n -by- n ($n \geq 2$) weighted shift matrices with weights $a_1, \dots, a_{n-1}, 0$ and $a_{n-1}, \dots, a_1, 0$, respectively, then $W(A) = W(B)$.*

Proof. Since $W(A)$ and $W(B)$ are circular discs centered at the origin by Proposition 3 (3), we need only check that $w(A) = w(B)$. By Lemma 2 (2), we may assume that $a_j \geq 0$ for all j . Let $x = [x_1, \dots, x_n]^T$ be a unit vector with nonnegative components such that $w(A) = \langle Ax, x \rangle$. Then

$$w(A) = \sum_{j=1}^{n-1} a_j x_{j+1} x_j = \langle By, y \rangle \leq w(B),$$

where $y = [x_n, \dots, x_1]^T$. Similarly, we have $w(B) \leq w(A)$. Thus $w(A) = w(B)$ as asserted. \square

As indicated by the referee, the preceding lemma can also be proven by noting, under $a_j \geq 0$ for all j , that $\text{Re } A$ and $\text{Re } B$ are unitarily equivalent:

$J(\operatorname{Re} A) = (\operatorname{Re} B)J$, where $J = [J_{ij}]_{i,j=1}^n$ is the n -by- n skew identity matrix with

$$J_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise,} \end{cases}$$

and hence $w(A) = \|\operatorname{Re} A\| = \|\operatorname{Re} B\| = w(B)$.

Proposition 7. *Let A be an n -by- n ($n \geq 3$) weighted shift matrix with weights a_1, \dots, a_n . If $|a_1| = \dots = |a_{n-3}|$ and $\partial W(A)$ has a line segment, then $|a_{n-2}| = |a_n| \neq 0$.*

Proof. By Lemma 2 (2), we may assume that $a_j \geq 0$ for all j . Since $\partial W(A)$ has a line segment, we even have $a_j > 0$ by Proposition 3 (3). Let A_1 and A_2 be the $(n - 1)$ -by- $(n - 1)$ weighted shift matrices with weights $a_1, \dots, a_{n-3}, a_{n-2}, 0$ and $a_1, \dots, a_{n-3}, a_n, 0$, respectively. Then $A_1 = A[n]$ and $W(A_2) = W(A_3)$, where A_3 is the $(n - 1)$ -by- $(n - 1)$ weighted shift matrix with weights $a_n, a_{n-3}, \dots, a_1, 0$, by Lemma 6. Since $a_1 = \dots = a_{n-3}$, by Lemma 2 (1), A_3 is unitarily equivalent to $A[n - 1]$. Thus $W(A_3) = W(A[n - 1])$. Note that the existence of a line segment on $\partial W(A)$ guarantees that $W(A[n]) = W(A[n - 1])$ by the necessity part of Theorem 1. We conclude that $W(A_1) = W(A_2)$. Therefore, $a_{n-2} = a_n$ by Lemma 5 (2). \square

From Proposition 7, we can derive the following for weighted shift matrices of size 3 or 4: (1) a 3-by-3 weighted shift matrix A with weights a_1, a_2, a_3 is such that $\partial W(A)$ contains a line segment if and only if $|a_1| = |a_2| = |a_3| \neq 0$, and (2) if the 4-by-4 weighted shift matrix A with weights a_1, a_2, a_3, a_4 is such that $\partial W(A)$ contains a line segment, then $|a_1| = |a_3| \neq 0$ and $|a_2| = |a_4| \neq 0$. The necessity in (1) and (2) is a consequence of Proposition 7 and Lemma 2 (1). The sufficiency in (1) has already been proven in Proposition 4. Note that the condition in (2) is actually also sufficient, but its proof has to wait until the proving of Theorem 1 (cf. Proposition 12).

The next proposition is the major step in proving the sufficiency of Theorem 1.

Proposition 8. *Let A be an n -by- n ($n \geq 3$) weighted shift matrix with nonzero weights a_1, \dots, a_n , and let $\theta = (\pi + \sum_{j=1}^n \arg a_j)/n$.*

- (1) *If $W(A[j - 1]) = W(A[j]) = W(A[j + 1]) = B(0; r)$ for some $j, 1 \leq j \leq n$ ($A[0] \equiv A[n]$ and $A[n + 1] \equiv A[1]$) and some $r > 0$, then r is either the largest or the second largest eigenvalue of $\operatorname{Re}(e^{-i\theta}A)$.*
- (2) *If $W(A[1]) = \dots = W(A[n]) = B(0; r)$ ($r > 0$), then $r = w_0(A)$ is the largest eigenvalue of $\operatorname{Re}(e^{-i\theta}A)$ with multiplicity at least two.*

For the proof, we need the following lemma.

Lemma 9. *Let A be an n -by- n ($n \geq 5$) weighted shift matrix with nonzero real weights a_1, \dots, a_n . For*

$1 \leq j \leq n - 2$, let $B = \operatorname{Re} A[n]$ be partitioned as $\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}$ with A_j, B_j, C_j and D_j of sizes j -by- j, j -by- $(n - j - 1), (n - j - 1)$ -by- j and $(n - j - 1)$ -by- $(n - j - 1)$, respectively. If λ is the maximum eigenvalue of B , then $a_2^2, \dots, a_{n-3}^2 \lambda^2 = 4^{n-4} \det(\lambda I_{n-3} - A_{n-3}) \det(\lambda I_{n-3} - D_2)$.

Proof. Since the a_j 's are nonzero, $W(A[n])$ properly contains $W(A[j + 1, \dots, n])$ for any $j, 1 \leq j \leq n - 2$, by Lemma 5 (3). Hence λ , being the radius of the circular disc $W(A[n])$, does not belong to $W(A[j + 1, \dots, n])$. In particular, λ is not an eigenvalue of $A_j = \operatorname{Re} A[j + 1, \dots, n]$ and therefore $\lambda I_j - A_j$ is invertible for all $j, 1 \leq j \leq n - 2$. Similarly, the same is true for $\lambda I_{n-j-1} - D_j$. Thus

$$\begin{aligned} 0 &= \det(\lambda I_{n-1} - B) \\ &= \det(\lambda I_j - A_j) \det((\lambda I_{n-j-1} - D_j) - (-C_j)(\lambda I_j - A_j)^{-1}(-B_j)) \end{aligned}$$

$$\begin{aligned}
 &= \det(\lambda I_j - A_j) \det \left((\lambda I_{n-j-1} - D_j) - \frac{1}{4} a_j^2 \frac{\det(\lambda I_{j-1} - A_{j-1})}{\det(\lambda I_j - A_j)} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \right) \\
 &= \det(\lambda I_j - A_j) (\det(\lambda I_{n-j-1} - D_j) - \frac{1}{4} a_j^2 \frac{\det(\lambda I_{j-1} - A_{j-1})}{\det(\lambda I_j - A_j)} \det(\lambda I_{n-j-2} - D_{j+1})),
 \end{aligned}$$

from which we obtain

$$a_j^2 = 4 \frac{\det(\lambda I_j - A_j) \det(\lambda I_{n-j-1} - D_j)}{\det(\lambda I_{j-1} - A_{j-1}) \det(\lambda I_{n-j-2} - D_{j+1})}$$

for $2 \leq j \leq n - 3$. Taking the product of the a_j^2 's yields

$$\begin{aligned}
 a_2^2, \dots, a_{n-3}^2 \lambda^2 &= 4^{n-4} \frac{\det(\lambda I_{n-3} - A_{n-3}) \det(\lambda I_{n-3} - D_2)}{\det(\lambda I_1 - A_1) \det(\lambda I_1 - D_{n-2})} \lambda^2 \\
 &= 4^{n-4} \det(\lambda I_{n-3} - A_{n-3}) \det(\lambda I_{n-3} - D_2)
 \end{aligned}$$

since A_1 and D_{n-2} are both the 1-by-1 zero matrix. \square

Proof of Proposition 8. (1) We may assume, by Lemma 2 (1), that $W(A[n - 1]) = W(A[n]) = W(A[1]) = B(0; r)$. Also, by Lemma 2 (2), A is unitarily equivalent to $e^{i\theta} C$, where C is the n -by- n weighted shift matrix with weights $|a_1|, \dots, |a_{n-1}|, -|a_n|$. Then $w_0(A) = w_0(C)$ is in $\partial W(C)$ by Proposition 3 (4) and $W(A[j]) = W(C[j])$ for all j . Thus $w_0(C)$ is the maximum eigenvalue of $\text{Re } C$ and r is the maximum eigenvalue of $\text{Re } C[j]$ for $j = 1, n - 1$ and n . We now expand the determinant of $rI_n - \text{Re } C$ by minors along its n th row to obtain

$$\begin{aligned}
 &\det(rI_n - \text{Re } C) \\
 &= \frac{1}{2} |a_n| (-1)^{n+1} d_{n1} - \frac{1}{2} |a_{n-1}| (-1)^{2n-1} d_{n,n-1} + r \det(rI_{n-1} - \text{Re } C[n]) \\
 &= \frac{1}{2} |a_n| (-1)^{n+1} d_{n1} - \frac{1}{2} |a_{n-1}| (-1)^{2n-1} d_{n,n-1},
 \end{aligned}$$

where $(-1)^{n+j} d_{nj}$ denotes the cofactor of the (n, j) -entry of $\text{Re } C$ in $\text{Re } C, j = 1, n - 1$. The expansion of the determinant d_{n1} (resp., $d_{n,n-1}$) along its first row (resp., its last row) yields

$$\begin{aligned}
 d_{n1} &= \frac{(-1)^{n-1}}{2^{n-1}} |a_1, \dots, a_{n-1}| + \frac{(-1)^n}{2} |a_n| \det C_1 \\
 (\text{resp., } d_{n,n-1} &= \frac{1}{2^{n-1}} |a_1, \dots, a_{n-2} a_n| - \frac{1}{2} |a_{n-1}| \det C_2),
 \end{aligned}$$

where

$$C_1 = \begin{bmatrix} r & -|a_2|/2 & & & \\ -|a_2|/2 & r & & \ddots & \\ & & \ddots & \ddots & \\ & & & -|a_{n-2}|/2 & \\ & & & -|a_{n-2}|/2 & r \end{bmatrix}$$

$$\left(\text{resp., } C_2 = \begin{bmatrix} r & -|a_1|/2 & & & \\ -|a_1|/2 & r & & \ddots & \\ & & \ddots & \ddots & -|a_{n-3}|/2 \\ & & & -|a_{n-3}|/2 & r \end{bmatrix} \right).$$

Hence

$$\begin{aligned} \det(rI_n - \text{Re } C) &= \frac{1}{2^n} |a_1, \dots, a_n| - \frac{1}{4} |a_n|^2 \det C_1 + \frac{1}{2^n} |a_1, \dots, a_n| - \frac{1}{4} |a_{n-1}|^2 \det C_2 \\ &= \frac{1}{2^{n-1}} |a_1, \dots, a_n| - \frac{1}{4} |a_n|^2 \det C_1 - \frac{1}{4} |a_{n-1}|^2 \det C_2. \end{aligned} \tag{i}$$

On the other hand, let

$$D_j = \begin{bmatrix} r & -|a_j|/2 & & & \\ -|a_j|/2 & r & & \ddots & \\ & & \ddots & \ddots & -|a_{n+j-5}|/2 \\ & & & -|a_{n+j-5}|/2 & r \end{bmatrix}$$

for $j = 1, 2$ and 3 . Since $\det(rI_{n-1} - \text{Re } C[n]) = 0$, expanding this determinant along its first row (resp., its last row) yields $r \det C_1 = (|a_1|^2/4) \det D_3$ (resp., $r \det C_2 = (|a_{n-2}|^2/4) \det D_1$). Similarly, from $\det(rI_{n-1} - \text{Re } C[1]) = 0$ (resp., $\det(rI_{n-1} - \text{Re } C[n-1]) = 0$), we obtain $r \det C_1 = (|a_{n-1}|^2/4) \det D_2$ (resp., $r \det C_2 = (|a_n|^2/4) \det D_2$). Since r is the maximum eigenvalue of $\text{Re } C[j]$ for $j = 1, n - 1$ and n , we have $\det C_j \geq 0$ for $j = 1$ and 2 , and $\det D_j \geq 0$ for $j = 1, 2$ and 3 . Thus (i) becomes

$$\begin{aligned} \det(rI_n - \text{Re } C) &= \frac{1}{2^{n-1}} |a_1, \dots, a_n| - \frac{1}{4} |a_n|^2 \frac{|a_{n-1}|^2}{4r} \det D_2 - \frac{1}{4} |a_{n-1}|^2 \frac{|a_n|^2}{4r} \det D_2 \\ &= \frac{1}{2^{n-1}} |a_1, \dots, a_n| - \frac{1}{8r} |a_{n-1} a_n|^2 \frac{|a_{n-2}|}{|a_n|} (\det D_1)^{1/2} \frac{|a_1|}{|a_{n-1}|} (\det D_3)^{1/2} \\ &= \frac{1}{2^{n-1}} |a_1, \dots, a_n| - \frac{1}{8r} |a_1 a_{n-2} a_{n-1} a_n| (\det D_1 \cdot \det D_3)^{1/2} = 0 \end{aligned}$$

by Lemma 9. Hence $\det(rI_n - \text{Re } (e^{-i\theta} A)) = 0$. Since r is the maximum eigenvalue of $\text{Re } (e^{-i\theta} A)[1]$, this shows that it is either the largest or the second largest eigenvalue of $\text{Re } (e^{-i\theta} A)$.

(2) From our assumption and the proof of (1), we have $\det(rI_{n-1} - \text{Re } C[j]) = \det(rI_n - \text{Re } C) = 0$ for all $j, 1 \leq j \leq n$. Thus if $p(z) = \det(zI_n - \text{Re } C)$, then $p'(r) = \sum_{j=1}^n \det(rI_{n-1} - \text{Re } C[j]) = 0$ (cf. [6, p. 43, Problem 4]). This shows that the eigenvalue r of $\text{Re } C$ has (algebraic) multiplicity at least two or, equivalently, $\dim \ker(rI_n - \text{Re } C) \geq 2$. Since $B(0; r) = W(C[n]) \subseteq W(C)$, we have $r \leq w_0(C)$. If $r < w_0(C)$, then we deduce from the facts that $w_0(C)$ is the maximum eigenvalue of $\text{Re } C$ and $\dim \ker(rI_n - \text{Re } C) \geq 2$ that $B(0; r) = W(C[n]) = W(C[n - 1, n])$. This contradicts Lemma 5 (3) since the a_j 's are nonzero. Hence we must have $r = w_0(C) = w_0(A)$, which is the largest eigenvalue of $\text{Re } (e^{-i\theta} A)$ with multiplicity at least two.

To prove the sufficiency of Theorem 1, we need the following condition for the line segment on the boundary of a numerical range. It is from [4, Lemma 1.4]. \square

Lemma 10. *Let A be an n -by- n ($n \geq 2$) matrix. Then $\partial W(A)$ has a line segment on the line $x \cos \theta + y \sin \theta = d$ if and only if d is the maximum eigenvalue of $\text{Re } (e^{-i\theta} A)$, which has unit eigenvectors x_1 and x_2 such that $\text{Im} \langle e^{-i\theta} A x_1, x_1 \rangle \neq \text{Im} \langle e^{-i\theta} A x_2, x_2 \rangle$.*

Lemma 11. Let A be an n -by- n ($n \geq 2$) weighted shift matrix with nonzero real weights a_1, \dots, a_n . Then $\partial W(A)$ has a line segment on the line $x = d$ if and only if d is the maximum eigenvalue of $\text{Re } A$ with multiplicity at least two.

Proof. In view of Lemma 10, we need only prove the sufficiency part. Since $\dim \ker(dI_n - \text{Re } A) \geq 2$, there are real vectors $b = [0 \ b_2 \ \dots \ b_n]^T$ and $c = [c_1 \ 0 \ c_3 \ \dots \ c_n]^T$ in $\ker(dI_n - \text{Re } A)$ with $b_2, c_1 \neq 0$. Then we obtain $a_1 b_2 + a_n b_n = 0, 2db_2 = a_2 b_3, 2db_j = a_{j-1} b_{j-1} + a_j b_{j+1}$ for $3 \leq j \leq n-1$, and $a_{n-1} b_{n-1} = 2db_n$ (resp., $2dc_1 = a_n c_n, a_1 c_1 + a_2 c_3 = 0, 2dc_3 = a_3 c_4, 2dc_j = a_{j-1} c_{j-1} + a_j c_{j+1}$ for $4 \leq j \leq n-1$, and $a_n c_1 + a_{n-1} c_{n-1} = 2dc_n$). Simple computations show that

$$\begin{aligned} a_n b_n c_1 &= -a_1 b_2 c_1 = a_2 b_2 c_3 = (2db_3 - a_3 b_4) c_3 \\ &= a_3 (b_3 c_4 - b_4 c_3) = a_3 b_3 c_4 - (2dc_4 - a_4 c_5) b_4 \\ &= a_4 (b_4 c_5 - b_5 c_4) \\ &= \dots \\ &= a_{n-1} (b_{n-1} c_n - b_n c_{n-1}). \end{aligned}$$

Letting $x_1 = (b + ic) / \|b + ic\|$ and $x_2 = b / \|b\|$, we have

$$\begin{aligned} \text{Im} \langle Ax_1, x_1 \rangle &= \frac{1}{\|b + ic\|^2} \left(-a_1 b_2 c_1 + a_2 b_2 c_3 + \sum_{j=3}^{n-1} a_j (b_j c_{j+1} - b_{j+1} c_j) + a_n b_n c_1 \right) \\ &= -\frac{na_1 b_2 c_1}{\|b + ic\|^2} \neq 0 = \text{Im} \langle Ax_2, x_2 \rangle. \end{aligned}$$

Our assertion follows from Lemma 10. \square

We are now ready to prove the sufficiency part of Theorem 1.

Proof of Theorem 1. Assume that $a_j \neq 0$ and $W(A[j]) = B(0; r)$ for all $j, 1 \leq j \leq n$. By Lemma 2 (2), A is unitarily equivalent to $e^{i\psi} C$, where C is the n -by- n weighted shift matrix with weights $|a_1|, \dots, |a_{n-1}|, -|a_n|$ and $\psi = (\pi + \sum_{j=1}^n \arg a_j) / n$. By Proposition 8 (2), $r = w_0(C) = w_0(A)$ is the largest eigenvalue of $\text{Re } C$ with multiplicity at least two. Lemma 11 then implies that $\partial W(C)$ has a line segment on the line $x = r$. Thus $\partial W(A)$ has a line segment on $x \cos \psi + y \sin \psi = r = w_0(A)$. \square

The next proposition characterizes those 4-by-4 weighted shift matrices A with $\partial W(A)$ containing a line segment in terms of the weights of A . It was worked out by Gau and the second author some years ago.

Proposition 12. Let A be a 4-by-4 weighted shift matrix with weights a_1, \dots, a_4 . Then the following conditions are equivalent:

- (1) $\partial W(A)$ has a line segment,
- (2) $|a_1| = |a_3| \neq 0$ and $|a_2| = |a_4| \neq 0$,
- (3) A is unitarily equivalent to $\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix}$, where $b_1 b_2 = -c_1 c_2 \neq 0$ and $|b_1|^2 + |b_2|^2 = |c_1|^2 + |c_2|^2$, and
- (4) A is unitarily equivalent to $\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \oplus i \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}$ with $b_1, b_2 \neq 0$.

In this case, $W(A)$ is the convex hull of the two (orthogonal) ellipses E_1 and E_2 (may degenerate to line segments if $|b_1| = |b_2|$) with E_1 having foci $\pm(b_1 b_2)^{1/2}$ and minor axis of length $||b_1| - |b_2||$ and $E_2 = iE_1$. In particular, $\partial W(A)$ has four line segments.

Proof. (1) \Leftrightarrow (2). Since the characteristic polynomial of $\text{Re } A[1]$ is

$$\det(zI_3 - \text{Re } A[1]) = z^3 - \frac{1}{4} (|a_2|^2 + |a_3|^2) z,$$

we have $w(A[1]) = \|\text{Re } A[1]\| = (|a_2|^2 + |a_3|^2)^{1/2}/2$. Similarly, we obtain values of $w(A[j])$ for $2 \leq j \leq 4$. Thus the equivalence of (1) and (2) follows from Theorem 1 and Proposition 3 (3).

(2) \Rightarrow (3). Since $\det(zI_4 - A) = z^4 - a_1 a_2 a_3 a_4$, the eigenvalues of A are $\alpha_j \equiv (a_1 a_2 a_3 a_4)^{1/4} \omega_4^j$, $0 \leq j < 4$. Their respective eigenvectors can be computed to be (multiples of) $x_j \equiv [1 \ \alpha_j/a_1 \ \alpha_j^2/(a_1 a_2) \ \alpha_j^3/(a_1 a_2 a_3)]^T$, $0 \leq j < 4$. Note that

$$\langle x_j, x_k \rangle = 1 + \frac{1}{|a_1|^2} \alpha_j \bar{\alpha}_k + \frac{1}{|a_1 a_2|^2} (\alpha_j \bar{\alpha}_k)^2 + \frac{1}{|a_1 a_2 a_3|^2} (\alpha_j \bar{\alpha}_k)^3$$

for any j and k . From this, it is easy to verify that

$$\langle x_1, x_2 \rangle = \langle x_1, x_4 \rangle = \langle x_3, x_2 \rangle = \langle x_3, x_4 \rangle = 0.$$

Let $y_1 = x_1 - x_3 = [0 \ 2\alpha_0/a_1 \ 0 \ 2\alpha_0^3/(a_1 a_2 a_3)]^T$, $y_2 = x_1 + x_3 = [2 \ 0 \ 2\alpha_0^2/(a_1 a_2) \ 0]^T$, $y_3 = x_2 - x_4 = [0 \ 2i\alpha_0/a_1 \ 0 \ -2i\alpha_0^3/(a_1 a_2 a_3)]^T$ and $y_4 = x_2 + x_4 = [2 \ 0 \ -2\alpha_0^2/(a_1 a_2) \ 0]^T$, and let M be the subspace of \mathbb{C}^4 spanned by y_1 and y_2 . Since $Ay_1 = Ax_1 - Ax_3 = \alpha_1 x_1 - \alpha_3 x_3$ and $Ay_2 = Ax_1 + Ax_3 = \alpha_1 x_1 + \alpha_3 x_3$, and M is also spanned by x_1 and x_3 , we have $AM \subseteq M$. A simple computation shows that $A^* y_1 = (\bar{a}_4 \alpha_0^3 / (a_1 a_2 a_3)) y_2$ and $A^* y_2 = (|a_1|^2 / \alpha_0) y_1$, where the assumptions that $|a_1| = |a_3|$ and $|a_2| = |a_4|$ are used. This shows that $A^* M \subseteq M$. Thus M is a reducing subspace of A . Moreover, it is easily seen that M^\perp is spanned by y_3 and y_4 , and $\langle y_1, y_2 \rangle = \langle y_3, y_4 \rangle = \langle Ay_j, y_j \rangle = 0$ for all j .

Therefore, A is unitarily equivalent to a matrix of the form $\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix} \equiv B \oplus C$ on $M \oplus M^\perp$.

Since x_1 and x_3 are in M , α_1 and α_3 are eigenvalues of B . Hence

$$-b_1 b_2 = \det B = \alpha_1 \alpha_3 = \alpha_0^{1/2} \omega_4^2 = -\alpha_0^{1/2}.$$

A similar argument with C yields $-c_1 c_2 = \alpha_0^{1/2}$. It follows that $b_1 b_2 = -c_1 c_2$.

To prove $|b_1|^2 + |b_2|^2 = |c_1|^2 + |c_2|^2$, note that simple computations give

$$b_1 = \left\langle A \frac{y_2}{\|y_2\|}, \frac{y_1}{\|y_1\|} \right\rangle = \alpha_0 \frac{\|y_1\|}{\|y_2\|},$$

$$b_2 = \left\langle A \frac{y_1}{\|y_1\|}, \frac{y_2}{\|y_2\|} \right\rangle = \alpha_0 \frac{\|y_2\|}{\|y_1\|},$$

$$c_1 = \left\langle A \frac{y_4}{\|y_4\|}, \frac{y_3}{\|y_3\|} \right\rangle = i\alpha_0 \frac{\|y_3\|}{\|y_4\|},$$

and

$$c_2 = \left\langle A \frac{y_3}{\|y_3\|}, \frac{y_4}{\|y_4\|} \right\rangle = i\alpha_0 \frac{\|y_4\|}{\|y_3\|},$$

and $\|y_1\| = \|y_3\|$ and $\|y_2\| = \|y_4\|$. Thus

$$\begin{aligned}
 |b_1|^2 + |b_2|^2 &= |\alpha_0|^2 \left(\frac{\|y_1\|^2}{\|y_2\|^2} + \frac{\|y_2\|^2}{\|y_1\|^2} \right) \\
 &= |\alpha_0|^2 \left(\frac{\|y_3\|^2}{\|y_4\|^2} + \frac{\|y_4\|^2}{\|y_3\|^2} \right) = |c_1|^2 + |c_2|^2
 \end{aligned}$$

as asserted.

(3) \Rightarrow (4). Note that $\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}$ (resp., $\begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix}$) is unitarily equivalent to $\begin{bmatrix} (b_1 b_2)^{1/2} & \| |b_1| - |b_2| \| \\ 0 & -(b_1 b_2)^{1/2} \end{bmatrix}$ (resp., $\begin{bmatrix} (c_1 c_2)^{1/2} & \| |c_1| - |c_2| \| \\ 0 & -(c_1 c_2)^{1/2} \end{bmatrix}$). From the assumption in (3), we have $(b_1 b_2)^{1/2} = \pm i(c_1 c_2)^{1/2}$ and $\| |b_1| - |b_2| \| = \| |c_1| - |c_2| \|$. Thus $\begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix}$ is unitarily equivalent to $i \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}$, and (4) follows.

(4) \Rightarrow (1). Since $W \left(\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \right)$ is the elliptic disc with foci $\pm (b_1 b_2)^{1/2}$ and minor axis of length $\| |b_1| - |b_2| \|$, that is, $W \left(\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \right) = E_1^\wedge$ and $W \left(i \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \right) = (iE_1)^\wedge$, it is obvious that $\partial W(A)$ contains four line segments. This also proves our assertion on $W(A)$, completing the proof. \square

For $n > 4$, we can use the same arguments as in the proof of (1) \Leftrightarrow (2) above to obtain conditions in terms of the weights. They turn out to be too complicated to be useful.

In a forthcoming paper [14] by the first author, more specific information is obtained for the numerical ranges of weighted shift matrices with periodic weights.

We conclude this paper by stating a theorem on the numerical ranges of matrices with an analogous structure, namely, the nilpotent matrices of the form

$$A = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 & a_n \\ & 0 & a_2 & \ddots & & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & 0 & a_{n-1} \\ & & & & & 0 \end{bmatrix} \tag{ii}$$

with weights a_1, \dots, a_n . Note that A and the weighted shift matrix with weights a_1, \dots, a_n (a_n real) have the same real parts, which explains why (almost) all results in this paper for the latter have their analogs for the former. The only difference is that in the present case A is unitarily equivalent to $\omega_{n-2}A$ and hence $W(A)$ has the $(n - 2)$ -symmetry property. The full details can be found in the first author's Ph.D. dissertation [15].

Theorem 13. *Let A be an n -by- n ($n \geq 3$) matrix of the form (ii). Then $\partial W(A)$ has a line segment if and only if the a_j 's are nonzero and $W(A[1]) = \cdots = W(A[n])$. In this case, $W(A[j])$ is the circular disc centered at the origin with radius $w_0(A)$, the line segment lies on one of the lines $x \cos \theta_k + y \sin \theta_k = w_0(A)$, where*

$\theta_k = ((2k + 1)\pi - \arg a_n + \sum_{j=1}^{n-1} \arg a_j) / (n - 2)$, $0 \leq k < n - 2$, and there are exactly $n - 2$ line segments on $\partial W(A)$.

The following is an easy corollary.

Corollary 14. *Let A (resp., B) be the n -by- n ($n \geq 3$) weighted shift matrix (resp., nilpotent matrix of the form (ii)) with weights a_1, \dots, a_n . Then*

- (1) $w(A) = w(B)$,
- (2) $w_0(A) = w_0(B)$, and
- (3) $\partial W(A)$ has a line segment if and only if $\partial W(B)$ has.

A study of the matrix of the form (ii) with $a_1 = \dots = a_n = 1$ was made in [3, Proposition 3.2].

Acknowledgments

The authors thank the (anonymous) referees for their incisive comments. One of them pointed out [8], the Ph.D. dissertation of Issos, to us, which contains some relevant information.

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