Interior Penalty Discontinuous Galerkin Methods with Implicit Time-Integration Techniques for Nonlinear Parabolic Equations

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We prove existence and numerical stability of numerical solutions of three fully discrete interior penalty discontinuous Galerkin methods for solving nonlinear parabolic equations. Under some appropriate regularity conditions, we give the $l^2(H^1)$ and $l^\infty(L^2)$ error estimates of the fully discrete symmetric interior penalty discontinuous Galerkin–scheme with the implicit θ-schemes in time, which include backward Euler and Crank–Nicolson finite difference approximations. Our estimates are optimal with respect to the mesh size h. The theoretical results are confirmed by some numerical experiments. © 2012 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 29: 1341–1366, 2013

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I. INTRODUCTION

Although the definition of discontinuous polynomial spaces emerged in the 1970s [1–3], it was in the recent decade that discontinuous Galerkin (DG) methods have become attractive as a powerful simulation tool for solving partial differential equations (see e.g., Refs. [4–8]). The mixed DG and the primal DG are two main families of DG. The local discontinuous Galerkin scheme (LDG) [9] is a representative of the mixed DG. The primal DG method depends on the appropriate choice of penalty terms for the discontinuous shape functions and has a different treatment of the diffusion term [3, 10, 11], which can be referred to as interior penalty discontinuous galerkin (IPDG) methods. There are four primal DG versions, symmetric interior penalty galerkin (SIPG) [3, 8], nonsymmetric interior penalty galerkin (NIPG) [8, 12], Incomplete Interior Penalty Galerkin (IIPG) [8], and Oden–Babuska–Baumann DG (OBB-DG) methods [11].

DG methods possess a few important features over other types of finite element methods. For example, they naturally handle inhomogeneous boundary conditions and interelement continuity with a weak enforcement; they allow the use of nonuniform and unstructured meshes and have local conservation properties. In addition, they appear to be nonoscillatory in the presence of high gradients and rough solutions. But they seem to introduce a relatively large number of degrees of freedom over interelements. Fortunately, two-level and multilevel preconditioners tend to remedy this disadvantage [13, 14].

The hp version [15] works efficiently and the accuracy is achieved if one geometrically refines the mesh by grading toward the corners of the polygonal boundary where in general singularities of the exact solution occur, and if one appropriately chooses the polynomial degree p on each subdivision. Arnold [16] first analyzed a semidiscrete IPDG method for solving nonlinear parabolic boundary value problems and stated optimal order error estimates in the energy and L^2 norms. For reaction-diffusion equations, Georgoulis and Süli [17] presented fully hp-optimal error bounds in the energy norm by introducing an augmented Sobolev space. Optimal convergence in $L^2(L^2)$ for SIPG has also been established for reactive transport in porous media, but sometimes NIPG and IIPG do not have $L^2(L^2)$ optimality [8]. For NIPG and IIPG, $l^\infty(L^2)$ optimal error estimates were given for a three-dimensional (3D) parabolic equation in a rectangular domain [18] and the L^2 optimality was established for polynomials of odd degrees for 1D elliptic equations [19]. Rivière and Wheeler [20] derived a priori error estimates in the $l^2(H^1)$ and $l^\infty(L^2)$ norms for a fully discrete NIPG scheme with a θ scheme for nonlinear parabolic equations. Recently, Ohm et al. [21] obtained an optimal $l^2(H^1)$ and $L^\infty(L^2)$ error estimates of a semidiscrete SIPG scheme for nonlinear parabolic equations, and generalized the $l^\infty(L^2)$ error estimate to the backward Euler SIPG method in Ref. [22]. But there are no related numerical experiments presented.

We consider the following nonlinear parabolic equation:

$$u_t - \nabla \cdot (a(x, u)\nabla u) = f(x, u), \quad \text{in} \quad \Omega \times (0, T), \tag{1.1}$$

$$a(x, u)\nabla u \cdot \mathbf{n} = 0, \quad \text{in} \quad \partial\Omega \times (0, T), \tag{1.2}$$

$$u|_{t=0} = \psi(x), \quad \text{on} \quad \Omega \times \{0\}, \tag{1.3}$$

where Ω is an open interval in $\mathbb{R}^1$, or a convex polygonal domain in $\mathbb{R}^2$, $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$, and $T > 0$ is arbitrary but fixed. The Eq. (1.1), supplemented with the boundary condition (1.2) and the initial condition (1.3), describes the diffusion of a chemical species of the concentration u in a porous medium with a source term $f(x, u)$.

We assume that a and f are uniformly Lipschitz continuous with respect to the variable u, namely, there exist positive constants L_a and L_f such that

$$|a(x, u_1) - a(x, u_2)| \leq L_a |u_1 - u_2|, \quad \text{for } u_1, u_2 \in \mathbb{R}, \tag{1.4}$$

$$|f(x, u_1) - f(x, u_2)| \leq L_f |u_1 - u_2|, \quad \text{for } u_1, u_2 \in \mathbb{R}. \tag{1.5}$$

Moreover, we assume that for any compact set S in $\mathbb{R}$, there exist positive constants K_0 and K_1 such that

$$0 < K_0 \leq a(x, p) \leq K_1, \quad 0 < K_0 \leq \frac{\partial}{\partial p} a(x, p) \leq K_1, \quad \text{in } \Omega \times S. \tag{1.6}$$

For the sake of simplicity, we also assume that

$$f(\cdot, 0) = 0. \tag{1.7}$$

In our study, u is assumed to be a strong solution. That is, $u \in C^2(\overline{\Omega} \times [0, T])$, a solution of the problem (1.1)–(1.3), satisfies the regularity conditions below:

$$\begin{cases} u, u_t \in L^\infty(0, T; H^s(\Omega)), & \text{for some } s \geq 2; \\ \nabla u \in L^\infty(\Omega \times (0, T)). \end{cases} \tag{1.8}$$

In this work, the IPDG schemes for approximating solutions of the problem (1.1)–(1.3) are analyzed, and for a fully discrete θ scheme in time the optimal $l^\infty(L^2)$ error estimates are derived under some appropriate regularity conditions. Our analysis focuses on the following issues which have not yet been adequately considered in the literature:

- Based on an implicit θ scheme, existence of solutions of the fully discrete IPDG schemes will be proven.
- When implicit θ time-integration techniques are considered to avoid rigid stability conditions associated with the mesh size, numerical stability of these fully discrete IPDG schemes will be analyzed.
- In the hp version, optimal $l^2(H^1)$ and $l^\infty(L^2)$ error estimates will be given for the fully discrete SIPG scheme with implicit time-integration schemes.
- Some numerical results are presented in our work. As we know, few numerical results from an implicit time-stepping IPDG scheme have been published for solving nonlinear parabolic equations.

The content of this article is summarized as follows. In Section II, we recall a few definitions and the formulation of the interior penalty DG formulations in a semidiscrete form and in fully discrete forms with a θ scheme. For the semidiscrete DG scheme, we state some properties for IPDG schemes in Section III, and derive existence and numerical stability for the fully discrete IPDG schemes in Sections IV and V, respectively. In Section VI, we give a unified analysis of optimal a priori error estimates in the $l^2(H^1)$ and $l^\infty(L^2)$ norms for the fully-discrete SIPG scheme with implicit θ time-integration techniques. Finally, the numerical results are presented to show the effectiveness of the DG methods in Section 7.

II. THE DG METHOD

We subdivide the domain Ω into elements $E_1, E_2, \ldots, E_{N_h}$, where E_i is an interval in 1D or a triangle in 2D and N_h is the number of all elements. Here, $h > 0$ denotes the maximal diameter of all elements. For each $h > 0$, we write the resulting subdivision in the form:

$$\mathcal{E}_h := \{E_1, E_2, \ldots, E_{N_h}\}.$$

Note that $\bar{\Omega} = \bigcup_{i=1}^{N_h} \bar{E}_i$, for each $h > 0$. We set, for each element of $\mathcal{E}_h$,

$$\begin{cases} h_i = \text{the diameter of } E_i,\ 1 \leq i \leq N_h, \\ \rho_i = \text{the radius of the largest ball inscribed in } E_i,\ 1 \leq i \leq N_h. \end{cases} \tag{2.1}$$

We impose a regularity assumption on the mesh $\mathcal{E}_h$, that is, there exists a constant $\zeta > 0$, independent of h, such that

$$\frac{h}{\min \rho_i} \leq \zeta.$$

Moreover, we assume that the mesh $\mathcal{E}_h$ is quasiuniform: there exists a constant $\tau > 0$, independent of h, such that

$$\frac{h}{\min h_i} \leq \tau. \tag{2.2}$$

We introduce the set $\mathcal{F}_h$ of edges of the mesh $\mathcal{E}_h$:

$$\mathcal{F}_h := \{e_1, e_2, \ldots, e_{P_h}, e_{P_h+1}, \ldots, e_{M_h}\},$$

where

$$\begin{cases} e_i \subset \Omega, & \text{if } 1 \leq i \leq P_h, \\ e_i \subset \partial\Omega, & \text{if } P_h + 1 \leq i \leq M_h. \end{cases}$$

On each e_i ($1 \leq i \leq M_h$) of $\mathcal{E}_h$, we fix a unit outer normal vector $\mathbf{n}_i$:

$$\mathbf{n}_i = \begin{cases} \text{the unit normal vector on } e_i, \text{ pointing from } E_k \text{ to } E_j, \text{ if } e_i = \partial E_k \cap \partial E_j, \\ \text{the unit normal vector on } e_i, \text{ pointing outward of } \Omega, \text{ if } P_h + 1 \leq i \leq M_h, \end{cases}$$

then we denote the average and jump operators below: For $v \in H^s(\mathcal{E}_h)$, $s > \frac{1}{2}$,

$$\{v\}_{e_i} := \begin{cases} \frac{1}{2}(v|_{E_j})|_{e_i} + \frac{1}{2}(v|_{E_k})|_{e_i}, \text{ if } e_i = \partial E_j \cap \partial E_k,\ 1 \leq i \leq P_h, \\ (v|_{E_k})|_{e_i}, \text{ if } e_i = \partial E_k \cap \partial\Omega,\ P_h + 1 \leq i \leq M_h. \end{cases}$$

$$[v]_{e_i} := \begin{cases} (v|_{E_k})|_{e_i} - (v|_{E_j})|_{e_i}, \text{ if } e_i = \partial E_j \cap \partial E_k,\ 1 \leq i \leq P_h, \\ (v|_{E_k})|_{e_i}, \text{ if } e_i = \partial E_k \cap \partial\Omega,\ P_h + 1 \leq i \leq M_h. \end{cases}$$

For brevity, we drop the subscript e_i of these two operators throughout this article.

Along this article, the H^m Sobolev norm on ω is defined by $\| \cdot \|_{m,\omega}$ for a positive integer m, that is,

$$\| \cdot \|_{m,\omega} := | \cdot |_{H^m(\omega)}, \quad \forall 0 \leq m < \infty,\ \forall \omega \subset \mathbb{R}^1 (\text{or } \mathbb{R}^2). \tag{2.3}$$

Note that, by default, $H^0(\omega)$ denotes $L^2(\omega)$ with the L^2 inner product $(\cdot, \cdot)$ and $\| \cdot \|_{\infty, \omega}$ is the standard L^∞-norm on ω. Then using (2.3), we introduce the broken Sobolev space for any real number s:

$$H^s(\mathcal{E}_h) = \left\{ v \in L^2(\Omega) :\ v|_{E_i} \in H^s(E_i),\ 1 \leq i \leq N_h \right\},$$

which is equipped with the broken space norm:

$$|||v|||_s := |v|_{H^s(\mathcal{E}_h)} = \left(\sum_{i=1}^{N_h} \|v\|_{s,E_i}^2 \right)^{1/2}.$$

Also, given a time interval $[a, b]$, we use the broken Sobolev $L^2(H^s)$ and $L^\infty(H^s)$ norms:

$$|||v|||_{L^2(a,b;H^s)}^2 = \int_a^b |||v(\cdot, t)|||_s^2 \, dt, \quad |||v|||_{L^\infty(a,b;H^s)} = \operatorname*{ess\,sup}_{t \in (a,b)} |||v(\cdot, t)|||_s.$$

We also introduce a space of test functions

$$\mathcal{D}_r(\mathcal{E}_h) = \left\{ v \in L^2(\Omega) : v|_{E_i} \in \mathbb{P}_r(E_i), \ 1 \le i \le N_h \right\}, \tag{2.4}$$

where, for each $1 \le i \le N_h$,

$$\mathbb{P}_r(E_i) := \{\text{the space of polynomials of (total) degree at most } r \text{ on } E_i\}.$$

It is clear that

$$\mathcal{D}_r(\mathcal{E}_h) \subset H^1(\mathcal{E}_h) \subset L^2(\Omega).$$

However, since the test functions in $\mathcal{D}_r(\mathcal{E}_h)$ are discontinuous along the edges $e_i, 1 \le i \le P_h$ in Ω, we notice that $\mathcal{D}_r(\mathcal{E}_h) \not\subset H^1(\Omega)$. We thus introduce the interior penalty term $J_0^\sigma : \mathcal{D}_r(\mathcal{E}_h) \times \mathcal{D}_r(\mathcal{E}_h) \to \mathbb{R}$ in the form:

$$J_0^\sigma(v, w) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \int_{e_k} [v][w] ds, \tag{2.5}$$

which penalizes the jump of the functions across the edges e_k, $1 \le k \le P_h$. Here, the penalty parameter σ_k is a nonnegative real number to be chosen and $|e_k|$ is the Lebesgue measure of the edge e_k. It is easy to see that

$$|e_i| \le h_i \le h, \quad \forall \, 1 \le i \le N_h. \tag{2.6}$$

We also define the energy norm on $\mathcal{D}_r(\mathcal{E}_h)$ throughout this article:

$$\|v\|_{\mathrm{DG}} = \left(\sum_{k=1}^{N_h} \|\nabla v\|_{0,E_k}^2 + J_0^\sigma(v, v) \right)^{1/2}, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{2.7}$$

Aiming to study the strong solution $u \in C^2(\overline{\Omega} \times [0, T])$, satisfying the regularity conditions (1.8), of the problem (1.1)–(1.3), we proceed element by element as appears in Ref. [23]. As a result, we obtain the following consistent weak formulation of the problem (1.1)–(1.3):

Find $u(t) \in H^s(\mathcal{E}_h), s > \frac{3}{2}$, such that

$$(u_t, v) + A_\epsilon(u; u, v) = (f(u), v), \quad \forall v \in H^s(\mathcal{E}_h), \tag{2.8}$$

$$(u(0), v) = (\psi, v), \tag{2.9}$$

where $A_\epsilon(\rho; v, w)$ is bilinear in the last two terms:

$$
\begin{aligned}
A_\epsilon(\rho; v, w) = \sum_{j=1}^{N_h} \int_{E_j} a(\rho) \nabla v \cdot \nabla w \, dx - \sum_{k=1}^{P_h} \int_{e_k} \{a(\rho) \nabla v \cdot \mathbf{n}_k\}[w] \, ds \\
+ \epsilon \sum_{k=1}^{P_h} \int_{e_k} \{a(\rho) \nabla w \cdot \mathbf{n}_k\}[v] \, ds + J_0^\sigma(v, w), \quad v, w \in H^s(\mathcal{E}_h).
\end{aligned}
\tag{2.10}
$$

Here, the parameter ϵ in A_ϵ may take the value -1, 0 or 1; see Remark 2.1.

To discretize the problem (2.8), we first introduce the following notations. For any smooth function $\phi : \Omega \times [0, T] \to \mathbb{R}$, we set

$$
\phi^n := \phi(x, t_n), \quad t_n = n \triangle t \quad \forall 0 \le n \le N,
\tag{2.11}
$$

where N is a positive integer and $\triangle t = T/N$. Then for an arbitrary but fixed $0 \le \theta \le 1$, we set

$$
\phi_\theta^n = \frac{1-\theta}{2} \phi^n + \frac{1+\theta}{2} \phi^{n+1}, \quad 0 \le n \le N-1.
\tag{2.12}
$$

To discrete (2.8), by (2.10)–(2.12), we use the DG method for the spatial variable, and the $\theta-$scheme for the time variable. Now, the fully discrete IPDG formulation is to seek a sequence $\{u_h^n\}_{n\ge 0}^N$ of functions in $\mathcal{D}_r(\mathcal{E}_h)$ such that $\forall n \ge 0$,

$$
\left(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v \right) + A_\epsilon(u_{h,\theta}^n; u_{h,\theta}^n, v) = (f(u_{h,\theta}^n), v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h),
\tag{2.13}
$$

$$
u_h^0 = \widetilde{u}_{0h},
\tag{2.14}
$$

where $\widetilde{u}_{0h}$ is a L^2 projection of ψ onto $\mathcal{D}_r(\mathcal{E}_h)$ and $u_{h,\theta}^n = \frac{1-\theta}{2} u_h^n + \frac{1+\theta}{2} u_h^{n+1}$.

Remark 2.1. Note that for a fixed function ρ, A_ϵ appearing in (2.10) is symmetric if $\epsilon = -1$, and is nonsymmetric if $\epsilon = 0$ or 1. Moreover, depending on the value of ϵ, the DG method considered in (2.13) is referred to SIPG if $\epsilon = -1$; NIPG if $\epsilon = 1$; or IIPG if $\epsilon = 0$. For the choice of penalty parameters σ_k of these discontinuous formulations, see Georgoulis and Süli [17], Rivière et al. [12], and the references therein.

Remark 2.2. If $\theta = 0$, (2.13) yields the Crank–Nicolson DG approximation; If $\theta = 1$, (2.13) becomes the backward Euler DG approximation.

III. SOME ESTIMATES OF THE IPDG SCHEMES

In this section, we mainly state some approximation results in the space of polynomials of degree r, which will be used later. And we denote by C a generic positive constant.

Lemma 3.1. *Assume that $u \in H^s(\Omega)$, for $s \geq 2$ and let $r \geq 2$ and assume that $\bar{a}$ is a given positive constant. Then, there exists an interpolant $\hat{u} \in \mathcal{D}_r(\mathcal{E}_h)$ of u satisfying that (see Ref. [24])*

$$\int_{e_k} \{\bar{a} \nabla(\hat{u} - u) \cdot \mathbf{n}_k\} ds = 0, \quad \forall k = 1, \dots, P_h, \tag{3.1}$$

$$\|\hat{u} - u\|_{\infty, E_j} \leq C \frac{h^\mu}{r^{s-1}} \|u\|_{s, E_j}, \quad \forall E_j \in \mathcal{E}_h, \tag{3.2}$$

$$\|\nabla(\hat{u} - u)\|_{0, E_j} \leq C \frac{h^{\mu-1}}{r^{s-1}} \|u\|_{s, E_j}, \quad \forall E_j \in \mathcal{E}_h, \tag{3.3}$$

$$\|\nabla^2(\hat{u} - u)\|_{0, E_j} \leq C \frac{h^{\mu-2}}{r^{s-2}} \|u\|_{s, E_j}, \quad \forall E_j \in \mathcal{E}_h, \tag{3.4}$$

$$\|\hat{u} - u\|_{0, E_j} \leq C \frac{h^\mu}{r^{s-1}} \|u\|_{s, E_j}, \quad \forall E_j \in \mathcal{E}_h, \tag{3.5}$$

$$\|\nabla \hat{u}\|_{\infty, e_k} \leq C \|\nabla u\|_{\infty, E_i \cup E_j}, \quad for\ e_k = \partial E_i \cap \partial E_j, \tag{3.6}$$

where $\mu = \min(r + 1, s)$ and C is independent of h, s and r.

Lemma 3.2. *For each $E_k \in \mathcal{E}_h$, let e_k be an edge of E_k and let $\mathbf{n}_k$ be a unit vector normal to e_k. Then, there exists a positive constant C depending only on τ and r [defined in (2.2) and (2.4)] such that the following two trace inequalities are valid Ref. [16]:*

$$\|v\|_{0, e_k}^2 \leq C(h_{E_k}^{-1} \|v\|_{0, E_k}^2 + h_{E_k} \|\nabla v\|_{0, E_k}^2), \quad \forall v \in H^1(E_k), \tag{3.7}$$

$$\left\| \frac{\partial v}{\partial \mathbf{n}_k} \right\|_{0, e_k}^2 \leq C(h_{E_k}^{-1} \|\nabla v\|_{0, E_k}^2 + h_{E_k} \|\Delta v\|_{0, E_k}^2), \quad \forall v \in H^2(E_k). \tag{3.8}$$

Lemma 3.3. *For each $E_k \in \mathcal{E}_h$ and $v \in \mathbb{P}_r(E_k)$, let e_k be an edge of E_k and let $\mathbf{n}_k$ be a unit vector normal to e_k. Then, there exists a positive constant C depending only on r and τ such that the following two local inverse inequalities hold Ref. [16]:*

$$\|\nabla^j v\|_{0, E_k} \leq C h_{E_k}^{-j} \|v\|_{0, E_k}, \quad \forall 0 \leq j \leq r, \left\| \frac{\partial v}{\partial \mathbf{n}_k} \right\|_{0, e_k} \leq C h_{E_k}^{-\frac{1}{2}} \|\nabla v\|_{0, E_k}. \tag{3.9}$$

Squaring the second inequality (3.9), multiplying it by $|e_k|$ and summing for $k = 1$ to P_h, we obtain

Lemma 3.4. *There exists a positive constant $C_{\Omega, \tau}$ depending on Ω and τ such that*

$$\sum_{k=1}^{P_h} |e_k| \left\| \left\{ \frac{\partial v}{\partial \mathbf{n}_k} \right\} \right\|_{0, e_k}^2 \leq C_{\Omega, \tau} \|\|\nabla v\|\|_0^2, \quad \forall v \in H^1(\mathcal{E}_h). \tag{3.10}$$

We define a new bilinear form:

$$A_{\epsilon, \lambda}(\rho; v, w) = A_\epsilon(\rho; v, w) + \lambda(v, w),$$

with a positive real number λ.

In Ref. [25], we note that A_ϵ is coercive. If the positive constant δ is such that $\frac{K_0}{2C_{\Omega,\tau}} > \delta > \frac{(1-\epsilon)^2 K_1^2}{4 \min_k \{\sigma_k\}}$ and the penalty parameters σ_k satisfy $\min_k \{\sigma_k\} > C_{\Omega,\tau} \frac{(1-\epsilon)^2 K_1^2}{2K_0}$, then for any $v, \rho \in \mathcal{D}_r(\mathcal{E}_h)$,

$$A_\epsilon(\rho; v, v) \geq \alpha_0 \left(\||\nabla v\||_0^2 + \sum_{k=1}^{P_h} |e_k| \left\| \left\{ \frac{\partial v}{\partial \mathbf{n}_k} \right\} \right\|_{0,e_k}^2 + J_0^\sigma(v, v) \right), \tag{3.11}$$

where $\alpha_0 = \min\{\frac{K_0}{2}, \frac{K_0}{2C_{\Omega,\tau}} - \delta, 1 - \frac{(1-\epsilon)^2 K_1^2}{4\delta \min\{\sigma_k\}}\}$. Thus, if we consider an NIPG method ($\epsilon = 1$), then we can choose $\sigma_k > 0$ and $\alpha_0 = \min\{K_0, 1\}$. For the SIPG and IIPG methods, the penalty parameters σ_k will be chosen sufficiently large, as the ratio $\frac{K_1^2}{K_0}$ or $\frac{K_1^2}{\delta}$ becomes extremely large. Based on the estimate (3.11) and the definition of $A_{\epsilon,\lambda}$, then we have the following lemma:

Lemma 3.5.   *Assume that $\frac{K_0}{2C_{\Omega,\tau}} > \delta > \frac{(1-\epsilon)^2 K_1^2}{4 \min\{\sigma_k\}}$ and $\min_k\{\sigma_k\} > C_{\Omega,\tau} \frac{(1-\epsilon)^2 K_1^2}{2K_0}$, then there exists a positive constant κ independent of h and r such that*

$$A_{\epsilon,\lambda}(\rho; v, v) \geq \kappa \|v\|_{DG}^2, \quad \forall \rho, v \in \mathcal{D}_r(\mathcal{E}_h). \tag{3.12}$$

Here, one can choose $\kappa = \alpha_0$. The following lemma can be proven similarly as in Ref. [25].

Lemma 3.6.   *There exists a positive constant C independent of h and r such that*

$$|A_{\epsilon,\lambda}(\rho; v, w)| \leq C \|v\|_{DG} \|w\|_{DG}, \quad \forall \rho, v, w \in \mathcal{D}_r(\mathcal{E}_h). \tag{3.13}$$

By Lemma 3.6, we notice that A_ϵ is continuous: there exists a constant $\beta_0 > 0$ such that

$$A_\epsilon(\rho; v, u) \leq \beta_0 \|v\|_{DG} \|u\|_{DG}, \quad \forall \rho, v, u \in \mathcal{D}_r(\mathcal{E}_h). \tag{3.14}$$

The Aubin–Nitsche lift technique is well suited to the analysis of the DG method for linear problems, because the SIPG scheme is symmetric. But for the nonlinear parabolic equation, we will use the following projection and lemma as in Ref. [21]. Let $u \in H^2(\Omega)$. The Galerkin projection $\pi_h u \in \mathcal{D}_r(\mathcal{E}_h)$ of u is defined by requiring that

$$A_{-1,\lambda}(u; u, v) = A_{-1,\lambda}(u; \pi_h u, v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{3.15}$$

$\pi_h u$ is a function mapping from $(0, T)$ onto $\mathcal{D}_r(\mathcal{E}_h)$ and its unique existence follows by the Lax–Milgram Theorem. We denote

$$(\pi_h u)_t = \frac{\partial}{\partial t}(\pi_h u),$$

and state the following estimates.

Lemma 3.7. *For the SIPG scheme ($\epsilon = -1$) and r, $s \geq 2$, there exists a constant C satisfying*

$$\|u - \pi_h u\|_{\mathrm{DG}} \leq C \frac{h^{\mu-1}}{r^{s-2}} \|\|u\|\|_s, \tag{3.16}$$

$$\|\|u - \pi_h u\|\|_0 \leq C \frac{h^{\mu}}{r^{s-2}} \|\|u\|\|_s, \tag{3.17}$$

$$\|u_t - (\pi_h u)_t\|_{\mathrm{DG}} \leq C \frac{h^{\mu-1}}{r^{s-2}} (\|\|u\|\|_s + \|\|u_t\|\|_s), \tag{3.18}$$

$$\|\|u_t - (\pi_h u)_t\|\|_0 \leq C \frac{h^{\mu}}{r^{s-2}} (\|\|u\|\|_s + \|\|u_t\|\|_s), \tag{3.19}$$

where $\mu = \min(r + 1, s)$.

A straightforward modification of the analysis of Theorem 4.1 in Ref. [21] yields the proof of Lemma 3.7, and we omit the proof.

IV. EXISTENCE OF A FULLY DISCRETE SOLUTION

The existence of the fully discrete IPDG formulation (2.13) is to find a sequence $\{u_h^n\}_{n=0}^N$ of functions in $\mathcal{D}_r(\mathcal{E}_h)$. We need the following Lemma in Ref. [26] to show the existence of a fully discrete solution u_h^{n+1} in (2.13).

Lemma 4.1. *Let X be a finite dimensional Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\lceil \cdot \rceil$ and let P be a continuous mapping from X into itself such that*

$$(P(\xi), \xi) > 0, \quad \text{for } \lceil \xi \rceil = K > 0, \tag{4.1}$$

then there exists $\bar{\xi} \in X$, $\lceil \bar{\xi} \rceil \leq K$ such that $P(\bar{\xi}) = 0$.

Let M be the dimension of $\mathcal{D}_r(\mathcal{E}_h)$ corresponding to the number of degrees of freedom (DOF). We choose a basis $\{\phi_i\}_{i=1}^M$ of $\mathcal{D}_r(\mathcal{E}_h)$ made of polynomials $\phi_i^{E_j}$ with $supp\, \phi_i^{E_j} \subset E_j$, $E_j \in \mathcal{E}_h$, and degree of $\phi_i^{E_j}$ less than r, for $1 \leq i \leq M$, $1 \leq j \leq N_h$. Then, thanks to Lemma 4.1, we state and prove the following existence result of a discrete solution u_h^n, $0 \leq n \leq N$, of (2.13).

Theorem 4.1. *Under the assumptions of Lemma 3.5, and if*

$$\triangle t < \frac{1}{2L_f}, \tag{4.2}$$

then there exists a solution $u_h^{n+1} \in \mathcal{D}_r(\mathcal{E}_h)$, $0 \leq n \leq N - 1$ of the parabolic equation (2.13) in weak form.

Proof. We first observe that $u_h^0 \in \mathcal{D}_r(\mathcal{E}_h)$ by the definition of u_h^0 in (2.14).

To complete the proof, we assume that there exists $u_h^n \in \mathcal{D}_r(\mathcal{E}_h)$, $0 \leq n \leq l$, solution of the IPDG formulation (2.13). Then, to show the existence of $u_h^{n+1} \in \mathcal{D}_r(\mathcal{E}_h)$, we consider (2.13) with $n = l$. Aiming to find $u_h^{l+1} = \sum_{i=1}^M \xi_i \phi_i$, we set a mapping

$$P(\xi) = (P_1(\xi), P_2(\xi), \ldots, P_M(\xi)), \tag{4.3}$$

where

$$\xi = (\xi_1, \xi_2, \ldots, \xi_M)^T,$$

$$P_i(\xi) := \left(\frac{u_h^{l+1} - u_h^l}{\triangle t}, \phi_i\right) + A_\epsilon(u_{h,\theta}^l; u_{h,\theta}^l, \phi_i) - (f(u_{h,\theta}^l), \phi_i), \quad 1 \le i \le M.$$

We apply Lemma 4.1 with $X = \mathbb{R}^M$ which is equipped with the inner product $(\cdot, \cdot)$ and the l^2 norm $|\cdot|_{l^2}$. Then, we find

$$
\begin{aligned}
(P(\xi), \xi) &= \sum_{i=1}^M P_i(\xi)\xi_i \\
&= \left(\frac{u_h^{l+1} - u_h^l}{\triangle t}, u_h^{l+1}\right) + A_\epsilon(u_{h,\theta}^l; u_{h,\theta}^l, u_h^{l+1}) - \left(f(u_{h,\theta}^l), u_h^{l+1}\right) \\
&= \left(\frac{u_h^{l+1} - u_h^l}{\triangle t}, u_h^{l+1}\right) - \left(f\left(\frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}\right), u_h^{l+1}\right) \\
&\quad + A_\epsilon\left(\frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}; \frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}, u_h^{l+1}\right).
\end{aligned}
$$

Assume that $\|u_h^l\|_{\mathrm{DG}}$ is finite. The operator P is continuous, and there remains to check (4.1); for this purpose, we consider the scalar product $(P(\xi), \xi)$. We have

$$\left(\frac{u_h^{l+1} - u_h^l}{\triangle t}, u_h^{l+1}\right) = \frac{1}{2\triangle t}(\||u_h^{l+1}\||_0^2 - \||u_h^l\||_0^2 + \||u_h^{l+1} - u_h^l\||_0^2),$$

and

$$
\begin{aligned}
&A_\epsilon\left(\frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}; \frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}, u_h^{l+1}\right) \\
&= \frac{1-\theta}{2}A_\epsilon\left(\frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}; u_h^l, u_h^{l+1}\right) + \frac{1+\theta}{2}A_\epsilon\left(\frac{1-\theta}{2}u_h^l + \frac{1+\theta}{2}u_h^{l+1}; u_h^{l+1}, u_h^{l+1}\right) \\
&\ge -\frac{1-\theta}{2}\beta_0\|u_h^l\|_{\mathrm{DG}}\|u_h^{l+1}\|_{\mathrm{DG}} + \frac{1+\theta}{2}\alpha_0\|u_h^{l+1}\|_{\mathrm{DG}}^2 \\
&\ge \frac{1+\theta}{2}\alpha_0\|u_h^{l+1}\|_{\mathrm{DG}}^2 - \frac{1-\theta}{4}\alpha_0\|u_h^{l+1}\|_{\mathrm{DG}}^2 - \frac{1-\theta}{4\alpha_0}\beta_0^2\|u_h^l\|_{\mathrm{DG}}^2 \\
&\ge \frac{1+3\theta}{4}\alpha_0\|u_h^{l+1}\|_{\mathrm{DG}}^2 - \frac{1-\theta}{4\alpha_0}\beta_0^2\|u_h^l\|_{\mathrm{DG}}^2,
\end{aligned}
$$

where the first inequality is derived by the coercivity (3.11) and the continuity (3.14) and the second inequality holds by the Cauchy–Schwarz inequality. We bound the third term

$$\left| \left(f\left(\frac{1-\theta}{2} u_h^l + \frac{1+\theta}{2} u_h^{l+1} \right), u_h^{l+1} \right) \right| = \left| \left(f\left(\frac{u_h^l + u_h^{l+1}}{2} + \theta \frac{u_h^{l+1} - u_h^l}{2} \right), u_h^{l+1} \right) \right|$$

$$\leq L_f \left\| \left| \frac{u_h^l + u_h^{l+1}}{2} + \theta \frac{u_h^{l+1} - u_h^l}{2} \right| \right\|_0 \| u_h^{l+1} \|_0$$

$$\leq \frac{L_f}{2} \| u_h^{l+1} \|_0^2 + \frac{L_f}{8} \| u_h^l + u_h^{l+1} + \theta(u_h^l - u_h^{l+1}) \|_0^2$$

$$\leq \frac{L_f}{2} \| u_h^{l+1} \|_0^2 + \frac{L_f}{4} \| u_h^l + u_h^{l+1} \|_0^2 + \frac{\theta^2 L_f}{4} \| u_h^{l+1} - u_h^l \|_0^2$$

$$\leq L_f \| u_h^{l+1} \|_0^2 + \frac{L_f}{2} \| u_h^l \|_0^2 + \frac{\theta^2 L_f}{4} \| u_h^{l+1} - u_h^l \|_0^2$$

$$\leq L_f \| u_h^{l+1} \|_0^2 + C_0^2 \frac{L_f}{2} \| u_h^l \|_{\mathrm{DG}}^2 + \frac{\theta^2 L_f}{4} \| u_h^{l+1} - u_h^l \|_0^2.$$

The last inequality is derived by a generalization of Poincaré inequality to the broken Sobolev space $H^1(\mathcal{E}_h)$ (see Ref. [23]). Then, we have

$$(P(\xi), \xi) \geq \frac{1}{2\Delta t}(1 - 2L_f \Delta t)\| u_h^{l+1} \|_0^2 + \frac{1}{2\Delta t}\left(1 - \frac{\theta^2}{2} L_f \Delta t \right) \| u_h^{l+1} - u_h^l \|_0^2$$

$$+ \frac{1+3\theta}{4} \alpha_0 \| u_h^{l+1} \|_{\mathrm{DG}}^2 - \left(\frac{\beta_0^2(1-\theta)}{4\alpha_0} + \frac{C_0^2 L_f}{2} + \frac{C_0^2}{2\Delta t} \right) \| u_h^l \|_{\mathrm{DG}}^2.$$

Therefore, using the assumption (4.2) gives

$$(P(\xi), \xi) \geq \frac{1+3\theta}{4} \alpha_0 \| u_h^{l+1} \|_{\mathrm{DG}}^2 - \left(\frac{\beta_0^2(1-\theta)}{4\alpha_0} + \frac{C_0^2 L_f}{2} + \frac{C_0^2}{2\Delta t} \right) \| u_h^l \|_{\mathrm{DG}}^2. \qquad (4.4)$$

We need to find a suitable $\xi \in \mathbb{R}^M$ to make the right hand side of (4.4) positive. To do this, because $u_h^{l+1} = \sum_{i=1}^M \xi_i \phi_i$, we set

$$\xi = (\xi_1, 0, 0, \ldots, 0) \in \mathbb{R}^M,$$

and write

$$\| u_h^{l+1} \|_{\mathrm{DG}} = \| \xi_1 \phi_1 \|_{\mathrm{DG}} = |\xi_1| \| \phi_1 \|_{\mathrm{DG}}.$$

Then, we choose $\xi_1 = K > 0$ large enough so that

$$\frac{1+3\theta}{4} \alpha_0 K^2 - \left(\frac{\beta_0^2(1-\theta)}{4\alpha_0} + \frac{C_0^2 L_f}{2} + \frac{C_0^2}{2\Delta t} \right) \| u_h^l \|_{\mathrm{DG}}^2 > 0.$$

This shows that there exists a vector $\xi \in \mathbb{R}^M$ with $|\xi|_{l^2} = K > 0$ such that $(P(\xi), \xi) > 0$. By Lemma 4.1, this implies the existence of $u_h^{l+1} \in \mathcal{D}_r(\mathcal{E}_h)$, which completes the proof. ∎

V. NUMERICAL STABILITY OF THE FULLY DISCRETE IPDG SCHEMES

Now, we prove a new stability result for these fully discrete IPDG schemes (2.13) and (2.14).

Theorem 5.1.   *Under the same assumptions of Lemma 3.5, and let the source term f satisfy the conditions (1.5) and (1.7). If a time step $\triangle t$ satisfies that*

$$\triangle t < \frac{1}{2L_f}, \tag{5.1}$$

then numerical solution of the fully discrete problem (2.13) and (2.14) is stable in the following sense:

$$\||u_h^n\||_0^2 \le e^{4L_f T} \||u_h^0\||_0^2, \quad \forall\, 0 \le n \le N, \tag{5.2}$$

$$\triangle t \sum_{n=0}^{N-1} \|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \le \frac{1}{2\alpha_0} e^{4L_f T} \||u_h^0\||_0^2, \quad \theta \in [0,1]. \tag{5.3}$$

Proof.   Taking $v = u_{h,\theta}^n$ in (2.13), we get

$$\left(\frac{u_h^{n+1} - u_h^n}{\triangle t}, u_{h,\theta}^n \right) + A_\epsilon(u_{h,\theta}^n; u_{h,\theta}^n, u_{h,\theta}^n) = (f(u_{h,\theta}^n), u_{h,\theta}^n). \tag{5.4}$$

We recall the following two identities:

$$2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2,$$

$$2(a - b, b) = |a|^2 - |b|^2 - |a - b|^2.$$

Using the two identities, it is easily proved that

$$\left(\frac{u_h^{n+1} - u_h^n}{\triangle t}, u_{h,\theta}^n \right) = \frac{1}{2\triangle t} \left(\||u_h^{n+1}\||_0^2 - \||u_h^n\||_0^2 + \theta\||u_h^{n+1} - u_h^n\||_0^2 \right). \tag{5.5}$$

Using (5.5), (3.11) and the Cauchy–Schwarz inequality, we infer from (5.4) that

$$\frac{1}{2\triangle t} \left(\||u_h^{n+1}\||_0^2 - \||u_h^n\||_0^2 + \theta\||u_h^{n+1} - u_h^n\||_0^2 \right) + \alpha_0 \|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \le \||f(u_{h,\theta}^n)\||_0 \||u_{h,\theta}^n\||_0. \tag{5.6}$$

On the other hand, using (1.5) and (1.7), we estimate the right hand side of (5.6)

$$\||f(u_{h,\theta}^n)\||_0 \||u_{h,\theta}^n\||_0 \le L_f \||u_{h,\theta}^n\||_0^2 \tag{5.7}$$

$$\le L_f \left\| \frac{1+\theta}{2} u_h^{n+1} + \frac{1-\theta}{2} u_h^n \right\|_0^2$$

$$\le \frac{L_f}{4} \||(u_h^{n+1} + u_h^n) + \theta(u_h^{n+1} - u_h^n)\||_0^2$$

$$\le \frac{L_f}{2} \||u_h^{n+1} + u_h^n\||_0^2 + \frac{\theta^2}{2} L_f \||u_h^{n+1} - u_h^n\||_0^2$$

$$\le L_f \||u_h^{n+1}\||_0^2 + L_f \||u_h^n\||_0^2 + \frac{\theta^2}{2} L_f \||u_h^{n+1} - u_h^n\||_0^2.$$

Combining (5.6) and (5.7), after multiplying by $2\triangle t$ on both sides, we find

$$(1 - 2L_f\triangle t)\||u_h^{n+1}\||_0^2 + \theta(1 - \theta L_f\triangle t)\||u_h^{n+1} - u_h^n\||_0^2 + 2\alpha_0\triangle t \|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \tag{5.8}$$
$$\leq (1 + 2L_f\triangle t)\||u_h^n\||_0^2,$$

Due to the assumption that $\triangle t < \frac{1}{2L_f}$, the above inequality (5.8) can be rewritten in the form:

$$\||u_h^{n+1}\||_0^2 + A\||u_h^{n+1} - u_h^n\||_0^2 + B\|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \leq D\||u_h^n\||_0^2, \tag{5.9}$$

where

$$A = \frac{\theta(1 - \theta L_f\triangle t)}{1 - 2L_f\triangle t} > 0, \quad B = \frac{2\alpha_0\triangle t}{1 - 2L_f\triangle t} > 0, \quad D = \frac{1 + 2L_f\triangle t}{1 - 2L_f\triangle t} > 0.$$

Multiplying (5.9) by D^{-n} and summing on n from $n = 0$ to $m - 1$ for all $0 \leq m \leq N$, we get

$$D^{-(m-1)}\||u_h^m\||_0^2 + \sum_{n=0}^{m-1} AD^{-n}\||u_h^{n+1} - u_h^n\||_0^2 + \sum_{n=0}^{m-1} BD^{-n}\|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \leq D\||u_h^0\||_0^2. \tag{5.10}$$

Because $D > 1$, (5.10) gives that

$$\||u_h^m\||_0^2 + \sum_{n=0}^{m-1} A\||u_h^{n+1} - u_h^n\||_0^2 + \sum_{n=0}^{m-1} B\|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \leq D^N\||u_h^0\||_0^2, \quad \forall 1 \leq m \leq N. \tag{5.11}$$

Due to the fact that $\frac{1+v}{1-v} \leq e^{2v}, \quad \forall 0 < v < 1$, we observe that

$$D^N\||u_h^0\||_0^2 \leq e^{4L_f N\triangle t}\||u_h^0\||_0^2 \leq e^{4L_f T}\||u_h^0\||_0^2. \tag{5.12}$$

Combining (5.11) and (5.12), we find that

$$\||u_h^m\||_0^2 \leq e^{4L_f T}\||u_h^0\||_0^2, \quad \forall 1 \leq m \leq N, \tag{5.13}$$

$$\sum_{n=0}^{m-1} B\|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \leq e^{4L_f T}\||u_h^0\||_0^2, \quad \forall 1 \leq m \leq N. \tag{5.14}$$

The above second inequality follows

$$\triangle t \sum_{n=0}^{N-1} \|u_{h,\theta}^n\|_{\mathrm{DG}}^2 \leq \frac{1}{2\alpha_0}(1 - 2L_f\triangle t)e^{4L_f T}\||u_h^0\||_0^2 \leq \frac{1}{2\alpha_0}e^{4L_f T}\||u_h^0\||_0^2,$$

which concludes the theorem. ∎

Remark 5.1.   Under the same condition as that in the existence Theorem 4.1, we have proved numerical stability for the fully discrete implicit IPDG methods. Also, the analogous proof can be given for a fully discrete explicit IPDG scheme. If the source term $f = f(x, u)$ is locally Lipschitz continuous in its argument u as in Ref. [27], one can give a similar numerical stability.

VI. ERROR ESTIMATES OF THE FULLY DISCRETE SIPG SCHEME

In this section, we restrict our attention to the SIPG case. Choosing $\epsilon = -1$ in (2.13) and no constraints on grid sizes and time steps required, we will show an error estimate of the implicit time stepping SIPG method, while the time derivative is discretized in time by the θ scheme.

Define the fully discrete $l^\infty(L^2)$ and $l^2(H^1)$ norms

$$\||v_h\||_{l^\infty(L^2)} = \max_{j=0,\dots,N} \||v_h^j\||_0, \quad \||v_h\||_{l^2(H^1)} = \left(\sum_{j=0}^{N-1} \||\nabla v_h^j\||_0^2 \right)^{1/2}. \tag{6.1}$$

Using the notation (2.12), we set

$$t_\theta^j = \frac{1-\theta}{2} t_j + \frac{1+\theta}{2} t_{j+1}, \quad 0 \le \theta \le 1, 0 \le j \le N - 1. \tag{6.2}$$

Then, we first give the following lemmas.

Lemma 6.1.   *For a sufficiently regular $u = u(x,t)$, we consider $\pi_h u^j \in \mathcal{D}_r(\mathcal{E}_h)$ where $\pi_h u^j = \pi_h u(t_j)$, $0 \le j \le N$. Then we have*

$$\frac{\pi_h u^{j+1} - \pi_h u^j}{\triangle t} = (\pi_h u)_t(t_\theta^j) + \triangle t \rho_{j,\theta}, \quad 0 \le j \le N - 1, \forall x \in \Omega, \tag{6.3}$$

where for $t^* \in (t_\theta^j, t_{j+1})$, $t^{**} \in (t_j, t_\theta^j)$,

$$\rho_{j,\theta} = \frac{1}{2} \left(\left(\frac{1-\theta}{2} \right)^2 - \left(\frac{1+\theta}{2} \right)^2 \right) (\pi_h u)_{tt}(t_\theta^j) + \frac{1}{6} \left(\frac{1-\theta}{2} \right)^3 \triangle t (\pi_h u)_{ttt}(t^*)$$

$$+ \frac{1}{6} \left(\frac{1+\theta}{2} \right)^3 \triangle t (\pi_h u)_{ttt}(t^{**}),$$

and for $s \ge 2$,

$$\||\rho_{j,\theta}\||_0 \le C_1 \||u_{tt}\||_{L^\infty(t_j, t_{j+1}; H^s)}.$$

In the case $\theta = 0$, we also have

$$\||\rho_{j,\theta}\||_0 \le C_2 \triangle t \||u_{ttt}\||_{L^\infty(t_j, t_{j+1}; H^s)}, \quad s \ge 2,$$

where C_1 and C_2 are two constants independent of u, $\pi_h u$, $\triangle t$ and h.

   Proof.   By applying the Taylor expansion to $\pi_h u$ at $t = t_\theta^j$, (6.3) easily follows. More results can be found similarly in Ref. [20].   ∎

Lemma 6.2.   *For any edge e_k of $\mathcal{F}_h$ and any element E_k of $\mathcal{E}_h$, $\nabla \pi_h u$ in the L^∞ norm is bounded by a positive constant C depending on u and independent of h, $\triangle t$, r, and s, that is,*

$$\|\nabla \pi_h u(t)\|_{\infty, e_k} < C, \quad \forall t \in [0, T], \tag{6.4}$$

$$\|\nabla \pi_h u(t)\|_{\infty, E_k} < C, \quad \forall t \in [0, T]. \tag{6.5}$$

Proof. We consider a given edge $e_k = \partial E_m \cap \partial E_n$ and $E_{mn} = E_m \cup E_n$, then we have by using Lemmas 3.1–3.3

$$
\begin{aligned}
\|\nabla \pi_h u\|_{\infty,e_k} &\leq \|\nabla \hat{u}\|_{\infty,e_k} + \|\nabla(\pi_h u - \hat{u})\|_{\infty,e_k} \\
&\leq C\|\nabla u\|_{\infty,E_{mn}} + Ch^{-\frac{1}{2}}\|\nabla(\pi_h u - \hat{u})\|_{0,e_k} \\
&\leq C\|\nabla u\|_{\infty,E_{mn}} + Ch^{-\frac{1}{2}}\left(h^{-\frac{1}{2}}\|\nabla(\pi_h u - \hat{u})\|_{0,E_{mn}} + h^{\frac{1}{2}}\|\nabla^2(\pi_h u - \hat{u})\|_{0,E_{mn}}\right) \\
&\leq C\|\nabla u\|_{\infty,E_{mn}} + Ch^{-1}\|\nabla(\pi_h u - \hat{u})\|_{0,E_{mn}} \\
&\leq C\|\nabla u\|_{\infty,E_{mn}} + \frac{c}{h}\left(\|\nabla(u - \pi_h u)\|_{0,E_{mn}} + \|\nabla(u - \hat{u})\|_{0,E_{mn}}\right) \\
&\leq C\|\nabla u\|_{\infty,E_{mn}} + \frac{c}{h}\left(\||\nabla(u - \pi_h u)\||_0 + \||\nabla(u - \hat{u})\||_0\right) \\
&\leq C\|\nabla u\|_{\infty,E_{mn}} + C\frac{h^{\mu-2}}{r^{s-2}}\||u\||_s \\
&< \infty,
\end{aligned}
\tag{6.6}
$$

which leads to (6.4).

For $u \in H^2(\Omega)$, one can bound

$$
\begin{aligned}
\|\nabla \pi_h u\|_{\infty,E_k} &\leq \|\nabla \pi_h u - \nabla u\|_{\infty,E_k} + \|\nabla u\|_{\infty,E_k} \\
&\leq C(\text{meas}(E_k))^{-\frac{1}{2}}\|\nabla \pi_h u - \nabla u\|_{0,E_k} + \|\nabla u\|_{\infty,E_k} \\
&\leq C\frac{h^{\mu-2}}{r^{s-2}}\||u\||_s + \|\nabla u\|_\infty \\
&< C.
\end{aligned}
$$

Consequently, (6.5) is proven.   ∎

We now state a priori $l^\infty(L^2)$ and $l^2(H^1)$ error estimates, which is optimal, for the finite element system (2.13) given by the fully discrete implicit time-stepping DG method. Note that our result below is a generation of Theorem 4.2 in Ref. [21], which is not involved in any time discretization. However, as appears in VIII, the main idea in Refs. [16, 21] can be adapted in our current situation.

Theorem 6.1. *Let the solution u of the nonlinear parabolic equation (1.1)–(1.3) satisfy the regularity properties*

$$
\nabla u \in L^\infty(\Omega \times (0,T)), \quad \text{and} \quad u, u_t \in L^\infty(0,T;H^s(\Omega)), \quad s \geq 2.
$$

In addition, we assume that for $\theta \in (0,1]$,

$$
\frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T;H^s(\Omega)),
$$

and in the case $\theta = 0$,

$$\frac{\partial^3 u}{\partial t^3} \in L^\infty(0, T; H^s(\Omega)).$$

Moreover, we assume that the initial condition $u_h^0(x, 0) = \widetilde{u}_{0h} \in \mathcal{D}_r(\mathcal{E}_h)$ satisfies

$$\|u_h^0 - \pi_h u^0\|_0 \leq C \frac{h^\mu}{r^{s-2}} \||\psi\||_s, \quad \mu = \min(r + 1, s). \tag{6.7}$$

Under the same assumptions of Theorem 4.1, then $\{u_h^j\}_{j=1}^N$ the numerical solution of the fully-discrete time SIPG scheme ($\epsilon = -1$) in (2.13) satisfies

$$\||u_h - u\||_{l^\infty(L^2)}^2 + h^2 \triangle t \||u_h - u\||_{l^2(H^1)}^2$$

$$\leq C \frac{h^{2\mu}}{r^{2s-4}} \left(\||\psi\||_s^2 + \triangle t \sum_{j=0}^N \left(\||u(t_j)\||_s^2 + \||u_t(t_j)\||_s^2 \right) \right)$$

$$+ \triangle t^2 \sum_{j=0}^{N-1} \triangle t \||u_{tt}\||_{L^\infty(t_j, t_{j+1}; H^s)}^2, \quad 0 < \theta \leq 1, \tag{6.8}$$

and

$$\||u_h - u\||_{l^\infty(L^2)}^2 + h^2 \triangle t \||u_h - u\||_{l^2(H^1)}^2$$

$$\leq C \frac{h^{2\mu}}{r^{2s-4}} \left(\||\psi\||_s^2 + \triangle t \sum_{j=0}^N \left(\||u(t_j)\||_s^2 + \||u_t(t_j)\||_s^2 \right) \right)$$

$$+ \triangle t^3 \sum_{j=0}^{N-1} \triangle t \||u_{ttt}\||_{L^\infty(t_j, t_{j+1}; H^s)}^2, \quad \theta = 0, \tag{6.9}$$

where $\mu = \min(r + 1, s)$, $r \geq 2$ and C is depending on u and independent of h, r, and $\triangle t$.

Remark 6.1.   We conclude that in the case $0 < \theta \leq 1$, the error is $O(\frac{h^{2\mu}}{r^{2s-4}} + \triangle t^2)$, so as to get an optimal convergence order, we need the restriction $\triangle t^2 = O(\frac{h^{2\mu}}{r^{2s-4}})$. The similar restriction $\triangle t^3 = O(\frac{h^{2\mu}}{r^{2s-4}})$ appears to the other case $\theta = 0$. Due to $\mu > 1$, so we can choose the case $\triangle t \leq Ch^2$ numerically.

Remark 6.2.   For the fully implicit SIPG schemes, in addition to the regularity setting (1.8), Theorem 6.1 requires the solution u satisfying $u_{tt} \in L^\infty(0, T; H^s(\Omega))$ and $u_{ttt} \in L^\infty(0, T; H^s(\Omega))$ corresponding to the cases of $0 < \theta \leq 1$ and $\theta = 0$, respectively, to assure the temporal accuracy. Compared to the fully implicit SIPG schemes, in the interior of each element, the fully explicit SIPG scheme in Ref. [25] only requires a less smooth solution satisfying $u \in L^2(0, T; H^s(\Omega))$, $u_t \in L^2(0, T; H^{s-1}(\Omega))$ and $u_{tt} \in L^\infty(0, T; H^1(\Omega))$. It appears that the fully implicit schemes need more regularity assumptions in time than the fully explicit one.

TABLE I. Convergence order of the SIPG method (2.13) with Crank-Nicolson discretization in time.

	$\|error\|_{l^\infty(L^2)}$			$\|error\|_{L^2(H^1)}$		
	h	$\frac{h}{2}$	Order	h	$\frac{h}{2}$	Order
$r = 1$	$5.78E-02$	$1.12E-02$	2.37	$3.62E-01$	$1.13E-01$	1.68
$r = 2$	$3.43E-03$	$1.92E-04$	4.16	$5.19E-02$	$1.31E-02$	1.98
$r = 3$	$3.40E-04$	$2.53E-05$	3.75	$1.79E-02$	$2.13E-03$	3.07
$r = 4$	$3.89E-05$	$1.28E-06$	4.92	$3.67E-03$	$1.78E-04$	4.37

Here, the mesh size is $h = 0.288$ and the time step is $\triangle t = 0.0001$.

VII. NUMERICAL RESULTS

In this section, we present some numerical experiments to illustrate the performance of our fully discrete SIPG methods for nonlinear parabolic equations. We will also show the performance of Euler backward and Crank–Nicolson schemes in the time discretization.

We consider the following nonlinear equation on the domain $\Omega = (0, 1)^2$

$$u_t - \nabla \cdot (u^2 \nabla u) = f(x, u), \quad \text{in } \Omega \times (0, T). \tag{7.1}$$

The exact solution is given by $u = (2 + \cos(\pi x)\cos(\pi y))exp(-t)$. The initial and boundary conditions and the right-hand side function f can be obtained by using the exact solution. The final time $T = 1$ is taken and the penalty parameter σ_k is uniformly defined by $\sigma_k = 20(r+1)^2$ on each edge.

The error of the solution is evaluated over the domain Ω in the $L^2(0, T; H^1(\Omega))$ seminorm

$$\|error\|_{L^2(H^1)}^2 = \triangle t \|\|u_h - u\|\|_{l^2(H^1)}^2.$$

We use the SIPG method in various quasiuniform meshes for spatial discretization respectively, and the backward Euler and Crank–Nicolson methods for time integration with a uniform time step $\triangle t$. We compute the errors corresponding to the exact solution $u(x; t)$ in the $l^\infty(L^2)$ and $L^2(0, T; H^1(\Omega))$ norms. Tables I–III demonstrate the errors and order of convergence of the fully discrete schemes, which confirm optimality as shown in Theorem 6.1. Table III shows that convergence order of the numerical solution decreases in the $l^\infty(L^2)$ norm and almost keeps invariant in the $L^2(0, T; H^1(\Omega))$ norm, while θ increases from 0 to 1. Figure 1 illustrates that the isolines of the numerical solutions do not have spurious overshoots as the polynomial degree increases.

To weaken the condition of $a(u)$, we consider another analytic solution $u = \sin(\pi x)\sin(\pi y)\exp(-t)$, which satisfies the Neumann boundary condition, but does not satisfy the condition $a(u) > 0$ on the boundary. From Tables IV–VI, we observe that these IPDG

TABLE II. Convergence order of the SIPG method (2.13) with Euler backward discretization in time.

	$\|error\|_{l^\infty(L^2)}$			$\|error\|_{L^2(H^1)}$		
	h	$\frac{h}{2}$	Order	h	$\frac{h}{2}$	Order
$r = 1$	$5.79E-02$	$1.12E-02$	2.37	$3.62E-01$	$1.13E-01$	1.68
$r = 2$	$3.45E-03$	$2.01E-04$	4.10	$5.19E-02$	$1.31E-02$	1.98
$r = 3$	$3.34E-04$	$6.91E-05$	2.27	$1.79E-02$	$2.13E-03$	3.07
$r = 4$	$7.02E-05$	$3.32E-06$	4.40	$3.67E-03$	$1.76E-04$	4.38

Here the mesh size is $h = 0.288$ and the time step is $\triangle t = 0.0001$.

TABLE III.   Convergence order of the SIPG method (2.13) with θ schemes in time for second-order polynomial approximation ($r = 2$).

	$\|error\|_{l^\infty(L^2)}$			$\|error\|_{L^2(H^1)}$		
	h	$\frac{h}{2}$	Order	h	$\frac{h}{2}$	Order
$\theta = 0$	$3.40E-03$	$1.91E-04$	4.15	$5.17E-02$	$1.31E-02$	1.98
$\theta = 0.1$	$3.43E-03$	$2.02E-04$	4.09	$5.17E-02$	$1.31E-02$	1.98
$\theta = 0.2$	$3.45E-03$	$2.42E-04$	3.83	$5.18E-02$	$1.31E-02$	1.98
$\theta = 0.3$	$3.47E-03$	$3.00E-04$	3.53	$5.18E-02$	$1.31E-02$	1.98
$\theta = 0.4$	$3.50E-03$	$3.66E-04$	3.26	$5.18E-02$	$1.31E-02$	1.98
$\theta = 0.5$	$3.52E-03$	$4.37E-04$	3.01	$5.18E-02$	$1.31E-02$	1.98
$\theta = 0.6$	$3.55E-03$	$5.09E-04$	2.80	$5.19E-02$	$1.31E-02$	1.99
$\theta = 0.7$	$3.58E-03$	$5.83E-04$	2.62	$5.19E-02$	$1.31E-02$	1.99
$\theta = 0.8$	$3.61E-03$	$6.58E-04$	2.46	$5.19E-02$	$1.31E-02$	1.99
$\theta = 0.9$	$3.64E-03$	$7.34E-04$	2.31	$5.20E-02$	$1.31E-02$	1.99
$\theta = 1$	$3.67E-03$	$8.10E-04$	2.18	$5.20E-02$	$1.31E-02$	1.99

Here, the mesh size is $h = 0.288$ and the time step is $\triangle t = 0.01$.
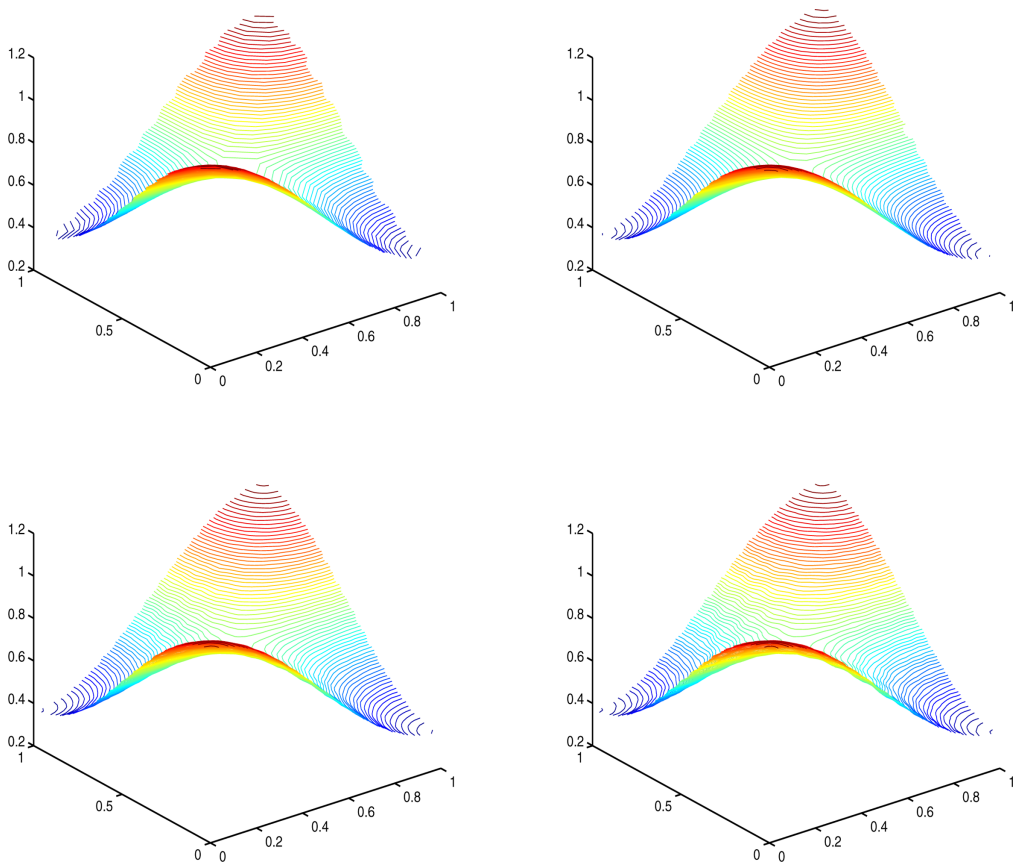

FIG. 1.   Isolines of the numerical solutions for P^1(top left), P^2(top right), P^3(bottom left), and P^4(bottom right) elements on the mesh, SIPG formulation with Crank–Nicolson time discretization. [Color figure can be viewed in this online issue, which is available at wileyonlinelibrary.com.]

TABLE IV.   Convergence order of the SIPG method (2.13) with Crank–Nicolson discretization in time.

	$\|error\|_{l^\infty(L^2)}$			$\|error\|_{L^2(H^1)}$		
	h	$\frac{h}{2}$	Order	h	$\frac{h}{2}$	Order
$r = 1$	$4.55E - 02$	$1.03E - 02$	2.14	$3.44E - 01$	$1.31E - 01$	1.39
$r = 2$	$6.91E - 03$	$6.50E - 04$	3.41	$1.24E - 01$	$2.41E - 02$	2.36
$r = 3$	$2.56E - 03$	$1.77E - 04$	5.32	$7.91E - 02$	$1.22E - 02$	2.70
$r = 4$	$4.78E - 04$	$1.31E - 05$	5.19	$2.78E - 02$	$1.61E - 03$	4.11

Here, the mesh size is $h = 0.288$ and the time step is $\triangle t = 0.001$.

TABLE V.   Convergence order of the SIPG method (2.13) with Euler backward discretization in time.

	$\|error\|_{l^\infty(L^2)}$			$\|error\|_{L^2(H^1)}$		
	h	$\frac{h}{2}$	Order	h	$\frac{h}{2}$	Order
$r = 1$	$4.55E - 02$	$1.03E - 02$	2.14	$3.45E - 01$	$1.32E - 01$	1.39
$r = 2$	$6.86E - 03$	$6.45E - 04$	3.41	$1.24E - 01$	$2.43E - 02$	2.35
$r = 3$	$2.56E - 03$	$1.88E - 04$	3.77	$7.89E - 02$	$1.23E - 02$	2.68
$r = 4$	$4.94E - 04$	$1.76E - 05$	4.81	$2.80E - 02$	$1.61E - 03$	4.12

Here, the mesh size is $h = 0.288$ and the time step is $\triangle t = 0.001$.

schemes are convergent and confirm the effectiveness of the fully discrete schemes. Table VI shows that convergence order of the numerical solution decreases in the $l^\infty(L^2)$ norm and almost keeps invariant in the $L^2(0, T; H^1(\Omega))$ norm, while θ increases from 0 to 1. In Fig. 2, the approximate solutions do not have spurious overshoots in P^1 element and appear more smooth on the boundary with increasing polynomial degree r.

VIII. CONCLUSIONS

We have analyzed the fully discrete IPDG method with a class of implicit θ schemes in time for a class of nonlinear parabolic equations in 1D or 2D. The existence of numerical solutions and

TABLE VI.   Convergence order of the SIPG method (2.13) with θ schemes in time for second-order polynomial approximation ($r = 2$).

	$\|error\|_{l^\infty(L^2)}$			$\|error\|_{L^2(H^1)}$		
	h	$\frac{h}{2}$	Order	h	$\frac{h}{2}$	Order
$\theta = 0$	$6.72E - 03$	$6.53E - 04$	3.36	$1.21E - 01$	$2.37E - 02$	2.35
$\theta = 0.1$	$6.67E - 03$	$6.49E - 04$	3.36	$1.21E - 01$	$2.38E - 02$	2.35
$\theta = 0.2$	$6.62E - 03$	$6.56E - 04$	3.34	$1.21E - 01$	$2.38E - 02$	2.35
$\theta = 0.3$	$6.58E - 03$	$6.90E - 04$	3.25	$1.22E - 01$	$2.39E - 02$	2.35
$\theta = 0.4$	$6.53E - 03$	$7.76E - 04$	3.07	$1.22E - 01$	$2.39E - 02$	2.35
$\theta = 0.5$	$6.49E - 03$	$8.95E - 04$	2.86	$1.22E - 01$	$2.40E - 02$	2.35
$\theta = 0.6$	$6.45E - 03$	$1.03E - 03$	2.65	$1.23E - 01$	$2.40E - 02$	2.35
$\theta = 0.7$	$6.42E - 03$	$1.17E - 03$	2.46	$1.23E - 01$	$2.41E - 02$	2.35
$\theta = 0.8$	$6.38E - 03$	$1.31E - 03$	2.28	$1.23E - 01$	$2.42E - 02$	2.35
$\theta = 0.9$	$6.34E - 03$	$1.46E - 03$	2.12	$1.23E - 01$	$2.42E - 02$	2.35
$\theta = 1$	$6.31E - 03$	$1.61E - 03$	1.97	$1.24E - 01$	$2.43E - 02$	2.34

Here, the mesh size is $h = 0.288$ and the time step is $\triangle t = 0.01$.
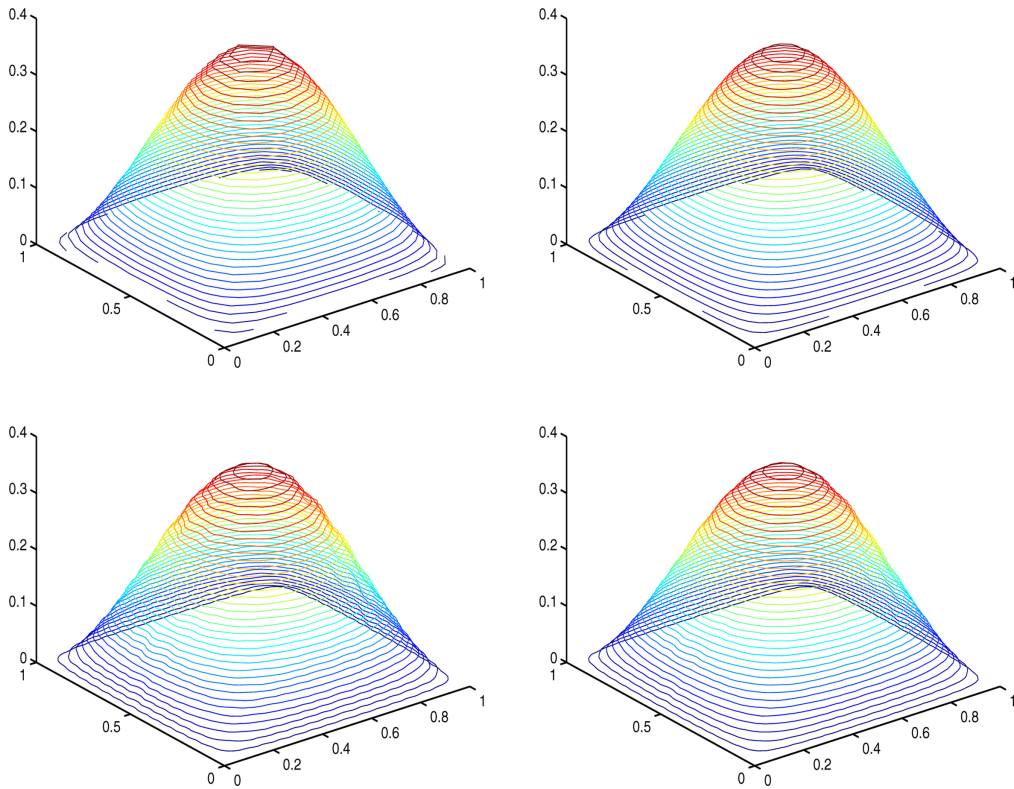
FIG. 2. Isolines of the numerical solutions for P^1(top left), P^2(top right), P^3(bottom left), and P^4(bottom right) elements on the mesh, SIPG formulation with Crank–Nicolson time discretization. [Color figure can be viewed in this online issue, which is available at wileyonlinelibrary.com.]

the numerical stability of these fully discrete IPDG schemes are proven, and they have the same restricted condition in time step. It is interesting that the stability condition is not associated with the mesh size. Using a nonlinear elliptic projection and the interpolant approximation for the SIPG scheme, we have proved a priori error estimates in the discrete $l^2(H^1)$ seminorm and in the $l^\infty(L^2)$ norm, which are optimal in h. The numerical results have confirmed the presented theory. The approximation spaces are considered on a quasiuniform triangular mesh, but our results can be extended to quadrilateral meshes. Analogously, error estimates and stability analysis can also be carried out in 3D.

APPENDIX: PROOF OF THEOREM 6.1

Proof.　To this end, we proceed in the following steps.　　　　　　　　　■

Step 1 (representation of an identity for $\pi_h u^j - u_h^j$).
For $n = j$, we write (2.13) in an equivalent form

$$\left(\frac{u_h^{j+1} - u_h^j}{\triangle t}, v\right) + A_{-1,\lambda}(u_{h,\theta}^j; u_{h,\theta}^j, v) = (f(u_{h,\theta}^j), v) + \lambda(u_{h,\theta}^j, v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \quad \text{(A.1)}$$

At $t = t_\theta^j$, defined by (6.2), we notice that the exact solution $u(t_\theta^j)$ satisfies that $\forall\, v \in \mathcal{D}_r(\mathcal{E}_h)$,

$$\left(\frac{\partial u(t_\theta^j)}{\partial t}, v\right) + A_{-1,\lambda}(u(t_\theta^j); u(t_\theta^j), v) = (f(u(t_\theta^j)), v) + \lambda(u(t_\theta^j), v). \tag{A.2}$$

Subtracting (A.1) from (A.2), we find

$$\left(\frac{\partial u(t_\theta^j)}{\partial t}, v\right) + A_{-1,\lambda}(u(t_\theta^j); u(t_\theta^j), v) - A_{-1,\lambda}(u_{h,\theta}^j; u_{h,\theta}^j, v) - \left(\frac{u_h^{j+1} - u_h^j}{\Delta t}, v\right)$$
$$= (f(u(t_\theta^j)) - f(u_{h,\theta}^j), v) + \lambda(u(t_\theta^j) - u_{h,\theta}^j, v), \quad \forall\, v \in \mathcal{D}_r(\mathcal{E}_h). \tag{A.3}$$

By taking $\hat{u} \in \mathcal{D}_r(\mathcal{E}_h)$ as an interpolant of u, which satisfies Lemma 3.1, we define

$$\eta_\theta^j = u(t_\theta^j) - \pi_h u(t_\theta^j), \quad \beta_\theta^j = \pi_h u(t_\theta^j) - \hat{u}_\theta^j, \quad \xi_\theta^j = \pi_h u(t_\theta^j) - u_{h,\theta}^j.$$

We also write $\xi^j = \pi_h u^j - u_h^j$. Then, it follows from (A.3)

$$\left(\frac{\xi^{j+1} - \xi^j}{\Delta t}, v\right) + A_{-1,\lambda}(u_{h,\theta}^j; \xi_\theta^j, v)$$
$$= \left(\frac{\pi_h u^{j+1} - \pi_h u^j}{\Delta t} - u_t(t_\theta^j), v\right) + A_{-1,\lambda}(u_{h,\theta}^j; \pi_h u(t_\theta^j), v) - A_{-1,\lambda}(u(t_\theta^j); u(t_\theta^j), v) \tag{A.4}$$
$$+ (f(u(t_\theta^j)) - f(u_{h,\theta}^j), v) + \lambda(u(t_\theta^j) - u_{h,\theta}^j, v).$$

Thanks to Lemma 6.1, it is easy to see that

$$\frac{\pi_h u^{j+1} - \pi_h u^j}{\Delta t} - u_t(t_\theta^j) = (\pi_h u)_t(t_\theta^j) - u_t(t_\theta^j) + \Delta t \rho_{j,\theta} = -(\eta_t)_\theta^j + \Delta t \rho_{j,\theta}. \tag{A.5}$$

Moreover, using (3.15) and the identity that $A_{-1,\lambda}(u(t_\theta^j); \eta_\theta^j, v) = 0$, we get

$$A_{-1,\lambda}(u_{h,\theta}^j; \pi_h u(t_\theta^j), v) - A_{-1,\lambda}(u(t_\theta^j); u(t_\theta^j), v)$$
$$= \sum_{k=1}^{N_h} \int_{E_k} (a(u_{h,\theta}^j) - a(u(t_\theta^j)))\nabla \pi_h u(t_\theta^j)\nabla v\, dx$$
$$- \sum_{k=1}^{P_h} \int_{e_k} \left\{(a(u_{h,\theta}^j) - a(u(t_\theta^j)))\nabla \pi_h u(t_\theta^j) \cdot \mathbf{n}_k\right\} [v]\, ds \tag{A.6}$$
$$- \sum_{k=1}^{P_h} \int_{e_k} \left\{(a(u_{h,\theta}^j) - a(u(t_\theta^j)))\nabla v \cdot \mathbf{n}_k\right\} [\pi_h u(t_\theta^j)]\, ds.$$

Substituting (A.5) and (A.6) into (A.4), and taking $v = \xi_\theta^j$, we obtain

$$
\left(\frac{\xi^{j+1} - \xi^j}{\triangle t}, \xi_\theta^j \right) + A_{-1,\lambda}(u_{h,\theta}^j; \xi_\theta^j, \xi_\theta^j)
$$

$$
= \left(-(\eta_t)_\theta^j, \xi_\theta^j \right) + \sum_{k=1}^{N_h} \int_{E_k} (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \pi_h u(t_\theta^j) \nabla \xi_\theta^j dx
$$

$$
- \sum_{k=1}^{P_h} \int_{e_k} \left\{ (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \pi_h u(t_\theta^j) \cdot \mathbf{n}_k \right\} [\xi_\theta^j] ds \tag{A.7}
$$

$$
- \sum_{k=1}^{P_h} \int_{e_k} \left\{ (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \xi_\theta^j \cdot \mathbf{n}_k \right\} [\pi_h u(t_\theta^j)] ds
$$

$$
+ (f(u(t_\theta^j)) - f(u_{h,\theta}^j), \xi_\theta^j) + \lambda(u(t_\theta^j) - u_{h,\theta}^j, \xi_\theta^j) + \triangle t(\rho_{j,\theta}, \xi_\theta^j)
$$

$$
\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
$$

Step 2 (estimates on I_i, $1 \le i \le 7$). Now, we derive the bounds for each I_i, $1 \le i \le 7$. We bound the first term I_1 by the Cauchy-Schwarz inequality

$$
|I_1| = \left| \left(-(\eta_t)_\theta^j, \xi_\theta^j \right) \right| \le \|\|(\eta_t)_\theta^j\|\|_0 \|\|\xi_\theta^j\|\|_0 \le C \left(\|\|(\eta_t)_\theta^j\|\|_0^2 + \|\|\xi_\theta^j\|\|_0^2 \right). \tag{A.8}
$$

Using (6.5), we estimate the term I_2:

$$
|I_2| \le C \sum_{k=1}^{N_h} \int_{E_k} |u_{h,\theta}^j - u(t_\theta^j)| |\nabla \pi_h u(t_\theta^j) \cdot \nabla \xi_\theta^j| dx
$$

$$
\le C \|\nabla \pi_h u(t_\theta^j)\|_{\infty, \mathcal{E}_h} \sum_{k=1}^{N_h} \left(\|\eta_\theta^j\|_{0,E_k} + \|\xi_\theta^j\|_{0,E_k} \right) \|\nabla \xi_\theta^j\|_{0,E_k} \tag{A.9}
$$

$$
\le C \left(\|\|\eta_\theta^j\|\|_0^2 + \|\|\xi_\theta^j\|\|_0^2 \right) + \alpha \|\|\nabla \xi_\theta^j\|\|_0^2.
$$

To deal with I_3, we consider a given edge $e_k = \partial E_m \cap \partial E_n$ and $E_{mn} = E_m \cup E_n$. Then using (1.4) and (6.4), we first notice that

$$
\left| \int_{e_k} \left\{ (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \pi_h u(t_\theta^j) \cdot \mathbf{n}_k \right\} [\xi_\theta^j] ds \right|
$$

$$
\le C \|\nabla \pi_h u(t_\theta^j)\|_{\infty, e_k} \|\{u(t_\theta^j) - u_{h,\theta}^j\}\|_{0,e_k} \|[\xi_\theta^j]\|_{0,e_k}
$$

$$
\le C \left(\|\{\eta_\theta^j\}\|_{0,e_k} + \|\{\xi_\theta^j\}\|_{0,e_k} \right) \|[\xi_\theta^j]\|_{0,e_k} \tag{A.10}
$$

$$
\le \alpha \frac{\sigma_k}{|e_k|} \|[\xi_\theta^j]\|_{0,e_k}^2 + C \left(\|\eta_\theta^j\|_{0,E_{mn}}^2 + h^2 \|\nabla \eta_\theta^j\|_{0,E_{mn}}^2 + \|\xi_\theta^j\|_{0,E_{mn}}^2 \right),
$$

and hence, a bound of the third term I_3 follows:

$$
\begin{aligned}
|I_3| &= \left| \sum_{k=1}^{P_h} \int_{e_k} \left\{ (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \pi_h u(t_\theta^j) \cdot \mathbf{n}_k \right\} [\xi_\theta^j] ds \right| \\
&\leq \alpha J_0^\sigma(\xi_\theta^j, \xi_\theta^j) + C \sum_{k=1}^{N_h} (\|\eta_\theta^j\|_{0,E_{mn}}^2 + h^2 \|\nabla \eta_\theta^j\|_{0,E_{mn}}^2 + \|\xi_\theta^j\|_{0,E_{mn}}^2) \\
&\leq \alpha \|\xi_\theta^j\|_{\mathrm{DG}}^2 + C (\|\!|\eta_\theta^j\|\!|_0^2 + h^2 \|\!|\nabla \eta_\theta^j\|\!|_0^2 + \|\!|\xi_\theta^j\|\!|_0^2).
\end{aligned} \tag{A.11}
$$

To estimate I_4, we observe that

$$
\begin{aligned}
&\left| \int_{e_k} \left\{ (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \xi_\theta^j \cdot \mathbf{n}_k \right\} [\eta_\theta^j] ds \right| \\
&\leq C \|\nabla \xi_\theta^j\|_{\infty,e_k} \|\{u_{h,\theta}^j - u(t_\theta^j)\}\|_{0,e_k} \|[\eta_\theta^j]\|_{0,e_k} \\
&\leq C h^{-\frac{1}{2}} \|\nabla \xi_\theta^j\|_{0,e_k} \left(\|\{\eta_\theta^j\}\|_{0,e_k} + \|\{\xi_\theta^j\}\|_{0,e_k} \right) \left(h^{-\frac{1}{2}} \|\eta_\theta^j\|_{0,E_{mn}} + h^{\frac{1}{2}} \|\nabla \eta_\theta^j\|_{0,E_{mn}} \right) \\
&\leq C h^{-1} \|\nabla \xi_\theta^j\|_{0,e_k} \left(h^{-\frac{1}{2}} \|\eta_\theta^j\|_{0,E_{mn}} + h^{\frac{1}{2}} \|\nabla \eta_\theta^j\|_{0,E_{mn}} + h^{-\frac{1}{2}} \|\xi_\theta^j\|_{0,E_{mn}} \right) \\
&\quad \times \left(\|\eta_\theta^j\|_{0,E_{mn}} + h \|\nabla \eta_\theta^j\|_{0,E_{mn}} \right) \\
&\leq C h^{-\frac{3}{2}} \|\nabla \xi_\theta^j\|_{0,e_k} \left(\|\eta_\theta^j\|_{0,E_{mn}} + h \|\nabla \eta_\theta^j\|_{0,E_{mn}} + \|\xi_\theta^j\|_{0,E_{mn}} \right) \\
&\quad \times \left(\|\!|\eta_\theta^j\|\!|_0 + h \|\!|\nabla \eta_\theta^j\|\!|_0 \right) \\
&\leq C h^{-2} \|\nabla \xi_\theta^j\|_{0,E_{mn}} \left(\|\eta_\theta^j\|_{0,E_{mn}} + h \|\nabla \eta_\theta^j\|_{0,E_{mn}} + \|\xi_\theta^j\|_{0,E_{mn}} \right) h^2 \|\!|u(t_\theta^j)\|\!|_2 \\
&\leq C \|\nabla \xi_\theta^j\|_{0,E_{mn}} \left(\|\eta_\theta^j\|_{0,E_{mn}} + h \|\nabla \eta_\theta^j\|_{0,E_{mn}} + \|\xi_\theta^j\|_{0,E_{mn}} \right),
\end{aligned} \tag{A.12}
$$

and thus, we get the bound of the fourth term I_4

$$
\begin{aligned}
|I_4| &= \left| \sum_{k=1}^{P_h} \int_{e_k} \left\{ (a(u_{h,\theta}^j) - a(u(t_\theta^j))) \nabla \xi_\theta^j \cdot \mathbf{n}_k \right\} [\eta_\theta^j] ds \right| \\
&\leq \alpha \|\xi_\theta^j\|_{\mathrm{DG}}^2 + C \left(\|\!|\eta_\theta^j\|\!|_0^2 + h^2 \|\!|\nabla \eta_\theta^j\|\!|_0^2 + \|\xi_\theta^j\|_0^2 \right).
\end{aligned} \tag{A.13}
$$

The terms I_5, I_6, I_7 are easy to estimate:

$$
|I_5| = \left| (f(u(t_\theta^j)) - f(u_{h,\theta}^j), \xi_\theta^j) \right| \leq C \|u(t_\theta^j) - u_{h,\theta}^j\|_0 \|\xi_\theta^j\|_0 \leq C \left(\|\!|\eta_\theta^j\|\!|_0^2 + \|\!|\xi_\theta^j\|\!|_0^2 \right). \tag{A.14}
$$

$$
|I_6| = \left| \lambda(u(t_\theta^j) - u_{h,\theta}^j, \xi_\theta^j) \right| \leq C \|u(t_\theta^j) - u_{h,\theta}^j\|_0 \|\xi_\theta^j\|_0 \leq C \left(\|\!|\eta_\theta^j\|\!|_0^2 + \|\!|\xi_\theta^j\|\!|_0^2 \right). \tag{A.15}
$$

$$
|I_7| = \left| \Delta t(\rho_{j,\theta}, \xi_\theta^j) \right| \leq C \left(\Delta t^2 \|\!|\rho_{j,\theta}\|\!|_0^2 + \|\!|\xi_\theta^j\|\!|_0^2 \right). \tag{A.16}
$$

Step 3 (summing up). To make the estimates on I_i, $1 \leq i \leq 7$ useful, we first notice the following inequality holds true:

$$\frac{1}{2\triangle t} \left(\||\xi^{j+1}\||_0^2 - \||\xi^j\||_0^2 \right) \leq \left(\frac{\xi^{j+1} - \xi^j}{\triangle t}, \xi_\theta^j \right).$$

Then, using the bounds of $I_1, I_2, \ldots, I_7$ and the coercivity (3.12) of $A_{-1,\lambda}$, we obtain from (A.7)

$$\frac{1}{2\triangle t} \left(\||\xi^{j+1}\||_0^2 - \||\xi^j\||_0^2 \right) + \kappa \|\xi_\theta^j\|_{\text{DG}}^2$$

$$\leq C \left(\||(\eta_t)_\theta^j\||_0^2 + \||\xi_\theta^j\||_0^2 + \||\eta_\theta^j\||_0^2 + h^2 \||\nabla \eta_\theta^j\||_0^2 + \triangle t^2 \||\rho_{j,\theta}\||_0^2 \right) \qquad \text{(A.17)}$$

$$+ \alpha \left(\||\nabla \xi_\theta^j\||_0^2 + \||\xi_\theta^j\||_0^2 + \|\xi_\theta^j\|_{\text{DG}}^2 \right).$$

By using the definition of $\| \cdot \|_{\text{DG}}$, we infer from (A.17) that

$$\frac{1}{2\triangle t} \left(\||\xi^{j+1}\||_0^2 - \||\xi^j\||_0^2 \right) + \frac{\kappa}{2} \|\xi_\theta^j\|_{\text{DG}}^2$$

$$\leq C \left(\||(\eta_t)_\theta^j\||_0^2 + \||\xi_\theta^j\||_0^2 + \||\eta_\theta^j\||_0^2 + h^2 \||\nabla \eta_\theta^j\||_0^2 + \triangle t^2 \||\rho_{j,\theta}\||_0^2 \right), \qquad \text{(A.18)}$$

for a sufficiently small α.

Multiplying (A.18) by $2\triangle t$ and summing up for $j = 0, \ldots, m-1, \forall 1 \leq m \leq N$, we find

$$\||\xi^m\||_0^2 - \||\xi^0\||_0^2 + \triangle t \kappa \sum_{j=0}^{m-1} \||\nabla \xi_\theta^j\||_0^2$$

$$\leq C \triangle t \sum_{j=0}^{m} \||\xi^j\||_0^2 + C \triangle t \sum_{j=0}^{m-1} \left(\||(\eta_t)_\theta^j\||_0^2 + \||\eta^j\||_0^2 + h^2 \||\nabla \eta^j\||_0^2 \right) \qquad \text{(A.19)}$$

$$+ C \triangle t^3 \sum_{j=0}^{m-1} \||\rho_{j,\theta}\||_0^2, \quad \forall 1 \leq m \leq N.$$

By discrete Gronwall's lemma, we find that, for sufficiently small $\triangle t$,

$$\||\xi^m\||_0^2 + C \triangle t \sum_{j=0}^{m-1} \||\nabla \xi_\theta^j\||_0^2$$

$$\leq C \||\xi^0\||_0^2 + C \triangle t \sum_{j=0}^{m-1} \left(\||(\eta_t)_\theta^j\||_0^2 + \||\eta^j\||_0^2 + h^2 \||\nabla \eta^j\||_0^2 \right) \qquad \text{(A.20)}$$

$$+ C \triangle t^3 \sum_{j=0}^{m-1} \||\rho_{j,\theta}\||_0^2.$$

Under the assumption (6.7), we obtain, for $\theta \in (0, 1]$, that

$$\||\xi^m\||_0^2 + C\triangle t \sum_{j=0}^{m-1} \||\nabla \xi_\theta^j\||_0^2$$

$$\leq C\frac{h^{2\mu}}{r^{2s-4}} \||\psi\||_s^2 + C\frac{h^{2\mu}}{r^{2s-4}}\triangle t \sum_{j=0}^m \left(\||u(t_j)\||_s^2 + \||u_t(t_j)\||_s^2\right) \tag{A.21}$$

$$+ C\triangle t^2 \sum_{j=0}^{m-1} \triangle t \||u_{tt}\||_{L^\infty(t_j,t_{j+1};H^s)}^2, \quad \forall 1 \leq m \leq N,$$

and, in the case $\theta = 0$,

$$\||\xi^m\||_0^2 + C\triangle t \sum_{j=0}^{m-1} \||\nabla \xi_\theta^j\||_0^2 \leq C\frac{h^{2\mu}}{r^{2s-4}} \||\psi\||_s^2 + C\frac{h^{2\mu}}{r^{2s-4}}\triangle t \sum_{j=0}^m \left(\||u(t_j)\||_s^2 + \||u_t(t_j)\||_s^2\right)$$

$$+ C\triangle t^3 \sum_{j=0}^{m-1} \triangle t \||u_{ttt}\||_{L^\infty(t_j,t_{j+1};H^s)}^2, \quad \forall 1 \leq m \leq N.$$

$$\tag{A.22}$$

Inequalities (A.21) and (A.22) conclude the proof with the use of Lemma 3.7. ∎

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