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A note on optimal pebbling of hypercubes

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Abstract A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a distribution δ of pebbles lets us move at least one pebble to each vertex by applying pebbling moves repeatedly(if necessary), then δ is called a pebbling of G. The *optimal pebbling number* $f'(G)$ of G is the minimum number of pebbles used in a pebbling of G. In this paper, we improve the known upper bound for the optimal pebbling number of the hypercubes Q_n. Mainly, we prove for large n, $f'(Q_n) = O(n^{3/2}(\frac{4}{3})^n)$ by a probabilistic argument.

Keywords Optimal pebbling · Hypercubes

Dedicated to Professor Gerald J. Chang on the occasion of his 60th birthday.

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1 Introduction

Let G be a graph such that $V(G) = \{v_1, v_2, \ldots, v_p\}$. By a *distribution* of pebbles on G we mean a function $\delta : V(G) \to N \cup \{0\}$ and for clarity, we use $(\delta_{v_1}, \delta_{v_2}, \ldots, \delta_{v_p})$ to denote δ, where δ_v is the number of pebbles distributed on $v \in V(G)$. The *support* S_δ of δ is defined as the set of vertices v in $V(G)$ such that $\delta_v > 0$. Therefore the number of pebbles used in G is $\sum_{v \in S_\delta} \delta_v$ and denoted by δ_G.

A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a distribution δ of pebbles lets us move at least one pebble to each vertex v by applying pebbling moves repeatedly(if necessary), then δ is called a pebbling of G. The *optimal pebbling number* $f'(G)$ of G is the minimum number of pebbles used in a pebbling of G. Note here that the *pebbling number* $f(G)$ of G is the minimum number of pebbles k such that any distribution of k pebbles is a pebbling of G. See Chung (1989), Lemke and Kleitman (1989) for references.

The problem of pebbling graph was first proposed by J. Lagarias and M. Saks as a tool for solving a number theoretic problem by Lemke and Kleitman (1989). Since then, quite a few of work has been done by F.R.K. Chung (1989), Guzman, Moews (1992), Pachter et al. (1995), Clarke et al. (1997) and Herscovici and Higgins (1998). Its dual concept, the optimal pebbling number of a graph G was first introduced by Pachter et al. (1995) and the following results are notable.

Theorem 1 (Pachter et al. 1995) *Let P be a path with $3t + r$ vertices with $0 \le r \le 2$. Then $f'(P) = 2t + r$.*

Shiue and Fu (2009) extend the study and they find the optimal pebbling number for the caterpillar.

Theorem 2 (Shiue 1999) *For any two graphs G and H, $f'(G \times H) \le f'(G)f'(H)$.*

Theorem 3 (Fu and Shiue 2000) *Let T_h^m be a complete m-ary tree with height h. Then $f'(T_h^m) = 2^h$ for each $m \ge 3$, and $f'(T_h^2) = \min\{\sum_{i=0}^h 2^i x_i \mid \sum_{i=0}^h (2^i - \frac{1}{3})x_i \ge \frac{1}{3} \cdot 2^{h+1}, x_0 \in \{0, 1, 2, 3\}$ and $x_i \in \{0, 2\}$, where $i = 1, 2, \ldots, h\}$.*

Theorem 4 (Moews 1998) *Let Q_n be the hypercube defined by $Q_n = Q_{n-1} \times K_2$. Then $f'(Q_n) = (\frac{4}{3})^{n+O(\log n)}$.*

In fact, the upper bound of $f'(Q_n)$ obtained by Moews (1998) is as follows.

Corollary 5 (Moews 1998) $f'(Q_n) \le 2(\frac{4}{3})^n n^2$.

In what follows, we shall improve this upper bound by using a probabilistic argument.

2 Main result

Let the *covering radius* of a subset W of $V(G)$ be the smallest positive integer d such that all vertices v of G are at distance no more than d from a vertex of W. Then, it is clear that we can place 2^d pebbles on each vertex of W and obtain a pebbling of G with $2^d|W|$ pebbles. Therefore, the choice of W determines an upper bound for $f'(Q_n)$. Let $d(v, W)$ be the minimum values of $d(v, w)$ for $w \in W$. Then, we have the following lemma.

Lemma 6 *Suppose* $0 < \beta < \frac{1}{2}$ *and* W *is a* k*-element subset of* $V(Q_n)$. *Let* p *be the probability for a vertex* $v \in V(Q_n)$ *satisfying that* $d(v, W) > \beta n$. *Then* $p \le [1 - 2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}]^k$.

Proof Set $\mathcal{U}_k = \{S\colon\ S \text{ is a } k\text{-element subset of } V(Q_n)\}$ and for each $v \in V(Q_n)$ $S_v = \{u \in V(Q_n)\colon\ d(u, v) > \beta n\}$. Then $|S_v| = \sum_{i=\beta n+1}^{n}\binom{n}{i} = 2^n - \sum_{i=0}^{\beta n}\binom{n}{i}$. It is easy to see that $d(v, W) > \beta n$ for $W \in \mathcal{U}_k$ if and only if $W \subseteq S_v$. Let $\mathcal{F}_v = \{W\colon\ W \text{ is a } k\text{-element subset of } S_v\}$, $q = |V(Q_n)| = 2^n$ and $r = |S_v|$. Then

$$
\begin{aligned}
p &= \frac{|\mathcal{F}_v|}{|\mathcal{U}_k|} = \frac{\binom{r}{k}}{\binom{q}{k}} \\
&= \frac{r(r-1)(r-2)\cdots(r-k+1)}{q(q-1)(q-2)\cdots(q-k+1)} \\
&\le \left(\frac{r}{q}\right)^k \\
&= \left[1 - 2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}\right]^k.
\end{aligned}
$$

□

Now, we are ready to prove the main result.

Theorem 7 $f'(Q_n) = O(n^{\frac{3}{2}}(\frac{4}{3})^n)$.

Proof Suppose $0 < \beta < \frac{1}{2}$. Let W be a randomly chosen k-element subset of $V(Q_n)$. Let p be the probability for a randomly and uniformly chosen $v \in V(Q_n)$ satisfying that $d(v, W) > \beta n$. Set X the number of vertices in $V(Q_n)$ at distance greater than βn from W. If the probability $Pr[X = 0] > 0$, then there exists a k-element subset of $V(Q_n)$ with covering radius at most βn.

It is clear that the expectation $E[X] = 2^n p$. By Markov's inequality, $Pr[X \ge 1] \le E[X] = 2^n p$. Hence, $Pr[X = 0] = 1 - Pr[X \ge 1] \ge 1 - 2^n p$. Moreover, by Lemma 6, we have

$$
p \le \left[1 - 2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}\right]^k.
$$

So,

$$Pr[X=0] \geq 1 - 2^n p \geq 1 - 2^n \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k.$$

By directed counting,

$$1 - 2^n \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k > 0$$

if and only if

$$-\log \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right] > \frac{n \log 2}{k}.$$

From elementary calculus, we have

$$\begin{aligned} -\log \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right] &= 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i} + \frac{1}{2}\left[2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^2 + \cdots \\ &\geq 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}. \end{aligned}$$

Combining with the estimation in MacWillians and Sloane (1977) (see p. 310),

$$\sum_{i=0}^{\beta n} \binom{n}{i} \geq 2^{nH(\beta)} \left[8n\beta(1-\beta)\right]^{-\frac{1}{2}} \geq 2^{nH(\beta)} (2n)^{-\frac{1}{2}},$$

we obtain

$$-\log \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right] \geq 2^{-n[1-H(\beta)]} (2n)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} n^{-\frac{1}{2}} 2^{-n[1-H(\beta)]},$$

where $H(t) = -t \log_2 t - (1-t)\log_2(1-t)$ for $0 \leq t \leq \frac{1}{2}$. Now, by letting

$$k \geq (\sqrt{2} \log 2) n^{\frac{3}{2}} 2^{n[1-H(\beta)]},$$

it is easy to see that

$$\frac{1}{\sqrt{2}} n^{-\frac{1}{2}} 2^{-n[1-H(\beta)]} \geq \frac{n \log 2}{k}$$

and then

$$Pr[X=0] \geq 1 - 2^n \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k > 0.$$

In this case, $f'(Q_n) \leq k 2^{\beta n}$. Since $T(\beta) = 1 + \beta - H(\beta)$, where $0 < \beta < \frac{1}{2}$, attains its minimum at $\beta = \frac{1}{3}$ with $T(\frac{1}{3}) = \log_2(\frac{4}{3})$, we have

$$f'(Q_n) = O\left(n^{\frac{3}{2}} 2^{n \log_2(\frac{4}{3})}\right) = O\left(n^{\frac{3}{2}} \left(\frac{4}{3}\right)^n\right).$$

This concludes the proof. □

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