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Semi-analytical solution of groundwater flow in a leaky aquifer system subject to bending effect

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SUMMARY

The bending of aquitard like a plate due to aquifer pumping and compression is often encountered in many practical problems of subsurface flow. This reaction will have large influence on the release of the volume of water from the aquifer, which is essential for the planning and management of groundwater resources in aquifers. However, the groundwater flow induced by pumping in a leaky aquifer system is often assumed that the total stress of aquifer maintains constant all the time and the mechanical behavior of the aquitard formation is negligible. Therefore, this paper devotes to the investigation of the effect of aquitard bending on the drawdown distribution in a leaky aquifer system, which is obviously of interest in groundwater hydrology. Based on the work of Wang et al. (2004) this study develops a mathematical model for investigating the impacts of aquitard bending and leakage rate on the drawdown of the confined aquifer due to a constant-rate pumping in the leaky aquifer system. This model contains three equations; two flow equations delineate the transient drawdown distributions in the aquitard and the confined aquifer, while the other describes the vertical displacement in response to the aquitard bending. For the case of no aquitard bending, this new solution can reduce to the Hantush Laplace-domain solution (Hantush, 1960). On the other hand, this solution without the leakage effect can reduce to the time domain solution of Wang et al. (2004). The results show that the aquifer drawdown is influenced by the bending effect at early time and by the leakage effect at late time. The results of sensitivity analysis indicate that the aquifer compaction is sensitive only at early time, causing less amount of water released from the pumped aquifer than that predicted by the traditional groundwater theory. The dimensionless drawdown is rather sensitive to aquitard's hydraulic conductivity at late time. Additionally, both the hydraulic conductivity and thickness of the aquifer are the most sensitive parameters in influencing the predicted dimensionless drawdown.

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1. Introduction

Hantush (1960) presented a Laplace-domain solution of transient drawdown, herein called as the Hantush solution, in a leaky aquifer system. His solution was obtained under the assumptions that (1) water was pumped at a constant-flux rate from a fully penetrating well; (2) the aquifer formation was isotropic and homogenous; (3) the leakage rate from the aquitard was proportional to the drawdown at any point; (4) the storages in both the aquitard and aquifer were considered; and (5) the hydraulic head in the layer supplying leakage rate was constant. The Hantush solution was obtained by utilizing the Hankel and Laplace transforms with associated boundary conditions.

In the past, there were many studies focusing on the prediction of drawdown distribution in a confined aquifer system. For instance, Denis and Motz (1988) developed a solution for both transient and steady-state drawdowns in a coupled aquifer system. Their solution included the effects of evapotranspiration and water storage in a confining unit on the drawdown distribution. Based on force balance and transient flow equations, Yeh et al. (1996) developed a three-dimensional Galerkin finite element model for simulating the land displacements due to pumping. Li (2007) presented an analytic solution in terms of the velocity and displacement to delineate the aquifer horizontal movement caused by well recharge and discharge in a confined leaky aquifer. Yang and Yeh (2009) provided a mathematical model in a two-zone leaky aquifer and developed a transient solution in Laplace domain. Their mathematical model accounted for the influence of skin zone, finite radius well, and aquitard storage.

Hunt and Scott (2005) pointed out the difference among the modified Theis and the solutions of Hantush and Jacob (1955) and Boulton (1954, 1963) for transient flow toward the pumping well. They also provided a new solution solved for drawdowns in

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the top layer with a linearized free-surface condition. In addition, Hunt and Scott (2007) developed an approximate solution to describe water flow toward the pumping well in an aquifer–aquitard–aquifer system. They improved the previous studies of Hemker and Maas (1987) and made the solution much easier, efficient and able to investigate the constraints on approximate solution and physical behavior in the multi-layered aquifer systems. Hu and Chen (2008) compared the difference of the analytical and numerical models for transient flow to the pumping well in a confined-unconfined aquifer system. More literatures regarding well hydraulics modeling could be found in Yeh and Chang (in press), which gave a comprehensive review on the analytical and numerical models for various types of aquifer flow induced by the well pumping.

Zhan and Bian (2006) presented analytical/semi-analytical solutions for constant-flux and constant-drawdown pimping tests in a leaky aquifer with a fully penetrating vertical well to predict the leakage rate and volume. They generalized the solutions for finite-size aquifers with lateral impermeable boundaries, which might be practical for managing the multi-layered aquifers. Li and Neuman (2007) provided a semi-analytical solution in a five-layered aquifer system and evaluated the time-domain results through a numerical inversion algorithm. Note that Cheng (2000) gave a fairly detailed introduction on the leaky aquifer theory and multi-layered aquifer theory.

Various types of effect on the drawdown distribution in the aquifer system had been investigated in the past. For instance, the Noordbergum effect as well as the bending effect had been considered to explore their influence on pumping drawdown in an aquifer system. Kim and Parizek (1997) studied the Noordbergum effect which was produced due to the difference of poroelastic response to the hydraulic pumping stress in an aquifer system. They applied a linear poroelasticity theory to compare the difference caused by the Noordbergum effect in aquifer–aquitard–aquifer systems and single-layered aquifer systems. Wang et al. (2004) applied the thin plate theory on small deflection to the modified Theis equation and derived an analytical solution describing the drawdown distribution in confined aquifers. The results indicated that the aquifer drawdown under bending effect was larger than that predicted by the modified Theis solution, and the difference was obvious near the pumping well at early time. According to field test data, the aquifer drawdown was dramatically affected by the bending effect as water was pumped from a high compressible aquifer layer.

The total stress of the aquifer system is generally assumed to be constant in traditional groundwater theory. Based on previous studies, the aquifer drawdown might be influenced by the bending effect at early pumping time (e.g., Wang et al., 2004) and the leakage effect (e.g., Yang and Yeh, 2009) at late pumping time. In reality, the neglect of those two effects might cause under- or overestimation in aquifer drawdown. Wang et al. (2004) found that the volume of water released from the aquifer estimated by the thin plate theory is less than that by the conventional groundwater theory at early pumping time. Based on the work of Wang et al. (2004) this paper develops a mathematical model describing the drawdown distribution with considering the aquitard bending effect in a leaky aquifer system. The pumping well is assumed to fully penetrate the confined aquifer in the system. The leakage rate from the aquitard is assumed to be proportional to the drawdown over the entire pumping period. The storages of both the aquitard and aquifer are considered in the model. This novel model consists of three governing equations: an equation representing the vertical displacement in response to the aquitard bending and two equations describing the drawdown distributions in the aquifer and the aquitard. The solution of the model in Laplace domain is developed by sequentially applying the Hankel transform and Laplace transform. The time-domain results are numerically computed by utilizing a Laplace inversion scheme called as the modified Crump method. The present solution is therefore capable of investigating the effects of aquitard bending and leakage rate on the drawdown distribution.

2. Analytical study

Fig. 1 illustrates the schematic representation of the cross-section of a leaky aquifer system subject to pumping with a constant flow rate. The aquifer is overlain by an aquitard and underlain by an impermeable layer. On the top of the aquitard, there is an unconfined aquifer of thin thickness and its water table remains unchanged over the entire period of pumping. The effect of aquitard storage is considered in the model and there is only vertical flow in the aquitard. The assumptions made within the model are as follows:

1. The aquifer and aquitard are homogeneous, isotopic, of constant thickness and infinite in radial extent.
2. The base rock is non-deformable.
3. The flow directions are horizontal in the confined aquifer and vertical in the aquitard.
4. The pumping well fully penetrates the confined aquifer and the effect of well radius is negligible.
5. The influence of bending effect on downward displacement only exists in the vertical direction.
6. The mechanical properties and total stress condition of the aquitard itself will not be influenced by the release of water from the aquitard.

2.1. Mathematical model

Based on those assumptions and with considering the aquitard bending effect, the governing equation describing the drawdown distribution in a leaky confined aquifer can be written as

$$T\left(\frac{\partial^2 s(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial s(r,t)}{\partial r}\right)+q_l=S\frac{\partial s(r,t)}{\partial t}+\frac{\partial}{\partial t}\Gamma \qquad (1)$$

where $s(r,t)$ is the aquifer drawdown, T is the transmissivity of the aquifer, S is the storage coefficient of the aquifer, q_l is the leakage rate of the aquitard, r is the radial distance from the centerline of

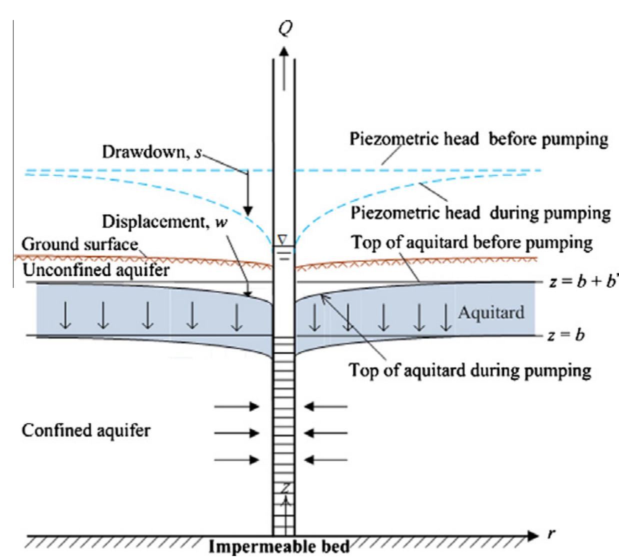

Fig. 1. Idealized representation of the well-flow system for a leaky confined aquifer.

the pumping well, t is the time from the start of the test, and Γ represents an additional amount of water released from aquifer when the aquifer system subject to aquitard bending. The leakage rate q_l equals $K'\partial s'(z,t)/\partial z$ where K' is hydraulic conductivity of the aquitard, $s'(z,t)$ is the drawdown in the aquitard, and z is the vertical distance from the bottom of impermeable layer. Note that $S = S_m + S_w$ where S_m (=$\gamma_w b\alpha$) represents the storage due to aquifer compaction, S_w (=$\gamma_w bn\beta_w$) represents the storage due to water expansion, γ_w is the specific weight of water, b is the thickness of the aquifer, α is the compressibility of aquifer matrix, β_w is the compressibility of water, and n is the porosity of the aquifer. Additionally, $w(r,t)$ represents the vertical downward displacement in response to the aquitard bending and equals the reduction of aquifer thickness, therefore, Γ is $w(r,t) - S_m s(r,t)$ (Wang et al., 2004). Note that Eq. (1) can reduce to Wang et al.'s flow equation (2004, Eq. (20)) if neglecting the leakage term.

The initial drawdown of the aquifer is assumed zero, i.e.

$$s(r,0) = 0 \tag{2}$$

As r approaches infinity, the remote boundary is treated as a zero drawdown condition and specified as

$$s(\infty,t) = 0 \tag{3}$$

Based on Darcy's law, the boundary condition at the wellbore for a constant-flux pumping is expressed as

$$\lim_{r\to 0} -r\frac{\partial s(r,t)}{\partial r} = \frac{Q}{2\pi T} \tag{4}$$

where Q is a constant rate and positive for pumping.

2.1.1. Aquitard

With considering the aquitard storage effect, the governing equation describing the drawdown in the aquitard is given as

$$b'K'\frac{\partial^2 s'(z,t)}{\partial z^2} = S'\frac{\partial s'(z,t)}{\partial t} \tag{5}$$

where $s'(z,t)$ is the drawdown in the aquitard, z is the vertical distance from the bottom of impermeable layer, b' is the thickness of the aquitard, and S' is the storage coefficient of the aquitard.

The initial and boundary conditions are:

$$s'(z,0) = 0 \tag{6}$$

$$s'(z,t) = s(r,t), z = b \tag{7}$$

and

$$s'(z,t) = 0, z = b + b' \tag{8}$$

2.1.2. Vertical downward displacement

The traditional groundwater theory on estimating the release of water from a unit horizontal area of the aquifer generally considers the aquifer matrix compressibility and water expansion only. When the confining unit is subject to the bending effect, the volume of water released from the confined aquifer is different from that computed via the conventional groundwater theory (Wang et al., 2004). Since the thickness of confining unit is very small when compared to the dimensions of the radial extent of the aquitard and thus treated as a thin plate. The thin plate theory (Boresi et al., 1993) is then adopted to describe the vertical deflection caused by the bending effect on the aquitard. The thin plate theory of small-deflection is also employed to describe the vertical displacement when the leaky aquifer system is under the bending effect and aquifer compression.

The deformation is assumed elastic and occurs only in the vertical direction. The deflection of the aquitard (i.e., vertical downward displacement) is equal to the reduction of confined aquifer thickness (Δb) as a result of increasing effective stress, which can be specified as

$$w(r,t) = b\alpha\Delta\sigma' \tag{9}$$

where the change of effective stress ($\Delta\sigma'$) is equal to the change of total stress ($\Delta\sigma$) minus the change of pore pressure (ΔP), i.e., $\Delta\sigma' = \Delta\sigma - \Delta P$ (Bear, 1988). The change of pore pressure associated with aquifer drawdown is

$$\Delta P = -\gamma_w s(r,t) \tag{10}$$

With the relationships of $S_m = \gamma_w b\alpha$ and Eqs. (9) and (10), the change of total stress can be expressed as

$$\Delta\sigma = \frac{\gamma_w}{S_m}w(r,t) - \gamma_w s(r,t) \tag{11}$$

The governing equation accounting for the aquitard bending is of the form as (Wang et al., 2004, Eq. (17))

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w(r,t) = -\Delta\sigma \tag{12}$$

where D is the flexural rigidity of the aquitard defined as (Boresi et al., 1993)

$$D = \frac{Eb'^3}{12(1-\nu^2)} \tag{13}$$

In Eq. (13), E and ν are Young's modulus and Poisson's ratio of the aquitard, respectively.

With Eq. (11), Eq. (12) can be written as

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w(r,t) = -\frac{\gamma_w}{S_m}w(r,t) + \gamma_w s(r,t) \tag{14}$$

The associated initial and boundary conditions are

$$w(r,0) = 0 \tag{15}$$

$$w(\infty,t) = 0 \tag{16}$$

and

$$\frac{\partial w(0,t)}{\partial r} = 0 \tag{17}$$

2.2. Semi-analytical solutions

2.2.1. Aquitard flow

Laplace transform is defined as $F(p) = L\{f(t)\} = \int_0^\infty e^{-pt}f(t)dt$, where p is the Laplace variable. Application of Laplace transform to remove time parameters in Eqs. (5), (7), and (8) leads to

$$\frac{d^2\tilde{s}'(z,p)}{dz^2} - \frac{\eta^2}{b'^2}\tilde{s}'(z,p) = 0, \quad \eta^2 = p\frac{S'b'}{K'} \tag{18}$$

$$\tilde{s}'(z,p) = \tilde{s}(r,p), \quad z = b \tag{19}$$

and

$$\tilde{s}'(z,p) = 0, \quad z = b + b' \tag{20}$$

where $\tilde{s}'(z,p)$ is the drawdown of the aquitard in Laplace domain and $\tilde{s}(r,p)$ is the drawdown of the aquifer in Laplace domain. Note that the aquitard property, K'/b', is known as the leakage coefficient or leakance and its inverse, b'/K', is called the hydraulic resistance.

The general solution of Eq. (18) is

$$\tilde{s}'(z,p) = c_1\sinh\left(\frac{\eta}{b'}z\right) + c_2\cosh\left(\frac{\eta}{b'}z\right) \tag{21}$$

where c_1 and c_2 are undetermined constants. Introducing Eq. (21) into Eqs. (19) and (20), respectively, leads the undetermined constants in Eq. (21) to be

$$c_1 = \frac{-\cosh\left(\frac{\eta}{b'}(b+b')\right)}{\sinh(\eta)}\tilde{s}(r,p) \tag{22}$$

and

$$c_2 = \frac{\sinh\left(\frac{\eta}{b'}(b+b')\right)}{\sinh(\eta)}\tilde{s}(r,p) \tag{23}$$

The solution for aquitard drawdown can be obtained by substituting Eqs. (22) and (23) into Eq. (21). After some manipulations, the aquitard drawdown in Laplace domain can then be expressed as

$$\tilde{s}'(z,p) = \frac{\sinh\left(\frac{\eta}{b'}(b+b'-z)\right)}{\sinh(\eta)}\tilde{s}(r,p) \tag{24}$$

Acquiring the mass conservation, the leakage rate from the aquitard into the confined aquifer can be expressed as

$$\tilde{q}_l = K'\frac{d\tilde{s}'(z,p)}{dz}\bigg|_{z=b} \tag{25}$$

Substituting Eq. (24) into Eq. (25), the leakage rate entering the confined aquifer can be obtained as

$$\tilde{q}_l = -\frac{K'}{b'}\eta\coth(\eta)\tilde{s}(r,p) \tag{26}$$

2.2.2. Drawdown and vertical downward displacement

The relationship between the solutions of aquifer drawdown and vertical downward displacement can be found by applying the Hankel transform and Laplace transform to Eqs. (1) and (14). The solutions for the aquifer drawdown and vertical downward displacement are obtained, respectively, as

$$\tilde{s}(r,p) = \frac{Q}{2\pi Tp}\int_0^\infty \frac{\lambda(\beta)}{\lambda(\beta)\beta^2 + \lambda(\beta)\frac{\eta}{\xi^2}\coth(\eta) + p\beta^2}J_0(\beta r)\beta d\beta \tag{27}$$

and

$$\tilde{w}(r,p) = \frac{Q}{2\pi Tp}\int_0^\infty \times \frac{S_m}{\left(\beta^2 + \frac{\eta}{\xi^2}\coth(\eta)\right)(1+c\beta^4) + p\frac{S}{T}\left(1+\frac{S_w}{S}c\beta^4\right)}J_0(\beta r)\beta d\beta \tag{28}$$

where β is the Hankel transform parameter, $c = S_m D/\gamma_w$, $J_0(\cdot)$ is the Bessel function of first kind of order zero, ξ is the leakage factor defined as $\xi = \sqrt{b'T/K'}$, and $\lambda(\beta) = T\beta^2(1+c\beta^4)/(S+S_w c\beta^4)$. The detailed development for the Laplace-domain solutions is presented in Appendix A.

3. Special cases

Two cases are studied in this paper; one neglects the bending effect, while the other ignores the leakage effect. In other words, these two cases illustrated in Sections 3.1 and 3.2 will indicate that both Hantush' solution (1960) and Wang et al.'s solution (2004) might be considered as special cases of the present solution.

3.1. Neglecting bending effect

When the parameter c is equal to zero, it represents the case that the formation is not rigid. Under this circumstance, Eq. (27) for the aquifer drawdown reduces to

$$\tilde{s}(r,p) = \frac{Q}{2\pi Tp}\int_0^\infty \frac{\beta}{\beta^2+\varphi^2}J_0(\beta r)d\beta \tag{29}$$

According to McLachlan (1955, p. 203, Eq. (200)) that $K_0(\varphi r) = \int_0^\infty \beta J_0(\beta r)d\beta/(1+\beta^2)$, Eq. (29) can be expressed as

$$\tilde{s}(r,p) = \frac{Q}{2\pi Tp}K_0(\varphi r) \tag{30}$$

where $\varphi = \sqrt{(\eta/\xi^2)\coth(\eta) + pS/T}$. Eq. (30) is identical to the Laplace domain solution given by Hantush (1960, Eq. (40)) for the leaky flow problem. Note that Hantush (1960) gave only small- and large-time solutions because the inversion of Eq. (30) to the time domain may not be possible.

3.2. Neglecting leakage effect

For the case where there is no leakage in the aquifer system, the hydraulic conductivity of the aquitard can be considered to be zero. As a result, Eq. (27) reduces to

$$\tilde{s}(r,p) = \frac{Q}{2\pi Tp}\int_0^\infty \frac{1}{\beta}\frac{\lambda(\beta)}{\lambda(\beta)+p}J_0(\beta r)d\beta \tag{31}$$

Taking the inverse Laplace transform of Eq. (31), the corresponding solution in time domain is

$$s(r,t) = \frac{Q}{2\pi T}\int_0^\infty \frac{1-e^{-\lambda(\beta)t}}{\beta}J_0(\beta r)d\beta \tag{32}$$

which is exactly the same as the solution presented in Wang et al. (2004, Eq. (26)).

4. Results and discussion

Eqs. (27) and (28) are in terms of integrals and contain the hyperbolic cosine and Bessel functions. Their solutions in time domain may not be tractable via analytical inversions. Therefore, the time-domain results are obtained by using the numerical inversion routine DINLAP of IMSL (2003). This routine is developed based on a numerical algorithm presented by Crump (1976) and de Hoog et al. (1982). Note that this routine has been successfully utilized to groundwater study (e.g., Yang and Yeh, 2005). The integrals of Eqs. (27) and (28) for the integration range from zero to infinity are difficult to accurately evaluate since the complexity of the product of the hyperbolic cosine and Bessel functions appeared in the integrands. The hyperbolic cosine and Bessel functions can be approximated using the formulas given in Watson (1958) and Abramowitz and Stegun (1964). A numerical approach containing a root search scheme, the Gaussian quadrature, and the Shanks method (Shanks, 1955) can be used to compute Eqs. (27) and (28). This approach had been applied successfully to compute some complicated equations appeared in groundwater problems (e.g., Yang et al., 2006).

4.1. Case study: a hypothetical aquifer system

A hypothetical case is used to illustrate the impacts of aquitard bending and leakage in a leaky aquifer system. This system contains an unconfined aquifer on the top, a confined aquifer consisted of silty sand at the bottom, and an aquitard of clayey silt in between. The parameters of the aquifer and aquitard in the system are given in Table 1.

Table 1
Aquifer and aquitard properties in a leaky confined aquifer system.

Layer	Property	Value	Unit
Aquitard:	Aquitard thickness (b')	25	m
Clayey silt	Storage coefficient of the aquitard (S')	0.0001	
	Hydraulic conductivity of the aquitard (K')	0.004	m/day
	Young's modulus (E)	7×10^6	N/m^2
	Poisson's ratio (υ)	0.3	
	Flexural rigidity (D)	1×10^{10}	N m
Aquifer	Aquifer thickness (b)	50	m
Silty sand	Transmissivity (T)	200	m^2/day
	Storage coefficient of the aquifer (S)	3.91×10^{-3}	
	Compressibility of the elastic aquifer matrix (α)	7.88×10^{-9}	m^2/N
	Compressibility of water (β_w)	4.8×10^{-10}	m^2/N
	Storage coefficient of skeleton compression (S_m)	3.85×10^{-3}	
	Storage coefficient of water expansion (S_w)	5.87×10^{-5}	
	Pumping rate (Q)	1000	m^3/day
	Specific weight of water (γ_w)	9777	N/m^3
	Porosity (n)	0.25	

Fig. 2a and b shows the temporal drawdown distributions in the aquifer for $r = 5$ m and 10 m, respectively, at small and large pumping times. At small pumping time, the aquifer drawdown predicted from the modified Theis solution agrees with that from the Hantush solution as demonstrated in Fig. 2a. Additionally, the aquifer drawdown calculated from the present solution matches with that from the Wang et al. solution (2004). This result indicates that the water released from the bending aquifer system is less than that from the non-bending one at early pumping time. The differences in the drawdowns predicted by the present solution and the Hantush solution are 16.6 cm and 8.4 cm at $r = 5$ m and 10 m, respectively, when $t = 1$ min. Fig. 2a also shows that the effect of aquitard bending on the aquifer drawdown decreases with increasing pumping time and radial distance and becomes negligible after 20 min. This result is consistent with one of the conclusions given in Wang et al. (2004). For a period from 20 to 1000 min, there is no obvious difference in aquifer drawdown among those four solutions. Fig. 2b shows that the present solution matches with the Hantush solution and the modified Theis solution agrees with the Wang et al. solution after 1000 min. With considering the leakage effect, the time–drawdown curves approach a constant value for the cases with and without accounting for the bending effect. On the contrary, the aquifer drawdown keeps increasing when neglecting the leakage effect.

Fig 3 exhibits the spatial drawdown distributions for the aquifer system under pumping when subject to both the bending and leakage effects. The distance–drawdown curves indicate that the bending effect is significant near the pumping well at short pumping time. On the other hand, the influence of leakage rate on the drawdown curve is more obvious at long pumping time. These results indicate that neglecting the bending effect would lead to underestimation of drawdown at early time and ignoring the impact of leakage rate might result in over-estimation of drawdown at late time.

Fig. 4 demonstrates temporal displacement and drawdown curves at $r = 5$ m. As shown in the figure, the displacement is proportional to aquifer drawdown over the entire pumping period. In fact, the vertical downward displacement represented by Eq. (28) depends on the aquifer drawdown described by Eq. (27). Because the aquifer drawdown increases with pumping time, the vertical downward displacement therefore increases with pumping time.

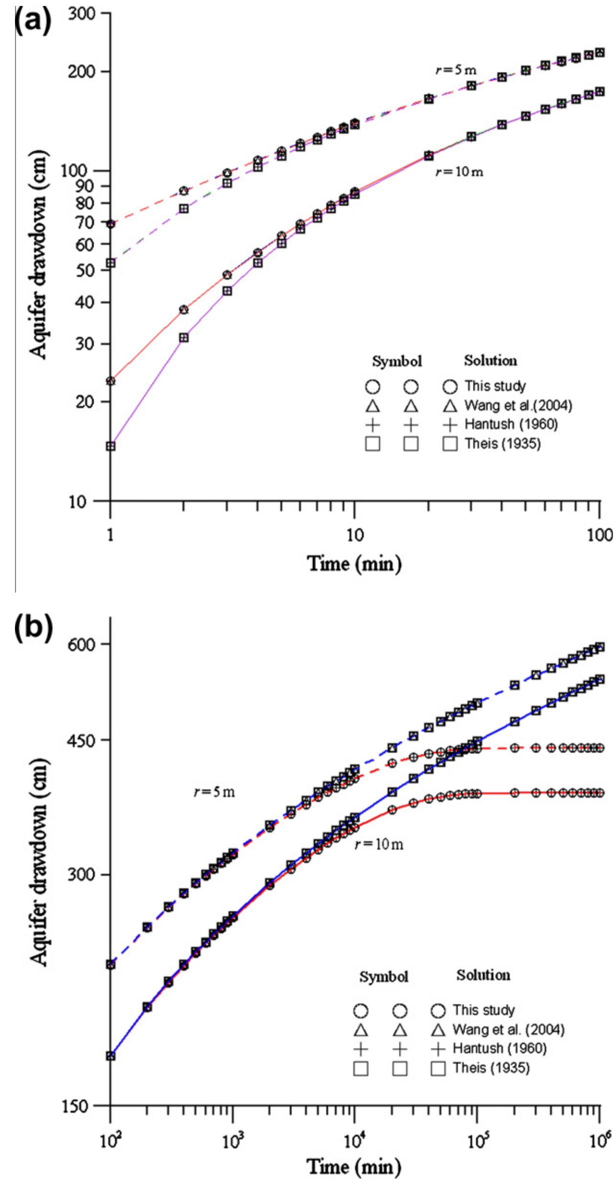

Fig. 2. The time–drawdown curves for $r = 5$ m and 10 m at (a) small time and (b) large time.

4.2. Analysis based on dimensionless parameters

For the convenience of discussion, the aquifer drawdown is expressed in terms of the dimensionless parameters s_D, t_D, and r_D, which are defined as $s_D = s/(Q/4\pi T)$, $t_D = 4Tt/(S \times r^2)$ and $r_D = \gamma_w r^4/(S_m \times D)$. The curves of s_D versus t_D at $r_D = 0.1$ are plotted in Fig. 5 for S_w/S ranging from zero to unity. The aquifer matrix is incompressible when S_w/S equals unity. Under this circumstance, the present solution reduces to the Hantush solution, which is independent of S_w/S. The aquifer drawdown increases with aquifer skeleton compression at early pumping time. As t_D is larger than 20, the present solution agrees with the Hantush solution.

4.3. Sensitivity analysis

Sensitivity analysis provides an indication for the model output (i.e., drawdown) in response to the changes in model parameters (Liou and Yeh, 1997). The model parameters often have different

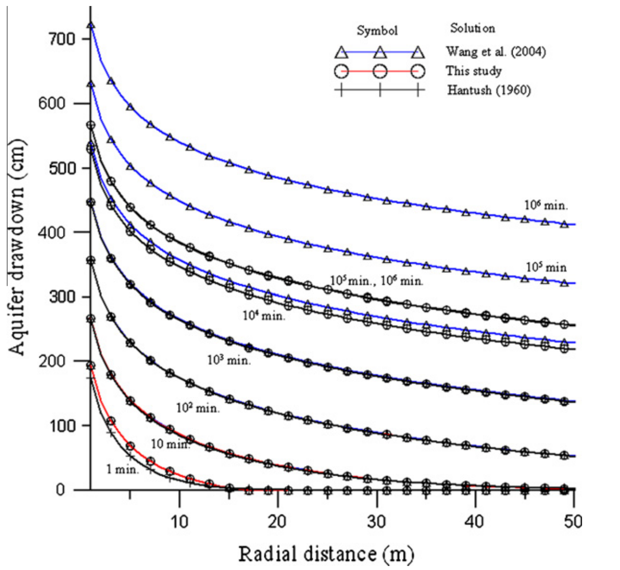

Fig. 3. The radial distance–drawdown curves for the influences of bending effect and leakage effect.

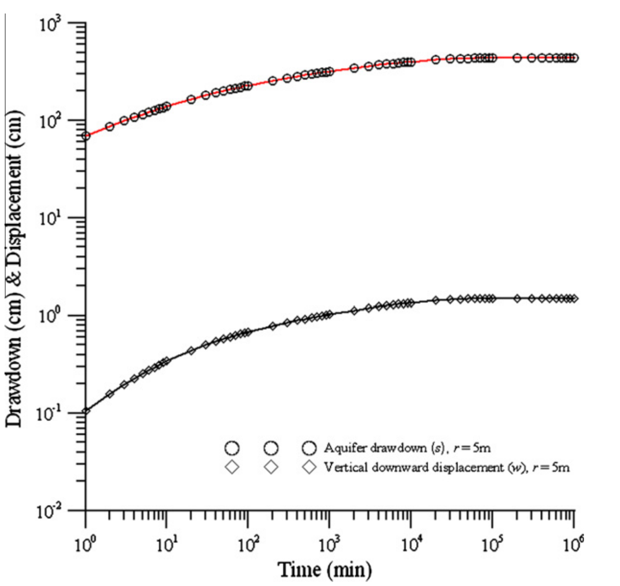

Fig. 4. The time–drawdown curves for aquifer drawdown and displacement at $r = 5$ m.

unit and magnitude. Thus, one may define the relative sensitivity as

$$X_{i,k} = \frac{\partial O_i / O_i}{\partial \Psi_k / \Psi_k} = \frac{\partial O_i}{\partial \Psi_k} \frac{\Psi_k}{O_i} \tag{33}$$

where $X_{i,k}$ is the relative sensitivity value, Ψ_k is the kth input parameter, and O_i is the model output. The values of $X_{i,k}$ are invariant to the magnitude of Ψ_k and O_i and therefore provide a useful means for the comparison of output sensitivities with respect to different input parameters. The derivative term in Eq. (33) can be approximated by the finite-difference formula as

$$\frac{\partial O_i}{\partial \Psi_k} = \frac{O_i(\Psi_k + \Delta\Psi_k) - O_i(\Psi_k)}{\Delta\Psi_k} \tag{34}$$

where $\Delta\Psi_k$ is a small increment selected as $10^{-3} \times \Psi_k$.

The parameters of the aquifer and aquitard used in the sensitivity analysis for the leaky aquifer subject to the bending effect are listed in Table 1. Fig. 6 shows the curves of dimensionless sensitivity for the aquifer drawdown to each of parameters b, b', K, K', α, β_w and S'. The aquifer thickness and hydraulic conductivity have obvious negative influences on the dimensionless aquifer drawdown. The effect of hydraulic conductivity increases with pumping time before 10^3 min and then keeps a constant magnitude after that time. The effect of aquifer thickness increases with pumping time before 20 min, then decreases with pumping time from 20 to 10^5 min, and keeps a constant magnitude after 10^5 min. For the aquitard storage coefficient and water compressibility, they have less impact on the dimensionless drawdown over the entire observation period. Before 20 min, the aquitard thickness and elastic aquifer matrix compressibility have positive influences on the dimensionless drawdown, which correspond to the result in Fig. 2a, demonstrating the impact of the bending effect. This result indicates the importance of bending effect in a leaky aquifer at

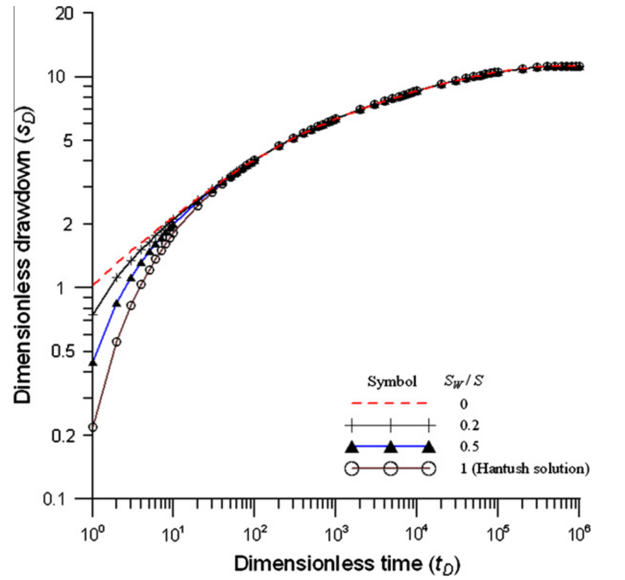

Fig. 5. The curves of dimensionless drawdown s_D versus dimensionless time t_D for several values of S_w/S at $r_D = 0.1$.

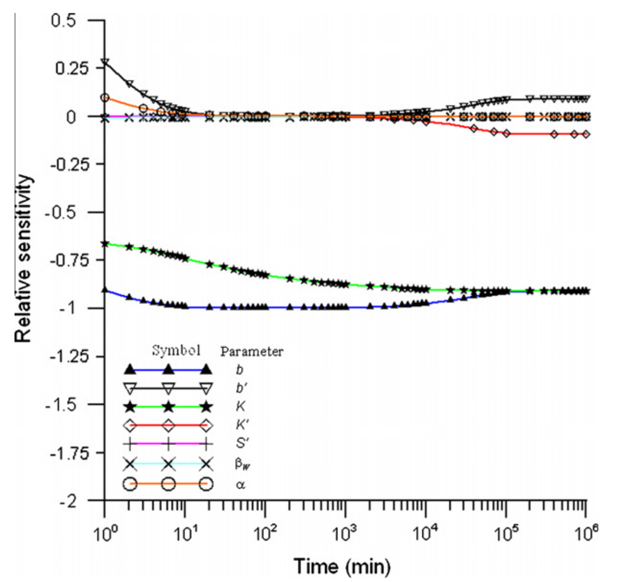

Fig. 6. Relative sensitivity curves for the aquifer drawdown to each of parameters b, b', K, K', α, β_w and S'.

early pumping time. For the period from 20 to 1000 min, the influences of parameters b', K', α, β_w and S' on the dimensionless drawdown are insignificant. After 1000 min, the elastic aquifer matrix compressibility still produces minor influence and the aquitard thickness impacts the dimensionless drawdown positively. The aquitard hydraulic conductivity has minor influence on dimensionless drawdown before 1000 min, and negative influence after that time.

According to the results of the sensitivity analysis, the thickness and hydraulic conductivity of the aquifer are the most important factors in influencing the dimensionless drawdown among the considered parameters. These two parameters present significant negative effect on the dimensionless drawdown distribution. Furthermore, the thickness of the aquitard shows small positive effect at early time and large at late time and the hydraulic conductivity of the aquitard presents a large negative effect at late time. The compressibility of aquifer matrix only reveals a slightly positive effect at early time. The compressibility of water and aquitard storage has almost no influence over the entire pumping time.

5. Conclusions

This paper provides an analytical framework for understanding how the drawdown distribution is influenced by both the bending effect and leakage effect in a leaky confined aquifer system subject to pumping at a fully penetrating well. The mathematical model developed herein involves three equations for describing the vertical displacement in response to the bending effect and the drawdown distributions in the aquifer and aquitard. The Laplace-domain solutions of this model are derived by sequentially applying the Hankel transform and Laplace transform. The corresponding time-domain results are obtained by employing the modified Crump method. The conclusions drawn from the present study are summarized as follows:

1. Contrary to the traditional groundwater theory, the present solution demonstrates a larger drawdown at early pumping time due to the effect of aquitard bending and a smaller drawdown at late pumping time due to the influence of leakage rate.
2. For the case of no aquitard leakage in a confined aquifer system, the present solution reduces to the Wang et al. solution (Wang et al., 2004), which accounts for the effect due to aquitard bending. Additionally, the Hantush solution (Hantush, 1960) has been shown as a special case of the present solution when neglecting the effect of aquitard bending.
3. Based on the thin plate theory, the vertical downward displacement is found proportional to the aquifer drawdown. The strength of bending effect fades out as the radial distance and pumping time increases. From the sensitivity analysis, it shows that the dimensionless drawdown is sensitive to the aquifer skeleton compression only at early time; therefore, neglecting the impact of bending effect leads to overestimate of water released from the aquifer system at short pumping period.

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Appendix A. Derivation of the Laplace-domain solutions to Eqs. (27) and (28)

Taking the Laplace transform of Eqs. (1) and (14) results in

$$T\left(\frac{d^2\tilde{s}(r,p)}{dr^2}+\frac{1}{r}\frac{d\tilde{s}(r,p)}{dr}\right)+\tilde{q}_l=pS\tilde{s}(r,p)+p\tilde{\Gamma}) \tag{A1}$$

and

$$D\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\tilde{w}(r,p)=-\frac{\gamma_w}{S_m}\tilde{w}(r,p)+\gamma_w\tilde{s}(r,p) \tag{A2}$$

where $\tilde{q}_l$ is the leakage rate of the aquitard in Laplace domain and $\tilde{\Gamma}$ is an additional amount of water released from aquifer in Laplace domain. The boundary conditions denoted by Eqs. (3), (4), (16), and (17) in Laplace domain are given as

$$\tilde{s}(\infty,p)=0 \tag{A3}$$

$$\lim_{r\to 0}\left(-r\frac{d\tilde{s}(r,p)}{dr}\right)=\frac{Q}{2\pi Tp} \tag{A4}$$

$$\tilde{w}(\infty,p)=0 \tag{A5}$$

and

$$\frac{d\tilde{w}(0,p)}{dr}=0 \tag{A6}$$

Substituting Eqs. (26) and (A4) into Eq. (A1) yields

$$T\left(\frac{d^2\tilde{s}(r,p)}{dr^2}+\frac{1}{r}\frac{d\tilde{s}(r,p)}{dr}\right)+(-\frac{K'}{b'}\eta\coth(\eta)-S_wp)\tilde{s}(r,p)-p\tilde{w}(r,p)=0 \tag{A7}$$

Note that Hankel transform of zero order is defined as $F(\beta)=\int_0^\infty rf(r)J_0(\beta r)dr$. Employing the Hankel transform to remove the radial distance variable in Eqs. (A2) and (A7) produces, respectively,

$$\bar{w}(\beta,p)(1+c\beta^4)=S_m\bar{s}(\beta,p) \tag{A8}$$

and

$$(T\beta^2+\frac{K'}{b'}\eta\coth(\eta)+S_wp)\bar{s}(\beta,p)+p\overline{w}(\beta,p)=\frac{Q}{2\pi p} \tag{A9}$$

where and the variables $\bar{s}(\beta,p)$ and $\bar{w}(\beta,p)$ are the aquifer drawdown and vertical downward displacement in Laplace–Hankel domain, respectively. Eq. (A8) can be written as

$$\bar{w}(\beta,p)=\frac{S_m}{1+c\beta^4}\bar{s}(\beta,p) \tag{A10}$$

Substituting Eq. (A10) into Eq. (A9), the aquifer drawdown can be expressed as

$$\bar{s}(\beta,p)=\frac{Q}{2\pi Tp}\frac{1+c\beta^4}{(\beta^2+\frac{\eta}{\xi^2}\coth(\eta))(1+c\beta^4)+p\frac{S}{T}(1+\frac{S_w}{S}c\beta^4)} \tag{A11}$$

The vertical downward displacement in a leaky aquifer system can now be obtained after substituting Eq. (A11) into Eq. (A10) and the result in Laplace–Hankel domain is

$$\bar{w}(\beta,p)=\frac{Q}{2\pi Tp}\frac{S_m}{(\beta^2+\frac{\eta}{\xi^2}\coth(\eta))(1+c\beta^4)+p\frac{S}{T}(1+\frac{S_w}{S}c\beta^4)} \tag{A12}$$

Utilizing the inverse Hankel transform of Eqs. (A11) and (A12), the Laplace-domain solutions of the aquifer drawdown and vertical downward displacement in a leaky aquifer with considering the bending effect can be found as Eqs. (27) and (28).

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