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The spanning laceability on the faulty bipartite hypercube-like networks

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ABSTRACT

A w-container $C(u,v)$ of a graph G is a set of w-disjoint paths joining u to v. A w-container of G is a w^*-container if it contains all the nodes of $V(G)$. A bipartite graph G is w^*-laceable if there exists a w^*-container between any two nodes from different parts of G. Let n and k be any two positive integers with $n \geqslant 2$ and $k \leqslant n$. In this paper, we prove that n-dimensional bipartite hypercube-like graphs are f-edge fault k^*-laceable for every $f \leqslant n-2$ and $f+k \leqslant n$.

1. Introduction

1.1. Basic graph definitions and notations

The research about interconnection networks is important for parallel and distributed computer systems. The layouts of processors and links in distributed computer systems are usually represented by a network structure. Computer network topologies are usually represented by graphs where nodes represent processors and edges represent links between processors. The containers of graphs do exist in information engineering design, telecommunication networks, and biological neural systems ([1,2] and its references). The study of w-container, w-wide distance, and their w^*-versions play a pivotal role in the design and the implementation of parallel routing and efficient information transmission in large scale networking systems. In bioinformatics and neuroinformatics, the existence as well as the structure of a w^*-container signifies the cascade effect in the signal transduction system and the reaction in a metabolic pathway.

For graph definitions and notations, we follow [3,4]. Let $G = (V, E)$ be a graph where V is a finite set and E is a subset of $\{(u,v) | (u,v)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set*. We use $n(G)$ to denote $|V|$. Two nodes u and v are *adjacent* if $(u,v) \in E$. For a node u, we use $N_G(u)$ to denote the *neighborhood* of u which is the set $\{v | (u,v) \in E\}$. For any node u of V, we denote the *degree* of u by $\deg_G(u) = |N_G(u)|$. A graph G is *k-regular* if $\deg_G(u) = k$ for every node u in G. A *path* P between nodes v_1 and v_k is a sequence of adjacent nodes, $\langle v_1, v_2, \ldots, v_k \rangle$, in which the nodes $v_1, v_2, \ldots, v_k$ are distinct except that possibly $v_1 = v_k$. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_1 \rangle$. The *length* of $P, l(P)$, is the number of edges in P. We also write the path P as $\langle v_1, v_2, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k \rangle$, where Q is the path $\langle v_i, v_{i+1}, \ldots, v_j \rangle$. Hence, it is possible to write a path as $\langle v_1, v_2, Q, v_2, v_3, \ldots, v_k \rangle$ if $l(Q) = 0$. Let $I(P) = V(P) - \{v_1, v_k\}$ be the set of the internal nodes of P. A set of paths $\{P_1, P_2, \ldots, P_k\}$ are *internally node-disjoint* (abbreviated as *disjoint*) if

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$I(P_i) \cap I(P_j) = \emptyset$ for any $i \neq j$. A path is a *hamiltonian path* if it contains all nodes of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct nodes of G [5]. A *cycle* is a path with at least three nodes such that the first node is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every node of G. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph G is *bipartite* if its node set can be partitioned into two subsets V_1 and V_2 such that every edge connects nodes between V_1 and V_2. A bipartite graph G is *hamiltonian laceable* if there is a hamiltonian path of G joining any two nodes from distinct bipartition [6]. A bipartite graph G is *k-edge fault hamiltonian laceable* if $G - F$ is hamiltonian laceable for any edge subset F of G with $|F| \leqslant k$.

A graph G is *k-connected* if there exists a set of k internally disjoint paths $\{P_1, P_2, \ldots, P_k\}$ between any two distinct nodes u and v. A subset S of $V(G)$ is a *cut set* if $G - S$ is disconnected. A *container* $C(u, v)$ between two distinct nodes u and v in G is a set of disjoint paths between u and v. A *w-container* $C_w(u, v)$ in a graph G is a set of w internally node-disjoint paths between u and v. The concepts of a container and of a wide distance were proposed by Hsu [2] to evaluate the performance of communication for an interconnection network. The *connectivity* of G, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Hence, a graph G is k-connected if $\kappa(G) \geqslant k$. It follows from Menger's Theorem [7] that there is a w-container for $w \leqslant k$ between any two distinct nodes of G if G is k-connected.

1.2. w^*-connected graphs and w^*-laceable graphs

In this paper, we are interested in a specific type of container. A *w^*-container* $C_{w^*}(u, v)$ in a graph G is a w-container such that every node of G is on some path in $C_w(u, v)$. A graph G is *w^*-connected* if there exists a w^*-container between any two distinct nodes in G. Obviously, we have the following remark.

Remark 1. $(1.a)$ a graph G is 1^*-connected if and only if it is hamiltonian connected [5], $(1.b)$ a graph G is 2^*-connected if it is hamiltonian, and $(1.c)$ an 1^*-connected graph except K_1 and K_2 is 2^*-connected.

The study of w^*-connected graph is motivated by the 3^*-connected graphs proposed by Albert et al. [8]. Some related works have appeared in [8,9]. Assume that the graph G is w^*-connected with $w \leqslant \kappa(G)$. The *spanning connectivity* of a graph $G, \kappa^*(G)$, is the largest integer k such that G is i^*-connected for every i with $1 \leqslant i \leqslant k$. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. In such case, the number $\kappa^*(G) = \kappa(G)$ is called the super spanning connectivity of G. In [10–13], some families of graphs are proved to be super spanning connected.

A bipartite graph is said to be *w^*-laceable* if there exists a w^*-container between any two nodes from different partite sets for some w with $1 \leqslant w \leqslant \kappa(G)$. Any bipartite w^*-laceable graph with $w \geqslant 2$ has the equal size of bipartition. We have the following remark.

Remark 2. $(2.a)$ an 1^*-laceable graph is also known as hamiltonian laceable graph [6], $(2.b)$ a graph G is 2^*-laceable if and only if it is hamiltonian, and $(2.c)$ an 1^*-laceable graph except K_1 and K_2 are 2^*-laceable.

The *spanning laceability* of a bipartite graph $G, \kappa^{*L}(G)$, is the largest integer k such that G is i^*-laceable for every i with $1 \leqslant i \leqslant k$. A graph G is *super spanning laceable* if $\kappa^{*L}(G) = \kappa(G)$. Recently, Chang et al. [14] proved that the n-dimensional hypercube Q_n is super spanning laceable for every positive integer n. It was proved in [11] that the n-dimensional star graph S_n is super spanning laceable if and only if $n \neq 3$.

1.3. Hypercube-like graphs H'_n

Among all interconnection networks proposed in the literature, the hypercube Q_n is one of the most popular topologies [14–17]. However, the hypercube does not have the smallest diameter for its resources. Various networks are proposed by twisting some pairs of links in hypercubes [18–21]. Because of the lack of the unified perspective on these variants, results of one topology are hard to be extended to others. To make a unified study of these variants, Vaidya et al. introduced the class of hypercube-like graphs [22]. We denote these graphs as H'-graphs. The class of H'-graphs, consisting of simple, connected, and undirected graphs, contains most of the hypercube variants.

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same number of nodes. A *1–1 connection* between G_0 and G_1 is defined as $E = \{(v, \phi(v)) | v \in V_0, \phi(v) \in V_1,$ and $\phi : V_0 \rightarrow V_1$ is a bijection$\}$. We use $G_0 \oplus G_1$ to denote $G = (V_0 \cup V_1, E_0 \cup E_1 \cup E)$. The operation "$\oplus$" may generate different graphs depending on the bijection ϕ. There are some studies on the operation "$\oplus$" [23,24]. Let $G = G_0 \oplus G_1$, and let x be any node in G. We use $\bar{x}$ to denote the unique node matched under ϕ.

Now, we can define the set of n-dimensional H'-graph, H'_n, as follows:

(1) $H'_1 = \{K_2\}$, where K_2 is the complete graph with two nodes.
(2) Assume that $G_0, G_1 \in H'_n$. Then $G = G_0 \oplus G_1$ is a graph in H'_{n+1}.

We can define the set of bipartite n-dimensional H'-graph, B'_n, as follows:

(1) $B'_1 = \{K_2\}$, where K_2 is the complete graph defined on $\{a, b\}$ with bipartition $V_0 = \{a\}$ and $V_1 = \{b\}$.
(2) For $i = 0, 1$, let G_i be a graph in B'_n with bipartition V_0^i and V_1^i. Let ϕ be a bijection between $V_0^0 \cup V_1^0$ and $V_0^1 \cup V_1^1$ such that $\phi(v) \in V_{1-i}^1$ if $v \in V_i^0$. Then $G = G_0 \oplus G_1$ is a graph in B'_{n+1}.

Every graph in H'_n is an n-regular graph with 2^n nodes, and every graph in B'_n contains 2^{n-1} nodes in each bipartition. Note that the n-dimensional hypercube $Q_n \in B'_n$.

Let G be a graph in H'_{n+1}. Then $G = G_0 \oplus G_1$ with both G_0 and G_1 in H'_n. Let u be a node in $V(G)$. Then u is a node in $V(G_i)$ for some $i = 0, 1$. We use $\bar{u}$ to denote the node in $V(G_{1-i})$ matched under ϕ. So $u = \bar{v}$ if $\bar{u} = v$.

In the following section, we give some properties about the bipartite n-dimensional hypercube-like graphs B'_n. Let n and k be any two positive integers with $n \geqslant 2$ and $k \leqslant n$. In Section 3 and Section 4, we prove that every B'_n is f-edge fault k^*-laceable for every $f \leqslant n-2$ and $f + k \leqslant n$. We give our conclusion in the final section.

2. Preliminaries

Park and Chwa [25] studied the hamiltonian laceability properties of the bipartite hypercube-like networks. Some results are listed as follows.

Theorem 1 [25]. *Every graph in B'_n is hamiltonian laceable, and every graph in B'_n is hamiltonian if $n \geqslant 2$.*

Theorem 2 [25]. *Suppose that $n \geqslant 2, i \in \{0, 1\}$, and G is a graph in B'_n with bipartition G_0 and G_1. Let $\{u_1, u_2\} \subseteq V(G_i)$ with $u_1 \neq u_2$, and $\{v_1, v_2\} \subseteq V(G_{1-i})$ with $v_1 \neq v_2$. Then there are two disjoint paths P_1 and P_2 of G such that (1) P_1 joins u_1 to v_1, (2) P_2 joins u_2 to v_2, and (3) $P_1 \cup P_2$ spans G.*

The fault-tolerance hamiltonian laceability of the bipartite hypercube-like networks is studied by Lin et al. in [26].

Theorem 3 [26]. *Let $n \geqslant 2$. Every graph in B'_n is $(n-2)$-edge fault hamiltonian laceable.*

Theorem 4 [26]. *Suppose that $n \geqslant 2, i \in \{0, 1\}$, and G is a graph in B'_n with bipartition G_0 and G_1. Let $z \in V(G_i)$, and $\{u, v\} \subseteq V(G_{1-i})$ with $u \neq v$. Then there is a hamiltonian path of $G - \{z\}$ joining u to v.*

3. The super spanning laceability of the graph in B'_n

Let n and k be any two positive integers with $n \geqslant 2$ and $k \leqslant n$. In this section, we show that every graph in B'_n is f-edge fault k^*-laceable for every $f \leqslant n-2$ and $f + k \leqslant n$. We give the concept of the spanning fan first. We note that there is another Menger-type Theorem. Let u be a node of G and $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of $V(G)$ not including u. An (u, S)-fan is a set of disjoint paths $\{P_1, P_2, \ldots, P_k\}$ of G such that P_i joins u to v_i for every $1 \leqslant i \leqslant k$ [27]. It is proved that a graph G is k-connected if and only if there exists an (u, S)-fan between any node u and any k-subset S of $V(G)$ such that $u \notin S$. With this observation, we define a *spanning fan* is a fan that spans a graph G. Naturally, we can study $\kappa^*_{fan}(G)$ as the largest integer k such that there exists a spanning (u, S)-fan between any node u and any k-node subset S with $u \notin S$. However, we defer such a study for the following reasons.

First, let S be a cut set of a graph G. Let u be any node of $V(G) - S$. It is easy to see that there is no spanning (u, S)-fan in G. Thus, $\kappa^*_{fan}(G) < \kappa(G)$ if G is not a complete graph.

Second, let G be a bipartite graph with bipartition $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ such that $|V_0| = |V_1|$. Let u be a node in V_i with $i \in \{0, 1\}, S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G) - \{u\}$, and $k \leqslant \kappa(G)$. Suppose that $|S \cap V_{1-i}| = r$. Without loss of generality, we assume that $\{v_1, v_2, \ldots, v_r\} \subset V_{1-i}$. Let $\{P_1, P_2, \ldots, P_k\}$ be any spanning (u, S)-fan of G. Then $l(P_i)$ is odd if $i \leqslant r$, and $l(P_i)$ is even if $r < i \leqslant k$. Let $l(P_i) = 2t_i + 1$ if $i \leqslant r$ and $l(P_i) = 2t_i$ if $i > r$. For $i \leqslant r$, there are $t_i - 1$ nodes of P_i in V_i other than u, and there are t_i nodes of P_i in V_{1-i}. For $i > r$, there are t_i nodes of P_i in V_i other than u, and there are t_i nodes of P_i in V_{1-i}. Thus, we have $|V_i| = 1 - r + \sum_{i=1}^{k} t_i$ and $|V_{1-i}| = \sum_{i=1}^{k} t_i$. Since $|V_i| = |V_{1-i}|, r = 1$. Thus, $r = 1$ is a natural requirement as we study the spanning fan of bipartite graphs with equal size of bipartition.

Theorem 5 [12]. *Suppose that n and k are two positive integers with $k \leqslant n$. Let G be a graph in B'_n with bipartition G_0 and G_1. There exists a spanning (u, S)-fan in G for any node u in $V(G_i)$ and any node subset S with $|S| = k \leqslant n$ such that $u \notin S$, and $|S \cap V(G_{1-i})| = 1$ with $i \in \{0, 1\}$.*

Lemma 1. *Suppose that*

1. $n \geqslant 2, f = n - 2$, *and* $i \in \{0, 1\}$,
2. G *is a graph in* B'_n *with bipartition* G_0 *and* G_1, *and*
3. $F \subset E(G)$ with $|F| = f$.

Then, for any $\{u, y\} \subseteq V(G_i)$ and $x \in V(G_{1-i})$ with $u \neq y$, there exists a spanning $(u, \{x, y\})$-fan in $G - F$.

Proof. By Theorem 3, there is a hamiltonian path $P = \langle x, P_1, u, P_2, y\rangle$ of $G - F$ joining x to y. Then $\{P_1, P_2\}$ is the spanning $(u, \{x, y\})$-fan of $G - F$. □

The following are the main results.

Theorem 6. *Suppose that*

1. $n \geqslant 2, k \leqslant n$, *and* $i \in \{0, 1\}$,
2. G *is a graph in* $B\prime_n$ *with bipartition* $G_0 = (V_0, E_0)$ *and* $G_1 = (V_1, E_1)$, *and*
3. $F \subset E(G)$ *with* $|F| + k \leqslant n$ *and* $|F| \leqslant n - 2$.

Then, for any $u \in V_i$ *and* $S \subseteq V(G) - \{u\}$ *with* $|S| = k$ *and* $|S \cap V_{1-i}| = 1$, *there exists a spanning* (u, S)*-fan in* $G - F$.

We prove the theorem by induction. However, the proof of the theorem is rather long. We prove it in the following section.

Theorem 7. *The bipartite n-dimensional hypercube-like graph* B'_n *is f-edge fault* k^**-laceable for* $f \leqslant n - 2$ *and* $f + k \leqslant n$.

Proof. Let G be a graph in B'_n with bipartition G_0 and G_1. Assume that $x \in V(G_i)$ and $y \in V(G_{1-i})$ for some $i \in \{0, 1\}$. Suppose that $F \subset E(G)$ with $|F| = f$ and $f \leqslant n - 2$. Let $S \subseteq V(G_i) - \{x\}$ adjacent to y in $G - F$ with $|S| = k - 1$ and $k \leqslant n - f$. We assume that $S = \{y_1, y_2, \ldots, y_{k-1}\}$. By Theorem 6, there exists a spanning $(x, S \cup \{y\})$-fan $\{P_1, P_2, \ldots, P_k\}$ in $G - F$ such that P_k joins x to y, and P_i joins x to y_i for $1 \leqslant i \leqslant k - 1$. Let $Q_i = \langle x, P_i, y_i, y\rangle$ for $1 \leqslant i \leqslant k - 1$. Thus, $\{P_k, Q_1, Q_2, \ldots, Q_{k-1}\}$ forms a k^*-container between x and y in $G - F$. The theorem is proved. □

4. Proof of Theorem 6

Let $G = G_0 \oplus G_1$ in B'_n with bipartition V_0^j and V_1^j for $j \in \{0, 1\}$. Thus, $V_0^0 \cup V_0^1$ and $V_1^0 \cup V_1^1$ form the bipartition of G. Assume that $|F| = f$. Let u be any node in $V_0^0 \cup V_0^1$ and $S = \{v_1, v_2, \ldots, v_k\}$ be any node subset in $G - \{u\}$ with v_1 being the unique node in $(V_1^0 \cup V_1^1) \cap S$. Without loss of generality, we assume that $u \in V_0^0$. For $n = 2$, we have G is isomorphic to a cycle with four nodes. Thus, this statement holds on $n = 2$. By Lemma 1, Theorem 3, and Theorem 5, this statement holds on $n = 3$. Thus, we assume that $n \geqslant 4$. By Lemma 1 and Theorem 3, this statement holds on $k \in \{1, 2\}$ and $f = n - 2$. By Theorem 5, this statement holds on $k \leqslant n$ and $f = 0$. Thus, we assume that $k \geqslant 3$ and $1 \leqslant f \leqslant n - 3$ with $k + f \leqslant n$. We set $T = S - \{v_1\}, F_j = F \cap E(G_j)$ for $j \in \{0, 1\}$, and $F_2 = F - (F_0 \cup F_1)$. Note that $|F| = |F_0| + |F_1| + |F_2|$ and $|F_{j\prime}| \leqslant n - 3$ for every $j\prime \in \{0, 1, 2\}$. Now we have the following cases.

Case 1. $|T \cap V_0^0| = |T|$.

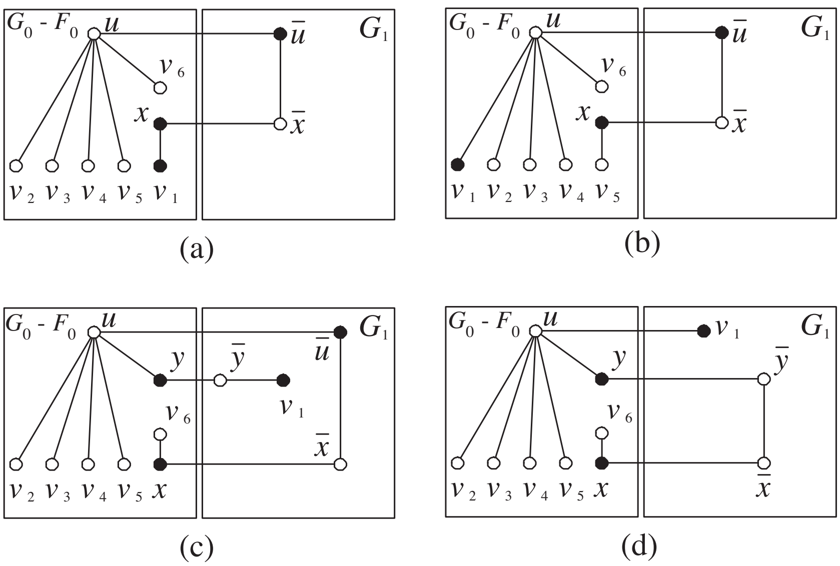

Fig. 1. Case 1.1 and Case 1.2. (Suppose that k = 6.).

Case 1.1. $|F_0| = |F|$ and $v_1 \in V_1^0$. Let $H = S - \{v_k\}$. We have $H \subset G_0, |H \cap V_1^0| = 1$, and $|H| = k - 1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of $G_0 - F_0$. Without loss of generality, we assume that P_i joins u to v_i for every $1 \leqslant i \leqslant k - 1$.

Suppose that $v_k \in V(P_1)$. Without loss of generality, we write P_1 as $\langle u, Q_1, v_k, x, Q_2, v_1\rangle$. Since $v_k \in V_0^0, x \in V_1^0$. (Note that $x = v_1$ if $l(Q_2) = 0$.) By Theorem 1, there is a hamiltonian path R of G_1 joining node $\bar{u} \in V_1^1$ to node $\bar{x} \in V_0^1$. We set $W_1 = \langle u, \bar{u}, R, \bar{x}, x, Q_2, v_1\rangle, W_i = P_i$ for every $2 \leqslant i \leqslant k - 1$, and $W_k = \langle u, Q_1, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$. See Fig. 1(a) for an illustration. Suppose that $v_k \in V(P_i)$ for some $2 \leqslant i \leqslant k - 1$. Without loss of generality, we assume that $v_k \in V(P_{k-1})$ and write P_{k-1} as $\langle u, Q_1, v_k, x, Q_2, v_{k-1}\rangle$. Since $v_k \in V_0^0, x \in V_1^0$. By Theorem 1, there is a hamiltonian path R of G_1 joining node $\bar{u} \in V_1^1$ to node $\bar{x} \in V_0^1$. We set $W_i = P_i$ for every $1 \leqslant i \leqslant k - 2, W_{k-1} = \langle u, \bar{u}, R, \bar{x}, x, Q_2, v_{k-1}\rangle$, and $W_k = \langle u, Q_1, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$. See Fig. 1(b) for an illustration.

Case 1.2. $|F_0| = |F|$ and $v_1 \in V_1^1$. We choose a node x in V_1^0. Let $H = (T \cup \{x\}) - \{v_k\}$. We have $H \subset G_0, |H \cap V_1^0| = 1$, and $|H| = k - 1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of $G_0 - F_0$ such that P_1 joins u to x, and P_i joins u to v_i for every $2 \leqslant i \leqslant k - 1$. Note that $\bar{u} \in V_1^1$ and $\bar{x} \in V_0^1$. Without loss of generality, we assume that $v_k \in V(P_1)$. Let $P_1 = \langle u, Q_1, y, v_k, Q_2, x\rangle$. Since $v_k \in V_0^0$, we have $y \in V_1^0$ and $\bar{y} \in V_0^1$.

Suppose that $v_1 \neq \bar{u}$. By Theorem 2, there are two disjoint paths R_1 and R_2 in G_1 such that (1) R_1 joins $\bar{y}$ to v_1, (2) R_2 joins $\bar{u}$ to $\bar{x}$, and (3) $R_1 \cup R_2$ spans G_1. We set $W_1 = \langle u, Q_1, y, \bar{y}, R_1, v_1\rangle, W_i = P_i$ for every $2 \leqslant i \leqslant k - 1$, and $W_k = \langle u, \bar{u}, R_2, \bar{x}, x, Q_2^{-1}, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$. See Fig. 1(c) for an illustration.

Suppose that $v_1 = \bar{u}$. By Theorem 4, there is a hamiltonian path R of $G_1 - \{v_1\}$ joining $\bar{y}$ to $\bar{x}$. We set $W_1 = \langle u, \bar{u} = v_1\rangle, W_i = P_i$ for every $2 \leqslant i \leqslant k - 1$, and $W_k = \langle u, Q_1, y, \bar{y}, R, \bar{x}, x, Q_2^{-1}, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$. See Fig. 1(d) for an illustration.

Case 1.3. $|F_0| < |F|$ and $v_1 \in V_1^0$. Since $|F_0| < |F| = f$, we have $k + |F_0| \leqslant k + f - 1 \leqslant n - 1$. By induction, there is a spanning (u, S)-fan $\{P_1, P_2, \ldots, P_k\}$ of $G_0 - F_0$. Without loss of generality, we assume that P_i joins u to v_i for every $1 \leqslant i \leqslant k$. Since $|V(G_0)| = 2^{n-1}$ and $\cup_{i=1}^k P_i$ span G_0, we have $\sum_{i=1}^k |E(P_i)| = 2^{n-1} - 1$. Since $2^{n-1} - 1 > 3n - 8 > 2(f - 1) + k$ if $n \geqslant 3$, there exists an edge (x, y) in $\cup_{i=1}^k E(P_i)$ such that $(x, \bar{x}) \notin F_2$ and $(y, \bar{y}) \notin F_2$. Without loss of generality, we assume that $(x, y) \in E(P_j)$ for some $1 \leqslant j \leqslant k$. Let $P_j = \langle u, R_1, x, y, R_2, v_j\rangle$. Note that $u = x$ if $l(R_1) = 0$ and $y = v_j$ if $l(R_2) = 0$. Since x and y are adjacent, x and y are in distinct bipartition of G_0. Moreover, $\bar{x}$ and $\bar{y}$ are in distinct bipartition of G_1. By Theorem 3, there is a hamiltonian path W of $G_1 - F_1$ joining $\bar{x}$ to $\bar{y}$. We set $W_i = P_i$ for every $i \in \{1, 2, \ldots, k\} - \{j\}$ and set $W_j = \langle u, R_1, x, \bar{x}, W, \bar{y}, y, R_2, v_j\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$.

Case 1.4. $|F_0| < |F|$ and $v_1 \in V_1^1$. Since $|V_1^0| = 2^{n-2} > n - 3$ if $n \geqslant 3$, there exists a node $x \in V_1^0$ such that $(x, \bar{x}) \notin F_2$. Let $H = T \cup \{x\}$. Since $|F_0| < |F| = f$, we have $k + |F_0| \leqslant k + f - 1 \leqslant n - 1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_k\}$ of $G_0 - F_0$. Without loss of generality, we assume that P_1 is joining u to x and P_i is joining u to v_i for every $2 \leqslant i \leqslant k$. Since $x \in V_1^0$, we have $\bar{x} \in V_0^1$. By Theorem 3, there is a hamiltonian path R of $G_1 - F_1$ joining $\bar{x}$ to v_1. We set $W_1 = \langle u, P_1, x, \bar{x}, R, v_1\rangle$ and $W_i = P_i$ for every $2 \leqslant i \leqslant k$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$.

Case 2. $|T \cap V_0^1| = 1$. We assume that $v_k \in V_0^1$. Note that $\bar{u} \in V_1^1$.

Case 2.1. $|F_0| = |F|$ and $v_1 \in V_1^0$. Let $H = S - \{v_k\}$. We have $H \subset G_0, |H \cap V_1^0| = 1$, and $|H| = k - 1$. By induction, there is a spanning (u, H)-fan $\{W_1, W_2, \ldots, W_{k-1}\}$ of $G_0 - F_0$. By Theorem 1, there is a hamiltonian path R of G_1 joining $\bar{u}$ to v_k. We set $W_k = \langle u, \bar{u}, R, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of $G - F$. See Fig. 2(a) for an illustration.

Case 2.2. $|F_0| = |F|$ and $v_1 \in V_1^1$. By Theorem 1, there is a hamiltonian path R of G_1 joining v_1 to v_k. We write R as $\langle v_1, R_1, \bar{u}, x, R_2, v_k\rangle$. Note that $v_1 = \bar{u}$ if $l(R_1) = 0$ and $x = v_k$ if $l(R_2) = 0$. Since $\bar{u} \in V_1^1$, we have $x \in V_0^1$ and $\bar{x} \in V_1^0$. Let $H = (T \cup \{\bar{x}\}) - \{v_k\}$. Thus $H \subset G_0, |H \cap V_1^0| = 1$, and $|H| = k - 1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of $G_0 - F_0$ such that P_1 joins u to $\bar{x}$, and P_i joins u to v_i for every $2 \leqslant i \leqslant k - 1$. We set

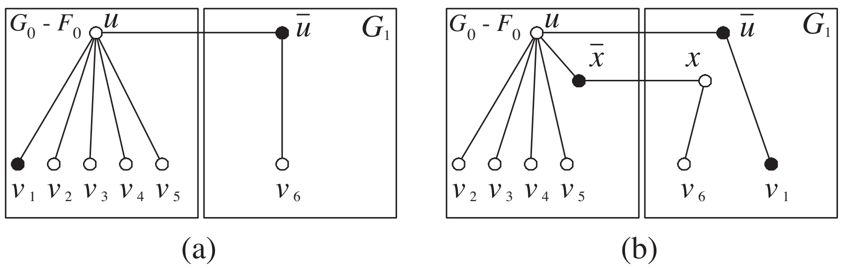

Fig. 2. Case 2.1 and Case 3.2. (Suppose that k = 6.).

$W_1 = \langle u, \bar{u}, R_1^{-1}, v_1 \rangle, W_i = P_i$ for every $2 \leqslant i \leqslant k-1$, and $W_k = \langle u, P_1, \bar{x}, x, R_2, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the (u,S)-fan of $G - F$. See Fig. 2(b) for an illustration.

Case 2.3. $|F_0| < |F|$ and $v_1 \in V_1^0$. Since $|V_0^0| = 2^{n-2} > n > k + f - 1$ if $n \geqslant 4$, there exists a node x in $V_0^0 - (T \cup \{u\})$ such that $(x, \bar{x}) \notin F_2$. Let $H = (S \cup \{x\}) - \{v_k\}$. Obviously, $H \subset G_0, |H \cap V_0^1| = 1$, and $|H| = k$. Since $|F_0| < |F| = f$, we have $k + |F_0| \leqslant k + f - 1 \leqslant n - 1$. By induction, there is a spanning (u,H)-fan $\{P_1, P_2, \ldots, P_k\}$ of $G_0 - F_0$ such that P_i joins u to v_i for every $1 \leqslant i \leqslant k-1$ and P_k joins u to x. By Theorem 1, there is a hamiltonian path R of $G_1 - F_1$ joining $\bar{x}$ to v_k. We set $W_i = P_i$ for every $1 \leqslant i \leqslant k-1$ and $W_k = \langle u, P_k, x, \bar{x}, R, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u,S)-fan of $G - F$.

Case 2.4. $|F_0| < |F|$ and $v_1 \in V_1^1$. Since $|V_0^0| = 2^{n-2} > n > k + f - 1$ if $n \geqslant 4$, there exists a node x in $V_0^0 - (T \cup \{u\})$ such that $(x, \bar{x}) \notin F_2$. Let $F' = \{(y, \bar{x}) | y \in G_1 \text{ and } (y, \bar{y}) \in F_2\}$. We have $|F_1 \cup F'| \leqslant |F_1| + |F_2| \leqslant |F| = f < n - 3$. By Theorem 1, there is a hamiltonian path R of $G_1 - (F_1 \cup F')$ joining v_1 to v_k. Without loss of generality, we write R as $\langle v_1, R_1, \bar{x}, z, R_2, v_k \rangle$. Note that $v_1 = \bar{x}$ if $l(R_1) = 0$ and $z = v_k$ if $l(R_2) = 0$. Since $x \in V_0^0$, we have $\bar{x} \in V_1^1, z \in V_0^1$, and $\bar{z} \in V_1^0$. Let $H = (T \cup \{x, \bar{z}\}) - \{v_k\}$. Obviously, $H \subset G_0, |H \cap V_1^0| = 1$, and $|H| = k$. Since $|F_0| < |F| = f$, we have $k + |F_0| \leqslant k + f - 1 \leqslant n - 1$. By induction, there is a spanning (u,H)-fan $\{P_1, P_2, \ldots, P_k\}$ of $G_0 - F_0$. Without loss of generality, we assume that P_1 joins u to x, P_2 joins u to $\bar{z}$, and P_i joins u to v_{i-1} for every $3 \leqslant i \leqslant k$. We set $W_1 = \langle u, P_1, x, \bar{x}, R_1^{-1}, v_1 \rangle, W_i = P_{i+1}$ for every $2 \leqslant i \leqslant k-1$, and $W_k = \langle u, P_2, \bar{z}, z, R_2, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the (u,S)-fan of $G - F$.

Case 3. $|T \cap V_0^1| \geqslant 2$ and $|T \cap V_0^0| \geqslant 1$. We have $n \geqslant k + 1 = |S| + 1 \geqslant 5$. Assume that $A = T \cap V_0^0 = \{v_2, v_3, \ldots, v_t\}$ and $B = T \cap V_0^1 = \{v_{t+1}, v_{t+2}, \ldots, v_k\}$ for some $2 \leqslant t \leqslant k-2$.

Case 3.1. $|F_0| = |F|$. Since $(n-1)|A| \leqslant (n-1)(n-3) < 2n^{n-2}$ if $n \geqslant 5$, there exists a node x in V_0^1 such that $v_1 \notin N_{G_1}(x)$ and $\bar{v}_i \notin N_{G_1}(x)$ for $2 \leqslant i \leqslant t$. By induction, there is a spanning $(x, B \cup \{\bar{u}\})$-fan $\{P_1, P_2, \ldots, P_{k-t+1}\}$ of G_1 such that $P_1 = \langle x, x_1, P'_1, \bar{u} \rangle$ joins x to $\bar{u}$, and $P_i = \langle x, x_i, P'_i, v_{t+i-1} \rangle$ joins x to v_{t+i-1} for every $2 \leqslant i \leqslant k - t + 1$.

Case 3.1.1. $v_1 \in V_1^0$. We set $H = A \cup \{v_1\} \cup \{\bar{x}_i | 2 \leqslant i \leqslant k - t\}$. Let $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ be a spanning (u,H)-fan of $G_0 - F_0$ such that Q_i joins u to v_i for every $1 \leqslant i \leqslant t$, and Q_j joins u to $\bar{x}_{j-t+1}$ for every $t+1 \leqslant j \leqslant k-1$. We set $W_i = Q_i, W_j = \langle u, Q_j, \bar{x}_{j-t+1}, x_{j-t+1}, P'_{j-t+1}, v_j \rangle$, and $W_k = \langle u, \bar{u}, P_1^{-1}, x, P_{k-t+1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u,S)-fan of $G - F$.

Case 3.1.2. $v_1 \in V_1^1$ and $v_1 \in V(P_1)$. We write $P_1 = \langle x, R_1, y, v_1, R_2, \bar{u} \rangle$. We set $H = A \cup \{\bar{x}_i | 2 \leqslant i \leqslant k - t\} \cup \{\bar{y}\}$. Let $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ be a spanning (u,H)-fan of $G_0 - F_0$ such that Q_1 joins u to $\bar{y}$, Q_j joins u to v_j for every $2 \leqslant j \leqslant t$, and $Q_{j\prime}$ joins u to $\bar{x}_{j\prime-t+1}$ for every $t+1 \leqslant j\prime \leqslant k-1$. We set $W_1 = \langle u, \bar{u}, R_2^{-1}, v_1 \rangle, W_j = Q_j$, $W_{j\prime} = \langle u, Q_{j\prime}, \bar{x}_{j\prime-t+1}, x_{j\prime-t+1}, P'_{j\prime-t+1}, v_{j\prime} \rangle$, and $W_k = \langle u, Q_1, \bar{y}, y, R_1^{-1}, x, P_{k-t+1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u,S)-fan of $G - F$.

Case 3.1.3. $v_1 \in V_1^1$ and $v_1 \in V(P_i)$ for some $2 \leqslant i \leqslant k - t + 1$. Without loss of generality, we assume that $v_1 \in V(P_2)$. Let $P_2 = \langle x, R_1, v_1, y, R_2, v_{t+1} \rangle$. We set $H = A \cup \{\bar{x}_i | 3 \leqslant i \leqslant k - t + 1\} \cup \{\bar{y}\}$. Let $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ be a spanning (u,H)-fan of $G_0 - F_0$ such that Q_1 joins u to $\bar{y}$, Q_j joins u to v_j for every $2 \leqslant j \leqslant t$, and $Q_{j\prime}$ joins u to $\bar{x}_{j\prime-t+1}$ for every $t+2 \leqslant j\prime \leqslant k$. We set $W_1 = \langle u, \bar{u}, P_1^{-1}, x, R_1, v_1 \rangle, W_j = Q_j$, $W_{t+1} = \langle u, Q_1, \bar{y}, y, R_2, v_{t+1} \rangle$, and $W_{j\prime} = \langle u, Q_{j\prime}, \bar{x}_{j\prime-t+1}, x_{j\prime-t+1}, P'_{j\prime-t+1}, v_{j\prime} \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u,S)-fan of $G - F$.

Case 3.2. $|F_0| < |F|$ and $n = 5$. We have $|F| = 1$ and $k = 4$. Thus, $|F_0| = 0$ and $|F_1| + |F_2| = 1$. Moreover, $A = \{v_2\}$ and $B = \{v_3, v_4\}$.

Case 3.2.1. $|F_1| = 0$ and $v_1 \in V_1^0$. Since $|V_1^1| = 8 > 3$, there exist two distinct nodes x_1 and x_2 in $V_1^1 - \{\bar{u}, \bar{v}_2\}$ such that $(x_1, \bar{x}_1) \notin F_2$ and $(x_2, \bar{x}_2) \notin F_2$. By Theorem 2, there are two disjoint paths P_1 and P_2 in G_1 such that P_i joins x_i to v_{i+2} for $i \in \{1, 2\}$, and $P_1 \cup P_2$ spans G_1. Let $\{Q_1, Q_2, Q_3, Q_4\}$ be a spanning $(u, \{v_1, v_2, \bar{x}_1, \bar{x}_2\})$-fan of G_0 such that Q_i joins u to v_i for $1 \leqslant i \leqslant 2$, and Q_j joins u to $\bar{x}_{j-2}$ for $3 \leqslant j \leqslant 4$. We set $W_i = \langle u, Q_{i+2}, \bar{x}_i, x_i, P_i, v_{i+2} \rangle$ for every $1 \leqslant i \leqslant 2$. Then $\{Q_1, Q_2, W_1, W_2\}$ forms a spanning (u,S)-fan of $G - F$.

Case 3.2.2. $|F_1| = 0$ and $v_1 \in V_1^1$. Since $|V_0^1| = 8$, there exists a node x in $V_0^1 - \{v_3, v_4\}$ such that $\bar{v}_2 \notin N_{G_1}(x)$ and $(x, \bar{x}) \notin F_2$. We set $F\prime = \{(x, y) | y \in N_{G_1}(x) \text{ and } (y, \bar{y}) \in F_2\}$. We have $|F\prime| \leqslant 1$. By induction, there is a spanning $(x, \{v_1, v_3, v_4\})$-fan $\{P_1, P_2, P_3\}$ of G_1 such that P_1 joins x to v_1 and P_i joins x to v_{i+1} for $2 \leqslant i \leqslant 3$. We set $P_1 = \langle x, x_1, R_1, v_1 \rangle$ and $P_i = \langle x, x_i, R_i, v_{i+1} \rangle$ for every $2 \leqslant i \leqslant 3$. Let $\{Q_1, Q_2, Q_3, Q_4\}$ be a spanning $(u, \{v_2, \bar{x}, \bar{x}_2, \bar{x}_3\})$-fan of G_0 such that Q_1 joins u to $\bar{x}$, Q_2 joins u to v_2, and Q_i joins u to $\bar{x}_{i-1}$ for $3 \leqslant i \leqslant 4$. We set $W_1 = \langle u, Q_1, \bar{x}, x, P_1, v_1 \rangle, W_2 = Q_2$, and $W_i = \langle u, Q_i, \bar{x}_{i-1}, x_{i-1}, R_{i-1}, v_i \rangle$ for every $3 \leqslant i \leqslant 4$. Then $\{W_1, W_2, W_3, W_4\}$ forms a spanning (u,S)-fan of $G - F$.

Case 3.2.3. $|F_1| = 1$ and $v_1 \in V_1^0$. We have $|F_2| = 0$. By Theorem 3, there is a hamiltonian path P of $G_1 - (F_1 \cup \{(v_3, \bar{v}_2)\})$ joining $\bar{u}$ to v_4. We set $P = \langle \bar{u}, P_1, v_3, x, P_2, v_4 \rangle$. Let $\{Q_1, Q_2, Q_3\}$ be a spanning $(u, \{v_1, v_2, \bar{x}\})$-fan of G_0 such that Q_i joins u to v_i for $1 \leqslant i \leqslant 2$ and Q_3 joins u to $\bar{x}$. Let $W_1 = \langle u, \bar{u}, P_1, v_3 \rangle$ and $W_2 = \langle u, Q_3, \bar{x}, x, P_2, v_4 \rangle$. Then $\{Q_1, Q_2, W_1, W_2\}$ forms a spanning (u, S)-fan of $G - F$.

Case 3.2.4. $|F_1| = 1$ and $v_1 \in V_1^1$. Since $|V_0^1| = 8 > 6$, there exists a node x in $V_0^1 - \{v_3, v_4\}$ such that $\bar{v}_2 \notin N_{G_1}(x)$. By induction, there exists a $(x, \{v_1, v_3, v_4\})$-fan $\{P_1, P_2, P_3\}$ of $G_1 - F_1$ such that P_1 joins x to v_1, and P_i joins x to v_{i+1} for $2 \leqslant i \leqslant 3$. Without loss of generality, we write $P_i = \langle x, y_{i-1}, R_{i-1}, v_{i-1} \rangle$ for $2 \leqslant i \leqslant 3$. Let $\{Q_1, Q_2, Q_3, Q_4\}$ be a $(u, \{\bar{x}, v_2, \bar{y}_1, \bar{y}_2\})$-fan of G_0 such that Q_1 joins u to $\bar{x}, Q_2$ joins u to v_2, and Q_i joins u to $\bar{y}_{i-2}$ for $3 \leqslant i \leqslant 4$. We set $W_1 = \langle u, Q_1, \bar{x}, x, P_1, v_1 \rangle, W_2 = Q_2$, and $W_i = \langle u, Q_i, \bar{y}_{i-2}, y_{i-2}, R_{i-2}, v_i \rangle$ for $3 \leqslant i \leqslant 4$. Then $\{W_1, W_2, W_3, W_4\}$ forms a spanning (u, S)-fan of $G - F$.

Case 3.3. $|F_0| < |F|$ and $n \geqslant 6$. Since $(n-1)(f + |A|) \leqslant (n-1)(f + k - 3) \leqslant (n-1)(n-3) < 2^{n-2}$ if $n \geqslant 6$, there exists a node x in V_0^1 such that $(x, \bar{x}) \notin F_2, \bar{v}_i \notin N_{G_1}(x)$ for every $2 \leqslant i \leqslant t$, and $(y, \bar{y}) \notin F_2$ for every $y \in N_{G_1}(x)$.

Case 3.3.1. $v_1 \in V_1^0$. Since $|A| + f < 2^{n-2}$ if $n \geqslant 6$, there exists a node y in V_1^1 such that $(y, \bar{y}) \notin F_2$ and $\bar{y} \notin A$.

Suppose that $x \notin B$. By induction, there is a spanning $(x, B \cup \{y\})$-fan $\{P_1, P_2, \ldots, P_{k-t+1}\}$ of $G_1 - F_1$ such that P_i joins x to v_{t+i} for every $1 \leqslant i \leqslant k - t$ and P_{k-t+1} joins x to y. Without loss of generality, we set $P_i = \langle x, x_i, R_i, v_{t+i} \rangle$ for every $1 \leqslant i \leqslant k - t - 1$. We set $H = A \cup \{\bar{x}_i | 1 \leqslant i \leqslant k - t - 1\} \cup \{\bar{y}\}$. Let $\{Q_1, Q_2, \ldots, Q_k\}$ be a spanning (u, H)-fan of $G_0 - F_0$ such that Q_i joins u to v_i for every $1 \leqslant i \leqslant t, Q_j$ joins u to $\bar{x}_{i-t}$ for every $t + 1 \leqslant j \leqslant k - 1$, and Q_k joins u to $\bar{y}$. We set $W_i = Q_i$ for every $1 \leqslant i \leqslant t, W_j = \langle u, Q_j, \bar{x}_{j-t}, x_{j-t}, R_{j-t}, v_j \}$ for every $t + 1 \leqslant j \leqslant k - 1$, and $W_k = \langle u, Q_k, \bar{y}, y, P_{k-t+1}^{-1}, x, P_{k-t}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Suppose that $x \in B$. We assume that $x = v_k$. This case is similar to the above.

Case 3.3.2. $v_1 \in V_1^1$. Suppose that $x \notin B$. By induction, there is a spanning $(x, B \cup \{v_1\})$-fan $\{P_1, P_2, \ldots, P_{k-t+1}\}$ of $G_1 - F_1$ such that P_i joins x to v_{t+i} for every $1 \leqslant i \leqslant k - t$ and P_{k-t+1} joins x to v_1. Without loss of generality, we set $P_i = \langle x, x_i, R_i, v_{t+i} \rangle$ for every $1 \leqslant i \leqslant k - t$. We set $H = A \cup \{\bar{x}_i | 1 \leqslant i \leqslant k - t\} \cup \{\bar{x}\}$. Let $\{Q_1, Q_2, \ldots, Q_k\}$ be a spanning (u, H)-fan of $G_0 - F_0$ such that Q_1 joins u to $\bar{x}, Q_i$ joins u to v_i for every $2 \leqslant i \leqslant t$, and Q_j joins u to $\bar{x}_{i-t}$ for every $t + 1 \leqslant j \leqslant k$. We set $W_1 = \langle u, Q_1, \bar{x}, x, P_{k-t+1}, v_1 \rangle, W_i = Q_i$ for every $2 \leqslant i \leqslant t$, and $W_j = \langle u, Q_j, \bar{x}_{j-t}, x_{j-t}, R_{j-t}, v_j \rangle$ for every $t + 1 \leqslant j \leqslant k$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Suppose that $x \in B$. We assume that $x = v_k$. The case is similar to the above.

Case 4. $|T \cap V_0^1| = k - 1$. With Case 2, we consider that $|T| \geqslant 2$.

Case 4.1. $|F_0| = |F|$. We have $|F_1| = 0$ and $|F_2| = 0$. Let x be a node in $V_0^1 - T$. By induction, there exists a spanning $(x, T \cup \{\bar{u}\})$-fan $\{P_1, P_2, \ldots, P_k\}$ of G_1 such that P_1 joins x to $\bar{u}$, and P_i joins x to v_i for every $2 \leqslant i \leqslant k$. We set $P_i = \langle x, x_i, R_i, v_i \rangle$ for every $2 \leqslant i \leqslant k$.

Case 4.1.1. $v_1 \in V_1^0$. We set $H = \{v_1\} \cup \{\bar{x}_i | 2 \leqslant i \leqslant k - 1\}$. Let $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ be a spanning (u, H)-fan of $G_0 - F_0$ such that Q_1 joins u to v_1, and Q_i joins u to $\bar{x}_i$ for every $2 \leqslant i \leqslant k - 1$. We set $W_1 = Q_1, W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \leqslant i \leqslant k - 1$, and $W_k = \langle u, \bar{u}, P_1^{-1}, x, P_k, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Case 4.1.2. $v_1 \in V_1^1$ and $v_1 \in V(P_1)$. We set $P_1 = \langle x, Z_1, y, v_1, Z_2, \bar{u} \rangle$. Let $H = \{\bar{y}\} \cup \{\bar{x}_i | 2 \leqslant i \leqslant k - 1\}$. Thus, there exists a spanning (u, H)-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ in $G_0 - F_0$ such that Q_1 joins u to $\bar{y}$, and Q_i joins u to $\bar{x}_i$ for every $2 \leqslant i \leqslant k - 1$. We set $W_1 = \langle u, \bar{u}, Z_2^{-1}, v_1 \rangle, W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \leqslant i \leqslant k - 1$, and $W_k = \langle u, Q_1, \bar{y}, y, Z_1^{-1}, x, P_k, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Case 4.1.3. $v_1 \in V_1^1$ and $v_1 \in V(P_i)$ for some $2 \leqslant i \leqslant k$. Without loss of generality, we assume that $v_1 \in V(P_k)$. We set $P_k = \langle x, Z_1, v_1, y, Z_2, v_k \rangle$. Let $H = \{\bar{y}\} \cup \{\bar{x}_i | 2 \leqslant i \leqslant k - 1\}$. Thus, there exists a spanning (u, H)-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ in $G_0 - F_0$ such that Q_1 joins u to $\bar{y}$, and Q_i joins u to $\bar{x}_i$ for every $2 \leqslant i \leqslant k - 1$. We set $W_1 = \langle u, \bar{u}, P_1^{-1}, x, Z_1, v_1 \rangle, W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \leqslant i \leqslant k - 1$, and $W_k = \langle u, Q_1, \bar{y}, y, Z_2, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Case 4.2. $|F_0| < |F|$ and $|F_1| < |F|$. Since $|V_0^1| = 2^{n-2} > (n-1)(n-4) + (n-2) \geqslant (n-1)|F| + (k-2)$ if $n \geqslant 4$, there exists a node x in V_0^1 such that $(x, \bar{x}) \notin F_2$ and $|\{(z, \bar{z}) | z \in N_{G_1}(x)\} \cap F_2| \leqslant 1$.

Case 4.2.1. $v_1 \in V_1^0$. There exists a node $y \in V_1^1 - \{x, \bar{u}\}$ such that $(y, \bar{y}) \notin F_2$. Let $\{P_1, P_2, \ldots, P_k\}$ be a spanning $(x, T \cup \{y\})$-fan of $G_1 - F_1$ such that P_1 joins x to y and P_i joins x to v_i for every $2 \leqslant i \leqslant k$. We set $P_i = \langle x, x_i, R_i, v_i \rangle$ for every $2 \leqslant i \leqslant k$. Since $|\{(z, \bar{z}) | z \in N_{G_1}(x)\} \cap F_2| \leqslant 1$, we assume that $(x_i, \bar{x}_i) \notin F_2$ for every $2 \leqslant i \leqslant k-1$. We set $H = \{v_1, \bar{y}\} \cup \{\bar{x}_i | 2 \leqslant i \leqslant k-1\}$. By induction, there is a spanning (u, H)-fan $\{Q_1, Q_2, \ldots, Q_k\}$ of $G_0 - F_0$ such that Q_1 joins u to v_1, Q_i joins u to $\bar{x}_i$ for every $2 \leqslant i \leqslant k-1$, and Q_k joins u to $\bar{y}$. Let $W_1 = Q_1, W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \leqslant i \leqslant k-1$, and $W_k = \langle u, Q_k, \bar{y}, y, P_1^{-1}, x, P_k, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Case 4.2.2. $v_1 \in V_1^1$. Let $\{P_1, P_2, \ldots, P_k\}$ be a spanning (x, S)-fan of $G_1 - F_1$ such that P_i joins x to v_i for every $1 \leqslant i \leqslant k$. We set $P_i = \langle x, x_i, R_i, v_i \rangle$ for every $1 \leqslant i \leqslant k$. We choose an index r in $\{1, 2, \ldots, k-1\}$ such that $(x_i, \bar{x}_i) \notin F_2$ for every $i \in \{1, 2, \ldots, k-1\} - \{r\}$. We set $H = \{\bar{x}_i | i \in \{1, 2, \ldots, k-1\} - \{r\}\} \cup \{\bar{x}\}$. By induction, there is a spanning (u, H)-fan $\{Q_1, Q_2, \ldots, Q_k\}$ of $G_0 - F_0$ such that Q_i joins u to $\bar{x}_i$ for every $i \in \{1, 2, \ldots, k\} - \{r\}$, and Q_r joins u to $\bar{x}$. Let $W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $i \in \{1, 2, \ldots, k\} - \{r\}$ and $W_r = \langle u, Q_r, \bar{x}, x, P_r, v_r \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Case 4.3. $|F_1| = |F|$. We have $|F_0| = 0$ and $|F_2| = 0$. Let x_1 be a node in $V_1^1 - \{\bar{u}, v_1\}$. By Theorem 3, there is a hamiltonian path P of $G_1 - F_1$ joining x_1 to v_k. We set $P = \langle x_1, P_1, v_2, x_2, P_3, v_3, \ldots, x_{k-1}, P_{k-1}, v_k \rangle$. Note that $x_i \in V_1^1$ for every $1 \leqslant i \leqslant k-1$.

Case 4.3.1. $v_1 \in V_1^0$. We set $H = \{v_1\} \cup \{\bar{x}_i | 1 \leqslant i \leqslant k-1\}$. By induction, there is a spanning (u, H)-fan $\{Q_1, Q_2, \ldots, Q_k\}$ of G_0 suck that Q_1 joins u to v_1 and Q_i joins u to $\bar{x}_{i-1}$ for every $2 \leqslant i \leqslant k$. We set $W_1 = Q_1$ and $W_i = \langle u, Q_i, \bar{x}_{i-1}, x_{i-1}, P_{i-1}, v_i \rangle$ for every $2 \leqslant i \leqslant k$. Then $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

Case 4.3.2. $v_1 \in V_1^1$. Suppose that v_1 is a node in $V(P_t)$ for some $1 \leqslant t \leqslant k$. We write $P_t = \langle x_t, R_1, v_1, x_k, R_2, v_t \rangle$. Let $H = \{\bar{x}_i | 1 \leqslant i \leqslant k\}$. By induction, there exists a spanning (u, H)-fan $\{Q_1, Q_2, \ldots, Q_k\}$ of G_0 where Q_i joins u to $\bar{x}_{i-1}$ for every $1 \leqslant i \leqslant k$. For every $j \in \{1, 2, \ldots, k-1\} - \{t\}$, we set $W_j = \langle u, Q_j, \bar{x}_j, x_j, P_j, v_{j+1} \rangle$. Let $W_t = \langle u, Q_t, \bar{x}_t, x_t, R_1, v_1 \rangle$ and $W_k = \langle u, Q_k, \bar{x}_k, x_k, R_2, v_t \rangle$. Thus, $\{W_1, W_2, \ldots, W_k\}$ forms a spanning (u, S)-fan of $G - F$.

5. Conclusion

Computer network topologies are usually represented by graphs where nodes represent processors and edges represent links between processors [28]. In practice, the processors or links in a network may be failure. Thus the fault-tolerant property become an important issue on network topologies. Many results have been proposed in literature [29–32,26]. In this paper, we have shown that n-dimensional bipartite hypercube-like graphs are f-edge fault k^*-laceable for every $f \leqslant n-2$ and $f + k \leqslant n$. Future work will try to study the fault-tolerant k^*-connectivity and k^*-laceability for some super spanning connected graphs and super spanning laceable graphs, respectively.

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