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Improvement on the estimation of constant-rate drawdown in large-diameter wells

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TECHNICAL NOTE

Improvement on the estimation of constant-rate drawdown in large-diameter wells

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Abstract Papadopulos and Cooper's (PC) solution can be used to describe the drawdown resulting from pumping with a constant rate at a large-diameter well. However, this solution is too complicated to be accurately evaluated due to the oscillatory nature of the Bessel functions. The PC approach resulted in tabulated values of dimensionless drawdown at the well with an accuracy of four or fewer digits for selected values of dimensionless storage coefficient *versus* dimensionless time. Some researchers have fitted the tabulated values with interpolation formulas that are easy to use in engineering applications. Those formulas may be more accurate if the tabulated values are computed with greater accuracy. In this study, we propose an efficient numerical procedure, including a root search scheme, to find the roots of the integrand, Gaussian quadrature for numerical integration, and Shanks transform to accelerate convergence of infinite series. The proposed procedure can evaluate the dimensionless drawdown with greater accuracy and is useful in practice if there is a need for high accuracy for the observation either at the well or in the aquifer at some distance from the pumping well

Key words constant-rate pumping; large-diameter well; drawdown; aquifers; wells; numerical approach

Amélioration de l'estimation du rabattement à débit de pompage constant dans les puits de grand diamètre

Résumé La solution de Papadopulos et Cooper (PC) peut être utilisée pour décrire le rabattement résultant d'un pompage à débit constant dans un puits de grand diamètre. Cette solution est cependant trop complexe pour être évaluée avec précision en raison de la nature oscillatoire des fonctions de Bessel. L'approche de PC conduit à des valeurs tabulées d'un rabattement adimensionnel au puits avec une précision de quatre chiffres au plus pour certaines valeurs du coefficient d'emmagasinement (sans dimension) en fonction d'un coefficient temporel adimensionnel. Certains chercheurs ont ajusté sur les valeurs tabulées des formules d'interpolation faciles à utiliser dans les applications d'ingénierie. Ces formules peuvent être plus précises si les valeurs du tableau sont calculées avec une meilleure précision. Dans cette étude, nous proposons une procédure numérique efficace, comprenant un système de recherche de racine permettant de déterminer les racines de la fonction à intégrer, une quadrature de Gauss pour l'intégration numérique, et une transformation de Shanks pour accélérer la convergence des séries infinies. La procédure proposée permet d'évaluer un rabattement adimensionnel avec une plus grande précision et est utile en pratique s'il est nécessaire d'atteindre une meilleure précision pour l'observation dans le puits ou dans l'aquifère à une certaine distance du puits de pompage.

Mots clefs pompage à débit constant; puits de grand diamètre; rabattement; aquifères; puits; approche numérique

INTRODUCTION

Large-diameter wells are widely used in many countries to meet the growing demand of water for domestic and irrigation uses. These wells are convenient for withdrawing large quantities of water from low-permeability aquifers. The drawdown solution in regard to large-diameter wells is different from that of small-diameter wells, because of the contribution of water from wellbore storage. Papadopulos and Cooper (1967) presented an exact solution (hereafter referred to as the PC solution) for describing the drawdown distribution when pumping in a large-diameter well. Their solution took into account the effects of finite well radius and the water stored within the wellbore, which were neglected in the Theis equation.

The PC solution contains an integral with the limits from zero to infinity and many terms of the products of zero-order and first-order Bessel functions. It can be transformed to an infinite series because the integrand is in terms of alternately oscillatory functions. Yet, the transformed series sometimes converges very slowly and may be very time-consuming in calculations. Under such circumstances, the PC solution requires a considerable amount of computing effort in numerical integrations to achieve the desired accuracy. Papadopulos and Cooper (1967) and Papadopulos (1967) gave tabulated values of dimensionless drawdown in the pumped well and the aquifer with four and three significant digits, respectively. Reed (1980) reported that Papadopulos (1967) developed two FORTRAN programs to compute the dimensionless drawdown in the test well and confined aquifer. He also reproduced the well function values, which were listed in Papadopulos and Cooper (1967) for dimensionless drawdown along the well and in Papadopulos (1967) for dimensionless drawdown at different radial distances. The methods employed in the computer programs include Simpson's rule for numerical integration, the Euler transform for accelerating the convergence of series summation, and the polynomial approximations to Bessel functions.

Considering the difficulty in calculating the PC solution, which requires numerical integration of an improper integral involving Bessel's functions, Swamee and Ojha (1995) proposed algebraic approximations to well functions, and these approximations were obtained from the tabulated values of well functions in Hantush (1964) and Papadopulos and Cooper (1967). Çimen (2001) derived an analytical solution for drawdown in a large-diameter well using independent variable transformation. This solution, however, contains improper integrals which are laborious to calculate and may obtain inaccurate results from the use of an improper numerical method for numerical integration. He further gave large time solutions for transient drawdowns at pumping and observation wells in confined aquifers with a large-diameter well. In addition, in order to estimate aquifer parameters conveniently, a number of studies have been devoted to developing an algebraic approximation to well drawdown based on the tabulated values in Papadopulos and Cooper (1967). Singh (2007a) proposed simple equations for estimating aquifer parameters, and these equations were obtained from the tabulated values given by Papadopulos and Cooper (1967). Singh (2007b) further developed a simple approximation of well function by fitting the tabulated values provided in Papadopulos and Cooper (1967). Nevertheless, the accuracy of the approximation depends on the algebraic formulas and the tabulated values from Papadopulos and Cooper (1967). The approximations might however not be accurate if the tabulated values do not maintain good accuracy.

Peng *et al*. (2002) proposed a unified numerical approach including a root search approach, the Gaussian quadrature, and the Shanks transform to compute the constant-head aquifer drawdown and wellbore flux. Their results were correct to five significant digits. This numerical approach has also been successfully applied to other groundwater problems, for example, by Yeh and Chang (2006), and Yeh and Yang (2006). The main objective of this paper is to compute the PC solution based on the numerical approach presented by Peng *et al*. (2002). Comparisons of the calculated dimensionless drawdowns from the present method and other methods demonstrate the significant improvement in the estimation of the PC solution.

DRAWDOWN SOLUTIONS

For pumping with a constant rate at a large-diameter well in a confined aquifer, the drawdown solution can be expressed as (Papadopulos and Cooper 1967):

$$s(u, \alpha) = \frac{Q}{4\pi T} W(u, \alpha, \rho) \tag{1}$$

with:

$$W(u,\alpha,\rho)=\int_0^\infty \frac{8\alpha}{\pi}\left[1-\exp\left(\frac{-\rho^2x^2}{4u}\right)\right] \times \frac{J_0(\rho x)A(x)-Y_0(\rho x)B(x)}{A^2(x)+B^2(x)}\frac{dx}{x^2} \tag{2}$$

$$A(x)=xY_0(x)-2\alpha Y_1(x) \tag{3}$$

and

$$B(x)=xJ_0(x)-2\alpha J_1(x) \tag{4}$$

where s is the drawdown in the aquifer at a distance r from the centre of well; $u=r^2S/4Tt$; $\alpha=r_w^2S/r_c^2$; $\rho=r/r_w$; t is time; T and S are the transmissibility and storage coefficient of aquifer, respectively; r_w is the radius of well screen; r_c is the radius of well casing; Q is a constant discharge rate; J_0 and Y_0 are zero-order Bessel functions of the first and second kind, respectively; J_1 and Y_1 are first-order Bessel functions of the first and second kind, respectively.

If $r=r_w$ (i.e. $\rho=1$) and $u_w=r_w^2S/4Tt$, the drawdown at the pumped well can be obtained from equation (1) using the relationship of $Y_0(x)J_1(x)-J_0(x)Y_1(x)=2/(\pi x)$ (Abramowitz and Stegun 1970) and then expressed as:

$$s(u_w,\alpha)=\frac{Q}{4\pi T}W(u_w,\alpha) \tag{5}$$

with:

$$W(u_w,\alpha)=\int_0^\infty \frac{32\alpha^2}{\pi^2}\left[1-\exp\left(\frac{-x^2}{4u_w}\right)\right] \times \frac{1}{A^2(x)+B^2(x)}\frac{dx}{x^3} \tag{6}$$

Swamee and Ojha (1995) presented the algebraic expression for the well function as:

$$W_{\mathrm{swa}}(u_w,\alpha)=\left\{\left[\ln\left(\frac{c_4}{u_w}+1\right)+\frac{0.7u_w^{0.41}}{\alpha^{0.25}}\right]^{-1} + \frac{u_w^2}{\alpha(u_w+0.45\alpha)}\right\} \tag{7}$$

with:

$$c_4=\exp(-0.577216) \tag{8}$$

Çimen (2001, equation (13)) used independent variable transform to derive an equation for calculating the drawdown in a large-diameter well, in our notation, as:

$$W_{\mathrm{c-exact}}(u_w,\alpha)=\frac{1}{u_w\left(\frac{1}{\alpha}+\int_{u_w}^{\infty}\frac{e^{-x}}{x^2}dx\Big/\int_{u_w}^{\infty}\frac{e^{-x}}{x}dx\right)} \tag{9}$$

Çimen also presented an approximate well function (equation (18) in Çimen 2001) at large time as:

$$W_{\mathrm{c-approx}}(u_w,\alpha)=\frac{1}{\frac{u_w}{\alpha}+\frac{0.98}{(-0.5772-\ln(u_w))}} \tag{10}$$

Furthermore, Singh (2007b) gave:

$$W_{\sin}(u_w,\alpha)=\frac{\alpha}{u_w}\left[1+c_1\left(\frac{\alpha}{u_w}\right)^{c_2}\right]^{-c_3} \tag{11}$$

with:

$$c_1=\frac{7}{5}\left[\ln\left(\frac{1}{\alpha}\right)\right]^{-1/8}-1 \tag{12}$$

$$c_2=\ln(\sqrt{3})\left[\ln\left(\frac{1}{\alpha}\right)\right]^{1/\pi} \tag{13}$$

$$c_3=\left(\frac{37\pi}{90}\right)^2\left[\ln\left(\frac{1}{\alpha}\right)\right]^{10/33} \tag{14}$$

NUMERICAL CALCULATION

The dimensionless drawdown, $s/(Q/4\pi T)$ in equations (1) or (5), can be calculated once the integral, i.e. $W(u, \alpha, \rho)$ or $W(u_w, \alpha)$, is computed. The integrand of $W(u, \alpha, \rho)$ or $W(u_w, \alpha)$ involves the products of zero-order and first-order Bessel functions in the numerator and/or denominator. The exponential function in the integrand, i.e. $1-\exp(1-x^2\rho^2/4u)$ or $1-\exp(-x^2/4u_w)$, is an accelerating factor and has a nature of rapidly progressing from zero to one, while x varies from zero to infinity. The effect of the accelerating factor may suddenly be increased when u/ρ^2 (or in terms of u_w) approaches zero. The integrand of $W(u, \alpha, \rho)$ has an alternatively oscillatory nature, while $W(u_w, \alpha)$ keeps a positive value for the whole range of integration. A numerical approach is therefore employed to transform the integral in equation (2) into a summation of infinite series and to accelerate the computation of the infinite series.

A plot of the integrand of $W(u, \alpha, \rho)$ *versus* x for the case that $u = 1$ and $\rho = 100$, while $\alpha = 10^{-1}$, 10^{-3} and 10^{-5}, is shown in Fig. 1. This indicates that the integrand depending on the value of α gives alternate oscillation along the horizontal axis and dies away quickly. The oscillation of the integrand is dramatic if α is large. Such an integrand can be effectively computed by the numerical approach of three steps described below:

– Step 1: A root search scheme is employed initially to find the consecutive roots of the integrand along the horizontal axis. A combination of bisection and Newton's methods is used to find the zeros of the integrand. A termination criterion in the process of finding zero is usually to set the relative error between the ith and $(i + 1)$th zeros, less than a very small value.
– Step 2: A 20-point Gaussian quadrature is then applied to perform the numerical integration for each area under the integrand between two consecutive roots within the range from zero to infinity. Since the integrand is an oscillatory function, each integration result is considered as a term of the alternating series. When applying the Gaussian quadrature, the integral $\int_a^b f(x)dx$ is transformed to another integral with the new integration interval of [–1,1] using the change of variable. The formula of the Gaussian quadrature is written as:

$$\int_{-1}^{1} f(\zeta)d\zeta = \sum_{i=1}^{n} W_i f(\zeta_i) \tag{15}$$

for n points, where W_i denotes the weighting factor and ζ_i represents the integration point i. Values of W_i and ζ_i can be found from a numerical analysis book, e.g. Burden and Faires (2001) or Gerald and Wheatley (2004).
– Step 3: Finally, the Shanks transform (Peng *et al*. 2002) is employed to accelerate the evaluation of the sum of the alternating series. The Shanks transform is a nonlinear iterative algorithm based on the sequence of partial sums and expressed as:

$$e_{k+1}(S_n) = e_{k-1}(S_{n+1}) + \frac{1}{e_k(S_{n+1}) - e_k(S_n)}, \quad k = 1, 2, \ldots \tag{16}$$

where S_n denotes a sequence of partial sums, $e_0(S_n) = S_n$ and $e_1(S_n) = [e_0(S_{n+1}) - e_0(S_n)]^{-1}$.

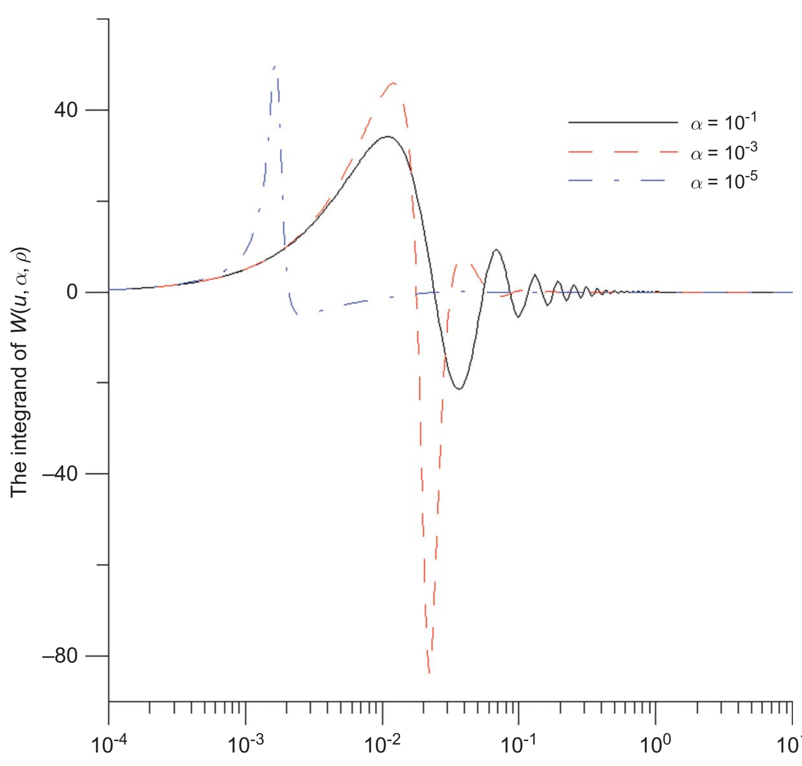

Fig. 1 Curves of the integrand of $W(u, \alpha, \rho)$ for $u = 1$ and $\rho = 10^2$ when $\alpha = 10^{-1}$, 10^{-3} and 10^{-5}.

A convergence criterion is required when applying the Shanks transform to calculate a given series. A convergence factor, ε, is defined as:

$$\left|\frac{e_{2r+2}(S_{n-1}) - e_{2r}(S_n)}{e_{2r+1}(S_{n-1})}\right| \le \varepsilon \qquad (17)$$

The running sum is terminated when the left-hand side in equation (17) is less than ε, a very small value.

Figure 2 shows the curves of the integrand of $W(u_w, \alpha, \rho)$ *versus* x for $u_w = 1$ and 10^{-8} when $\alpha = 10^{-1}$, 10^{-3} and 10^{-5}. This figure exhibits that the curves increase initially, reach a maximum at some values of x, then decrease rapidly and diminish when x is very large. A dual-peak pattern can be observed when u_w approaches zero. Both the 10-point and 20-point formulas of the Gaussian quadrature are used at the same time to carry out the numerical integration for each integrand. If the difference of these two results for any interval between the consecutive roots is greater than the prescribed criterion, then the interval will be divided into two portions. The same integration procedure is applied repeatedly to each portion until the convergence is met to ensure that the result bears the desired level of accuracy.

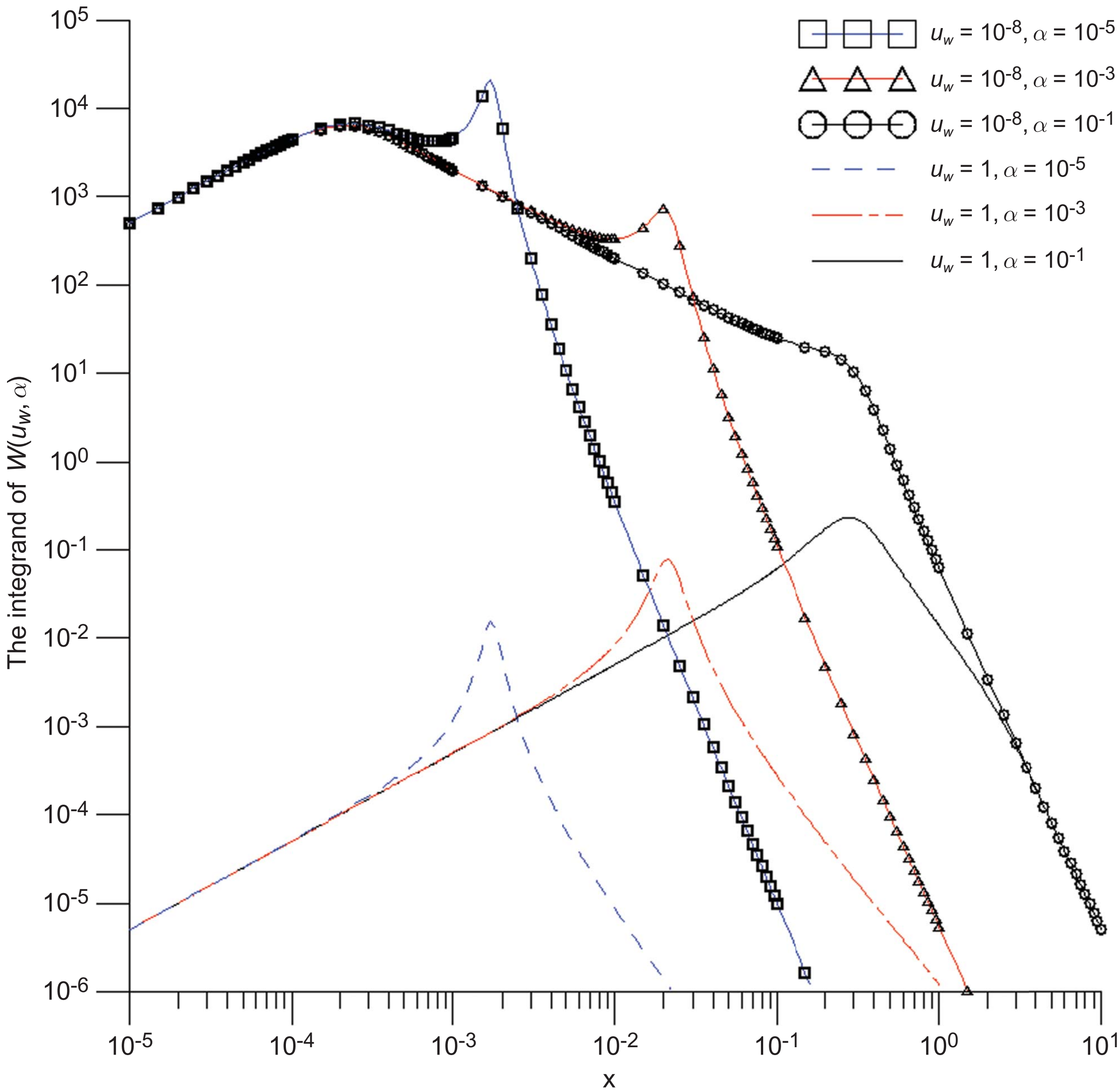

Fig. 2 Curves of the integrand of $W(u_w, \alpha)$ for $u_w = 1$ and 10^{-8} when $\alpha = 10^{-1}$, 10^{-3} and 10^{-5}.

RESULTS AND DISCUSSION

The termination criteria for root search, numerical integration and Shanks transform are chosen as 10^{-5}, which represents a result that is accurate to five significant digits when applying the proposed method to compute equations (1) and (5). Values of the well function $W(u_w, \alpha)$ computed by the proposed numerical approach are given in Table 1 for u_w ranging from 10^{-9} to 10 and α ranging from 10^{-5} to 10^{-1}. In addition, the values tabulated in Papadopulos and Cooper (1967), calculated from the exact solution in Çimen (2001, equation (13)) and from the approximate well functions in Swamee and Ojha (1995), Çimen (2001, equation (18)) and Singh (2007b) are also listed in Table 1.

As shown in Table 1, the well function is accurate to five significant digits using the proposed numerical approach. Compared with the tabulated values in Papadopulos and Cooper (1967), the difference in dimensionless drawdown is large when u_w is small. For example, the value of well function estimated by the proposed approach and that in Papadopulos and Cooper (1967) for $\alpha = 10^{-1}$ when $u_w = 10$ are 9.75520 $\times$ 10^{-3} and 9.755 $\times$ 10^{-3}, respectively. In addition, the value of well function for $\alpha = 10^{-1}$ when $u_w = 2 \times 10^{-9}$ is 2.01461 $\times$ 10^{1} using the proposed approach and 2.015 $\times$ 10^{1} in Papadopulos and Cooper (1967). Moreover, the difference is 2 $\times$ 10^{-7} when $u_w = 10$ and 3.9 $\times$ 10^{-3} when $u_w = 2 \times 10^{-9}$. Figure 3(a)–(e) plots the differences between the

Table 1 Values of well functions for u_w ranging from 10^{-9} to 10 and α ranging from 10^{-5} to 10^{-1}.

u_w	Proposed approach	PC	SO	C13	C18	S
$\alpha = 10^{-1}$						
1E+01	9.75520E-03	9.755E-03	1.00141E-02	9.90870E-03	1.00344E-02	9.87632E-03
1E+00	9.19130E-02	9.192E-02	9.84160E-02	9.36604E-02	1.21341E-01	9.38479E-02
5E-01	1.76721E-01	1.767E-01	1.93088E-01	1.79098E-01	8.19089E-02	1.80473E-01
2E-01	4.06155E-01	4.062E-01	4.67813E-01	4.04918E-01	3.41140E-01	4.12952E-01
1E-01	7.33537E-01	7.336E-01	9.00120E-01	7.16145E-01	6.40418E-01	7.40188E-01
5E-02	1.25948E+00	1.260E+00	1.63453E+00	1.19697E+00	1.10879E+00	1.25890E+00
2E-02	2.30327E+00	2.303E+00	2.96059E+00	2.11781E+00	2.03204E+00	2.28871E+00
1E-02	3.27530E+00	3.276E+00	3.93137E+00	2.98340E+00	2.92315E+00	3.28707E+00
5E-03	4.25485E+00	4.255E+00	4.75592E+00	3.91335E+00	3.89556E+00	4.36994E+00
2E-03	5.42069E+00	5.420E+00	5.71045E+00	5.12900E+00	5.17534E+00	5.75096E+00
1E-03	6.21241E+00	6.212E+00	6.39671E+00	5.99586E+00	6.08579E+00	6.67563E+00
5E-04	6.95947E+00	6.960E+00	7.07699E+00	6.81226E+00	6.93811E+00	7.48329E+00
2E-04	7.91136E+00	7.866E+00	7.97766E+00	7.82984E+00	7.99260E+00	8.40595E+00
1E-04	8.61765E+00	8.572E+00	8.66165E+00	8.56751E+00	8.75245E+00	9.03088E+00
5E-05	9.31794E+00	9.318E+00	9.34777E+00	9.28781E+00	9.49177E+00	9.62148E+00
2E-05	1.02389E+01	1.024E+01	1.02573E+01	1.02239E+01	1.04501E+01	1.03824E+01
1E-05	1.09338E+01	1.093E+01	1.09468E+01	1.09251E+01	1.11668E+01	1.09605E+01
5E-06	1.16278E+01	1.163E+01	1.16372E+01	1.16228E+01	1.18795E+01	1.15515E+01
2E-06	1.25447E+01	1.255E+01	1.25509E+01	1.25423E+01	1.28183E+01	1.23632E+01
1E-06	1.32381E+01	1.324E+01	1.32426E+01	1.32367E+01	1.35271E+01	1.30063E+01
5E-07	1.39313E+01	1.393E+01	1.39347E+01	1.39306E+01	1.42352E+01	1.36785E+01
2E-07	1.48477E+01	1.485E+01	1.48500E+01	1.48473E+01	1.51707E+01	1.46163E+01
1E-07	1.55409E+01	1.554E+01	1.55426E+01	1.55407E+01	1.58782E+01	1.53663E+01
5E-08	1.62340E+01	1.623E+01	1.62353E+01	1.62339E+01	1.65856E+01	1.61538E+01
2E-08	1.71503E+01	1.705E+01	1.71512E+01	1.71503E+01	1.75207E+01	1.72560E+01
1E-08	1.78435E+01	1.784E+01	1.78441E+01	1.78434E+01	1.82280E+01	1.81391E+01
5E-09	1.85366E+01	1.854E+01	1.85371E+01	1.85366E+01	1.89353E+01	1.90672E+01
2E-09	1.94529E+01	1.945E+01	1.94532E+01	1.94529E+01	1.98703E+01	2.03670E+01
1E-09	2.01461E+01	2.015E+01	2.01463E+01	2.01460E+01	2.05776E+01	2.14088E+01
$\alpha = 10^{-2}$						
1.0E+01	9.97495E-04	9.760E-04	1.00028E-03	9.99079E-04	1.00034E-03	9.99656E-04
1.0E+00	9.91366E-03	9.914E-03	1.00072E-02	9.93277E-03	1.01790E-02	9.97316E-03
5.0E-01	1.97440E-02	1.974E-02	2.00130E-02	1.97693E-02	1.74799E-02	1.99005E-02
2.0E-01	4.89017E-02	4.890E-02	5.00928E-02	4.88528E-02	4.77752E-02	4.94396E-02
1.0E-01	9.66416E-02	9.665E-02	1.00675E-01	9.61875E-02	9.46837E-02	9.79379E-02
5.0E-02	1.89594E-01	1.896E-01	2.03898E-01	1.87426E-01	1.85121E-01	1.92470E-01
2.0E-02	4.52897E-01	4.529E-01	5.27766E-01	4.40105E-01	4.36278E-01	4.59228E-01
1.0E-02	8.51999E-01	8.520E-01	1.08941E+00	8.09594E-01	8.05090E-01	8.58242E-01

(*Continued*)

Table 1 (Continued).

u_w	Proposed approach	PC	SO	C13	C18	S
5.0E-03	1.54014E+00	1.540E+00	2.15575E+00	1.41736E+00	1.41502E+00	1.52944E+00
2.0E-03	3.04290E+00	3.043E+00	4.28205E+00	2.66688E+00	2.67935E+00	2.93699E+00
1.0E-03	4.54441E+00	4.545E+00	5.78313E+00	3.89436E+00	3.93210E+00	4.31719E+00
5.0E-04	6.03093E+00	6.031E+00	6.87774E+00	5.21393E+00	5.28733E+00	5.75312E+00
2.0E-04	7.55728E+00	7.557E+00	7.95351E+00	6.86264E+00	6.98735E+00	7.41170E+00
1.0E-04	8.44337E+00	8.443E+00	8.66765E+00	7.95418E+00	8.11335E+00	8.39717E+00
5.0E-05	9.22872E+00	9.229E+00	9.35971E+00	8.91520E+00	9.10295E+00	9.18674E+00
2.0E-05	1.02010E+01	1.020E+01	1.02679E+01	1.00392E+01	1.02572E+01	1.00376E+01
1.0E-05	1.09138E+01	1.087E+01	1.09552E+01	1.08187E+01	1.10557E+01	1.06033E+01
5.0E-06	1.16173E+01	1.162E+01	1.16436E+01	1.15624E+01	1.18164E+01	1.11406E+01
2.0E-06	1.25402E+01	1.254E+01	1.25553E+01	1.25141E+01	1.27888E+01	1.18423E+01
1.0E-06	1.32357E+01	1.324E+01	1.32460E+01	1.32210E+01	1.35106E+01	1.23819E+01
5.0E-07	1.39301E+01	1.393E+01	1.39372E+01	1.39218E+01	1.42261E+01	1.29370E+01
2.0E-07	1.48471E+01	1.485E+01	1.48517E+01	1.48434E+01	1.51666E+01	1.37016E+01
1.0E-07	1.55406E+01	1.554E+01	1.55439E+01	1.55385E+01	1.58759E+01	1.43068E+01
5.0E-08	1.62339E+01	1.623E+01	1.62363E+01	1.62327E+01	1.65844E+01	1.49375E+01
2.0E-08	1.71503E+01	1.705E+01	1.71519E+01	1.71497E+01	1.75201E+01	1.58128E+01
1.0E-08	1.78434E+01	1.784E+01	1.78446E+01	1.78432E+01	1.82277E+01	1.65085E+01
5.0E-09	1.85366E+01	1.854E+01	1.85375E+01	1.85364E+01	1.89352E+01	1.72346E+01
2.0E-09	1.94529E+01	1.945E+01	1.94535E+01	1.94528E+01	1.98702E+01	1.82435E+01
1.0E-09	2.01461E+01	2.015E+01	2.01465E+01	2.01460E+01	2.05776E+01	1.90457E+01
$\alpha = 10^{-3}$						
1.0E+01	9.99719E-05	9.998E-05	1.00004E-04	9.99908E-05	1.00003E-04	9.99992E-05
1.0E+00	9.99128E-04	9.991E-04	1.00022E-03	9.99324E-04	1.00176E-03	9.99917E-04
5.0E-01	1.99741E-03	1.997E-03	2.00072E-03	1.99767E-03	1.97158E-03	1.99967E-03
2.0E-01	4.98883E-03	4.989E-03	5.00381E-03	4.98829E-03	4.97682E-03	4.99788E-03
1.0E-01	9.96554E-03	9.966E-03	1.00156E-02	9.96052E-03	9.94417E-03	9.99143E-03
5.0E-02	1.98918E-02	1.989E-02	2.00692E-02	1.98667E-02	1.98405E-02	1.99654E-02
2.0E-02	4.94940E-02	4.949E-02	5.05045E-02	4.93287E-02	4.92802E-02	4.97809E-02
1.0E-02	9.83413E-02	9.834E-02	1.02199E-01	9.77022E-02	9.76363E-02	9.91176E-02
5.0E-03	1.94478E-01	1.945E-01	2.09193E-01	1.92103E-01	1.92060E-01	1.96464E-01
2.0E-03	4.72475E-01	4.725E-01	5.55324E-01	4.59776E-01	4.60146E-01	4.78183E-01
1.0E-03	9.06879E-01	9.069E-01	1.18765E+00	8.64467E-01	8.66313E-01	9.15646E-01
5.0E-04	1.68782E+00	1.688E+00	2.48716E+00	1.55813E+00	1.56462E+00	1.68674E+00
2.0E-04	3.52280E+00	3.523E+00	5.38777E+00	3.07015E+00	3.09487E+00	3.40821E+00
1.0E-04	5.52588E+00	5.526E+00	7.52927E+00	4.63564E+00	4.68925E+00	5.17349E+00
5.0E-05	7.63069E+00	7.631E+00	8.97277E+00	6.36262E+00	6.45768E+00	7.01497E+00
2.0E-05	9.67644E+00	9.676E+00	1.01999E+01	8.50270E+00	8.65857E+00	9.04575E+00
1.0E-05	1.06802E+01	1.068E+01	1.09447E+01	9.85877E+00	1.00552E+01	1.01649E+01
5.0E-06	1.15033E+01	1.150E+01	1.16478E+01	1.09905E+01	1.12198E+01	1.10106E+01
2.0E-06	1.24934E+01	1.249E+01	1.25619E+01	1.22384E+01	1.25010E+01	1.18880E+01
1.0E-06	1.32114E+01	1.321E+01	1.32516E+01	1.30655E+01	1.33483E+01	1.24652E+01
5.0E-07	1.39174E+01	1.392E+01	1.39416E+01	1.38352E+01	1.41356E+01	1.30152E+01
2.0E-07	1.48418E+01	1.484E+01	1.48548E+01	1.48038E+01	1.51253E+01	1.37378E+01
1.0E-07	1.55378E+01	1.554E+01	1.55462E+01	1.55168E+01	1.58533E+01	1.42957E+01
5.0E-08	1.62324E+01	1.623E+01	1.62380E+01	1.62209E+01	1.65720E+01	1.48701E+01
2.0E-08	1.71496E+01	1.705E+01	1.71531E+01	1.71444E+01	1.75146E+01	1.56604E+01
1.0E-08	1.78431E+01	1.784E+01	1.78455E+01	1.78403E+01	1.82247E+01	1.62843E+01
5.0E-09	1.85364E+01	1.854E+01	1.85382E+01	1.85349E+01	1.89335E+01	1.69324E+01
2.0E-09	1.94528E+01	1.945E+01	1.94540E+01	1.94521E+01	1.98695E+01	1.78284E+01
1.0E-09	2.01461E+01	2.015E+01	2.01469E+01	2.01456E+01	2.05772E+01	1.85374E+01
$\alpha = 10^{-4}$						
1.0E+01	9.99884E-06	1.000E-05	1.00000E-05	9.99990E-06	1.00000E-05	1.00000E-05
1.0E+00	9.99903E-05	1.000E-04	1.00003E-04	9.99932E-05	1.00018E-04	9.99998E-05
5.0E-01	1.99973E-04	2.000E-04	2.00011E-04	1.99977E-04	1.99712E-04	1.99999E-04
2.0E-01	4.99887E-04	4.999E-04	5.00062E-04	4.99883E-04	4.99767E-04	4.99995E-04
1.0E-01	9.99654E-04	9.997E-04	1.00023E-03	9.99604E-04	9.99439E-04	9.99976E-04
5.0E-02	1.99891E-03	1.999E-03	2.00092E-03	1.99866E-03	1.99839E-03	1.99990E-03
2.0E-02	4.99490E-03	4.995E-03	5.00600E-03	4.99320E-03	4.99271E-03	4.99929E-03
1.0E-02	9.98321E-03	9.984E-03	1.00253E-02	9.97654E-03	9.97585E-03	9.99694E-03

(*Continued*)

Table 1 (Continued).

u_w	Proposed approach	PC	SO	C13	C18	S
5.0E-03	1.99437E-02	1.994E-02	2.01066E-02	1.99181E-02	1.99177E-02	1.99868E-02
2.0E-03	4.97138E-02	4.972E-02	5.07061E-02	4.95664E-02	4.95707E-02	4.99083E-02
1.0E-03	9.90049E-02	9.901E-02	1.02906E-01	9.84564E-02	9.84803E-02	9.96040E-02
5.0E-04	1.96502E-01	1.965E-01	2.11708E-01	1.94485E-01	1.94585E-01	1.98295E-01
2.0E-04	4.81376E-01	4.814E-01	5.69703E-01	4.70430E-01	4.71006E-01	4.88364E-01
1.0E-04	9.33977E-01	9.340E-01	1.24475E+00	8.96282E-01	8.98268E-01	9.51077E-01
5.0E-05	1.76812E+00	1.768E+00	2.70995E+00	1.64699E+00	1.65329E+00	1.80056E+00
2.0E-05	3.82783E+00	3.828E+00	6.31368E+00	3.36011E+00	3.38418E+00	3.83798E+00
1.0E-05	6.24510E+00	6.245E+00	9.16536E+00	5.22377E+00	5.27841E+00	6.09629E+00
5.0E-06	8.99091E+00	8.991E+00	1.10318E+01	7.35362E+00	7.45555E+00	8.56991E+00
2.0E-06	1.17352E+01	1.174E+01	1.24442E+01	1.00291E+01	1.02048E+01	1.13073E+01
1.0E-06	1.29100E+01	1.291E+01	1.32244E+01	1.16908E+01	1.19167E+01	1.27482E+01
5.0E-07	1.37775E+01	1.378E+01	1.39390E+01	1.30243E+01	1.32902E+01	1.37799E+01
2.0E-07	1.47859E+01	1.479E+01	1.48583E+01	1.44196E+01	1.47244E+01	1.48021E+01
1.0E-07	1.55092E+01	1.551E+01	1.55498E+01	1.53031E+01	1.56303E+01	1.54596E+01
5.0E-08	1.62176E+01	1.622E+01	1.62410E+01	1.61033E+01	1.64493E+01	1.60830E+01
2.0E-08	1.71434E+01	1.714E+01	1.71552E+01	1.70917E+01	1.74596E+01	1.69011E+01
1.0E-08	1.78399E+01	1.784E+01	1.78471E+01	1.78117E+01	1.81949E+01	1.75325E+01
5.0E-09	1.85348E+01	1.854E+01	1.85394E+01	1.85194E+01	1.89174E+01	1.81819E+01
2.0E-09	1.94521E+01	1.945E+01	1.94548E+01	1.94453E+01	1.98624E+01	1.90736E+01
1.0E-09	2.01457E+01	2.015E+01	2.01475E+01	2.01420E+01	2.05734E+01	1.97758E+01
$\alpha = 10^{-5}$						
1E+01	9.99946E-07	1.000E-06	1.00000E-06	9.99999E-07	1.00000E-06	1.00000E-06
1E+00	1.00002E-05	1.000E-05	1.00000E-05	9.99993E-06	1.00002E-05	1.00000E-05
5E-01	2.00006E-05	2.000E-05	2.00001E-05	1.99998E-05	1.99971E-05	2.00000E-05
2E-01	5.00016E-05	5.000E-05	5.00008E-05	4.99988E-05	4.99977E-05	5.00000E-05
1E-01	1.00002E-04	1.000E-04	1.00003E-04	9.99960E-05	9.99944E-05	1.00000E-04
5E-02	2.00001E-04	2.000E-04	2.00011E-04	1.99987E-04	1.99984E-04	2.00000E-04
2E-02	4.99979E-04	5.000E-04	5.00070E-04	4.99932E-04	4.99927E-04	4.99999E-04
1E-02	9.99893E-04	1.000E-03	1.00028E-03	9.99765E-04	9.99758E-04	9.99994E-04
5E-03	1.99956E-03	2.000E-03	2.00115E-03	1.99918E-03	1.99917E-03	1.99997E-03
2E-03	4.99713E-03	4.998E-03	5.00746E-03	4.99563E-03	4.99567E-03	4.99978E-03
1E-03	9.98998E-03	9.992E-03	1.00307E-02	9.98435E-03	9.98459E-03	9.99898E-03
5E-04	1.99646E-02	1.997E-02	2.01264E-02	1.99434E-02	1.99445E-02	1.99953E-02
2E-04	4.98090E-02	4.982E-02	5.08127E-02	4.96877E-02	4.96941E-02	4.99651E-02
1E-04	9.93095E-02	9.932E-02	1.03290E-01	9.88560E-02	9.88801E-02	9.98405E-02
5E-05	1.97487E-01	1.975E-01	2.13130E-01	1.95803E-01	1.95892E-01	1.99271E-01
2E-05	4.86071E-01	4.861E-01	5.78403E-01	4.76733E-01	4.77215E-01	4.94592E-01
1E-05	9.49196E-01	9.493E-01	1.28176E+00	9.16227E-01	9.17894E-01	9.75611E-01
5E-06	1.81639E+00	1.817E+00	2.86913E+00	1.70652E+00	1.71195E+00	1.89209E+00
2E-06	4.03193E+00	4.033E+00	7.09785E+00	3.57513E+00	3.59721E+00	4.27972E+00
1E-06	6.77769E+00	6.779E+00	1.06981E+01	5.69679E+00	5.74991E+00	7.26360E+00
5E-07	1.01252E+01	1.013E+01	1.30526E+01	8.21156E+00	8.31645E+00	1.09476E+01
2E-07	1.37046E+01	1.371E+01	1.46842E+01	1.14482E+01	1.16395E+01	1.54169E+01
1E-07	1.51267E+01	1.513E+01	1.55052E+01	1.34506E+01	1.37027E+01	1.77899E+01
5E-08	1.60503E+01	1.605E+01	1.62322E+01	1.50153E+01	1.53156E+01	1.94019E+01
2E-08	1.70784E+01	1.708E+01	1.71564E+01	1.65816E+01	1.69276E+01	2.08812E+01
1E-08	1.78069E+01	1.781E+01	1.78493E+01	1.75307E+01	1.79017E+01	2.17808E+01
5E-09	1.85178E+01	1.851E+01	1.85413E+01	1.83664E+01	1.87577E+01	2.26136E+01
2E-09	1.94451E+01	1.940E+01	1.94562E+01	1.93775E+01	1.97917E+01	2.36937E+01
1E-09	2.01421E+01	2.015E+01	2.01486E+01	2.01055E+01	2.05354E+01	2.45229E+01

Notes: PC: Papadopulos and Cooper (1967); SO: Swamee and Ojha (1995); C13: Çimen (2001, equation (13)); Çimen (2001, equation (18)); S: Singh (2007b).

well function values estimated by the proposed procedure and those in Papadopulos and Cooper (1967), Swamee and Ojha (1995), Çimen (2001, equations (13) and (18)) and Singh (2007b) for $u_w = 10^{-9} \sim 10^{1}$ when $\alpha = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}. As illustrated in Fig. 3, significant errors in Papadopulos and Cooper (1967) occur at $u_{\text{w}} = 2 \times 10^{-8}$ with a value of 1.003×10^{-1} when $\alpha = 10^{-1}$, 1.003×10^{-1} when $\alpha = 10^{-2}$ and 9.96×10^{-3} when $\alpha = 10^{-3}$. The largest error in Çimen (2001, equation (13)) is -3.415×10^{-1} at $u_w = 5 \times 10^{-3}$ when $\alpha = 10^{-1}$, -8.17×10^{-1} at $u_w = 5 \times 10^{-4}$ when $\alpha = 10^{-2}$, -1.26807 at $u_w = 5 \times 10^{-5}$

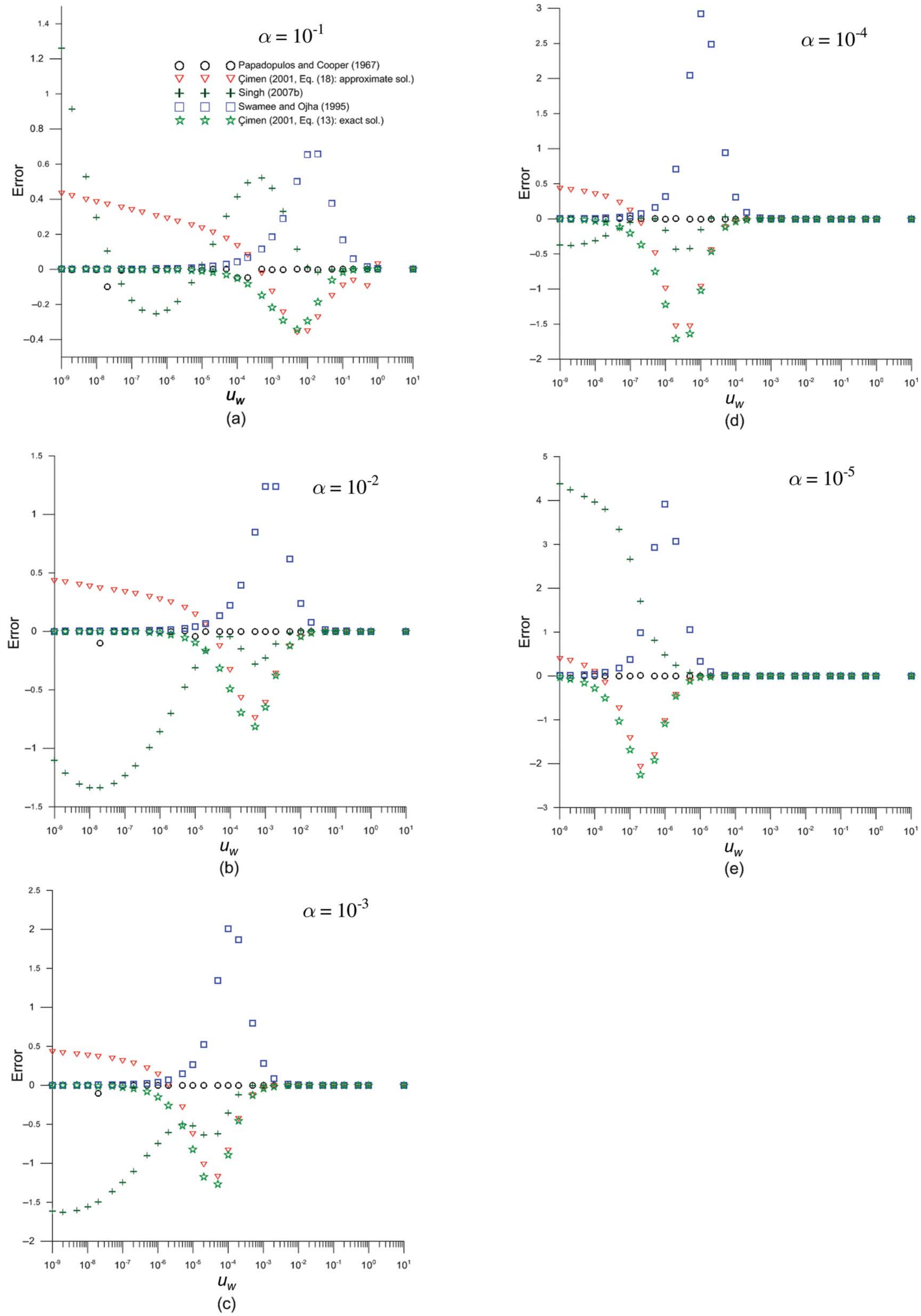

Fig. 3 Errors of the well function in Papadopulos and Cooper (1967) (○), Swamee and Ojha (1995) (□), Çimen (2001, equation (18)) (∇), Çimen (2001, equation (13)) (☆) and Singh (2007b) (+) for $u_w = 10^{-9}$–10^{1} when: (a) $\alpha = 10^{-1}$; (b) $\alpha = 10^{-2}$; (c) $\alpha = 10^{-3}$; (d) $\alpha = 10^{-4}$ and (e) $\alpha = 10^{-5}$.

Table 2 Values of dimensionless drawdown $s(u, \rho)/(Q/4\pi T)$ for u ranging from 10^{-5} to 10 and α ranging from 10^{-5} to 10^{-1}.

u	$\rho = 10$		$\rho = 50$		$\rho = 100$	
	Proposed approach	Papadopulos (1967)	Proposed approach	Papadopulos (1967)	Proposed approach	Papadopulos (1967)
$\alpha = 10^{-1}$						
2.E+00	2.32726E-02	2.40E-02	4.41736E-02	4.24 E-02	4.73378E-02	4.48 E-02
1.E+00	1.39911E-01	1.41E-01	2.10823E-01	2.09E-01	2.16768E-01	2.14 E-01
5.E-01	4.43625E-01	4.44E-01	5.51102E-01	5.49 E-01	5.57232E-01	5.55 E-01
2.E-01	1.12494E+00	1.13E+00	1.21687E+00	1.22 E-00	1.22101E+00	–
1.E-01	1.75668E+00	1.76E+00 E+0	1.81928E+00	–	1.82191E+00	–
5.E-02	2.42718E+00	2.43E+00	2.46575E+00	–	2.46731E+00	–
2.E-02	3.33479E+00	3.34E+00	3.35369E+00	–	3.35444E+00	–
1.E-02	4.02667E+00	4.03E+00	4.03737E+00	–	4.03778E+00	–
5.E-03	4.71983E+00	4.72E+00	4.72579E+00	–	4.72602E+00	–
2.E-03	5.63655E+00	5.64E+00	5.63926E+00	–	5.63936E+00	–
1.E-03	6.33000E+00	–	6.33147E+00	–	6.33153E+00	–
5.E-04	7.02335E+00	–	7.02415E+00	–	7.02418E+00	–
2.E-04	7.93981E+00	–	7.94016E+00	–	7.94018E+00	–
1.E-04	8.63303E+00	–	8.63322E+00	–	8.63323E+00	–
5.E-05	9.32622E+00	–	9.32632E+00	–	9.32633E+00	–
2.E-05	1.02425E+01	–	1.02426E+01	–	1.02426E+01	–
$\alpha = 10^{-2}$						
2.E+00	3.43316E-03	3.52E-03+00 E-03 E-03	2.04562E-02	2.03E-02	3.46739E-02	3.44E-02
1.E+00	2.68788E-02	2.69E-02	1.42479E-01	1.42E-01	1.91452E-01	1.91E-01
5.E-01	1.20726E-01	1.21E-01	4.65508E-01	4.65E-01	5.31401E-01	5.31E-01
2.E-01	5.12307E-01	5.12E-01	1.15672E+00	1.16E+00	1.20422E+00	1.20E+00
1.E-01	1.12122E+00	1.12E+00	1.78159E+00	1.78E+00	1.81152E+00	1.81E+00
5.E-02	1.95356E+00	1.95E+00	2.44370E+00	2.44E+00	2.46126E+00	2.46E+00
2.E-02	3.10533E+00	3.11E+00	3.34338E+00	3.34E+00	3.35163E+00	3.35E+00
1.E-02	3.90362E+00	3.90E+00	4.03168E+00	4.03E+00	4.03624E+00	–
5.E-03	4.65376E+00	4.65E+00	4.72269E+00	4.72+00	4.72518E+00	–
2.E-03	5.60738E+00	5.61E+00	5.63788E+00	5.64E+00	5.63899E+00	–
1.E-03	6.31428E+00	6.31E+00	6.33073E+00	–	6.33133E+00	–
5.E-04	7.01491E+00	7.01E+00	7.02375E+00	–	7.02408E+00	–
2.E-04	7.93611E+00	7.94E+00	7.93999E+00	–	7.94013E+00	–
1.E-04	8.63106E+00	–	8.63313E+00	–	8.63320E+00	–
5.E-05	9.32517E+00	–	9.32627E+00	–	9.32631E+00	–
2.E-05	1.02421E+01	–	1.02426E+01	–	1.02426E+01	–
$\alpha = 10^{-3}$						
2.E+00	3.60225E-04	3.70E-04	3.03289E-03	3.05E-03	8.40388E-03	8.38E-03
1.E+00	2.93644E-03	2.95E-03	2.92952E-02	2.81E-02	7.55830E-02	7.56E-02
5.E-01	1.42065E-02	1.42E-02	1.47453E-01	1.54E-01	3.23142E-01	3.23E-01
2.E-01	7.24228E-02	7.24E-02	6.48328E-01	6.59E-01	1.02374E+00	1.02E+00
1.E-01	2.00562E-01	2.01E-01	*1.35418E+00*	*1.38E+00*	1.69958E+00	1.70E+00
5.E-02	4.86997E-01	4.84E-01	*2.18500E+00*	*2.27E+00*	2.39793E+00	2.40E+00
2.E-02	1.31071E+00	1.31E+00	3.23122E+00	3.22E+00	3.32290E+00	3.32E+00
1.E-02	2.37563E+00	2.38E+00	3.97215E+00	3.96E+00	4.02064E+00	4.02E+00
5.E-03	3.68139E+00	3.68E+00	4.69090E+00	4.69E+00	4.71676E+00	4.72E+00
2.E-03	5.23004E+00	5.23E+00	5.62399E+00	5.63E+00	5.63529E+00	5.64E+00
1.E-03	6.13489E+00	6.13E+00	6.32331E+00	6.32E+00	6.32935E+00	–
5.E-04	6.92456E+00	6.92E+00	7.01980E+00	7.02E+00	7.02302E+00	–
2.E-04	7.89806E+00	7.90E+00	7.93828E+00	–	7.93968E+00	–
1.E-04	8.61104E+00	8.61E+00	8.63222E+00	–	8.63296E+00	–
5.E-05	9.31461E+00	9.31E+00	9.32579E+00	–	9.32618E+00	–
2.E-05	1.02376E+01	1.024E+01	1.02424E+01	–	1.02425E+01	–
$\alpha = 10^{-4}$						
2.E+00	3.67429E-05	3.73E-05	3.17005E-04	3.16E-04	9.64666E-04	9.56E-04
1.E+00	2.96913E-04	2.98E-04	3.23439E-03	3.32E-03	1.00867E-02	1.01E-02
5.E-01	1.44604E-03	1.45E-03	1.80096E-02	1.80E-02	5.61877E-02	5.62E-02
2.E-01	7.53325E-03	7.54E-03	1.02690E-01	1.03E-01	3.03641E-01	3.04E-01

(*Continued*)

Table 2 (Continued).

u	$\rho = 10$		$\rho = 50$		$\rho = 100$	
	Proposed approach	Papadopulos (1967)	Proposed approach	Papadopulos (1967)	Proposed approach	Papadopulos (1967)
1.E-01	2.15579E-02	2.16E-02	2.97345E-01	2.97E-01	7.92140E-01	7.92E-01
5.E-02	5.55210E-02	5.55E-02	7.29650E-01	7.30E-01	1.62114E+00	1.62E+00
2.E-02	1.73660E-01	1.74E-01	1.86614E+00	1.87E+00	2.94934E+00	2.95E+00
1.E-02	3.85757E-01	3.86E-01	3.08055E+00	3.08E+00	3.83982E+00	3.84E+00
5.E-03	8.12657E-01	8.13E-01	4.25344E+00	4.25E+00	4.62609E+00	4.63E+00
2.E-03	1.96728E+00	1.97E+00	5.46585E+00	5.47E+00	5.59719E+00	5.60E+00
1.E-03	3.44006E+00	3.44E+00	6.24422E+00	6.24E+00	6.30932E+00	6.31E+00
5.E-04	5.25859E+00	5.26E+00	6.97898E+00	6.98E+00	7.01246E+00	7.01E+00
2.E-04	7.32690E+00	7.33E+00	7.92095E+00	7.92E+00	7.93514E+00	7.94E+00
1.E-04	8.36882E+00	8.37E+00	8.62311E+00	8.62E+00	8.63057E+00	–
5.E-05	9.19909E+00	9.20E+00	9.32101E+00	9.32E+00	9.32493E+00	–
2.E-05	1.01905E+01	1.019E+01	1.02403E+01	1.024E+01	1.02420E+01	–
$\alpha = 10^{-5}$						
2.E+00	3.38657E-06	4.19E-06	3.25124E-05	3.21E-05	9.75050E-05	9.77E-05
1.E+00	2.93990E-05	3.07E-05	3.27542E-04	3.27E-04	1.04123E-03	1.04E-03
5.E-01	1.44533E-04	1.47E-04	1.84090E-03	1.84E-03	6.01563E-03	6.02E-03
2.E-01	7.56008E-04	7.61E-04	1.08395E-02	1.08E-02	3.60750E-02	3.61E-02
1.E-01	2.17145E-03	2.18E-03	3.30295E-02	3.30E-02	1.10016E-01	1.10E-01
5.E-02	5.62808E-03	5.65E-03	8.90313E-02	8.90E-02	2.92080E-01	2.92E-01
2.E-02	1.79061E-02	1.80E-02	2.89222E-01	2.89E-01	8.91220E-01	8.91E-01
1.E-02	4.08131E-02	4.09E-02	6.48649E-01	6.49E-01	1.80290E+00	1.80E+00
5.E-03	9.01160E-02	9.03E-02	1.34782E+00	1.35E+00	3.14045E+00	3.14E+00
2.E-03	2.46877E-01	2.47E-01	3.03158E+00	3.03E+00	5.00579E+00	5.01E+00
1.E-03	5.14097E-01	5.15E-01	4.74848E+00	4.75E+00	6.05996E+00	6.06E+00
5.E-04	1.03830E+00	1.04E+00	6.31278E+00	6.31E+00	6.89571E+00	6.90E+00
2.E-04	2.44721E+00	2.45E+00	7.71373E+00	7.71E+00	7.88796E+00	7.89E+00
1.E-04	4.27920E+00	4.28E+00	8.52419E+00	8.52E+00	8.60620E+00	8.61E+00
5.E-05	6.62625E+00	6.63E+00	9.27112E+00	9.21E+00	9.31224E+00	9.31E+00
2.E-05	9.35324E+00	9.36E+00	1.02195E+01	1.022E+01	1.02366E+01	1.024E+01

when $\alpha = 10^{-3}$, –1.7061 at $u_w = 2 \times 10^{-6}$ when $\alpha = 10^{-4}$ and 2.2564 at $u_w = 2 \times 10^{-7}$ when $\alpha = 10^{-5}$. The largest error in Çimen (2001, equation (18)) is 4.31514×10^{-1} at $u_w = 10^{-9}$ when $\alpha = 10^{-1}$, -7.436×10^{-1} at $u_w = 5 \times 10^{-4}$ when $\alpha = 10^{-2}$, –1.1731 at $u_w = 5 \times 10^{-5}$ when $\alpha = 10^{-3}$, –1.53536 at $u_w = 9 \times 10^{-5}$ when $\alpha = 10^{-4}$ and 2.06512 at $u_w = 2 \times 10^{-7}$ when $\alpha = 10^{-5}$. The largest error in Swamee and Ojha (1995) increases with decreasing u_w with a value of 6.57319×10^{-1} at $u_w = 2 \times 10^{-2}$ when $\alpha = 10^{-1}$, 1.23915 at $u_w = 2 \times 10^{-3}$ when $\alpha = 10^{-2}$, 2.00339 at $u_w = 10^{-4}$ when $\alpha = 10^{-3}$, 2.92026 at $u_w = 10^{-5}$ when $\alpha = 10^{-4}$ and 3.92039 at $u_w = 10^{-6}$ when $\alpha = 10^{-5}$. Except in the case of $\alpha = 10^{-1}$, the largest error occurs at the peak of the curve predicted from Çimen (2001, equation (18)). The errors for the solutions of Swamee and Ojha (1995) and Çimen (2001, equation (13)) appear in moderate times as shown in Fig. 3(a)–(e). In other words, the errors of these studies vanish when u_w is small (for late times) and large (for early times). The largest error in Singh (2007b) is 1.26269 at $u_w = 10^{-9}$ when $\alpha = 10^{-1}$, –1.33745 at $u_w = 2 \times 10^{-8}$ when $\alpha = 10^{-2}$, –1.62439 at $u_w = 2 \times 10^{-9}$ when $\alpha = 10^{-3}$, –0.42788 at $u_w = 2 \times 10^{-6}$ when $\alpha = 10^{-4}$ and 4.38082 at $u_w = 10^{-9}$ when $\alpha = 10^{-5}$. The patterns of the error distribution curves for Çimen (2001, equation (13)) and Çimen (2001, equation (18)) are slightly similar. However, the error calculated from Çimen (2001, equation (18)) is larger than that computed from Çimen (2001, equation (13)) when u_w is small. Additionally, Çimen's solution (2001, equation (13)) approaches the Theis equation for late times, as stated in Çimen (2001).

The differences in the values of well function estimated by the proposed procedure and those presented in the PC solution, Swamee and Ojha (1995), Çimen (2001, equation (18)), Çimen (2001, equation (13)) and Singh (2007b) can be changed as the differences in drawdown by multiplying $Q/(4\pi T)$. Assuming $Q = 432$ m^3/d and $T = 86.4$ m^2/d (Batu 1998, p. 199, example 4–13), the largest difference of drawdown is 3.99 cm at $u_w = 2 \times 10^{-8}$ when $\alpha = 10^{-1}$ in Papadopulos and Cooper (1967), 155.98 cm at $u_w = 10^{-6}$ when $\alpha = 10^{-5}$ in Swamee and Ojha (1995), 82.16 cm at $u_w = 2 \times 10^{-7}$ when $\alpha = 10^{-5}$ in Çimen (2001, equation (18)), 89.77 cm at

$u_w = 2 \times 10^{-7}$ when $\alpha = 10^{-5}$ in Çimen (2001, equation (18)) and 174.3 cm at $u_w = 10^{-9}$ when $\alpha = 10^{-5}$ in Singh (2007b). These drawdown differences are large when one is considering measurement accuracy of the order of millimetres. Moreover, these results reflect that the well function values given by Papadopulos and Cooper (1967) with four significant digits do not have enough accuracy in developing the approximate formulas as did those given in Swamee and Ojha (1995) and Singh (2007b).

Table 2 lists the dimensionless drawdown *versus u* in the range from 10^{-5} to 10 with α ranging from 10^{-5} to 10^{-1} for the dimensionless distance $\rho =$ 10, 50 and 100. Reed (1980) used Simpson's rule to compute the integral of Papadopulos and Cooper's solution, which may result in low accuracy, and a classical method of accelerating convergence such as the Euler transform is effective only in the case of slowly and monotonically decreasing series. Such numerical evaluation for the integral has the problem of slow convergence whenever ρ and/or u are large and therefore results in slightly poor accuracy. Papadopulos and Cooper gave at most three significant digits for the dimensionless drawdown (Reed 1980), which may contribute to the problem of lack of high accuracy in numerical calculations.

CONCLUSION

This study proposes a numerical approach for calculating the drawdown in a large-diameter well, which includes a root search scheme for finding the roots of the integrand, Gaussian quadrature for numerical integration, and Shanks transform for accelerating convergence of infinite series. This study demonstrates that some numerical approaches or approximate solutions yield inaccurate results in the evaluation of the well function, especially for small radial distance or large pumping time. The dimensionless drawdown for selected values of dimensionless distance and dimensionless time are expressed in tabulated forms, which may be practically useful if there is a need for high accuracy for the observation either at the well or some distance from the pumping well.

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