ELSEVIER

Information Processing Letters 52 (1994) 317-322

Information Processing Letters

The path-partition problem in block graphs [1]

Jing-Ho Yan, Gerard J. Chang *

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan

Communicated by D. Gries; received 3 December 1993; revised 12 August 1994

Abstract

We present a linear-time algorithm for the path-partition problem in block graphs.

Keywords: Algorithms; Graph theory; Path-partition

1. Introduction

A *path partition* of a graph G is a collection of vertex-disjoint paths that cover all vertices of the graph. The *path-partition problem* is to find the *path-partition number* $p(G)$: the minimum size of a path partition of G. Note that G has a Hamiltonian path iff $p(G) = 1$. Since the Hamiltonian path problem is NP-complete for planar graphs, bipartite graphs, and chordal graphs (see [6]), so is the path-partition problem. Bonucelli and Bovet [4] and Arikati and Pandu Rangan [2] gave linear-time algorithms for the path-partition problem in interval graphs, Skupien [8] gave a polynomial-time algorithm for forests, and Chang and Kuo [3] gave a linear-time algorithm for cographs, and Srikant et al. [9] gave linear-time algorithms for bipartite permutation graphs and block graphs.

Srikant et al. [9] did not prove the correctness of their algorithm for block graphs. In fact, as discussed in Section 4, it does not work for all block graphs. The purpose of this paper is to give a linear-time algorithm for the path-partition problem for all block graphs and prove its correctness. For technical reasons, we also consider the following variant problem, which is the path-partition problem with a side condition. For a fixed vertex v in G, a *v-path-partition* of G is a path partition in which v is an end-vertex of a path in the partition. The *v-path-partition problem* is to find the *v-path-partition number* $p_v(G)$: the minimum size of a v-path-partition of G.

We now review block graphs. A vertex x is a *cut-vertex* if deleting x and all edges incident to it increases the number of connected components. A *block* is a maximal connected subgraph without a cut-vertex. The intersection of two distinct blocks contains at most one vertex, and a vertex is a cut-vertex iff it is the intersection of two or more blocks. Consequently, a graph with one or more cut-vertices has at least two blocks. A *block graph* is a graph whose blocks are complete graphs.

* Corresponding author. Email: gjchang@math.nctu.edu.tw.

[1] Supported in part by the National Science Council under grant NSC83-0208-M009-050.

SSDI 0020-0190(94)00158-8

Suppose $G_1, G_2, \ldots, G_t$ are disjoint graphs and $v_i \in V(G_i)$ for $1 \leqslant i \leqslant t$. The *composition* of $G_1, G_2, \ldots, G_t$ is the graph $G = (V, E)$ obtained from the union of these graphs by making $\{v_1, v_2, \ldots, v_t\}$ a clique, i.e.

$$V = \bigcup_{1 \leqslant i \leqslant t} V(G_i) \quad \text{and} \quad E = \Big(\bigcup_{1 \leqslant i \leqslant t} E(G_i) \Big) \cup \{v_i v_j \mid 1 \leqslant i < j \leqslant t\}.$$

On the other hand, suppose B is a block of t vertices in a connected block graph G. Deleting all edges in B results in t connected block graphs whose composition is G. Repeatedly applying this inverse operation of composition to a connected block graph give trivial graphs. In other words, a connected block graph can be obtained from trivial graphs by a series of compositions.

2. Path partition in block graphs

We establish some basic theorems to be used later. Suppose P is a path partition of graph G. For any induced subgraph H of G, P_H denotes the path partition of H resulting from P when each vertex in $G - H$ is deleted from the path containing it in P.

Lemma 1. *The following relations hold.*
(a) $p(G - v) + 1 \geqslant p_v(G)$.
(b) $p_v(G) \geqslant p(G)$.
(c) $p(G - v) + 1 \geqslant p_v(G) \geqslant p(G - v)$.

Proof. $p(G - v) + 1 \geqslant p_v(G)$, since a path partition of $G - v$ together with the path v forms a v-path-partition of G. $p_v(G) \geqslant p(G)$, since a v-path-partition is a path partition. $p_v(G) \geqslant p(G - v)$, since the deletion of v from a path of a v-path-partition results in a path partition of $G - v$. □

Lemma 2. $p(G) > p(G - v)$ *iff* $p(G - v) + 1 = p_v(G) = p(G)$.

Proof. The lemma follows immediately from (a) and (b) of Lemma 1. □

Suppose $G_1, G_2, \ldots, G_t$ are disjoint graphs and $v_i \in V(G_i)$ for $1 \leqslant i \leqslant t$. Define

$$I = \{i \mid 1 \leqslant i \leqslant t \text{ and } p(G_i - v_i) + 1 = p_{v_i}(G_i) = p(G_i)\},$$

$$J = \{i \mid 1 \leqslant i \leqslant t \text{ and } p(G_i - v_i) = p_{v_i}(G_i) = p(G_i)\}.$$

By Lemma 1(c),

$$I \cup J = \{i \mid 1 \leqslant i \leqslant t \text{ and } p_{v_i}(G_i) = p(G_i)\}.$$

Theorem 3. *If G is the composition of $G_1, G_2, \ldots, G_t$, then*

$$p(G) = \begin{cases} \sum_{i=1}^{t} p(G_i) - |I| + 1 & \text{if } I \neq \emptyset \text{ and } J = \emptyset, \\ \sum_{i=1}^{t} p(G_i) - |I| - \lfloor |J|/2 \rfloor & \text{otherwise.} \end{cases}$$

Proof. For each $1 \leqslant i \leqslant t$, let P_i be an optimal path partition of G_i. For $i \in I$ (respectively, $i \in J$), we may assume that v_i is a path in P_i (respectively, there is a path q_i with an end-vertex v_i in P_i). For the case of $I \neq \emptyset$ and $J = \emptyset$, $P = (\bigcup_{1 \leqslant i \leqslant t} P_i) - (\bigcup_{i \in I}\{v_i\}) \cup \{q\}$ is a path partition of G, where q is the path formed by all vertices v_i with $i \in I$. Hence $p(G) \leqslant \sum_{i=1}^{t} p(G_i) - |I| + 1$. For the other cases, $P = (\bigcup_{1 \leqslant i \leqslant t} P_i) - (\bigcup_{i \in I}\{v_i\}) - (\bigcup_{i \in J}\{q_i\}) \cup \{r_1, r_2, \ldots, r_{\lceil |J|/2 \rceil}\}$ is a path partition of G, where each r_i except r_1 is the catenation of two q_i's and r_1 is formed by the remaining one or two q_i's together with all v_i's with $i \in I$. Hence

$$p(G) \leqslant \sum_{i=1}^{t} p(G_i) - |I| - |J| + \lceil |J|/2 \rceil = \sum_{i=1}^{t} p(G_i) - |I| - \lfloor |J|/2 \rfloor.$$

On the other hand, suppose P is an optimal path partition of G. A path in P is called *mixed* if it contains vertices in at least two different G_i's. We may assume that P is chosen to contain the fewest vertices in all mixed paths. Note that any mixed path q is of the form $q'q''q'''$, where q' or q''' is $\emptyset$ or a nontrivial path in some G_i with v_i as an end vertex and q'' is a sequence of some v_i's. It follows that the deletion of any vertex x in q'' is still a path, which we denote by $q - x$. Let

$$I' = \{i \mid v_i \text{ is the only vertex of } G_i \text{ that is in some mixed path } q\},$$

$$J' = \{i \mid G_i \text{ contains a nonempty } q_i' \text{ or } q_i''' \text{ of a mixed path } q_i\}.$$

Note that $I' \cap J' = \emptyset$. We claim that $I' \subseteq I$ and $J' \subseteq I \cup J$.

Suppose $i \in I' - I$. Let v_i be the only vertex of G_i that is in a mixed path q. Assume P_i is an optimal path partition of G_i. Consider the path partition $P' = P - P_{G_i - v_i} - \{q\} \cup P_i \cup \{q - v_i\}$. Then

$$|P'| = |P| - |P_{G_i - v_i}| + |P_i| \leqslant |P| - p(G_i - v_i) + p(G_i) \leqslant |P|$$

since $i \notin I$ implies $p(G_i) \leqslant p(G_i - v_i)$ by Lemma 2. Consequently, P' is another optimal path partition of G with fewer vertices in all mixed paths than P, a contradiction. So $I' \subseteq I$.

Suppose $i \in J' - (I \cup J)$. Without loss of generality, we may assume that G_i contains a nonempty q_i' for some mixed path q_i. Let P_i be an optimal path partition of G_i. Consider the path partition $P' = P - P_{G_i - V(q_i')} - \{q_i\} \cup P_i \cup \{q_i''q_i'''\}$. Then

$$|P'| = |P| - |P_{G_i - V(q_i')}| + |P_i| \leqslant |P| - (p_{v_i}(G_i) - 1) + p(G_i) \leqslant |P|$$

since $i \notin I \cup J$ implies $p_{v_i}(G_i) > p(G_i)$ by (b) of Lemma 1. Consequently, P' is another optimal path partition of G with fewer vertices than P in all mixed paths, a contradiction. So $J' \subseteq I \cup J$.

Now, assume P has m mixed paths. Then

$$\begin{aligned} p(G) &= |P| \\ &= \sum_{i \notin I' \cup J'} |P_{G_i}| + \sum_{i \in I'} |P_{G_i - v_i}| + \sum_{i \in J'} |P_{G_i - V(q_i')}| + m \\ &\geqslant \sum_{i \notin I' \cup J'} p(G_i) + \sum_{i \in I'} p(G_i - v_i) + \sum_{i \in J'} (p_v(G_i) - 1) + m \\ &= \sum_{i=1}^{t} p(G_i) - |I'| - |J'| + m. \end{aligned}$$

Note that the last equality follows from $p(G_i - v_i) + 1 = p(G_i)$ for $i \in I' \subseteq I$ and $p_v(G_i) = p(G_i)$ for $i \in J' \subseteq I \cup J$. For the case of $I \neq \emptyset$ and $J = \emptyset$, $I' \cup J' \subseteq I$. If $m = 0$, then $I' = J' = \emptyset$, so $-|I'| - |J'| + m = 0 \geqslant -|I| + 1$. If $m \geqslant 1$, then $-|I'| - |J'| + m \geqslant -|I| + 1$. Hence $p(G) \geqslant \sum_{i=1}^{n} p(G_i) - |I| + 1$. For the other cases, since each mixed path contains vertices in at most two G_i's with $i \in J'$, $-|I'| - |J'| + m \geqslant -|I'| - \lfloor |J'|/2 \rfloor \geqslant -|I'| - |J' \cap I| - \lfloor |J' \cap J|/2 \rfloor \geqslant -|I| - \lfloor |J|/2 \rfloor$. Hence $p(G) \geqslant \sum_{i=1}^{t} p(G_i) - |I| - \lfloor |J|/2 \rfloor$. □

An argument similar to that used above gives the solution to $p_v(G)$ for $v = v_t$. For this purpose, use

$$I^* = I \cap \{i \mid i \neq t\} \quad \text{and} \quad J^* = J \cap \{i \mid i \neq t\}.$$

Theorem 4. *If G is the composition of $G_1, G_2, \ldots, G_t$ and $v = v_t$ and G' is the composition of $G_1, G_2, \ldots, G_{t-1}$, then*

$$p_v(G) = \begin{cases} p(G') + p_{v_t}(G_t) & \text{if } p_{v_t}(G_t) = p(G_t - v_t), \\ \sum_{i=1}^{t-1} p(G_i) + p_{v_t}(G_t) - |I^*| - \lceil |J^*|/2 \rceil & \text{if } p_{v_t}(G_t) > p(G_t - v_t). \end{cases}$$

Proof. Suppose P is an optimal v-path-partition of G in which q is a path ending at v.

For the case $p_{v_t}(G_t) = p(G_t - v_t)$, we may assume that q is in G_t and hence $p_v(G) = p(G') + p_{v_t}(G_t)$. Otherwise, suppose q is not in G_t, i.e. q intersects G_{v_t} only at v_t. Let P_t be an optimal v_t-path-partition of G_t. Then $P' = P - \{q\} - P_{G_t - v_t} \cup \{q - v_t\} \cup P_t$ is another v_t-path-partition with

$$|P'| = |P| - |P_{G_t - v_t}| + |P_t| \leqslant |P| - p(G_t - v_t) + p_{v_t}(G_t) = |P|.$$

So P' is another optimal v-path-partition in which the only path ending at v_t is in G_t.

For the case $p_{v_t}(G_t) > p(G_t - v_t)$, we may assume that q intersects G_{v_t} only at v_t. Otherwise, P_{G_t} is a v_t-path-partition of G_t. Then we can replace P_{G_t} by v_t together with an optimal partition of $G_t - v_t$ to obtain an optimal v_t-path-partition in which q intersects G_{v_t} only at v_t. So $p_v(G) = p_v(G^*) + p(G_t - v_t)$, where $G^* = G - (G_t - v_t)$. Let G^{**} be the graph obtained from G^* by adding a new vertex v' adjacent to v only. Then $p_v(G^*) = p_{v'}(G^{**}) = p(G^{**})$, since a v-path-partition of G^* corresponds to a v'-path-partition of G^{**} and a path partition of G^{**} is a v'-path-partition of G^{**}. However, G^{**} is the composition of $G_1, G_2, \ldots, G_{t-1}, K_2$. So we may apply Theorem 3 to G^{**} by considering I^* to be I and $J^* \cup \{t\}$ to be J. Note that $J^* \cup \{t\} \neq \emptyset$ gives

$$p(G^{**}) = 1 + \sum_{i=1}^{t-1} p(G_i) - |I^*| - \lfloor (|J^*| + 1)/2 \rfloor = \sum_{i=1}^{t-1} p(G_i) - |I^*| - \lceil |J^*|/2 \rceil + 1.$$

Also, by Lemma 1, $p(G_t - v_t) + 1 = p_v(G_t)$. Hence the theorem holds. □

Theorem 5. *If G is the composition of $G_1, G_2, \ldots, G_t$ and $v = v_t$ and G' is the composition of $G_1, G_2, \ldots, G_{t-1}$, then $p(G - v) = p(G') + p(G_t - v_t)$.*

Proof. The theorem follows from the fact that $G - v$ is the disjoint union of G' and $G_t - v_t$. □

3. Algorithm

We are ready to give a linear-time algorithm for the path-partition problem in block graphs. We may consider only connected block graphs G. Our algorithm iteratively processes a block $B = \{v_1, v_2, \ldots, v_t\}$ with exactly one cut-vertex v_t of the current graph G'. Let $G_i - v_i$ be the graph that has been deleted from the original graph G so far and let G_i connect to G only at v_i. Then the composition of $G_1, G_2, \ldots, G_t$ connects to $G' - \{v_1, v_2 \ldots, v_{t-1}\}$ only at v_t. Theorems 3–5 can be used to compute p and p_v for this composition. After computing p and p_v,

Algorithm PPN

Find the path-partition number $p(G)$ of a connected block graph G:
for all $v_i \in V(G)$: $p_1(v_i) := 1$; /* for $p(G_i)$ */
$p_2(v_i) := 1$; /* for $p_{v_i}(G_i)$ */
$p_3(v_i) := 0$; /* for $p(G_i - v_i)$ */
while G contains more than one vertex **do**
choose a block $B = \{v_1, v_2, \ldots, v_t\}$ with exactly
one cut-vertex v_t or with no cut-vertex;
$I := \{i \mid 1 \leqslant i \leqslant t \text{ and } p_3(v_i) + 1 = p_2(v_i) = p_1(v_i)\}$;
$J := \{i \mid 1 \leqslant i \leqslant t \text{ and } p_3(v_i) = p_2(v_i) = p_1(v_i)\}$;
$I^* := I \cap \{i \mid 1 \leqslant i \leqslant t-1\}$;
$J^* = J \cap \{i \mid 1 \leqslant i \leqslant t-1\}$;
if $I \neq \emptyset$ **and** $J = \emptyset$
then $p_1(v_t) := \sum_{i=1}^{t} p_1(v_i) - |I| + 1$
else $p_1(v_t) := \sum_{i=1}^{t} p_1(v_i) - |I| - \lfloor |J|/2 \rfloor$;
if $I^* \neq \emptyset$ **and** $J^* = \emptyset$
then $p_1' := \sum_{i=1}^{t-1} p_1(v_i) - |I^*| + 1$
else $p_1' := \sum_{i=1}^{t-1} p_1(v_i) - |I^*| - \lfloor |J|/2 \rfloor$;
if $p_2(v_t) = p_3(v_t)$
then $p_2(v_t) := p_1' + p_2(v_t)$
else $p_2(v_t) := \sum_{i=1}^{t-1} p_1(v_i) + p_2(v_t) - |I^*| - \lceil |J^*|/2 \rceil$;
$p_3(v_t) := p_1' + p_3(v_t)$;
$G := G - \{v_1, v_2, \ldots, v_{t-1}\}$
end while;
let v be the only vertex in G;
output $p_1(v)$.

Fig. 1. Algorithm PPN.

delete $v_1, v_2, \ldots, v_{t-1}$ from G' and continue the same process until G' has only one vertex. The algorithm is shown in Fig. 1.

Theorem 6. *Algorithm PPN computes the path-partition number of a connected block graph in linear time.*

Proof. The correctness of the algorithm follows from Theorems 3-5. The algorithm takes only linear time since depth-first search can be used to find end blocks and each iteration of the while loop requires only $O(|B|)$ operations. □

4. Discussion

Srikant et al.'s [9] algorithm for the path-partition problem in block graphs is similar to ours except that it solves only the case when $I = \emptyset$ in all iterations (see the definition of I after Lemma 2). If $I \neq \emptyset$ at some iterations, their algorithm does not provide a correct answer. For instance, consider the block graph G in Fig. 2. The correct value of $p(G)$ is 5 with a path partition $\{P_1, P_2, P_3, P_4, P_5 \cup P_6\}$, while their algorithm gives 6 with a path partition $\{P_1, P_2, P_3, P_4, P_5, P_6\}$, where

$$P_1 = (1,2,3), \quad P_2 = (6,7,8), \quad P_3 = (9,10,11),$$
$$P_4 = (14,15,16), \quad P_5 = (4,5), \quad P_6 = (12,13).$$

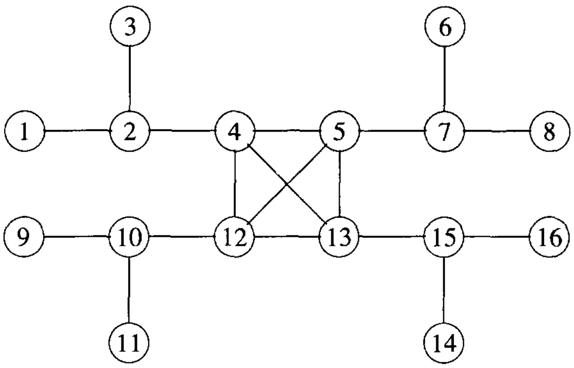

Fig. 2. A block graph G.

We explain why the two algorithms give different solutions. At some iteration, the subgraphs rooted at $4,5,12,13$, respectively, have optimal solutions $\{P_1,4\}$, $\{P_2,5\}$, $\{P_3,12\}$, $\{P_4,13\}$. Algorithm PPN merges $4,5,12,13$ to make a new path, since they are in a block. But their algorithm merges 4 and 5 to get P_5 and merges 12 and 13 to get P_6. In the case of there are many paths of single vertices in a block, these two methods give answers with a big gap.

Acknowledgments

The authors thank the referee and Professor David Gries for many constructive suggestions for the revision of this paper.

References

[1] A. Aho, J. Hopcroft and J.D. Ullman, *The Design and Analysis of Computer Algorithms* (Addison-Wesley, Reading, MA, 1974).

[2] S.R. Arikati and C. Pandu Rangan, Linear algorithm for optimal path cover problem on interval graphs, *Inform. Process. Lett.* **35** (1990) 149-153.

[3] G.J. Chang and D. Kuo, The $L(2,1)$-labeling problem on graphs, *SIAM Discrete Math.*, to appear.

[4] M.A. Bonuccelli and D.P. Bovet, Minimum node disjoint path covering for circular-arc graphs, *Inform. Process. Lett.* **8** (1979) 159-161.

[5] J.S. Deogun and G. Steiner, Hamiltonian cycle is polynomial on cocomparability graphs, *Discrete Appl. Math.* **39** (1992) 165-172.

[6] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs* (Academic Press, New York, 1980).

[7] W.L. Hsu, An $O(n^2 \log n)$ algorithm for the Hamiltonian cycle problem on circular-arc graphs, *SIAM J. Comput.* **21** (1992) 1026-1046.

[8] Z. Skupien, Path partitions of vertices and hamiltonity of graphs, in: *Proc. 2nd Czechoslovakian Symp. on Graph Theory*, Prague, 1974.

[9] R. Srikant, Ravi Sundaram, Karan Sher Singh and C. Pandu Rangan, Optimal path cover problem on block graphs and bipartite permutation graphs, *Theoret. Comput. Sci.* **115** (1993) 351-357.