Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

A flexible tree for evaluating guaranteed minimum withdrawal benefits under deferred life annuity contracts with various provisions

Sharon S. Yang [a,b], Tian-Shyr Dai [c,*]

[a] Department of Finance, National Central University, Taiwan
[b] Risk and Insurance Research Center, College of Commerce, National Chengchi University, Taiwan
[c] Department Information and Financial Management and Institute of Finance, National Chiao-Tung University, Taiwan

ARTICLE INFO

ABSTRACT

Valuing guaranteed minimum withdrawal benefit (GMWB) has attracted significant attention from both the academic field and real world financial markets. However, some popular provisions of GMWB contracts, like the deferred life annuity structure, rollup interest rate guarantees, and surrender options are hard to be evaluated analytically and are rarely addressed in the academic literature. This paper proposes a flexible tree model that can accurately evaluate the values and the fair insurance fees of GMWBs. The flexibility of our tree allows us to faithfully implement the aforementioned provisions without introducing significant numerical pricing errors. The mortality risk can also be easily incorporated into our pricing model. Our numerical results verify the robustness of our tree and demonstrate how the aforementioned provisions and the mortality risk significantly influence the values and the fair insurance fees of GMWBs.

1. Introduction

The variable annuity (VA)[1] is a popular insurance product sold in the U.S. retirement market. When people purchase a VA product, they either pay a lump sum or make periodic payments into a fund that is invested in an investment portfolio, such as a mutual fund. The account value of the fund accumulates in accordance with the performance of the investment portfolio. Policyholders can choose the investment portfolio and thus bear the investment risk. In recent years, granting the investment guarantee has become a popular design with VA products. With this design, the insurer guarantees a specified return on the policy's account value through various types of investment guarantees, such as guaranteed minimum death benefits (GMDBs), guaranteed minimum maturity benefits (GMMBs), guaranteed minimum income benefits (GMIBs), and guaranteed minimum withdrawal benefits (GMWBs). To refer to this broad class of guarantees, we employ the term GMXBs and note that, regardless of the type, the guarantee features of GMXBs provide downside risk protection to policyholders. These VA products have enjoyed great market success in the United States and Asia. Condron (2008) suggests that the guarantee

features account for the growing popularity of VA, as manifested in more than $1.35 trillion currently invested, a 50% increase over the previous five years. The products are also gaining popularity in international markets (Ledlie et al., 2008).

Granting GMXBs means that VA products contain embedded financial options. The various guarantees can be viewed as various types of exotic options, and the pricing of these exotic options has become a critical research focus. Brennan and Schwartz (1976, 1979) first priced unit-linked contracts with an asset guarantee (i.e., GMMB). The payoff of a GMMB is similar to that of an ordinary European option, so they derived closed-form pricing formulas by taking advantage of classic Black–Scholes assumptions. Milevsky and Posner (2001) regarded GMDB benefits as a Titanic option and presented closed-form solutions with a simplified exponential mortality model. Analytic solutions for valuing GMIBs appear in Boyle and Hardy (2003) and Ballotta and Haberman (2003, 2006). Among these investment guarantees, the GMWB guarantee has, however, attracted particularly significant attention and sales in recent years. A GMWB contract allows the policyholder to withdraw funds periodically for a contractually specified amount for a specified guaranteed withdrawal period, regardless of the performance of the underlying investment portfolio. When the contract expires, the holder can either redeem the remaining investment or convert it into a life annuity. Recent research thus addresses the pricing of GMWB contracts, starting with Milevsky and Salisbury (2006), who first introduce the concept of a Quanto Asian put for valuing GMWBs. Chen et al. (2008) then consider the

* Corresponding author.
E-mail addresses: syang@ncu.edu.tw (S.S. Yang), d88006@csie.ntu.edu.tw, cameldai@mail.nctu.edu.tw (T.-S. Dai).

[1] Also known as unit-linked products in the United Kingdom.

jump effect and employ a jump diffusion process to value GMWBs. Finally, Dai et al. (2008) instead provide a rigorous derivation of the singular stochastic control model for pricing variable annuities with GMWBs using the Hamilton–Jacobi–Bellman (HJB) equation. Bauer et al. (2008) also consider a universal pricing framework in which they can price various GMXBs consistently using simulation techniques.

Because the insurance policy entitles policyholders to terminate their contracts before the maturity date and receive a certain cash refund (called the surrender value), taking the surrender feature into account has become a mainstream tactic for valuing the equity-linked policies. Shen and Xu (2005) study fair valuations of equity-linked policies with interest rate guarantees in the presence of surrender options. Costabile et al. (2008) consider fair periodical premiums for equity-linked policies with a surrender option under a binomial model, and then tackle the problem of computing fair periodical premiums for an equity-linked policy with a maturity guarantee and an embedded surrender option. Regarding the recently developed GMWB contract, since its payoff is more complex than that of other guarantee types, it turns out that valuing a GMWB contract is much more difficult, especially if surrender is allowed. Milevsky and Salisbury (2006) assume that an optimal withdrawal policy seeks to maximize the annuity value by lapsing the product at an optimal time. Our paper extends their work by analyzing how the policyholders optimize their surrender decisions in an effort to strike a balance among losses of the time value due to delayed withdrawal, losses due to mortality risk, and early redemption penalties. Much of the literature has been concerned with the optimal withdrawal behavior as opposed to the surrender options. Our tree can be extended to model the optimal withdrawal without difficulty.

Most studies oversimplify the various provisions of the GMWB contract in order to make their pricing models tractable. However, these oversimplifications might result in significant pricing deviations as illustrated in the numerical experiments in Section 4. For example, most GMWBs are associated with deferred variable annuities, and guaranteed withdrawals normally take place after deferred periods. Different guaranteed withdrawal amounts might be designed for a deferred life annuity, such as a rollup interest rate guarantee. To the best of our knowledge, the existing literature assumes that the guaranteed withdrawal starts immediately, at the inception of the policy, even though this discrepancy could result in significantly different pricing results. Therefore, we investigate the effect of deferred periods and various guarantee designs on the fair charge numerically. Besides, the mortality improvements in recent years can affect the value of GMWBs. We also consider mortality improvements when valuing the GMWB contracts by incorporating mortality improvement factors into our tree model. In addition, we also analyze whether the presence of mortality risk, rollup interest rate guarantees, the volatility of the underlying investment, and the redemption penalty influence the value of surrender options.

Evaluating the fair charge for granting the GMWB is the key goal for the GMWB evaluation problem. This insurance fee is subtracted from the account value in return for the investment guarantee and provisions provided by the insurance company. Note that the value to hold a GMWB contract, abbreviated as the "value of the GMWB" for simplicity, decreases with the increment of the insurance fee. The fair charge is the fee that makes the value of the GMWB equal to the policyholder's initial investment. Complex provisions of GMWB contracts and the mortality risk prevent the fair charge from being analytically solved. On the other hand, evaluating the values of complex GMWB contracts with numerical methods could generate oscillating pricing results, which would result in no or multiple solutions for the fair charge.

The major contribution of this paper is that it develops an accurate numerical tree method to calculate the value of the GMWB and the fair charge. The flexible nature of the tree can help us to incorporate the mortality models into the tree, to combine the evolution of different account value processes during the deferred and the withdrawal periods, and to deal with optimal surrender and withdrawal decisions. In addition, to faithfully model various provisions of GMWB contracts without incurring significant numerical pricing errors, our tree also borrows the trinomial structures proposed by the stair tree (Dai, 2009) and the bino-trinomial tree (BTT; Dai and Lyuu, 2010). The stair tree uses trinomial structures to faithfully model the downward jump of stock prices due to discrete dividend payouts; this idea can be used to model the downward jump in the account value of GMWB contracts due to discrete withdrawal and the fair charge. It can also help us to adjust the tree structure in order to price GMWB more stably. To alleviate the price oscillation problem due to the nonlinearity errors (Figlewski and Gao, 1999), the BTT uses the trinomial structure to adjust the tree structure to coincide with "critical locations"—the locations where the function of the financial derivative value is highly nonlinear. This paper also uses the trinomial structure to make the tree coincide with certain critical locations caused by the periodical withdrawal guarantees listed in the GMWB contracts. Thus our tree can stably price the value of GMWBs without numerical errors and can thus find the fair charge stably.

The structure of this paper is as follows. In Section 2, we describe the GMWB contract and some important provisions, for instance, the deferred life annuity structure and the rollup interest rate guarantee design. The process of the GMWB account value and the payoff of the policyholder are then modeled according to the provisions. The required knowledge of the tree model is also reviewed in the same section. In Section 3, we construct a new tree model and implement the backward induction method to deal with the valuation of complicated provisions (e.g., the deferred annuity structure, rollup interest rate guarantees, and surrender options) and mortality risk in GMWB contracts. The structure of our tree is sophisticatedly designed to suppress numerical pricing errors in order to generate the stable value of GMWBs and the fair charge. The numerical results in Section 4 analyze how the presence of different provisions influences the fair charge and GMWB values. Section 5 concludes the paper.

2. The structure of the GMWB contract and tree models

2.1. Account dynamics of the GMWB contract with a deferred variable annuity

We assume a single-premium deferred variable annuity associated with the GMWB. The policyholder deposits an initial premium ω_0 in an account that is invested in a selected fund portfolio and is guaranteed the right to withdraw a specified amount from that account at each withdrawal date during the guaranteed withdrawal period. Let W_t denote the account value at time t for the GMWB contract, and initial account value $W_0 = \omega_0$. The account value changes according to the return on the invested fund portfolio and diminishes by the periodical withdrawals and payments of the insurance fee. Let the time intervals $[0, T_1]$ and $[T_1, T_2]$ denote the deferred period and withdrawal period, respectively, whereas T_2 is the maturity date. The policyholders are only allowed to make guaranteed withdrawals periodically during the withdrawal period. During the deferred period, the account value changes only in relation to the return on the underlying asset and the insurance fee. Thus the stochastic differential equation (SDE) of W_t during the deferred period is

$$dW_t = (r - \alpha) W_t dt + \sigma W_t dB_t, \quad 0 < t \le T_1, \quad (1)$$

where r denotes the risk-free interest rate, α is the continuous charge proportional to the account value, σ denotes the volatility of the account value, and B_t denotes a Brownian motion. Note that our tree can also model a discrete charge just as we model the discrete guaranteed withdrawal (introduced later). Here we follow the continuous charge setting proposed by Milevsky and Salisbury (2006) so we can compare our pricing results with those of Milevsky and Salisbury in our numerical experiments.

After the deferred period, the policyholder is allowed to withdraw periodically up to time T_2. To reflect the discrete withdrawal of the account value, the SDE of W_t during the withdrawal period is defined as follows:

$$dW_t = (r - \alpha) W_t dt + \sigma W_t dB_t, \quad T_1 \leq t \leq T_2 \tag{2}$$

$$W_\tau^+ = W_\tau^- - G, \quad \text{where } \tau \text{ denotes a withdrawal date.} \tag{3}$$

Eq. (2) claims that the account value still follows a log-normal diffusion process between two adjacent withdrawal dates. Eq. (3) suggests that the account value falls by the guaranteed withdrawal amount G due to the discrete withdrawal at a withdrawal date τ. In this paper, we use the superscripts "+" and "−" to distinguish the account value immediately after or before a certain time. For example, W_τ^- and W_τ^+ denote the account value immediately before and after the withdrawal date τ, respectively.

According to the SDE in Eqs. (2) and (3), we can express the account dynamics for the GMWB contracts during the deferred period and during the withdrawal period as

$$W_t = W_0 e^{(r - \alpha - 0.5\sigma^2)t + \sigma B_t}, \quad 0 \leq t \leq T_1$$
$$W_t = W_\tau e^{(r - \alpha - 0.5\sigma^2)(t - \tau) + \sigma(B_t - B_\tau)}, \quad T_1 \leq \tau \leq t \leq T_2, \tag{4}$$
given no withdrawal dates belong to $[\tau, t]$.

These expressions of the account dynamics in Eq. (4) represent a general setting for VA products embedded with GMWBs, which can also apply to the GMWB contract with an immediate life annuity if we set $T_1 = 0$.

2.2. Settings of the guaranteed withdrawal amount

In a typical guarantee for an immediate variable annuity contract, the total guaranteed withdrawal amount is commonly set to equal the initial investment, and the fixed guaranteed withdrawal amount is taken as a fixed percentage of the initial investment, that is, $G = \frac{\omega_0}{mT_2}$ (e.g., Milevsky and Salisbury, 2006; Dai et al., 2008; Chen et al., 2008), where m denotes the number of withdrawals per year. For a deferred variable annuity contract, the total guaranteed withdrawal amount ($W_{T_1}^+$), which is also the account value at the beginning of the withdrawal period, is determined as the maximum of a contract-specified value $C(T_1)$ and the account value at the end of the deferred period $W_{T_1}^-$. Here $C(T_1)$ can be interpreted as the lower bound of the total guaranteed withdrawal. Thus, the guaranteed withdrawal amount at each withdrawal date G can be calculated as

$$G = \frac{\text{Max}[C(T_1), W_{T_1}^-]}{m(T_2 - T_1)}. \tag{5}$$

One popular provision to determine the lower bound of the total guaranteed withdrawal $C(T_1)$ is a rollup interest rate guarantee. $C(T_1)$ is defined as the return on the initial investment with a rollup interest rate guaranteed interest rate i, as follows:

$$C(T_1) = \omega_0(1 + i)^{T_1}. \tag{6}$$

2.3. Surrender options and the optimal withdrawal provision

Milevsky and Salisbury (2006) study the GMWB contract that allows a policyholder to surrender his or her policy to redeem the policy value before the maturity date. A rational policyholder may surrender the policy early if the continuous value is less than the current policy value. The continuous value is determined by the present value of future expected cash flows generated from holding the GMWB contract. In practice, surrendering the policy early may incur early redemption penalties, so the early redemption value at time t is

$$G + (1 - k)(W_t - G),$$

where k denotes the proportional penalty charge. Policyholders redeem the contract early if they find that the loss of the time value due to postponing their withdrawal from the GMWB account and loss due to mortality risk (as we discuss subsequently) exceeds the early redemption penalty $k(W_t - G)$. Our paper will study how the value of the surrender option is influenced by other provisions, say, the rollup interest rate guarantee.

Many studies have carefully examined the optimal withdrawal behavior. That is, the policyholder is allowed to withdraw any amount from the account to maximize his or her benefit. Similar to the above, a withdrawal over the limit would also require the payment of some penalty charge. A later section will sketch how our tree model can be extended to solve this problem by using more state variables to keep the information required to analyze different withdrawal strategies as proposed in Hull and White (1993).

2.4. The impact of mortality improvement

The GMWB contract might also be terminated early if the policyholder dies before its maturity date. In this case, the account value is immediately returned to the policyholder, and the withdrawal guarantees annulled. In our numerical experiments, the mortality improvement is assumed to follow the GAR-94 table (see Society of Actuaries Group, 1995). Note that alternative mortality assumptions can be incorporated into our evaluation model without difficulty.

2.5. Dynamics of tree models

A tree model is a popular numerical pricing method that can describe the evolution of a stochastic process, for instance, the account value process in this paper. To fit the complex provisions of the GMWB contract, we create a novel tree that combines elements from the CRR tree (Cox et al., 1979), stair tree (Dai, 2009), and BTT (Dai and Lyuu, 2010). The core ideas of these three tree models are illustrated in Fig. 1. A tree divides a certain time interval $[0, T]$ into n equally spaced time steps and specifies the account value at each step. The length of each time step Δt equals T/n. The tree asymptotically converges with the account value process by matching the first and second moments of the account value process (Eqs. (1) and (2)) for each tree node (Duffie, 1996). Let us take the black nodes and branches in Fig. 1 that represent the structure of the CRR tree as an example. From an arbitrary node with account value W, the account value may move upward to Wu with probability p or downward to Wd with probability $1 - p$, where $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, and $p = \frac{e^{(r-\alpha)\Delta t} - d}{u - d}$, to match the first two moments of the account value process.

To model jumps in the account value due to discrete withdrawal (see Eq. (3)) without incurring significant numerical pricing errors, we incorporate the stair tree (Dai, 2009) and the BTT (Dai and Lyuu, 2010), which are constructed based on the CRR tree. For convenience, define the W-log price of W' as $\ln\left(\frac{W'}{W}\right)$, where W and W' denote two arbitrary account values. The W_0-log price for each node at time steps 1 and 2 appears in parentheses. The distance between two adjacent nodes at the same time step is
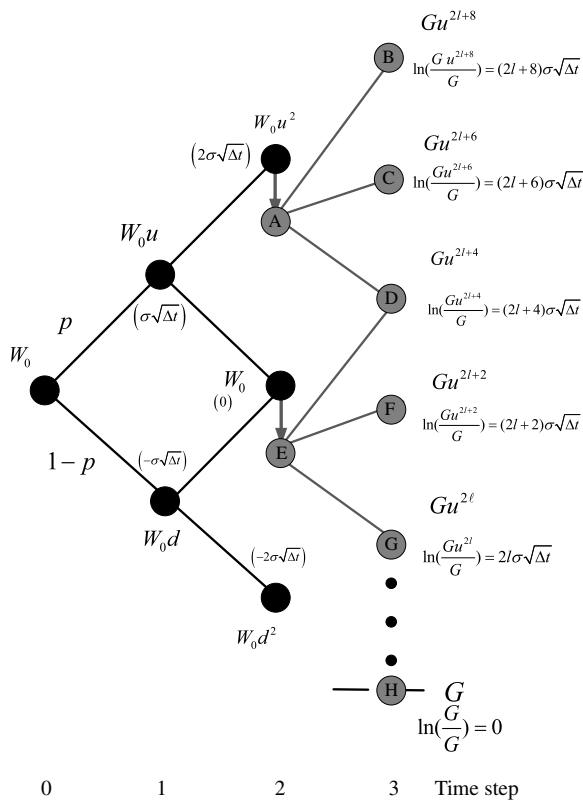
Fig. 1. Illustration of core ideas of the CRR tree, the stair tree, and the BTT tree. Note: the structure of the CRR tree is plotted in black. The initial account value is W_0. Account values can move either upward or downward, with multiplicative factors u and d and branching probabilities p and $1 - p$, respectively. The W_0-log price for each node at time steps 1 and 2 are in parentheses. Downward arrows connected to nodes A and E model the jumps of the account value. The trinomial structures (emitted from nodes A and E) connect to nodes B, C, D, F, and G at time step 3 without incurring an uncombined tree structure. Note that the outgoing branch from the lowest node at time step 2 is ignored for simplicity. The account value and the G-log price for each node at time step 3 are listed next to that node. We follow the core idea of the BTT by laying out the nodes at time step 3 to ensure that node H hits the critical location plotted by the thick gray line.

$2\sigma\sqrt{\Delta t}$ due to the nature of the structure of the CRR tree. To faithfully model the downward jump of the account value (marked by the downward arrows in time step 2) without incurring either numerical errors or an uncombined tree structure,[2] Dai (2009) inserts the trinomial structure (colored in gray in Fig. 1) to connect the after-jump nodes, say, nodes A and E at time step 2, to the nodes at time step 3. The nodes at time step 3 still follow the CRR tree structure; that is, the distance between two adjacent nodes at the same time step is $2\sigma\sqrt{\Delta t}$. Dai (2009) uses this property to prove that a valid trinomial structure can be constructed with feasible branching probabilities (as we detail subsequently). In addition, this property also ensures that the outgoing branches from the nodes at time step 3 can simply follow the CRR binomial structure without incurring an uncombined tree structure. Indeed, this trinomial structure is used in this paper to adjust the tree structure to meet certain provisions of the GMWBs.

Constructing a method that can evaluate the value of the GMWBs accurately and stably is a critical requirement for evaluating the fair charge. To make our tree generate stable pricing results, we need to suppress nonlinearity errors (see Figlewski and Gao, 1999) by borrowing the core idea of the BTT proposed in Dai

[2] Dai (2009) shows that existing tree models (apart from his stair tree) did not recombine after down jumps of the account value. The size of the tree grows significantly, which makes the computational problem intractable.

and Lyuu (2010); that is, we adjust the tree to make a node, says H in Fig. 1, to coincide with a critical location, such as the account value level G (marked by the thick dashed line) at a withdrawal date. Recall that the policyholder is permitted to withdraw G up to the maturity date even if the account value drops to zero at a withdrawal date. Thus, there should be a kink in the GMWB value function located at a line with account value G. Our numerical results suggest that coinciding with the kink does help our tree to generate stable pricing results.

3. Valuation of the GMWBs with a novel tree model

In this section, we discuss how we construct our tree to model the deferred period and other various provisions (e.g., rollup interest rate guarantees). To make the pricing results stable and accurate, our tree structure is designed to coincide with the critical locations, which alleviates the nonlinearity error problem. We also discuss how to incorporate the morality risk into our tree model. Finally, we explain how the backward induction procedure is designed to model the surrender option and the optimal withdrawal provision.

3.1. Construction of a new tree model for valuing GMWB with a deferred variable annuity

Assume that the GMWB contract begins at time 0 with account value W_0. The deferred period spans from time 0 to time T_1, and the withdrawal period spans from time T_1 to time T_2. A GMWB without a deferred period (as mentioned in most past studies) can be dealt with by setting $T_1 = 0$. We demonstrate the construction of our new tree model in the deferred period and withdrawal period separately. We focus on tree construction for the deferred period first, and then discuss the tree construction for the withdrawal period.

We illustrate a simple example of our tree model in Fig. 2, with the tree structure for the deferred period plotted in detail. The tree structure for the withdrawal period is marked by triangles emitted from nodes E, F, H, A, and C, as detailed in Fig. 3. To evaluate a GMWB without incurring significant pricing errors, we design our tree to adjust its structure and ensure that a node (e.g., node F in Fig. 2) coincides with the account value level $C(T_1)$ at time T_1. This critical location reflects the provision that ensures minimum total guaranteed withdrawal amounts in Eq. (5). If the account value at the end of the deferred period $W_{T_1}^-$ is greater than $C(T_1)$, as is the case for node E, then the total guaranteed withdrawal amounts and the account value at the beginning of the withdrawal period $W_{T_1}^+$ is set to the account value at the end of the withdrawal period $W_{T_1}^-$. Therefore the guaranteed withdrawal amount G for the tree emitted from node E is $\frac{W(E)}{m(T_2 - T_1)}$, where $W(X)$ denotes the account value for an arbitrary node X. Otherwise, the provision in Eq. (5) would reset the account value and the total guaranteed withdrawal amounts as $C(T_1)$. Therefore, the guaranteed withdrawal amount G would be $\frac{C(T_1)}{m(T_2 - T_1)}$, as is the case for nodes F, H, A and C.

To generate stable pricing results, our tree should have a node to coincide with the critical location—a kink in the GMWB value function located at the account value level $C(T_1)$ (black dashed line) at time T_1. To achieve this, we insert a trinomial structure following node S to adjust the position of the following truncated, even-step CRR tree (spanning from time steps 1 to 3 in this example) to hit $C(T_1)$ at time T_1 (i.e., node F in Fig. 2). This truncated CRR tree is designed to emanate from some crosses, say, nodes I, J, and K, at time step 1. Note that among the crosses, the W_0-log price of one cross (node I in this example) should be $g\left(= \ln\left(\frac{C(T_1)}{W_0}\right)\right)$ to ensure that this even-step truncated CRR tree
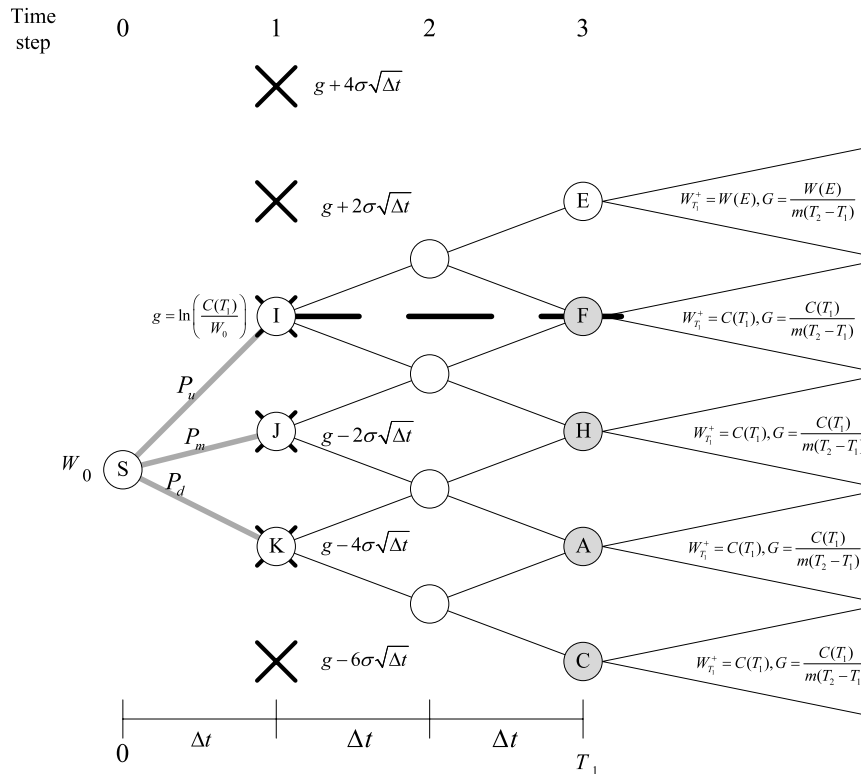
Fig. 2. A tree for pricing GMWB with the deferred period part plotted in detail. Note: the initial account value W_0 is represented by node S at time 0. The outgoing trinomial branches from node S are marked with thick gray lines, and the CRR binomial branches are marked using thin black lines. P_u, P_m, and P_d denote the outgoing branching probabilities from node S, and Δt denotes the length of each time step. The W_0-log price for each cross at time step 1 is listed next to the cross. The black dashed line denotes the value level $C(T_1)$. All the nodes located at time T_1 with the account value lower than $C(T_1)$ are colored in gray. The total guaranteed withdrawal amounts and the account value for these gray nodes are reset to $C(T_1)$. Each triangle following the node at time step 3 denotes one tree structure for the withdrawal period (see Fig. 3). The initial account value and guaranteed withdrawal for each tree structure represented by a triangle are denoted as $W_{T_1}^+$ and G, respectively, in that triangle.

coincides with node F.[3] The W_0-log price distance between two adjacent crosses must be $2\sigma\sqrt{\Delta t}$ due to the nature of the CRR tree mentioned in Section 2.2. This "$2\sigma\sqrt{\Delta t}$" property is applied to construct a valid trinomial structure (see Dai, 2009) depicted as follows. We choose three nodes from three adjacent crosses (e.g., I, J, and K) that will be connected by outgoing trinomial branches emanating from the root node S. These trinomial branches should possess two features. First, they should match the first and second moments of the account value process in Eq. (4). Second, the probabilities for the trinomial branches (i.e., P_u, P_m, and P_d) must be valid (i.e., probabilities should not be negative or greater than 1). Dai (2009) provides a node selection procedure to guarantee the above two features as follows. First, we calculate the W_0-log price for each cross at time step 1. The W_0-log price for a cross can be interpreted as the return on the account value from node S to that cross. Second, we find a unique cross whose return is nearest to the expected return on the asset value, $(r - \alpha - 0.5\sigma^2)\Delta t$ (see Eq. (4)). In our example, this node is J in Fig. 2. The outgoing trinomial branches from node S connect to node J and its adjacent nodes, I and K. The calculation of branching probabilities for this trinomial structure is provided in the Appendix.

The tree structure for the withdrawal period illustrated in Fig. 3 demonstrates two critical features of our tree models. First, the jumps of the account value due to discrete withdrawals are implemented faithfully, without uncombined tree structure and

any drastic growth in tree size. Second, the critical locations are hit by our tree to avoid unstable pricing results due to the nonlinearity error problem. To address the first issue, recall that the account value falls by a contractually specified amount G at each withdrawal date, due to the discrete withdrawal by the policyholder. To model the discrete jump of the account value, we borrow the idea of the stair tree from Dai (2009) and insert a downward jump (denoted by a downward arrow) and an "after-payment" node (marked by a small circle) into each node at the withdrawal date. The valid trinomial structure going from each "after-payment" node to the nodes at the next time step can be constructed by following the node selection procedure for constructing the branch for node S, as mentioned previously. Our tree is recombined; the tree structure emitted from a node such as F' overlaps the tree structure emitted from another node, such as H'. By contrast, implementing the downward jumps with a traditional tree structure would result in an uncombined structure. Dai (2009) suggests that the size and thus computational time of an unrecombined tree increases drastically with the number of downward jumps. Our tree efficiently prices a long GMWB contract (i.e., many withdrawal dates), because of its recombinative nature.

To reduce the nonlinearity errors, we adjust the tree structure to make certain nodes, such as A' and B', coincide with the value level G (black dashed line) at each withdrawal date. The guaranteed withdrawal G is paid to policyholders even if the account value cannot meet the obligation G, as in nodes D' and C'. The account values for those nodes whose account values do not exceed G become 0 after paying the guaranteed withdrawal, and no outgoing branches are required. Obviously, there is a kink in the GMWB value function at account value level G for each withdrawal date. The nonlinearity error can be suppressed by making some tree nodes, such as A' and B', hit the critical locations.

---

[3] Note that this does not imply that the outgoing trinomial branches from node S must connect to node I. If the initial account value W_0 is lower, the trinomial structure may connect to the other three nodes, say, J, K, and L. However, the resulting truncated CRR tree emanating from these three nodes still has a node F to coincide with the critical location.
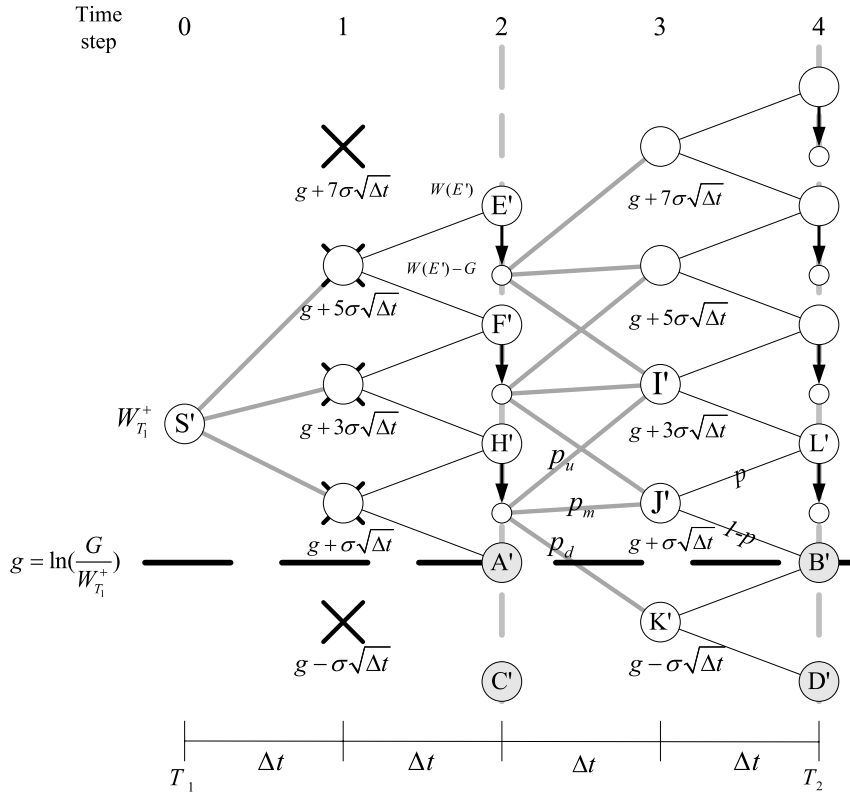
Fig. 3. Tree structure for the withdrawal period. Note: the root node with the initial account value $W_{T_1}^+$ is denoted by S'. The trinomial branches used to adjust the tree structure are marked with thick gray lines, and the CRR binomial branches are marked using thin black lines. Δt denotes the length of each time step. The black dashed line denotes the value level G. The $W_{T_1}^+$-log price for each cross at time step 1 and each node at time step 3 is listed next to that cross (or node). Withdrawal dates are marked by gray dashed vertical lines. The downward jumps of the account value at the withdrawal dates are marked by downward arrows. The small circles denote the account value after paying the guaranteed withdrawal G. The account values for the gray nodes at time steps 2 and 4 are no greater than G and become 0 after paying G to policyholders. Finally, p_u, p_m, and p_d denote the outgoing branching probabilities from node H', and p and $1 - p$ denote the upward and downward outgoing branching probabilities from node J'.

The construction of the tree structure for the withdrawal period illustrated in Fig. 3 (or the triangle illustrated in Fig. 2) proceeds as follows. Trinomial structures are inserted to make the tree hit the critical locations. The tree structure, between time steps 1 and 2, forms a truncated, odd-step CRR tree structure. To make this CRR tree hit the critical locations (node A'), the $W_{T_1}^+$-log price for each cross (or node) at time step 1 must have the form $g+j\sigma\sqrt{\Delta t}$, where g is defined as $\ln\left(G/W_{T_1}^+\right)$, and j is an odd integer. The outgoing trinomial branches from node S' connect to three adjacent crosses. The trinomial branch construction method is the same as the method for constructing the outgoing trinomial branches for node S in Fig. 2.

The tree structure between time steps 3 and 4 is again a truncated, odd-step CRR tree. To make the tree hit the critical locations (node B'), $W_{T_1}^+$-log prices for the nodes at time step 3 must take the form $g + j\sigma\sqrt{\Delta t}$, where j is an odd integer. At time step 2, the policyholder withdraws G from the account, and the account value jumps down. Take node E' as an example. The account value $W(E')$ decreases to $W(E') - G$ (small circle below node E') after the payment of G. A trinomial structure is constructed to connect the small circle to the three following nodes at time step 3. The procedure for constructing the trinomial branches is the same as that for constructing the branches for node S'. Note that there are no outgoing branches from nodes A' and C', because the account values for these nodes fall to 0 after withdrawal.
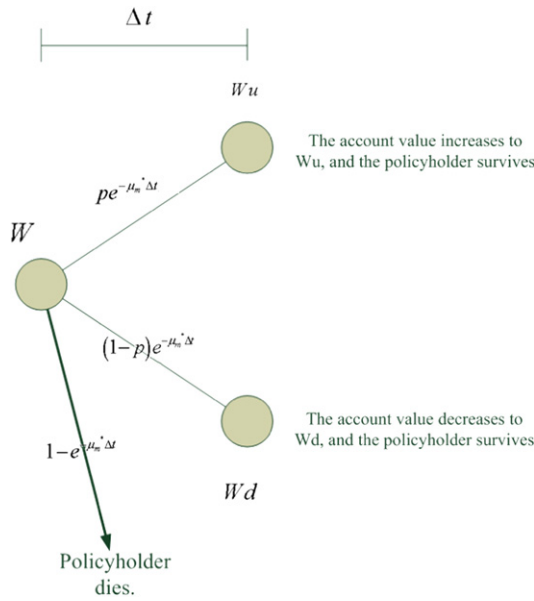
3.2. Incorporating mortality rate in the tree model

To capture the effect of mortality improvements on pricing GMWB contracts, we incorporate mortality factors into our tree

model. Let $p_{x_0}(t_0)$ denote the one-year survival probability and denote the mortality force for an x_0-year-old person in calendar year t_0 who reaches age $x_0 + 1$. Thus, the one-year survival probability can be calculated as $p_{x_0}(t_0) = \exp(-\mu_{x_0}(t_0))$ under a constant force of mortality assumption. We assume that the age-specific mortality rates are constant within bands of age and time, but may vary from one band to the next. Specifically, given any integer age x_0 and calendar year t_0, the n-year survival probability, $p_{x_0}(t_0, n)$, that an x_0-aged person in calendar year t_0 reaches age $x_0 + n$ is calculated as $p_{x_0}(t_0, n) = \prod_{j=0}^{n-1} \exp\left(-\mu_{x_0+j}(t_0 + j)\right)$.

The mortality risk can be incorporated into our tree model as demonstrated in Fig. 4. Let Δt denote the length of each time step in the tree. The probability of a policyholder dying within a time step is calculated as $1 - e^{-\mu_{x_0}(t_0)\Delta t}$. It is reasonable to assume that the account value process and mortality events are independent. The probability of the event whereby the account value moves upward and the policyholder survives can be obtained by directly multiplying the upward branch probability p by the survival probability $e^{-\mu_{x_0}(t_0)\Delta t}$. Similarly, the probability that the account value will move downward, given that the policyholder is alive, can be calculated as $(1 - p) e^{-\mu_{x_0}(t_0)\Delta t}$.

To evaluate the impacts of mortality risk on pricing the GMWB contract, we need to risk neutralize the mortality rates in our tree model. Wang (2000) proposes a transformation for pricing contingent claims even if they are not traded in the financial markets. Because contracts contingent on mortality rates are usually not traded in financial markets, Wang's transformation provides a way to value mortality-linked securities (Lin and Cox, 2005; Dowd et al., 2006; Kijima, 2006; Liao et al., 2007; Denuit et al., 2007). Specifically, the Wang transform convert the real-world mortality rates (without asterisks) into risk-neutral ones

Fig. 4. Incorporating mortality risk in the CRR binomial structure. Note: p and $1-p$ denote the upward and downward branching probabilities, respectively, under the CRR tree. The branching probabilities are listed on the branches.

(with asterisks) as follows. Let the distribution function of the mortality rate under the real-world probability for a person of age x who dies before age $x + m$ is $F_x(m)$.[4] The Wang transform converts $F_x(m)$ into the risk-neutral world distribution function $F_x^*(m)$ with a distortion operator:

$$F_x^*(m) = g\left[F_x(m)\right] = \Phi\left(\Phi^{-1}\left(F_x(m)\right) - \lambda\right), \tag{7}$$

where $F_x(m) = 1 - \prod_{j=0}^{n-1} \mu_{x_0+j}(t_0 + j)$, g is a distortion function with $g(0) = 0$, $g(1) = 1$, and $g'(0) = \infty$. Furthermore, Φ denotes the distribution function of the standard normal distribution, and λ denotes the market price of risk. Denuit et al. (2007) suggest that λ can be calibrated by letting the market price of the life annuity equal the expected present value of all future annuity cash flows paid to the annuitants under the risk-neutral probability. According to Eq. (7), the risk-neutral force of the mortality rate for a person aged $x + m(\mu_{x_0+m}^*(t_0 + m))$ can be solved recursively with the following equation:

$$\mu_{x_0+m}^*(t_0 + m) = -\ln\left(1 - F_x^*(m)\right) - \sum_{j=0}^{m-1} \mu_{x_0+j}^*(t_0 + j).$$

3.3. Backward induction procedure in the presence of a surrender option

A sophisticated backward induction procedure on our tree, as illustrated in Figs. 2–4, enables the faithful implementation of the GMWB contract with surrender options, mortality risk, and other common provisions. To demonstrate the backward induction procedure, we define $V(x)$ as the GMWB value at node x, $V_R(x)$ as the value to exercise the surrender option at x, $V_C(x)$ as the value to keep the policy (i.e., a continuous value) at x, and $W(x)$ as the account value at x.

We first evaluate the GMWB value for each node at maturity T_2. A GMWB holder is entitled to withdraw a contractually specified amount G at specified withdrawal dates (including T_2), even when the account value does not meet G. In addition, at time T_2, the

policyholder can receive the remaining funds (if any) from the account. Thus, the GMWB value at maturity is equal to

$$G + \max(W_{T_2} - G, 0) = \max(W_{T_2}, G).$$

For example, the GMWB values for nodes B' and D' are G, and the value for node L' is $W(L')$. If the policyholder does not surrender the GMWB, he or she can gain from the future cash flows generated by the policy. This value, or the continuous value, can be evaluated by taking the expectation of the present value of future cash flows. We take node J' in Fig. 3 as an example. Let $\mu_{J'}^*$ denote the mortality force at node J'. The probability that the policyholder will die during one time step is $1 - e^{-\mu_{J'}^* \Delta t}$. Because the account value immediately returns to the policyholder and the withdrawal guarantees are annulled if the policyholder dies, the holder receives either $W(L')$ if the account value goes up to reach node L' (with probability p) or $W(B')$ if the account value goes down to reach node B' with probability $1 - p$. Thus, the GMWB value at node J' when the policyholder dies during a time step is

$$\left(1 - e^{-\mu_{J'}^* \Delta t}\right)\left(pW(L') + (1-p)W(B')\right)e^{-r\Delta t}. \tag{8}$$

Similarly, the value contributed by the survival of a policyholder during a time step is

$$e^{-\mu_{J'}^* \Delta t}\left(pV(L') + (1-p)V(B')\right)e^{-r\Delta t}. \tag{9}$$

Thus, the GMWB value for continuing the policy at node J' (i.e., $V_C(J')$) is the sum of Eqs. (8) and (9). In addition, the GMWB value for continuing the policy at node S in Fig. 2 is calculated as

$$V_C(S) = \left(1 - e^{-\mu_S^* \Delta t}\right)(P_u W(I) + P_m W(J) + P_d W(K))e^{-r\Delta t}$$
$$+ e^{-\mu_S^* \Delta t}(P_u V(I) + P_m V(J) + P_d V(K))e^{-r\Delta t}.$$

In our numerical analysis, we can also analyze the effect of ignoring mortality risk on the fair charge by setting $\mu^* = 0$.

The policyholders may exercise the surrender option on the withdrawal dates to redeem the account value with the cost of the proportional penalty charge k. For example, the value of exercising the surrender option at node H' is

$$V_R(H') \equiv G + (1-k)(W(H') - G).$$

Because the continuous value at node H' is the sum of the value contributed by the death of a policyholder during a time step, plus the value contributed by the survival of the policyholder and the guaranteed withdrawal G, we have

$$V_C(H') \equiv \left(1 - e^{-\mu^* \Delta t}\right)\left(p_u W(I') + p_m W(J') + p_d W(K')\right)e^{-r\Delta t}$$
$$+ e^{-\mu^* \Delta t}\left(p_u V(I') + p_m V(J') + p_d V(K')\right)e^{-r\Delta t} + G.$$

The policyholder decides whether to surrender the policy to maximize his or her benefits; that is, the GMWB value at node H' is expressed as $V(H') = \max(V_R(H'), V_C(H'))$. Obviously, the policyholder optimizes his or her surrender strategy to strike a balance among the loss of time value due to the postponement of the withdrawal, the loss of mortality risk, and the early redemption penalty $k(W(H') - G)$. On the other hand, to evaluate the policy without surrender options, we would simply set $V(H') = V_C(H')$.

If the account value on the withdrawal dates cannot meet the guaranteed withdrawal, as in nodes A', B', C', and D' in Fig. 3, the account value is set to 0 after the withdrawal. Thus, the GMWB contract degenerates into a life annuity that might be canceled early due to the death of the policyholder; that is, the GMWB value for these nodes equals the sum of the present value of the cancelable annuity. Take node A' for example. The probability that the policyholder will not die before time step 4, with the condition that the holder is alive at time step 2, is $e^{-2\mu_m^* \Delta t}$. The value of the

GMWB at node A' is $e^{-2\mu_m^*\Delta t}Ge^{-2r\Delta t} + G$, where the first term is contributed by the cash flow paid at time step 4.

By applying the backward induction procedure from time T_2 back to time T_0, we can obtain the GMWB value at node S. The fair charge can be computed by picking a proper rate of insurance fee α to make the value of the GMWB equal to the initial investment. Since our tree can price the value of the GMWB smoothly without oscillations, our pricing results (for the GMWB value) would decrease monotonically with the increment of insurance fee α. Thus the solution for the fair charge is unique and can be easily solved by the standard root-finding procedure, say, the bisection method.
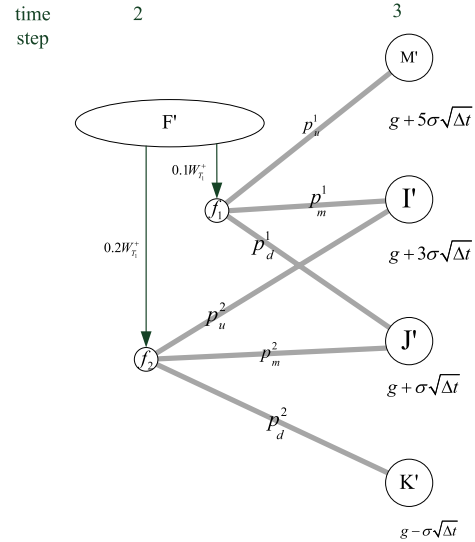
3.4. The optimal withdrawal provision

Our tree can be slightly modified to model the optimal withdrawal provision. A brief sketch is given to describe the required modifications. Other unchanged parts, for instance, the incorporation of the mortality risk, are ignored for simplicity. Since the policyholder is allowed to withdraw any amount from the account, our tree must deal with a new variable—the remaining guaranteed withdrawal amount—the amount that is guaranteed but is not withdrawn by the policyholder. Thus for each node X in the withdrawal period, more state variables, e.g., $V(X, W_{T_1}^+)$, $V(X, 0.9W_{T_1}^+)$, $V(X, 0.8W_{T_1}^+)\ldots$, are required to memorize the GMWB value at node X with the remaining guaranteed withdrawal amount $W_{T_1}^+$, $0.9W_{T_1}^+$, $0.8W_{T_1}^+$, and so on.[5] The tree structure is also slightly modified as illustrated in Fig. 5. For simplicity, we only draw the modification for part of the withdrawal period tree starting from node F' in Fig. 3. The layout of the nodes at time step 3 is the same as the layout in Fig. 3 to ensure that the tree can still coincide with the critical locations. To evaluate the GMWB value $V(F, Q)$, the GMWB value at node F with the remaining guaranteed withdrawal amount Q, we need to evaluate the GMWB value under different withdrawal strategies first. For example, the GMWB value given that the policyholder withdraws $0.1W_{T_1}^+$ can be evaluated as

$$
\begin{aligned}
V(f_1) \equiv & \left(1 - e^{-\mu^*\Delta t}\right)\left(p_u^1 W(M') + p_m^1 W(I') + p_d^1 W(J')\right)e^{-r\Delta t} \\
& + e^{-\mu_\Delta^* t}\left(p_u^1 V(M', Q - 0.1W_{T_1}^+) + p_m^1 V(I', Q - 0.1W_{T_1}^+) \right. \\
& \left. + p_d^1 V(J', Q - 0.1W_{T_1}^+)\right)e^{-r\Delta t} \\
& + 0.1W_{T_1}^+ - k\max\left(0.1W_{T_1}^+ - G, 0\right),
\end{aligned}
$$

where the first line denotes the GMWB value given that the holder dies during the time step, the second line denotes the GMWB value given that the holder survives during the time step, the first term in line three denotes the value withdrawn by the policyholder, and the second term denotes the over-withdrawal penalty. Similarly, the GMWB value given that the policyholder withdraws $0.2W_{T_1}^+$ can be evaluated as

$$
\begin{aligned}
V(f_2) \equiv & \left(1 - e^{-\mu^*\Delta t}\right)\left(p_u^2 W(I') + p_m^2 W(J') + p_d^2 W(K')\right)e^{-r\Delta t} \\
& + e^{-\mu_\Delta^* t}\left(p_u^2 V(I', Q - 0.2_{T_1}^+) + p_m^2 V(J', Q - 0.2W_{T_1}^+) \right. \\
& \left. + p_d^2 V(K', Q - 0.2W_{T_1}^+)\right)e^{-r\Delta t} \\
& + 0.2W_{T_1}^+ - k\max\left(0.2W_{T_1}^+ - G, 0\right).
\end{aligned}
$$

Fig. 5. Part of tree structure for modeling the optimal withdrawal provision. Note: this figure illustrates the modification for part of the tree emanating from node F' in Fig. 3. The policyholder may withdraw $0.1W_{T_1}^+$ or $10.2W_{T_1}^+$ from the, account and the account value drops to node f_1 or f_2, respectively. The outgoing trinomial structures for f_1 (with branching probabilities p_i^1, p_m^1, and p_d^1) and f_2 (with branching probabilities p_i^2, p_m^2, and p_d^2) connect to different nodes at the next time step. Other possible withdrawal amounts are ignored for simplicity. The $W_{T_1}^+$-log price for each node at time step 3 is listed next to that node.

We can add more nodes, say, f_3, f_4, and branches into Fig. 5 to model the GMWB value under different withdrawal strategies. Since the policyholder will optimize his or her withdrawal strategy, $V(F, Q)$ can be evaluated as $\max(V(f_i))$.

While the properties of the optimal withdrawal provision have been widely studied in much of the literature (see Chen et al., 2008; Dai et al., 2008), not so much attention has been given to the features of surrender options. Our numerical results will thus focus on the surrender options and other provisions that were not given much consideration in the past literature.

4. Numerical results

In this section, we first verify the accuracy and robustness of our proposed tree model. We then perform sensitivity analyses on the fair charges. The effects of various factors, such as surrender options, mortality risk, rollup interest rate guarantee designs, deferred annuities, discrete withdrawals, and early redemption penalties on the fair charge of the GMWB contract are analyzed numerically. In addition, our method can assess the joint effects of these factors on fair charges. For simplicity, we assume that a policyholder invests a single premium of 100 at time 0, the withdrawals are taken annually (i.e., $m = 1$ in Eq. (5)), and the risk-free rate is 3.25%, unless stated otherwise.

4.1. Accuracy of proposed tree model

To examine the accuracy of our new tree model, we use a Monte Carlo simulation as a proxy benchmark. The values of the GMWB generated by our new tree model and by the Monte Carlo simulation (denoted by MC) with 10,000,000 trials[6] without mortality risk are listed in Table 1. For convenience, we use the pair (T_1, T_2) to express the length of the deferred period T_1 and the time to maturity T_2. For the accuracy check, we present the result for

---

[5] We can add more state variables at each node to evaluate the GMWB value under different remaining guaranteed withdrawal amounts in more detail. Note that this also increases the computational time.

[6] The insurance fee α is set to 0 in this case.

Table 1
Accuracy of proposed tree model for valuing GMWBs without mortality risk.

	$G = 4$ $(T1, T2) = (0, 25)$			$G = 5$ $(T1, T2) = (0, 20)$			$G = 10$ $(T1, T2) = (0, 10)$		
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
TM(10)	106.255	113.271	120.232	106.732	113.723	120.655	107.368	113.638	119.874
TM(50)	106.243	113.224	120.133	106.723	113.677	120.562	107.363	113.625	119.841
TM(100)	106.243	113.222	120.127	106.723	113.676	120.557	107.362	113.623	119.840
TM(500)	106.243	113.220	120.125	106.723	113.675	120.555	107.361	113.622	119.837
TM(1000)	106.243	113.220	120.125	106.723	113.675	120.555	107.361	113.622	119.837
TM(5000)	106.243	113.220	120.124	106.723	113.675	120.555	107.361	113.622	119.837
MC	106.019	113.212	120.110	106.751	113.650	120.141	107.307	113.716	119.680
SE	0.296	0.733	1.812	0.239	0.545	1.223	0.132	0.245	0.431

Note: the numbers in parentheses in the first column denote the number of time steps in our tree model (TM). MC and SE denote the GMWB values estimated by the Monte Carlo method and the standard errors, respectively. The guaranteed withdrawal amounts for different guaranteed periods are determined by $G = \frac{\omega_0}{T_2}$, so we investigate the guaranteed withdrawal amounts of $G = 4$, 5 and 10.

Table 2
Accuracy of proposed tree model for valuing GMWBs with mortality risk.

	$G = 4$ $(T1, T2) = (0, 25)$			$G = 5$ $(T1, T2) = (0, 20)$			$G = 10$ $(T1, T2) = (0, 10)$		
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
TM	104.154	109.409	114.913	105.296	111.133	117.087	106.886	112.816	118.742
MC	103.938	108.630	114.952	105.580	110.749	117.036	106.828	112.536	118.303

Notes: TM and MC denote our tree model and the Monte Carlo simulation, respectively.

an immediate life annuity with different guaranteed withdrawal periods of 10, 20, and 25 years. The value of the GMWB increases with σ, because this value can be decomposed into an annuity plus an option, and the value of the option increases with σ. In addition, the GMWB value decreases with the increment of time to maturity T_2. This is because postponing withdrawal would result in the loss of time value. The pricing results generated by our tree model converge smoothly and quickly to the benchmark with the increment of the number of time steps in our tree, because our sophisticated tree structure design sharply reduces nonlinearity error. Therefore, our new tree model provides a precise and efficient method to evaluate GMWB contracts.

We consider the mortality factor in Table 2, where the annuitant is assumed to be a 65-year-old man. For illustrative purposes, we assume that the real world mortality rate follows the GAR-94 table, and the effect of mortality improvement can be captured by the mortality improvement factor.[7] Alternative mortality assumptions can also be incorporated into our tree model without difficulty. Again, the pricing results generated by our tree model are close to those generated by the Monte Carlo simulation. By comparing the results in Tables 1 and 2, we find that incorporating the mortality risk decreases the GMWB value. In addition, the impact of the mortality risk increases with the increment of the time to maturity T_2. For example, when the volatility is high ($\sigma = 0.4$), a long-term GMWB contract ($T_2 = 25$) will decrease the GMWB value by about 5 ($\approx 120.110 - 114.952$) while a short-term contract ($T_2 = 10$) will reduce the GMWB value by about ($1 \approx 119.680 - 118.303$).

4.2. Sensitivity analyses of the fair charge for GMWB contracts

Evaluating the fair charge (α) is important for an insurer to issue GMWB contracts. We detail the sensitivity analyses for the fair charge in this section; in Table 3, we investigate the pricing results of fair charges for the continuous withdrawal assumption

Table 3
Impact of discrete/continuous withdrawals of fair charges for GMWB contracts.

Method	$G = 4$ $(T1, T2) = (0, 25)$		$G = 5$ $(T1, T2) = (0, 20)$		$G = 10$ $(T1, T2) = (0, 10)$	
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$
TM	17 b.p.	50 b.p.	28 b.p.	75 b.p.	92 b.p.	214 b.p.
M&S	23 b.p.	60 b.p.	37 b.p.	90 b.p.	140 b.p.	271 b.p.

Note: M&S denotes the fair charges evaluated under the continuous withdrawal assumption by the method proposed in Milevsky and Salisbury (2006). TM denotes the fair charges evaluated under the annual withdrawal assumption by our tree. We follow the numerical settings in Milevsky and Salisbury (2006) by setting the risk-free rate as 5% in this table.

(which are evaluated by the method proposed in Milevsky and Salisbury (2006)) and the annual discrete withdrawal assumption (which are evaluated by our proposed tree model). In contrast to the argument that "the value of a continuous withdrawal formulation is close to the discrete withdrawal case if withdrawal intervals are less than one year" as proposed by Chen et al. (2008), our numerical results suggest that approximating the longer discrete withdrawal intervals, say, one year in this example, using the continuous withdrawal assumption may significantly overestimate the fair charge. For example, a guaranteed withdrawal amount $G = 4$ produces a fair charge of approximately 17 basis points (b.p.) for a lower volatility ($\sigma = 0.2$) investment and 50 b.p. for a higher volatility investment ($\sigma = 0.3$) in the discrete withdrawal setting. However, according to Milevsky and Salisbury (2006), the corresponding fair charges in a continuous withdrawal setting would be 23 b.p. for lower and 60 b.p. for higher volatility investments. In addition, for a higher guaranteed withdrawal amount, the fair charge grows much higher, and the overestimation phenomenon becomes more significant. This price discrepancy could explain why the current guaranteed charges in real-world financial markets are lower than the estimates produced by Milevsky and Salisbury (2006).[8]

We also analyze the fair charges for the GMWB contracts under an immediate or a deferred life annuity in Tables 4 and 5,

---

[7] With the GAR-94 table, the mortality rate for a person aged x in year $1994+n$ can be estimated as $q_x^{1994+n} \equiv q_x^{1994} (1 - AA_x)^n$, where AA_x is the annual improvement factor in the mortality rate for age x. Both q_x^{1994} and AA_x are provided by Society of Actuaries Group (1995).

[8] Milevsky and Salisbury (2006) note that recent GMWB products introduced in the market charged only 30–50 b.p., even though the underlying investment fund contained high-volatility investment choices.

Table 4
Fair charges for GMWB contracts for immediate life annuities.

		$G = 4$ $(T1, T2) = (0, 25)$			$G = 5$ $(T1, T2) = (0, 20)$		
		$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
No surrender option	No mortality risk	46 b.p.	102 b.p.	157 b.p.	66 b.p.	142 b.p.	216 b.p.
	With mortality risk	33 b.p.	78 b.p.	126 b.p.	54 b.p.	119 b.p.	185 b.p.
With surrender option	No mortality risk	46 b.p.	158 b.p.	395 b.p.	66 b.p.	224 b.p.	523 b.p.
	With mortality risk	33 b.p.	81 b.p.	208 b.p.	54 b.p.	143 b.p.	351 b.p.

Note: the risk-free rate is assumed to be 3.25%, and the proportional penalty charge k is 0.1.

Table 5
Fair charge for GMWB contracts for deferred life annuities.

		$(T1, T2) = (10, 25)$			$(T1, T2) = (10, 20)$		
		$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
No surrender option	No mortality risk	80 b.p.	170 b.p.	261 b.p.	98 b.p.	206 b.p.	316 b.p.
	With mortality risk	65 b.p	136 b.p	212 b.p	79 b.p.	166 b.p.	257 b.p.
With surrender option	No mortality risk	85 b.p.	248 b.p.	438 b.p.	102 b.p.	274 b.p.	475 b.p.
	With mortality risk	65 b.p.	150 b.p.	277 b.p.	79 b.p.	184 b.p.	326 b.p.

Note: the risk-free rate is assumed to be 3.25%, and the proportional penalty charge k is 0.1. i and m are set as 0 and 1 for the rollup interest rate guaranteed design defined in Eq. (5), respectively. $G = \frac{\text{Max}[C(T_1), W(T_1)]}{m(T_2 - T_1)}$.

respectively. In particular, we study the effects of incorporating surrender options and mortality risk on calculating the fair charge. Table 4 shows the fair charge under a discrete withdrawal assumption for the immediate annuity case priced by our proposed tree model, with a proportional penalty charge k of 0.1. The surrender option grants the policyholder the right to redeem the policy early at the cost of a higher fair charge. For example, the fair charge increases from 157 b.p. to 395 b.p. if $G = 4$, $\sigma = 0.4$, and the option premium is 238 b.p. ($=395 - 157$). In addition, the increment of the fair charge due to the presence of a surrender option tends to be higher if the guaranteed withdrawal amount or volatility of the account value is higher. For example, the option premium is 0 b.p. ($=46 - 46$) when $G = 4$ and $\sigma = 0.2$. Increasing the volatility σ from 0.2 to 0.4 would increase the option premium from 0 b.p. to 238 b.p. ($=395 - 157$). Increasing the guaranteed withdrawal amount from 4 to 5 would further increase the option premium from 238 b.p. to 307 b.p. ($=523 - 216$). Besides, taking mortality risk into account instead reduces the fair charge, because the possibility of death reduces the chances for policyholders to receive the guaranteed withdrawal. This effect could explain why ignoring the mortality risk leads to overestimates of the fair charge. Note that taking mortality risk into account also reduces the chances of policyholders exercising the surrender option and thus reduces the option premium. For example, the premium declines from 238 b.p. to 82 b.p. ($=208 - 126$) when $G = 4$ and $\sigma = 0.4$.

The above numerical analyses rely on the immediate guaranteed withdrawal assumption: guaranteed withdrawals are made at inception, and the total guaranteed withdrawal amount is set to the initial investment. The extant literature is similarly based on this assumption. We therefore investigate the impact of a deferred guaranteed withdrawal, which is more popular than the immediate guaranteed withdrawal one, on the fair charge (Table 5). For comparison purposes, we set the length of the deferred period to 10 years and consider two guaranteed withdrawal periods: 10 and 15 years. The numerical settings for Tables 4 and 5 are the same, except for the length of the guaranteed withdrawal period and the guaranteed withdrawal amount. The guaranteed withdrawal amount is based on the rollup interest rate guaranteed design defined in Eq. (5) with $i = 0\%$ and $m = 1$. By comparing Table 4 with Table 5, we find that replacing the immediate guaranteed withdrawal assumption with the deferred guaranteed withdrawal assumption significantly increases the values of fair charges. This is because the rollup interest rate guaranteed design provides

downside protection for the policyholder's investment during the deferred period. The effects of mortality risk and the surrender option on the fair charge for a deferred life annuity are similar to those for an immediate life annuity. Thus we can ignore these analyses for simplicity.

We further investigate how the rollup interest rate guaranteed design influences the fair charge. In Table 6, we illustrate that the fair charge for the GMWB contracts increases with the rollup guaranteed interest rate i. For example, if both mortality risk and surrender options are absent, the fair charge increases from 209 b.p. to 386 b.p. (or 512 b.p. to 796 b.p.) when the rollup interest rate i increases from 3% to 5% for the low volatility $\sigma = 0.2$ (or high volatility $\sigma = 0.4$) case. This is because a higher i provides a bigger guarantee on the policyholder's investment (see Eq. (6)). Note that the rollup interest rate guarantee design is very popular in variable life annuity markets, and the significant increment in the fair charge due to rollup interest rate guarantee design cannot be ignored. In addition, the impacts of the presence of mortality risk or the surrender option tend to be more significant as i increases. For example, given the volatility $\sigma = 0.2$ and the absence of mortality risk, the surrender option premium increases from 98 b.p. ($=307 - 209$) to 684 b.p. ($=1070 - 386$) when the rollup interest rate i increases from 3% to 5%. Therefore, a valuation method capable of simultaneously pricing the rollup interest rate guarantee and other provisions is essential for analyzing and designing GMWB contracts. Besides, the presence of mortality risk (the surrender option) will decrease (increase) the fair charge, as we showed previously.

It can be observed in Table 7 that a higher proportional penalty charge k reduces the fair charge. This is because a higher k would prevent the policyholder from surrendering the policy, which decreases the value of the surrender option. The decrement of the fair charge reflects the decrement of the value of the surrender option. Besides, it can be observed that the decrement amounts of the fair charge become more significant with the increment of the volatility σ and the absence of the mortality risk.

5. Conclusion

In recent years, variable annuities have emerged as key components of the retirement income system, largely because of their tax-deferred features. To mitigate some of the investment risk inherent in VA products, investment guarantees are commonly

Table 6
Impacts of rollup interest rate guaranteed design on the fair charge for GMWB contracts with surrender option and mortality risk.

Condition		$i = 0\%$			$i = 3\%$			$i = 5\%$		
		$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
No surrender option	No mortality risk	98 b.p.	206 b.p.	316 b.p.	209 b.p.	363 b.p.	512 b.p.	386 b.p.	594 b.p.	796 b.p.
	With mortality risk	79 b.p.	166 b.p.	257 b.p.	183 b.p.	301 b.p.	416 b.p.	329 b.p.	465 b.p.	598 b.p.
With surrender option	No mortality risk	102 b.p.	274 b.p.	475 b.p.	307 b.p.	630 b.p.	977 b.p.	1070 b.p.	1667 b.p.	2335 b.p.
	With mortality risk	79 b.p.	184 b.p.	326 b.p.	220 b.p.	431 b.p.	651 b.p.	646 b.p.	906 b.p.	1193 b.p.

Note: the guaranteed withdrawal amount at each withdrawal date is determined by Eq. (5): $G = \frac{\text{Max}[W_0(1+i)^{T_1}, W_{T_1}^-]}{m(T_2 - T_1)}$, where $T_1 = 10$, $T_2 = 2$, $m = 1$ (i.e., annual withdrawal).
The risk-free rate is assumed to be 3.25%, and the proportional penalty charge k is 0.1.

Table 7
Impacts of the proportional penalty charge on the fair charge for GMWB contracts with surrender option.

Condition	$k = 0\%$			$k = 60\%$		
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
No mortality risk	164 b.p.	359 b.p.	574 b.p.	98 b.p.	207 b.p.	316 b.p.
With mortality risk	125 b.p.	267 b.p.	423 b.p.	79 b.p.	166 b.p.	257 b.p.

Note: the risk free rate is 3.25% and $(T_1, T_2) = (10, 20)$.

embedded in them. These various guarantees can be viewed as different exotic options. Although some simple guarantees can be accurately evaluated by the analytical pricing formulas for exotic options, many complicated guarantees cannot be approximately priced without approximations or simplifications. These approximating formulas may significantly misprice the value and fair charge of GMWB contracts. To solve this mispricing problem, we propose a new tree model that faithfully implements some popular policy designs of GMWB contracts. Specifically, we extend the existing literature on GMWBs to deal with deferred withdrawals, rollup interest guaranteed designs, the mortality risk, and the surrender option, as well as take into account discrete withdrawal behavior in the valuation framework. To the best of our knowledge, this article is the first to analyze the impacts of these complex provisions on the valuation of GMWB contracts.

To model policy designs without incurring significant numerical pricing errors, we propose a new tree model that entails a sophisticated design. The flexibility of our new tree model enables us to model the GMWB associated with various provisions and mortality risk. The numerical analysis verifies the accuracy of our new tree model, in comparison with a Monte Carlo simulation. We show that the continuous withdrawal assumption to approximate discrete withdrawals would significantly overestimate the fair charge for GMWB contracts. Whereas existing research can only deal with the immediate guaranteed withdrawal (i.e., starting at policy inception), our method can deal with the more popular deferred guaranteed withdrawal and show that the latter provision would result in a much higher fair charge than the former one. Our analyses also show that ignoring the presence of a surrender option (mortality risk) would underestimate (overestimate) the fair charge. The changes in the premium of surrender options under different scenarios are also substantially analyzed. Finally, our proposed tree model can deal with rollup interest rate guarantee designs for GMWB contracts, which have been very popular in variable annuity markets. Our numerical results suggest that fair charges significantly increase with the increment of rollup guaranteed interest rates. Due to our substantial numerical analyses, developing a robust pricing method that can deal with various provisions of GMWB contracts and mortality risk is vital for risk management.

In light of our analysis, we suggest two areas for further research. Due to the trend in mortality improvement, it is necessary to consider the mortality risk in valuing variable annuity products. This research demonstrates the effect of mortality improvement on valuing GMWB contracts by employing a simple deterministic mortality model. The stochastic mortality model has been regarded as being more capable of capturing mortality risk. It is well worth extending our tree model to incorporate various mortality models to examine the effect of mortality risk on the valuation of GMWB contracts. Moreover, valuing the GMWB contracts under the stochastic interest rate assumption is also important especially for the long-duration insurance contract. Constructing a three-dimensional tree with a stochastic interest rate and account dynamic to deal with the valuation problem for the GMWBs should be the focus of a further study.

Appendix. Calculating branching probabilities

Define α, β, and γ as the W_0-log price of the account value at nodes I, J, and K in Fig. 2, respectively, minus the expected return of the account value $(r - \alpha - 0.5\sigma^2)\Delta t$, as follows:

$$\beta \equiv g - 2\sigma\sqrt{\Delta t} - (r - \alpha - 0.5\sigma^2)\Delta t.$$
$$\alpha \equiv g - (r - \alpha - 0.5\sigma^2)\Delta t = \beta + 2\sigma\sqrt{\Delta t}.$$
$$\gamma \equiv g - 4\sigma\sqrt{\Delta t} - (r - \alpha - 0.5\sigma^2)\Delta t = \beta - 2\sigma\sqrt{\Delta t}.$$

To match the first and second moments of the asset value process at node S, the branching probabilities P_u, P_m, and $\boldsymbol{P}_d$, connected to nodes I, J, and K, respectively, should satisfy the following three equalities:

$$P_u \cdot \alpha + P_m \cdot \beta + P_d \cdot \gamma = 0$$
$$P_u \cdot \alpha^2 + P_m \cdot \beta^2 + P_d \cdot \gamma^2 = \sigma^2 \Delta t,$$
$$P_u + P_m + P_d = 1$$

where the first two equalities match the first two moments of the logarithmic price process (see Eq. (4)), and the last equality ensures that these three branching probabilities sum to 1. By applying Cramer's rule, we can solve these three branching probabilities: $P_u = \frac{\Delta_u}{\Delta}$, $P_m = \frac{\Delta_m}{\Delta}$, and $P_d = \frac{\Delta_d}{\Delta}$, where $\Delta = (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$, $\Delta_u = (\beta \cdot \gamma + \sigma^2 \Delta t)(\gamma - \beta)$, $\Delta_m = (\alpha \cdot \gamma + \sigma^2 \Delta t)(\alpha - \gamma)$, and $\Delta_d = (\alpha \cdot \beta + \sigma^2 \Delta t)(\beta - \alpha)$. The proof in Dai (2009) suggests that his trinomial branches construction procedure used in this paper leads to feasible branching probabilities.

References

Ballotta, L., Haberman, S., 2003. Valuation of guaranteed annuity conversion options. Insurance: Mathematics & Economics 33, 87–108.

Ballotta, L., Haberman, S., 2006. The fair valuation problem of guaranteed annuity options: the stochastic mortality environment case. Insurance: Mathematics & Economics 38, 195–214.

Bauer, D., Kling, A., Russ, J., 2008. A universal pricing framework for guaranteed minimum benefits in variable annuities. ASTIN Bulletin 38, 621–651.

Brennan, M.J., Schwartz, E.S., 1976. The pricing of equity-linked life insurance Policies with an asset value guarantee. Journal of Financial Economics 3, 195–213.

Brennan, M.J., Schwartz, E.S., 1979. Alternative investment strategies for the issuers of equity linked life insurance policies with an asset value guarantee. Journal of Business 52, 63–93.

Boyle, P.P., Hardy, M.R., 2003. Guaranteed annuity options. ASTIN Bulletin 33, 125–152.

Chen, Z., Vetzal, K., Forsyth, P.H., 2008. The effect of modelling parameters on the value of GMWB guarantees. Insurance: Mathematics & Economics 43, 165–173.

Costabile, M., Massabó, I., Russo, E., 2008. A binomial model for valuing equity-linked policies embedding surrender options. Insurance: Mathematics & Economics 42, 873–886.

Condron, C.M., 2008. Variable annuities and the new retirement realities. Geneva Papers on Risk and Insurance Issues and Practice 33, 12–32.

Cox, J.C., Ross, S., Rubinstein, M., 1979. Option pricing: a simplified approach. Journal of Financial Economics 7, 229–264.

Dai, T.-S., 2009. Efficient option pricing on stocks paying discrete or path-dependent dividends with the stair tree. Quantitative Finance 9, 827–838.

Dai, M., Kowk, Y.K., Zong, J., 2008. Guaranteed minimum withdrawal benefit in variable annuities. Mathematical Finance 18, 595–611.

Dai, T.-S., Lyuu, Y.-D., 2010. The bino-trinomial tree: a simple model for efficient and accurate option pricing. Journal of Derivatives 17, 7–24.

Denuit, M., Devolder, P., Goderniaux, A., 2007. Securitization of longevity risk: pricing survivor bonds with wang transform in the lee-carter framework. Journal of Risk and Insurance 74, 87–113.

Dowd, K., Blake, D., Cairns, A.J.G., Dawson, P., 2006. Survivor wwaps. Journal of Risk and Insurance 73, 1–17.

Duffie, D., 1996. Dynamic Asset Pricing Theory, second ed. Princeton University Press, Princeton, NJ.

Figlewski, S., Gao, B., 1999. The adaptive mesh model: a new approach to efficient option pricing. Journal of Financial Economics 53, 313–351.

Hull, J., White, A., 1993. Efficient procedures for valuing european and american path-dependent options. Journal of Derivatives 1, 21–31.

Kijima, M., 2006. A multivariate extension of equilibrium pricing transforms: the multivariate esscher and wang transforms for pricing financial and insurance risks. ASTIN Bulletin 36, 269–283.

Ledlie, M.C., Corry, D.P., Finkelstein, G.J., Ritchie, A.J., Su, K., Wilson, D.C.E., 2008. Variable Annuities. Working Paper Presented to the Faculty of Actuaries. 17 March. Institute of Actuaries. 31 March.

Liao, H.H., Yang, S.S., Huang, I.H., 2007. The Design of Securitization for Longevity Risk: Pricing under Stochastic Mortality Model with Tranche Technique. Presented at the Third International Longevity Risk and Capital Market Solutions Symposium. Taipei.

Lin, Y., Cox, S.H., 2005. Securitization of mortality risks in life annuities. Journal of Risk and Insurance 72 (2), 227–252.

Milevsky, M.A., Posner, S.E., 2001. The titanic option: valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. Journal of Risk and Insurance 68 (1), 93–128.

Milevsky, M.A., Salisbury, T.S., 2006. Financial valuation of guaranteed minimum withdrawal benefits. Insurance: Mathematics & Economics 38, 21–38.

Shen, W., Xu, H., 2005. The valuation of unit-linked policies with or without surrender options. Insurance: Mathematics & Economics 36 (1), 79–92.

Society of Actuaries Group Annuity Valuation Table Task Force, 1995. Group Annuity Mortality Table and 1994 Group Annuity Reserving Table. Transactions of Society of Actuaries 47, 865–918.

Wang, S., 2000. A class of distortion operators for pricing financial and insurance risks. Journal of Risk and Insurance 67, 15–36.