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An algebraic approach to BCJ numerators

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Abstract: One important discovery in recent years is that the total amplitude of gauge theory can be written as BCJ form where kinematic numerators satisfy Jacobi identity. Although the existence of such kinematic numerators is no doubt, the simple and explicit construction is still an important problem. As a small step, in this note we provide an algebraic approach to construct these kinematic numerators. Under our Feynman-diagram-like construction, the Jacobi identity is manifestly satisfied. The corresponding color ordered amplitudes satisfy off-shell KK-relation and off-shell BCJ relation similar to the color ordered scalar theory. Using our construction, the dual DDM form is also established.

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1 Introduction

Recent studies have revealed that there are many new structures for scattering amplitudes unforeseen from lagrangian perspective. One of such examples is the color-kinematic duality discovered by Bern, Carrasco and Johansson [1] (BCJ). In the work it was conjectured that color-ordered amplitudes of gauge theories can be rearranged into a form where kinematic numerators satisfy the same Jacobi identities as the color part does (i.e, the part given by multiplication of structure constants of gauge group according to corresponding cubic Feynman diagrams). These forms (we will call BCJ-form) lead to very nontrivial linear relations among color ordered amplitudes,[1] thus we can reduce the number of independent amplitudes to $(n-3)!$. A further conjecture of color-kinematic dual form (BCJ-form) is that if we replace the color part by kinematic part in the BCJ-form, we will

[1] BCJ relations between color-ordered amplitudes has been proved in string theory in [2–5] and in field theory in [6–8] using on-shell recursion relations

get corresponding gravity amplitudes. The double-copy formulation of tree-level gravity amplitudes is equivalent[2] to the Kawai-Lewellen-Tye (KLT) relations [9, 10]. However, unitarity suggests that double-copy formulation may be generalized beyond tree-level and therefore provides an extremely useful aspect to understand or calculate gravity amplitudes at loop-levels. Recent discussions on loop-level can be found in [12–24]. Because these important applications to gravity amplitudes, the simple and explicit construction of kinematic numerators is very important. In this paper we show that assuming gauge symmetry provides enough degrees of freedom, it is possible to construct kinematic numerators as linear combinations of contributions coming from cubic graphs, with vertices given by generalization of the algebraic structure constant given in [25, 26]. This construction makes many algebraic relations between numerators, such as Jacobi identity, KK-relation and BCJ relations, manifest.

Another interesting consequence of color-kinematic duality is that gauge theory amplitudes may have different forms. Two such examples are the color-ordered decomposition $\mathcal{A}_{\rm tot} = \sum_{\sigma\in S_{n-1}} Tr(T^{\sigma_1}T^{\sigma_2}\dots T^{\sigma_n})\, A(\sigma)$ (which we will call "Trace form") and the form discovered by Del Duca, Dixon and Maltoni (DDM) [27] $\mathcal{A}_{\rm tot} = \sum_{\sigma\in S_{n-2}} f^{1\sigma_2 x_1} f^{x_1\sigma_3 x_2}\dots f^{x_{n-3}\sigma_{n-1}n}\, A(1,\sigma,n)$ (which we will call the "DDM form"). The equivalence of two forms gives another proof of Kleiss-Kuijf (KK) [28] relations of the color-ordered amplitudes.[3] Within the color-kinematic duality, it is natural to have the "Dual Trace form" and "Dual DDM form" as discussed in [11, 31]. However, unlike the Trace form and DDM form, the dual form does not have very simple construction for the dual color part. In this paper, we will give a partial construction of the dual color part.

This paper is organized as follows. In section 2 we introduce the Lie algebra of general diffeomorphism in Fourier basis. Upon the sum of cyclic permutations of the structure constant, we get Yang-Mills 3-point vertex. Section 3 is our main part where the construction of kinematic numerators is given. We start with two examples, the 4-point numerator and 5-point numerator, where explicit calculations are given. Then we give a general frame for our construction. In section 4 we discuss relations, such as KK and off-shell BCJ relations, among quantities defined in section 3. In section 5 we derive the dual DDM form using relations from previous section. A few comments on relations between different formulations of Yang-Mills amplitudes are given in section 6. After a short conclusion, a proof of KK relation using off-shell recursion relation is included in the appendix.

2 Generators and kinematic structure constant

Our starting point is a generalization of the diffeomorphism Lie algebra introduced by Bjerrum-Bohr, Damgaard, Monteiro and O'Connell [25, 26]. The generator is defined as

$$T^{k,a} \equiv e^{ik\cdot x}\partial_a, \tag{2.1}$$

[2] A proof can be found in [11].

[3] In [27], the DDM form was derived using the properties of Lie algebra. However, it can also be derived [29] using KLT formulation of Yang-Mills amplitude [30].

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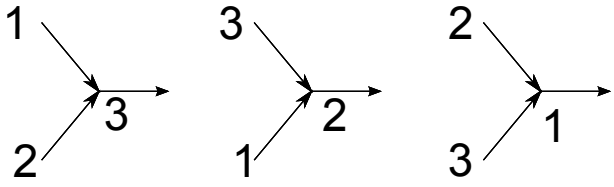

Figure 1. From left to right, three diagrams represent $f^{12}{}_3, f^{31}{}_2, f^{23}{}_1$ in eq. (2.5), where we used arrows to distinguish upper from lower indices.

with label (k, a), where k is a D-dimensional vector and a, a Lorentz index. The kinematic structure constant can be read out from commutator

$$\begin{aligned}[T^{k_1,a}, T^{k_2,b}] &= (-i)(\delta_a{}^c k_{1b} - \delta_b{}^c k_{2a})\, e^{i(k_1+k_2)\cdot x}\partial_c \\ &= f^{(k_1,a),(k_2,b)}{}_{(k_1+k_2,c)}\, T^{(k_1+k_2,c)}.\end{aligned} \tag{2.2}$$

In the following we shall use a shorthand notation by writing $f^{(k_1,a),(k_2,b)}{}_{(k_1+k_2,c)}$ as $f^{1_a,2_b}{}_{(1+2)^c}$. The upper and lower scripts of Lorentz indices a, b and c are introduced to distinguish whether the corresponding generators are contravariant or covariant under Lorentz symmetry. Jacobi identity coming from cyclic sum of the commutator $[[T^{k_1,a}, T^{k_2,b}], T^{k_3,c}]$ is given by

$$f^{1_a,2_b}{}_{(1+2)^e} f^{(1+2)_e,3_c}{}_{(1+2+3)^d} + f^{2_b,3_c}{}_{(2+3)^e} f^{(2+3)_e,1_a}{}_{(1+2+3)^d} + f^{3_c,1_a}{}_{(1+3)^e} f^{(1+3)_e,2_b}{}_{(1+2+3)^d} = 0. \tag{2.3}$$

To relate structure constants to Feynman rules, we need to lower or raise Lorentz indices by contracting with Minkowski metric. For example

$$f^{1_a,2_b}{}_{(1+2)_c} \equiv f^{1_a,2_b}{}_{(1+2)^e}\, \eta_{ec} = (-i)(\eta_{ac} k_{1b} - \eta_{bc} k_{2a}), \tag{2.4}$$

The index-lowered structure constant (2.4) does not enjoy cyclic symmetry. However summing over cyclic permutations of k_1, k_2 and $k_3 = -k_1 - k_2$ produces the familiar color ordered 3-point Yang-Mills vertex

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left(f^{1_a,2_b}{}_{-3_c} + f^{3_c,1_a}{}_{-2_b} + f^{2_b,3_c}{}_{-1_a}\right) \\ &\quad = \frac{i}{\sqrt{2}}\left[\eta_{ab}(k_1 - k_2)_c + \eta_{bc}(k_2 - k_3)_a + \eta_{ca}(k_3 - k_1)_b\right].\end{aligned} \tag{2.5}$$

Three terms at the left handed side of (2.5) can be represented by the three arrowed graphs in figure 1. In this representation, two upper indices a, b of $f^{ab}{}_c$ are denoted by arrows pointing towards the vertex while lower index c is denoted by an arrow leaving the vertex. These three terms are related to each other by counter-clockwise cyclic rotation, thus from the left to right, they represent $f^{12}{}_3, f^{31}{}_2, f^{23}{}_1$.

Note that when expressed in terms of index-lowered structure constants, Jacobi identity becomes

$$\begin{aligned}f^{1_a,2_b}{}_{(1+2)_e}\eta^{e\widetilde{e}} f^{(1+2)_{\widetilde{e}},3_c}{}_{(1+2+3)^d} + f^{2_b,3_c}{}_{(2+3)_e}\eta^{e\widetilde{e}} f^{(2+3)_{\widetilde{e}},1_a}{}_{(1+2+3)^d} \\ + f^{3_c,1_a}{}_{(1+3)_e}\eta^{e\widetilde{e}} f^{(1+3)_e,2_b}{}_{(1+2+3)^d} = 0.\end{aligned} \tag{2.6}$$

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When we interpret relations between numerators as Jacobi identities, the Minkowski metric $\eta^{e\widetilde{e}}$ comes from gluon propagator and connects two structure constants. In discussions below we neglect Lorentz indices of structure constants, which can be easily recovered from the context. Contraction of a structure constant $f^{1_a,2_b}{}_{3_c}$ with other structure constants should be understood as the same as contracting a tensor $f^{1,2}{}_3$ labelled by legs 1, 2, 3.

3 Construction of kinematic numerators

In this section we present an algorithm to construct the kinematic numerators that satisfy Jacobi identity as proposed by Bern, Carrasco and Johansson [1]. We demonstrate our method through 4-point and 5-point amplitudes, and then present the general picture for arbitrary n-point amplitudes.

3.1 kinematic numerators for 4-point amplitudes

For 4-point amplitudes, we consider two color-ordered ones $A(1234)$ and $A(1324)$, since rest of amplitudes can be obtained from these two with the Kleiss-Kuijf (KK) [28] relations. From the prescription of Bern, Carrasco and Johansson, 4-point color-ordered amplitudes can be divided into contributions of s, t and u-channels [1],[4]

$$A(1234) = \frac{n_s}{s} - \frac{n_u}{u}, \qquad A(1324) = -\frac{n_t}{t} + \frac{n_u}{u}. \tag{3.1}$$

Our goal is to construct kinematic (BCJ) numerators n_s, n_t, n_u that satisfy Jacobi identity $n_s + n_t + n_u = 0$. Let us first focus on amplitude $A(1234)$. From color-ordered Feynman rules, amplitude $A(1234)$ contains a s-channel and a u-channel graphs with only cubic vertices. Thus it is natural to attribute expressions coming from Feynman rules to numerators n_s and n_u respectively. In addition we have a contribution from color-ordered 4-point vertex

$$i\eta_{ac}\eta_{bd} - \frac{i}{2}(\eta_{ab}\eta_{cd} + \eta_{ad}\eta_{bc}) = \frac{i}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) + \frac{i}{2}(\eta_{ac}\eta_{bd} - \eta_{ab}\eta_{cd}), \tag{3.2}$$

(where the Lorentz indices of particles $1, 2, 3, 4$ are $a, b, c, d,$). We attribute the first and the second terms of (3.2) at the right handed side to n_s and n_u.[5] Using propagator $\frac{-i\eta_{\mu\nu}}{p^2}$, the s-channel numerator n_s^* can be read out

$$\begin{aligned} n_s^* = &-\frac{i}{2}\left[f^{1,2}{}_{-e} + (f^{e,1}{}_{-2} + f^{2,e}{}_{-1})\right] \cdot \left[f^{3,4}{}_{e} + (f^{e,3}{}_{-4} + f^{4,-e}{}_{-3})\right] \\ &+\frac{i}{2}s\,(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}), \end{aligned} \tag{3.3}$$

where we used equation (2.5) to express 3-point Yang-Mills vertex in terms of kinematic structure constants. A star sign of n_s^* was introduced to denote quantities that have

[4] We follow the sign convention such that n_s, n_t, n_u correspond to cyclic permutations of the three external particles with the fourth one fixed, as they appear in the Jacobi identity.

[5] The reason for assigning the first term with $\eta_{ad}\eta_{bc}$ to s-channel can be understood as collecting contributions that carry the same color dependence as s-channel graph from the complete 4-point Yang-Mills vertex.

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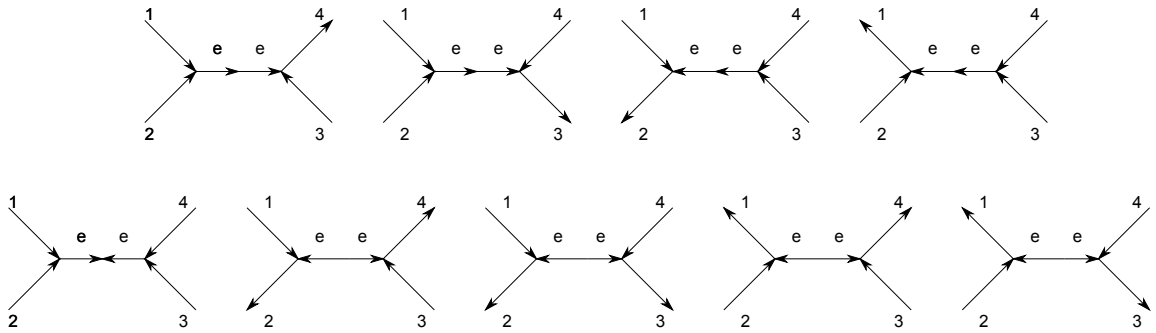

Figure 2. Graphical representation of contributions from eq. (3.4) where the first four graphs correspond to contributions from the first line of the equation, and the remaining five graphs, from the second line of the equation.

not been contracted with polarization vectors. Expanding product of kinematic structure constants in the first line yields the following nine terms

$$\begin{aligned} &f^{1,2}{}_{-e}(f^{e,3}{}_{-4}+f^{4,-e}{}_{-3})+(f^{e,1}{}_{-2}+f^{2,e}{}_{-1})f^{3,4}{}_{e} \\ &+f^{1,2}{}_{-e}f^{3,4}{}_{e}+(f^{e,1}{}_{-2}+f^{2,e}{}_{-1})(f^{e,3}{}_{-4}+f^{4,-e}{}_{-3}), \end{aligned} \tag{3.4}$$

which can be represented by the graphs in figure 2. From these graphs, several information can be read out.

First we note that contraction of the repeated index e leads to consistent arrow directions for internal lines in first four graphs but inconsistent arrow directions for internal line in the remaining five graphs. As we will see in discussions below, contributions from consistent contractions satisfy the Jacobi identity of kinematic structure constants $f^{ab}{}_{c}$ while inconsistent contractions do not. Because of this reason we shall split contributions from Feynman diagrams consisting of only cubic vertices into two groups: the "good ones" with consistent contractions and the "bad ones" with at least one inconsistent contractions.

Secondly, we note that a "good" graph can only have one outgoing arrow among all external particles. The unique external particle line carrying outgoing arrow plays an important role when we consider identities among graphs. In particular, we shall see that Jacobi identity is separately satisfied among graphs that have same outgoing leg.

Based on above observations, we can write numerator $n_s^* = G + X$ where G is contributions from good graphs and X is contributions from bad graphs and four-point vertex $\frac{i}{2}s\,(\eta_{ac}\eta_{bd}-\eta_{ad}\eta_{bc})$. As we will explain in section 3.1.1, remainder X can be eliminated through averaging procedure.

However, for the simple 4-point amplitude, we can do better by digging out some good part from the "bad contribution". We note that we are allowed to freely translate between upper and lower script structure constants through the identities

$$(f^{e,1}{}_{-2}+f^{2,e}{}_{-1})-f^{1,2}{}_{-e} = -i\eta_{ab}(-k_1+k_2)_e+\mathcal{O}(k_{1a},k_{2b}), \tag{3.5}$$

$$(f^{e,3}{}_{-4}+f^{4,-e}{}_{-3})-f^{3,4}{}_{e} = -i\eta_{cd}(-k_3+k_4)_e+\mathcal{O}(k_{3c},k_{4d}), \tag{3.6}$$

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where $\mathcal{O}_e(k_{1a}, k_{2b})$ denotes longitudinal term $\mathcal{O}_e(k_{1a}, k_{2b}) = -i(\eta_{ea}k_{2b} - \eta_{eb}k_{1a})$, and similarly does $\mathcal{O}_e(k_{3c}, k_{4d})$. Both longitudinal terms do not contribute when they are contracted with physical polarization vectors of external legs. Multiplying (3.5) with (3.6) we obtain the identity

$$\begin{aligned} & f^{1,2}{}_{-e} f^{3,4}{}_{e} + (f^{e,1}{}_{-2} + f^{2,e}{}_{-1})(f^{e,3}{}_{-4} + f^{4,-e}{}_{-3}) \\ & = f^{1,2}{}_{-e}(f^{e,3}{}_{-4} + f^{4,-e}{}_{-3}) + (f^{e,1}{}_{-2} + f^{2,e}{}_{-1}) f^{3,4}{}_{e} - (t-u)\eta_{ab}\eta_{cd} \\ & \quad + \mathcal{O}(k_{1a}, k_{2b}, k_{3c}, k_{4d}), \end{aligned} \tag{3.7}$$

where $\mathcal{O}(k_{1a}, k_{2b}, k_{3c}, k_{4d}) = \mathcal{O}_e(k_{3c}, k_{4d}) \cdot \mathcal{O}_e(k_{1a}, k_{2b}) - i\eta_{ab}\mathcal{O}_e(k_{3c}, k_{4d}) \cdot (-k_1 + k_2)_e - i\eta_{cd}\mathcal{O}_e(k_{1a}, k_{2b}) \cdot (-k_3 + k_4)_e$. Thus n_s^* is given by

$$\begin{aligned} n_s^* = & -i \left[f^{1,2}{}_{-e}(f^{e,3}{}_{-4} + f^{4,-e}{}_{-3}) + (f^{e,1}{}_{-2} + f^{2,e}{}_{-1}) f^{3,4}{}_{e} \right] \\ & + \frac{i}{2} \left[s\left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right) + (t-u)\eta_{ab}\eta_{cd} \right] + \mathcal{O}(k_{1a}, k_{2b}, k_{3c}, k_{4d}), \end{aligned} \tag{3.8}$$

and n_u^*, n_t^* can be derived from it by permutations of indices (123) $\to$ (312) and (123) $\to$ (231) respectively. To obtain the numerators n_s, n_u, n_t in equation (3.1), we just need to contract n_s^*, n_u^*, n_t^* with physical polarization vectors, thus the longitudinal terms $\mathcal{O}(k_{1a}, k_{2b}, k_{3c}, k_{4d})$ drop out.

Having obtained expressions (3.8) we want to check the Jacabi idenity $n_s + n_t + n_u = 0$. First we notice that after contraction, contributions from $\frac{i}{2}\left[s\left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right) + (t-u)\eta_{ab}\eta_{cd}\right]$ will be trivially zero under cyclic sum. To see contributions from the first line of equation (3.8) give zero, let us expand the first line of n_s^* into

$$f^{1,2}{}_{-e} f^{-e,3}{}_{-4} + f^{1,2}{}_{-e} f^{4,-e}{}_{-3} + f^{e,1}{}_{-2} f^{3,4}{}_{e} + f^{2,e}{}_{-1} f^{3,4}{}_{e}. \tag{3.9}$$

and write down corresponding terms of n_u^* by permutation (123) $\to$ (231)

$$f^{2,3}{}_{-e} f^{-e,1}{}_{-4} + f^{2,3}{}_{-e} f^{4,-e}{}_{-1} + f^{e,2}{}_{-3} f^{2,4}{}_{e} + f^{3,e}{}_{-2} f^{2,4}{}_{e}, \tag{3.10}$$

and similarly terms of n_t^* by permutation (123) $\to$ (312)

$$f^{3,1}{}_{-e} f^{-e,2}{}_{-4} + f^{3,1}{}_{-e} f^{4,-e}{}_{-2} + f^{e,3}{}_{-1} f^{2,4}{}_{e} + f^{1,e}{}_{-3} f^{2,4}{}_{e}. \tag{3.11}$$

When summing these three contributions (3.9), (3.10) and (3.11) together, the first terms from each contribution add up to zero because of the Jacobi identity derived from cyclic permutations of legs (123),

$$f^{1,2}{}_{-e} f^{-e,3}{}_{-4} + f^{2,3}{}_{-e} f^{-e,1}{}_{-4} + f^{3,1}{}_{-e} f^{-e,2}{}_{-4} = 0. \tag{3.12}$$

Adding up the rest three terms from each contribution again gives zero by following three Jacobi identities (3.13), (3.14) and (3.15),

$$f^{1,2}{}_{-e} f^{4,-e}{}_{-3} + f^{2,4}{}_{-e} f^{1,-e}{}_{-3} + f^{4,1}{}_{-e} f^{2,-e}{}_{-3} = 0, \tag{3.13}$$

$$f^{3,1}{}_{-e} f^{4,-e}{}_{-2} + f^{1,4}{}_{-e} f^{3,-e}{}_{-2} + f^{4,3}{}_{-e} f^{1,-e}{}_{-2} = 0, \tag{3.14}$$

$$f^{2,3}{}_{-e} f^{4,-e}{}_{-1} + f^{3,4}{}_{-e} f^{2,-e}{}_{-1} + f^{4,2}{}_{-e} f^{3,-e}{}_{-1} = 0. \tag{3.15}$$

which correspond to fixed leg $3, 2, 1$.

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It is easy to see that when expressed graphically, terms in equation (3.12) shall all have the outgoing arrow on leg 4, and similarly terms in equations (3.13), (3.14) and (3.15) shall all have the outgoing arrows on legs 3, 2 and leg 1 respectively. Thus Jacobi identity can be translated as the sum of three graphs related to each other by cyclic permutations with a fixed leg having outgoing arrow.

3.1.1 Eliminating contact terms

In the discussion above we demonstrated explicitly that contributions from cubic and quartic diagrams together give rise to numerators that satisfy the Jacobi identity. While "good parts" of these contributions satisfy the identity manifestly, the "bad parts", do not. For 4-point amplitudes since structure of amplitudes is simple, we were able to rewrite these "bad parts" into nicer forms. However this rewriting becomes rather difficult for higher point amplitudes, therefore we resort to an alternative way to solve the problem. The idea is the following. Since the numerator such as n_s is calculated from contracting n_s^* with polarization vectors, in a gauge theory we have the freedom to choose different gauges (i.e., different polarization vectors). Using this freedom we can eliminate "bad contributions" and keep only "good contributions", thus the final result will satisfy Jacobi identity manifestly.

Now we demonstrate the idea using 4-point amplitudes. For simplicity let us abuse the notation a bit by writing

$$n_s(q) = \epsilon_1^{a_1}(q_1)\dots\epsilon_n^{a_n}(q_n)n_s^* \equiv \epsilon(q)\cdot n_s^*\,, \tag{3.16}$$

where q represents the set of reference momenta $\{q_1, q_2, \dots, q_n\}$ collectively. Using the notation that the good contribution given by equation (3.9), (3.10) and (3.11) as $\tilde{n}_s^*$, $\tilde{n}_u^*$ and $\tilde{n}_t^*$, we have

$$\begin{aligned} n_s(q) &= -i\epsilon(q)\cdot\tilde{n}_s^* + \frac{i}{2}sX_s(q), \\ X_s(q) &\equiv \epsilon(q)\cdot X_s^* = \epsilon(q)\left\{(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) + \frac{(t-u)}{s}\eta_{ab}\eta_{cd}\right\} \end{aligned} \tag{3.17}$$

and similarly for $n_u(q), n_t(q), X_u(q)$, and $X_t(q)$. Thus the amplitudes are given by

$$\begin{aligned} A(1234) &= \epsilon(q)\cdot\left(\frac{-i\tilde{n}_s^*}{s} - \frac{-i\tilde{n}_u^*}{u} + \frac{i}{2}X_1^*\right), \qquad X_1^* = X_s^* - X_u^* \\ A(1324) &= \epsilon(q)\cdot\left(-\frac{-i\tilde{n}_t^*}{t} + \frac{-i\tilde{n}_u^*}{u} + \frac{i}{2}X_2^*\right), \qquad X_2^* = -X_t^* + X_u^*\,. \end{aligned} \tag{3.18}$$

In above expressions, good contributions into $\tilde{n}_i^*$, which satisfy Jacobi identity, have been separated from the bad contributions X_i^* manifestly. Having done the reorganization, next step is to eliminate the X_i^* parts. To realize it, we consider the average of above two color-ordered amplitudes over three different choices of gauges. Since $A(1234)$ is invariant under gauge choices, we can get rid of all X_i^* parts simultaneously if we impose following three conditions

$$T_3 : \begin{cases} 1 = c_1 + c_2 + c_3 \\ 0 = \sum_{i=1}^3 c_i\epsilon(q_i)\cdot X_1^* \\ 0 = \sum_{i=1}^3 c_i\epsilon(q_i)\cdot X_2^* \end{cases} \tag{3.19}$$

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By gauge invariance, the first condition guarantee that

$$A(1,2,3,4) = \frac{n_s}{s} - \frac{n_u}{u}, \quad A(1,3,2,4) = \frac{-n_t}{t} + \frac{n_u}{u} \tag{3.20}$$

where $n_s = \sum_{i=1}^{3} -ic_i\epsilon(q_i)\cdot\tilde{n}_s^*$ and similarly for n_u, n_t. Since each $\epsilon(q_i)\cdot\tilde{n}_{s,u,t}$ satisfies Jacobi identity, so do n_s, n_u, n_t. To see that there is indeed a solution for c_i, we simply need to show that the following matrix has nonzero determinant

$$\begin{pmatrix} 1 & 1 & 1 \\ \epsilon(q_1)\cdot X_1^* & \epsilon(q_2)\cdot X_1^* & \epsilon(q_3)\cdot X_1^* \\ \epsilon(q_1)\cdot X_2^* & \epsilon(q_2)\cdot X_2^* & \epsilon(q_3)\cdot X_2^* \end{pmatrix} . \tag{3.21}$$

This can be checked by explicit calculations.

3.1.2 KK vs. BCJ-independent basis

In previous section we have showed how to derive kinematic numerators n_s, n_t and n_u satisfying Jacobi identity by eliminating bad contributions. In the derivation we considered the analytic structures of two color ordered amplitudes $A(1234)$ and $A(1324)$, which serve as a basis when KK-relations [28] are taken into account. Since there were two remainders (i.e., the bad contribution part X) we need to introduce three c_i to achieve our goal. But could we do better by introducing fewer variables c_i?

Let us consider only $A(1234)$. To eliminate its remainder term, we only need to average over two different gauge choices. The constraint conditions for c_i are

$$T_2 : \begin{cases} 1 = \widetilde{c}_1 + \widetilde{c}_2 \\ 0 = \sum_{i=1}^{2} \widetilde{c}_i\epsilon(q_i)\cdot X_1^* \end{cases} \tag{3.22}$$

which have the solution $\widetilde{c}_1 = \frac{-\epsilon(q_2)\cdot X_1^*}{\epsilon(q_1)\cdot X_1^* - \epsilon(q_2)\cdot X_1^*}$ and $\widetilde{c}_2 = \frac{\epsilon(q_1)\cdot X_1^*}{\epsilon(q_1)\cdot X_1^* - \epsilon(q_2)\cdot X_1^*}$. Substituting them back, we have

$$A(1234) = \frac{n_s}{s} - \frac{n_u}{u}, \quad n_s = -i(\widetilde{c}_1\epsilon(q_1) + \widetilde{c}_2\epsilon(q_2))\cdot\tilde{n}_s^*, \qquad n_u = -i(\widetilde{c}_1\epsilon(q_1) + \widetilde{c}_2\epsilon(q_2))\cdot\tilde{n}_u^* \tag{3.23}$$

Having obtained these two numerators, we can *define* an amplitude using

$$\widetilde{A}(1324) = -\frac{-(n_s + n_u)}{t} + \frac{n_u}{u} \tag{3.24}$$

It is easy to check that the amplitude just defined satisfies fundamental BCJ relation [1] by construction,

$$s_{21}A(1234) + (s_{21} + s_{23})\widetilde{A}(1324) = 0, \tag{3.25}$$

Since the same relation is satisfied between physical amplitudes, $s_{21}A(1234) + (s_{21} + s_{23})A(1324) = 0$, we conclude that $\widetilde{A}(1324) = A(1324)$ and in particular, $n_t = -(n_s + n_u)$, i.e., the kinematic-dual Jacobi identity we would like to have.

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Above discussions show that, because of the BCJ relation for color-ordered amplitudes, we can use fewer c_i to eliminate remainders. After doing so, X_i in rest of the color-ordered amplitudes automatically disappear, i.e.,

$$\widetilde{c}_1\epsilon(q_1)\cdot X_2^* + \widetilde{c}_2\epsilon(q_2)\cdot X_2^* = 0\ . \tag{3.26}$$

Now we have developed two methods to eliminate remainders through averaging over KK or BCJ basis of amplitudes. We need to clarify the relation between these two methods. To do so, let us assume that we have solution (c_1, c_2, c_3) with gauge choice $\epsilon(q_3) = \alpha\epsilon(q_1) + \beta\epsilon(q_2)$. This gauge choice can be achieved if reference spinors of polarization vectors of three particles, for example, $2, 3, 4$ are same for gauge choices $\epsilon(q_1), \epsilon(q_2), \epsilon(q_3)$, but reference spinors of polarization vector of particle 1 satisfy the relation $\epsilon(q_3) = \alpha\epsilon(q_1) + \beta\epsilon(q_2)$. Putting it back to the second equation of T_3 given in (3.19) and comparing with the second equation of T_2 given in (3.22), we can write down the following solution for T_2,

$$\widetilde{c}_1 = c_1 + \alpha c_3 + y\epsilon(q_2)\cdot X_1^*, \qquad \widetilde{c}_2 = c_2 + \beta c_3 - y\epsilon(q_1)\cdot X_1^* \tag{3.27}$$

where y is determined by $\widetilde{c}_1 + \widetilde{c}_2 = 1$ to be $y = \frac{(\alpha+\beta-1)c_3}{(\epsilon(q_1)-\epsilon(q_2))\cdot X_1^*}$. It is easy to check that the above indeed constitutes a solution if we assume $\alpha + \beta - 1 = 0$. In this case (3.26) is automatically satisfied because of the third equation in (3.19). To see that indeed $\alpha+\beta = 1$, notice that the reference spinors of particle 1 have relation $\widetilde{\mu}_3 = a\widetilde{\mu}_1 + b\widetilde{\mu}_2$, so

$$\epsilon(\widetilde{\mu}_3) = \frac{\lambda_1\widetilde{\mu}_3}{[1|\widetilde{\mu}_3]} = \left(\frac{a\,[1|\widetilde{\mu}_1]}{a\,[1|\widetilde{\mu}_1] + b\,[1|\widetilde{\mu}_2]}\right)\frac{\lambda_1\widetilde{\mu}_1}{[1|\widetilde{\mu}_1]} + \left(\frac{b\,[1|\widetilde{\mu}_2]}{a\,[1|\widetilde{\mu}_1] + b\,[1|\widetilde{\mu}_2]}\right)\frac{\lambda_1\widetilde{\mu}_2}{[1|\widetilde{\mu}_2]} \Longrightarrow \alpha + \beta = 1$$

This explanation shows that solutions $(\widetilde{c}_1, \widetilde{c}_2)$ can be taken as a special case of solutions (c_1, c_2, c_3).

3.2 5-point numerators

For 5-point amplitudes KK relations reduce the number of independent color-ordered amplitudes to six. It was shown by Bern, Carrasco and Johansson [1] that these six amplitudes can be written into following forms with fifteen numerators suggested by possible cubic graphs:

$$\begin{aligned}
A(12345) &= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}},\\
A(14325) &= \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}},\\
A(13425) &= \frac{n_9}{s_{13}s_{45}} + \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}},\\
A(12435) &= \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}},\\
A(14235) &= \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}},\\
A(13245) &= \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}},
\end{aligned} \tag{3.28}$$

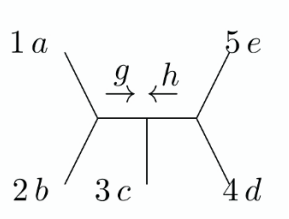

Figure 3. A Feynman diagram contributing to n_1^*

To find expressions for these n_i, as in the 4-point amplitudes, we divide contributions from Feynman rules to good contributions plus a remainder (bad contributions),

$$A^*(12345) = \frac{n_1^*}{s_{12}s_{45}} + \frac{n_2^*}{s_{23}s_{51}} + \frac{n_3^*}{s_{34}s_{12}} + \frac{n_4^*}{s_{45}s_{23}} + \frac{n_5^*}{s_{51}s_{34}} + X^*(12345)\ . \tag{3.29}$$

As before we use $*$ to denote quantities that have not been contracted with polarization vectors. The definition of n_i^* and X^* is the following. First we include all contributions that contain at least one 4-point vertex in Feynman diagrams to X^*. For remaining Feynman diagrams having only cubic vertices like figure 3 for example, we use (2.5) to translate 3-point vertices to kinematic structure constants, thus obtain [6]

$$\left[(f^{g,1}{}_{-2} + f^{2,g}{}_{-1}) + f^{1,2}{}_{-g}\right] \times (f^{-g,3}{}_{h} + f^{-h,-g}{}_{-3} + f^{3,-h}{}_{g}) \times \left[(f^{h,4}{}_{-5} + f^{5,h}{}_{-4}) + f^{4,5}{}_{-h}\right]\ . \tag{3.30}$$

Expanding (3.30) produces 27 terms, five terms among them have consistent arrows in the internal lines (see figure 4) (so they are good contributions). We assign these five terms to n_1^* and the rest to X^* (these bad contributions), thus we have

$$n_1^* = f^{1,2}{}_{-g} f^{g,3}{}_{h}(f^{h,4}{}_{5} + f^{5,h}{}_{4}) + f^{1,2}{}_{-g} f^{-h,-g}{}_{-3} f^{4,5}{}_{-h} + (f^{g,1}{}_{-2} + f^{2,g}{}_{-1}) f^{3,-h}{}_{g} f^{4,5}{}_{-h}. \tag{3.31}$$

It is worth noticing that five terms in n_1^* correspond to five possible assignments of single outgoing arrow to external legs in graphical representations. If we use $n_{1,k}^*$ to denote the consistent graph having leg k with outgoing arrow, for example $n_{1,3}^* = f^{1,2}{}_{-g} f^{-h,-g}{}_{-3} f^{4,5}{}_{-h}$, the numerator can be written as $n_1^* = \sum_{k=1}^{5} n_{1,k}^*$. It is straightforward to see that numerators from rest of channels can be written into similar structures. In particular, $(-n_{15}^*)$ is found to be the same as permutation $(123) \to (312)$ of n_1^*

$$-n_{15}^* = f^{3,1}{}_{-g} f^{g,2}{}_{h}(f^{h,4}{}_{5} + f^{5,h}{}_{4}) + f^{3,1}{}_{-g} f^{-h,-g}{}_{-2} f^{4,5}{}_{-h} + (f^{g,3}{}_{-1} + f^{1,g}{}_{-3}) f^{2,-h}{}_{g} f^{4,5}{}_{-h}, \tag{3.32}$$

and $(-n_4^*)$, the same as permutation $(123) \to (231)$ of n_1^*,

$$-n_4^* = f^{2,3}{}_{-g} f^{g,1}{}_{h}(f^{h,4}{}_{5} + f^{5,h}{}_{4}) + f^{2,3}{}_{-g} f^{-h,-g}{}_{-1} f^{4,5}{}_{-h} + (f^{g,2}{}_{-3} + f^{3,g}{}_{-2}) f^{1,-h}{}_{g} f^{4,5}{}_{-h}. \tag{3.33}$$

[6] For simplicity we neglect the overall factor $\frac{(-i)^2}{(\sqrt{2})^3}$, where $1/\sqrt{2}$ comes from (2.5) and $(-i)^2$ come from two propagators.

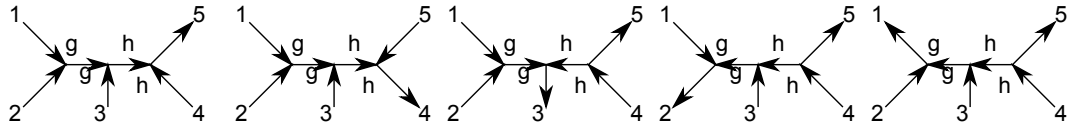

Figure 4. Five terms with consistent arrow directions contributing to n_1^*

When adding up (3.31), (3.32) and (3.33), terms with same outgoing leg will add to zero by Jacobi identity. Thus by our construction, we have Jacobi identity $n_1^* - n_{15}^* - n_4^* = 0$. Similar argument shows when we permute $(345) \to (534)$ we will produce $(-n_3^*)$ and when we permute $(345) \to (453)$ we will produce $(-n_{12}^*)$. Thus $n_1^* - n_3^* - n_{12}^* = 0$ is guaranteed by Jacobi identity of kinematic structure constants $f^{ab}{}_c$.

3.2.1 Eliminating contact terms

Having established the form (3.29) as well as similar expressions for other five amplitudes given in (3.28), we construct the n_i given in (3.28) by averaging over different choices of gauges. Just like for the 4-point amplitudes, we consider seven gauge choices denoted by $\epsilon(q_i)$ with $i = 1, \ldots, 7$ for polarization vectors under gauge choice $q_i = \{q_{1,i}, q_{2,i}, q_{3,i}, q_{4,i}, q_{5,i}\}$ and impose following seven equations for coefficients c_i, $i = 1, \ldots, 7$:

$$\sum_{i=1}^{7} c_i = 1, \qquad \sum_{i=1}^{7} c_i \epsilon(q_i) \cdot X_j = 0, \quad j = 1, \ldots, 6 \tag{3.34}$$

where six remainders X_j are those given in (3.29). After solving c_i from above equations, we can get n_i defined in (3.28) as following

$$n_i = \sum_{j=1}^{7} c_j \epsilon(q_j) \cdot n_i^* \ . \tag{3.35}$$

Since by our construction, n_i^* satisfy Jacobi identity even before contracting with polarization vectors and c_j are same for all fifteen n_i^*, n_i will too satisfy Jacobi identity.

In the prescription above, we use the KK-basis (i.e., the basis under KK-relation) and BCJ relations between amplitudes follow as a consequence of the Jacobi identities among n_i. However, if our focus is the construction of these n_i numerators, we can take another logic starting point using only BCJ-basis (i.e., the basis under BCJ relation). For 5-point amplitudes, we can take $A(12345)$ and $A(13245)$ as BCJ-basis and consider averaging over three different gauge choices

$$c_1 + c_2 + c_3 = 1, \qquad \sum_{i=1}^{3} c_i \epsilon(q_i) \cdot X^*(12345) = 0, \qquad \sum_{i=1}^{3} c_i \epsilon(q_i) \cdot X^*(13245) = 0 \tag{3.36}$$

By imposing these conditions we obtain

$$n_i = \sum_{i=1}^{3} c_i \epsilon_i \cdot n_i^*, \quad i = 1, 2, 3, 4, 5, 10, 11, 15 \ . \tag{3.37}$$

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Then we construct the remaining seven coefficients using Jacobi identities

$$\begin{aligned} &n_6 = n_{10} + n_1 - n_3 - n_4 + n_5, \qquad n_7 = n_2 - n_4, \qquad n_8 = -n_3 + n_5, \\ &n_9 = n_3 - n_5 + n_6, \qquad n_{12} = n_1 - n_3, \qquad n_{13} = n_1 + n_2 - n_3 - n_4 - n_6, \\ &n_{14} = -n_2 + n_4 + n_6 \end{aligned} \tag{3.38}$$

The relation between these two eliminating methods can be understood similarly to the 4-point example.

3.3 n-point numerators

Having above two examples, it is straightforward to see the structure of kinematic numerators for n-point amplitudes. Generically a color-ordered amplitude can be written as

$$A^* = \sum_i \frac{n_i^*}{D_i} + X^* \tag{3.39}$$

where the sum is taken over all cubic graphs. In this expression, n_i^* contain only contributions from cubic graphs that have consistent arrow directions. All other contributions from cubic graphs with inconsistent arrow directions as well as graphs with at least one 4-point vertex are assigned to X^* part. Furthermore, according to which external particle has been assigned with the outgoing arrow in graphical representation, we can divide kinematic numerator into

$$n_i^* = \sum_{k=1}^{n} n_{i,k}^*, \tag{3.40}$$

so that each $n_{i,k}^*$ is represented by a single graph. All these $n_{i,k}*$ will have Jacobi identities among themselves with different i but same fixed k.

Having expressions as in (3.39), we average over amplitudes to eliminate the remainder terms X^*. This can be done through averaging over either KK-basis or BCJ basis. The average coefficients c_i are determined by $N_1 = (n-2)! + 1$ equations

$$T_{KK} : \begin{cases} 1 = \sum_{i=1}^{N_1} c_i \\ 0 = \sum_{i=1}^{N_1} c_i \epsilon(q_i) \cdot X_j^*, \qquad j = 1, \ldots, (n-2)! \end{cases} \tag{3.41}$$

for KK-basis or $N_2 = (n-3)! + 1$ equations

$$T_{BCJ} : \begin{cases} 1 = \sum_{i=1}^{N_2} \widetilde{c}_i \\ 0 = \sum_{i=1}^{N_2} \widetilde{c}_i \epsilon(q_i) \cdot X_j^*, \qquad j = 1, \ldots, (n-3)! \end{cases} \tag{3.42}$$

for BCJ-basis. After the averaging we have $n_j = \sum_{i=1}^{N} c_i \epsilon(q_i) \cdot n_j^*$. Other n_i's which do not show up in the KK-basis or BCJ-basis can be constructed from various relations including Jacobi identities. From either method we can construct the numerators proposed by Bern, Carrasco and Johansson in [1].

A technical issue concerning the above averaging procedure is the existence of solution for equation (3.41) and (3.42). The existence for lower point amplitudes can be checked by explicit calculations, but we have not find a proof for general n. In this paper, we will assume their existence.

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4 Fundamental BCJ relations

In previous section, we have shown how to construct the kinematic numerator satisfying the Jacobi identity by averaging over different gauge choices. An important step is to separate contributions from Feynman diagrams to two parts

$$A^* = \sum_i \frac{\sum_{k=1}^n n^*_{i,k}}{D_i} + X^*, \tag{4.1}$$

where each $n^*_{i,k}$ can be represented by a single consistent arrow graph with only cubic vertices. Effectively, we can treat these graphs as if they were built from the Feynman rules with only cubic vertices, where the coupling is given by kinematic structure constant $f^{ab}{}_c$. From this point of view we can define an n-point color-ordered amplitude for given k as

$$A^*_{n;k} = \sum_i \frac{n^*_{i,k}}{D_i}. \tag{4.2}$$

The physical amplitude is given by linear combination of these fixed-k amplitudes

$$A = \sum_{i=1}^{N} c_i \epsilon(q_i) \cdot \sum_{k=1}^{n} A^*_{n;k}. \tag{4.3}$$

An important feature of formula (4.3) is that the part $\sum_{i=1}^N c_i\epsilon(q_i)$ coming from averaging procedure does not depend on the color ordering of external particles.

The amplitudes defined in (4.2) contain similar algebraic structure as these amplitudes $A_n^{(color)}$ of color-dressed scalar theory considered in [29]. In that paper we have shown that amplitudes $A_n^{(color)}$ satisfy color-order reversed relations, U(1) decoupling relations, KK-relations and both on-shell and off-shell BCJ relations. Because of the similarity between amplitudes $A_n^{(color)}$ and $A^*_{n;k}$, it is natural to ask if the $A^*_{n;k}$ defined by (4.2) obey these same identities. We can not make the naive conclusion since there are differences between these two theories. First the kinematic coupling constant $f^{ab}{}_c$ here is only antisymmetric between a, b while group structure constant f^{abc} of U(N) is totally antisymmetric. In addition, $f^{ab}{}_c$ depends on kinematics while f^{abc} is independent of momenta. Bearing these in mind, we discuss properties of new amplitudes in this section.

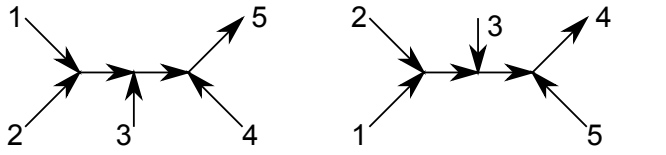
4.1 The color-order reversed relation

Since each $n^*_{i,k}$ is given by single graph, it is easy to analyze it directly. Under the color-order reversing, each cubic vertex will gain a minus sign coming from $f^{ab}{}_c = -f^{ba}{}_c$ (See figure 5a for example). For n-points amplitudes, there are $(n-2)$ cubic vertices and $(n-3)$ propagators, thus we will get a sign $(-)^{n-2}$, i.e., we do have

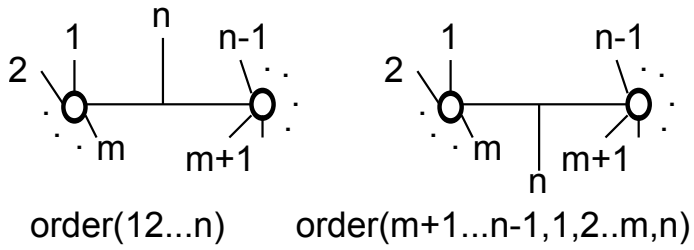
$$A^*_{n;k}(123\ldots n) = (-)^n A^*_{n;k}(n\ldots 321)\ . \tag{4.4}$$

To see U(1)-decoupling relation

$$\sum_{cyclic} A^*_{n;k}(C(1,2,\ldots,n-1),n) = 0 \tag{4.5}$$



(a) reversing the color ordering in a five point graph



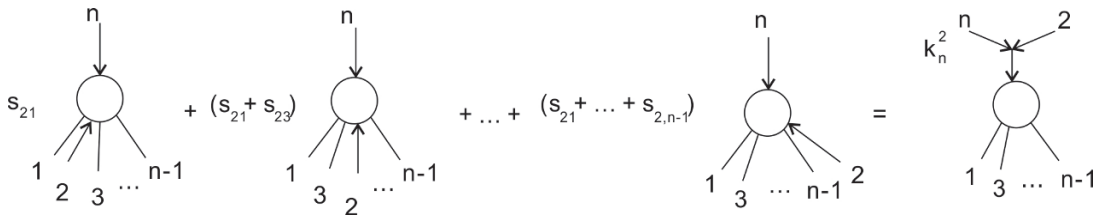
(b) two typical terms in U(1)-decoupling relation

Figure 5. Demonstration of color-order reversed relation (part (a)) and the U(1)-decoupling relation (part (b))

is satisfied, we draw two typical terms in the cyclic sum in figure 5b). These two terms have same denominator and same numerator up to a sign since the only difference between them is the reversing of vertex connecting n, thus contributing $(-)$ sign. However, the left term belongs to color ordering $(123,\ldots,n)$ while the right term belongs to color ordering $(m+1,\ldots,n-1,1,2,\ldots,m,n)$, thus we can see the general pair-by-pair cancellation in U(1) identity given in (4.5).

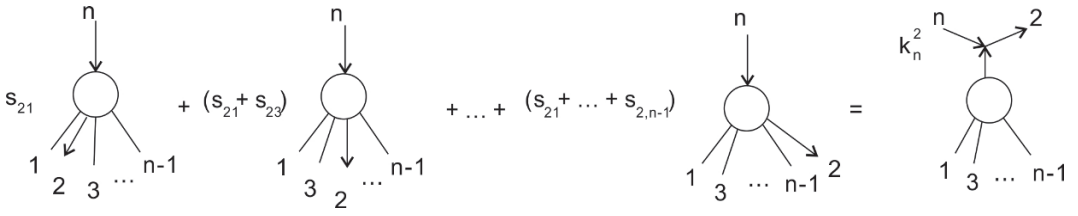
4.2 The off-shell and on-shell BCJ relation

Just like the color-dressed scalar field theory, the $A^*_{n;k}$ satisfies a similar off-shell BCJ relation, which can be represented graphically by



(4.6)

with the momentum of particle n taken off-shell. Depending on the arrow directions of $2, n$ we have another two similar relations



(4.7)

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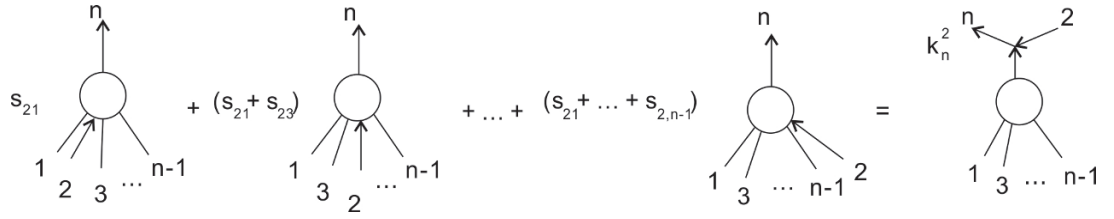
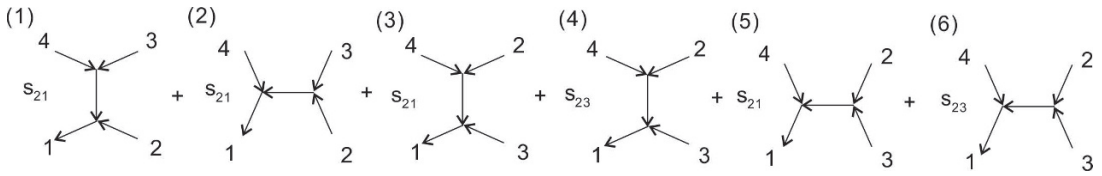
and



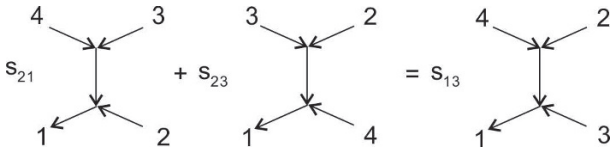
(4.8)

There three relations show that the off-shell BCJ relations are, as in the case of color-order reversed and U(1)-decoupling relations, independent of the choice of arrow directions. The proof of relations (4.6) is similar to the proof given in [29] for color-dressed scalar theory.

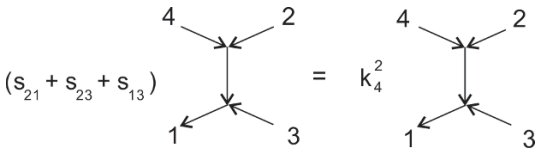
The case of $n = 3$ is trivially true from momentum conservation. For $n = 4$, the left handed side of relation (4.6) consists of following sum of graphs,



We note that graphs (2) and (5) cancel due to antisymmetry of the structure constant. Using Jacobi identity, graphs (1) and (6) combine to produce



which, when added to the rest two graphs (3) and (4), produces the result as claimed using the on-shell conditions of particles $1, 2, 3$.



Similar manipulations can be done for (4.7) and (4.8).

Having proven the $n = 4$ example, let us consider, for example, relation (4.8) for general n. We divide contributions to any amplitude $A^*_{n;k}$ into the two sub-amplitudes that share same cubic vertex with leg n. (See part figure 5b as an illustration.) i.e.,

$$A^*_{n;k} = \sum_{\#(n_L)=1}^{\#(n_L)=n-2} A_L(\{n_L\}; P_L) V_3(n, P_L, P_R) A_R(-P_R; \{n_R\}) \tag{4.9}$$

where the number of legs in set n_L can be $1, 2, \ldots, (n-2)$, and we used $V_3(n, P_L, P_R)$ to denote the cubic vertex that connects leg n to the two sub-amplitudes. Using this

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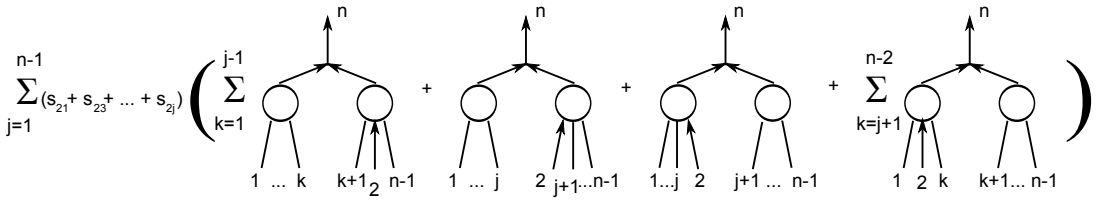
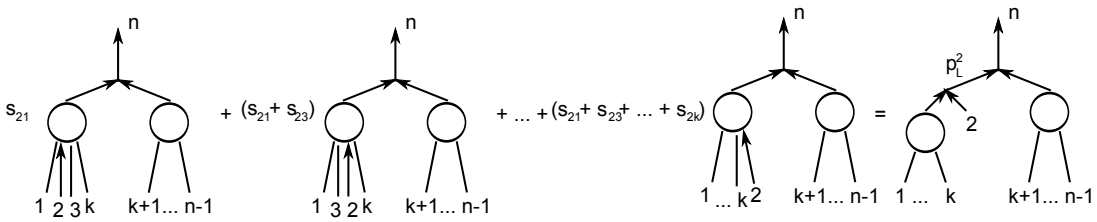
decomposition, the left handed side of (4.8) can be expressed by following graphs



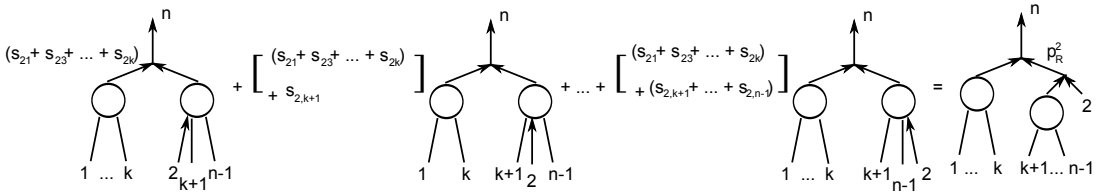
(4.10)

We can categorize terms in (4.10) according to whether leg 2 belongs the left or right sub-amplitude. When the 2 belongs to the left, the summation is given by



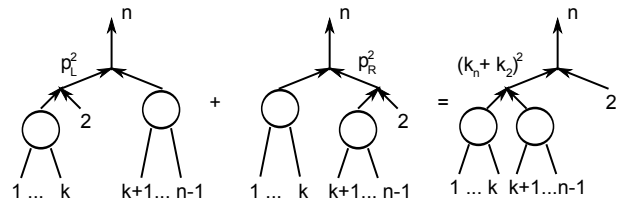
(4.11)

where we used the off-shell BCJ relation for left part with fewer points. The value of k in sum (4.11) can be $1, 3, 4, \ldots, n-2$. Similarly, when leg 2 belongs to the right sub-amplitude, the summation is given by



(4.12)

The above sum can be split into two parts. First there are terms carrying the common factor $\sum_{i=1}^{k} s_{2i}$, and their sum $(\sum_{i=1}^{k} s_{2i})(A_R(2, k+1, \ldots, n-1, P_R) + A(k+1, 2, \ldots, n-1, P_R) + \ldots + A(k+1, \ldots, n-1, 2, P_R)) = 0$ by U(1)-decoupling identity. The remaining part can be simplified by off-shell BCJ relation for fewer points. The value of k for sum (4.12) can be $1, 3, 4, \ldots, n-2$. The sum given in (4.11) and (4.12) can be further combined to



(4.13)

by using the Jacobi identity derives from permuting the internal 4-point tree $\{n, 2, \{1, \ldots, k\}, \{k+1, \ldots, n-1\}\}$. When we sum over k, we get $(k_n + k_2)^2 V_3(P_L, 2, n) A^*_{n-1}(1, 3, \ldots, n-1, P_L)$. Finally, result given in (4.13) is combined with

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term $(\sum_{j=1}^{n-1} s_{2j})V_3(P_L, 2, n)A^*_{n-1}(1, 3, \ldots, n-1, P_L)$ coming from the decomposition of $A^*_n(1, 3, 4, \ldots, n-1, 2, n)$ according to (4.9) (which is the boundary term that has been neglected in the sum (4.11) and (4.12)). Putting together, we have the same graph multiplied by $(k_n + k_2)^2 - 2k_2 \cdot k_n = k_n^2$, which is exactly the right handed side of (4.8). In other words, we have proved the off-shell BCJ relation for $A^*_{n;k}$ amplitudes defined in (4.2). Taking the on-shell limit $k_n^2 \to 0$, we get the familiar on-shell BCJ relation.

4.3 The KK-relation

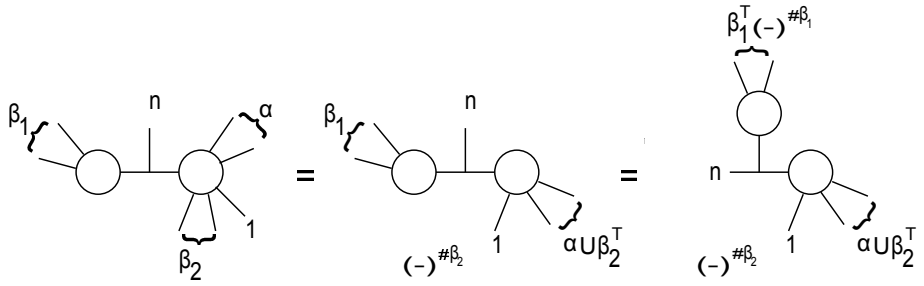
The KK-relation found originally in [28] for gauge theory is given by

$$A_n(\beta_1, \ldots, \beta_r, 1, \alpha_1, \ldots, \alpha_s, n) = (-1)^r \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\}^T)} A_n(1, \{\sigma\}, n), \tag{4.14}$$

where the sum is over all permutations keeping relative ordering inside the set α and the set β^T (where the T means the set β with its order reversed), but at the same time allowing all relative orderings between sets α and β. We show that relation (4.14) still holds if we replace A_n by the fixed k amplitude $A^*_{n;k}$ defined in (4.2).

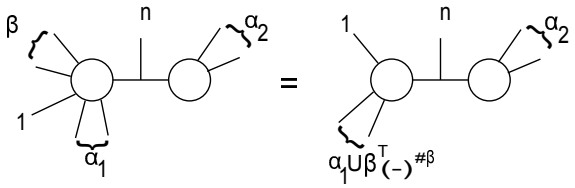
When $\{\alpha\}$ is empty set (4.14) reduces to the color-order reversed relation (4.4), while when there is only one leg in the set $\{\beta\}$ or the set $\{\alpha\}$, (4.14) reduces to the U(1)-decoupling identity (4.5). Thus the color-order reversed relation and the U(1)-decoupling identity are just two special cases of KK relation. Since KK-relations coincide with these two relations for $n \leq 5$, the starting point of our induction proof is checked.

Now we give the proof. Using the graphical representation, when the set $\{\alpha\}$ is not empty, there are two types of graphs depending on if 1 is at the left or right handed side of n. When leg 1 is at the right sub-amplitude as described by the left graph of (4.15)



(4.15)

we can use KK-relation of the right sub-amplitude to get the middle graph of (4.15). After that we reverse the ordering of sub-amplitude β_1 and flip it to the right hand side of n. The final result is the last graph of (4.15). When leg 1 is at the left handed side of n as given by the left graph of (4.16),



(4.16)

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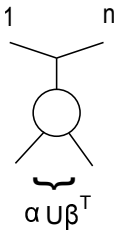
we can use KK-relation for the left sub-amplitude to get the right graph of (4.16). When we combine results from (4.16) and (4.15), we find they are nothing but the graphical representation of right handed side of equation (4.14) except contributions from following graphs



(4.17)

These contributions are nothing, but[7]

$$V_3(n1P)\left(\sum_{\{\sigma\}\in P(O\{\alpha\}\cup O\{\beta^T\})} A^*_{n-1}(P_{1,n},\{\sigma\})\right)=0\ . \tag{4.18}$$

Sum inside the bracket of (4.18) to be zero can be proved by exactly same method as that given in [29] after using two times of KK-relation for $(n-1)$-point amplitudes.

5 Kinematic ordering of gauge theory amplitude

As proposed in [30] and proved in [29], the full gauge theory amplitude can be represented by the manifestly $(n-2)!$ symmetric KLT formula (which was found in [34])

$$\mathcal{A}_n = (-)^n\sum_{\gamma,\beta}\frac{\widetilde{A}(n,\gamma(2,\ldots,n-1),1)\mathcal{S}[\gamma(2,\ldots,n-1)|\beta(2,\ldots,n-1)]_{p_1}A(1,\beta(2,\ldots,n-1),n)}{s_{123\ldots(n-1)}} \tag{5.1}$$

where leg k_n has to be taken off-shell prior to the summation and the full amplitude is given by the limit $k_n^2\to 0$. The momentum kernel $\mathcal{S}$ is defined as

$$\mathcal{S}[i_1,\ldots,i_k|j_1,j_2,\ldots,j_k]_{p_1}=\prod_{t=1}^{k}(s_{i_t1}+\sum_{q>t}^{k}\theta(i_t,i_q)s_{i_ti_q})\,, \tag{5.2}$$

where $\theta(i_t,i_q)$ is zero when pair (i_t,i_q) has the same ordering in both set $\mathcal{I},\mathcal{J}$ and otherwise it is one. In the KLT formulation above, one copy of the amplitudes $\widetilde{A}$ is calculated from the color-dressed scalar theory discussed in [29] and other copy A is the familiar color-ordered gauge theory amplitude.

To calculate the sum in numerator of (5.1), let us consider the following sum for given fixed ordering of γ, for example, $\gamma(2,\ldots,n-1)=(2,3,\ldots,n-1)$,

$$\sum_{\{i\}\in S_{n-2}}\mathcal{S}[2,3,\ldots n-1|i_2,i_3,\ldots i_{n-1}]_{k_1}\,A_n(1,\,i_2,i_3,\ldots i_{n-1};n) \tag{5.3}$$

[7] Generalized U(1)-decoupling equation (4.18) has been written down in [35].

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where the semicolon is used to emphasize that leg n is taken off-shell. For amplitudes given by

$$A_n = \sum_{i=1}^{N} c_i \epsilon(q_i) \cdot \left(\sum_{k=1}^{n} A^*_{n;k} \right) \tag{5.4}$$

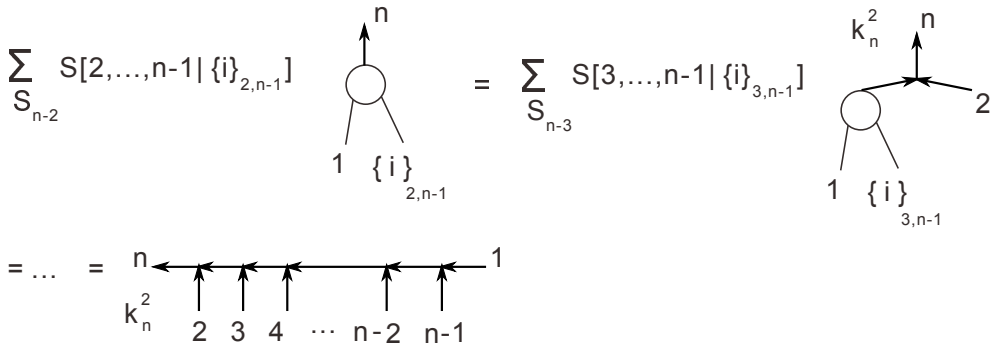
since the part $\sum_{i=1}^{N} c_i \epsilon(q_i)$ is *same for all color orderings*, using the definition of function $\mathcal{S}$ we see that the sum in (5.3) can be written as[8]

$$\begin{aligned} &\sum_{\{j\} \in S_{n-3}} \mathcal{S}[3, \dots n-1 | j_3, \dots j_{n-1}] \\ &\times \Big[s_{21} A_n(1, 2, j_3 \dots j_{n-1}; n) + (s_{21} + s_{2j_3}) \tilde{A}_n(1, j_3, 2, \dots j_{n-1}; n) + \dots \Big] \\ &\quad = \frac{p_n^2}{p_{n2}^2} V_3(2nc) \sum_{\{j\} \in S_{n-3}} \mathcal{S}[3, \dots n-1 | j_3, \dots j_{n-1}] \tilde{A}_{n-1}(1, j_3, \dots; c). \end{aligned} \tag{5.5}$$

where $\{j\}$ is the set defined by deleting leg 2 from the set $\{i\}$. In the last line we have used off-shell BCJ relation (4.8) as well as the form (5.4). The sum over the new $\mathcal{S}$ can be done similarly and we obtain

$$\frac{p_n^2}{p_{n2}^2} V_3(2nc) \frac{p_{2n}^2}{p_{n23}^2} V_3(3cc_1) \sum_{\{j\} \in S_{n-4}} \mathcal{S}[4, \dots n-1 | j_3, \dots j_{n-1}] \tilde{A}_{n-1}(1, j_4, \dots; c_1) \tag{5.6}$$

Repeatedly reducing the number of legs contained in the amplitude one by one for $A^*_{n;k}$ we arrive at the graphical representation



(5.7)

Putting this result back to amplitudes given by (5.4), the KLT formula (5.1) produces

[8] In this form, we have used V_3, which is not exactly right since we have not included the factor $\sum_{i=1}^{N} c_i \epsilon(q_i)$.

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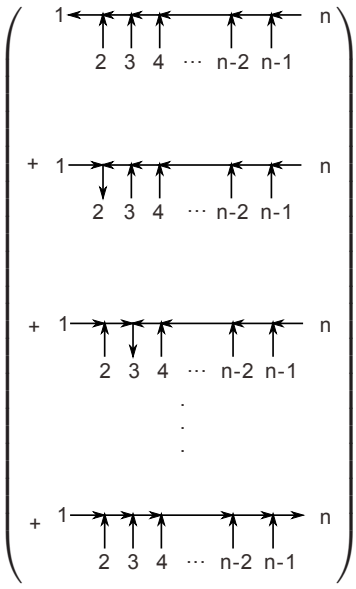
naturally the following expression

$$\mathcal{A}_n = \sum_{\gamma(23\ldots(n-1))\in S_{n-2}} \tilde{A}(1\gamma n)\sum_{j=1}^{N} c_j\epsilon(q_j)\cdot$$



(5.8)

The graph at the right handed side of (5.8) is very similar to the chain of U(N) group structure constant given in [27]. The manipulation demonstrated above can obviously be applied to KLT relation of gravity theory, so graviton amplitudes can be ordered by the same kinematic structure constants.

6 Various forms of amplitudes

From recent progresses we saw that amplitudes of gauge theory can be expressed in following three formulations [1, 27]:

$$\text{double}-\text{copy form}: \quad \mathcal{A}_{\text{tot}} = \sum_i \frac{c_i n_i}{D_i} \tag{6.1}$$

$$\text{Trace form}: \quad \mathcal{A}_{\text{tot}} = \sum_{\sigma\in S_{n-1}} \text{Tr}(T^{\sigma_1}\ldots T^{\sigma_n})A(\sigma) \tag{6.2}$$

$$\text{DDM form}: \quad \mathcal{A}_{\text{tot}} = \sum_{\sigma\in S_{n-2}} c_{1|\sigma(2,\ldots,n-1)|n}A(1,\sigma,n) \tag{6.3}$$

where A are color ordered amplitudes, T^a is the matrix of fundamental representation of U(N) group and $c_i, c_{1|\sigma(2,\ldots,n-1)|n}$ are constructed using the structure constants f^{abc}. For example, we have

$$c_{1|\sigma(2,\ldots,n-1)|n} = f^{\sigma_1\sigma_2 x_1}f^{x_1\sigma_3 x_2}\ldots f^{x_{n-3}\sigma_{n-1}n} \tag{6.4}$$

The transformation from double-copy formulation to DDM was shown in [29] using the KLT relation, while the transformation from DDM to Trace was given in [27] where the following two properties of Lie algebra of U(N) gauge group were essential

$$\text{Property One}: \quad (f^a)_{ij} = f^{aij} = \text{Tr}(T^a[T^i, T^j]), \tag{6.5}$$

$$\text{Property Two}: \quad \sum_a \text{Tr}(XT^a)\text{Tr}(T^aY) = \text{Tr}(XY) \tag{6.6}$$

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A special feature of double-copy formulation is that both c_i and n_i satisfy the Jacobi identity in corresponding Feynman diagrams with only cubic vertices. Because of this duality, it is natural to exchange the role between c_i and n_i and consider the following two dual formulations

$$\text{Dual Trace form}: \quad \mathcal{A}_{\text{tot}} = \sum_{\sigma\in S_{n-1}} \tau_{\sigma_1\ldots\sigma_n}\widetilde{A}(\sigma) \tag{6.7}$$

$$\text{Dual DDM form}: \quad \mathcal{A}_{\text{tot}} = \sum_{\sigma\in S_{n-2}} n_{1|\sigma(2,\ldots,n-1)|n}\widetilde{A}(1,\sigma,n) \tag{6.8}$$

where $\widetilde{A}$ is color ordered scalar theory with f^{abc} as cubic coupling constants (see the references [29, 30]) and τ is required to be cyclic invariant. Indeed, the Dual-DDM was given in [11] while the Dual-Trace-form was conjectured in [31] with explicit constructions given for the first few lower-point amplitudes and a general construction was given in [26]. Although the existence of above two dual formulations were established, a systematic Feynman rule-like prescription to the coefficients τ and n is not known at this moment. Our result (5.8) $n_{1\sigma n}$ for dual-DDM-form serves as a small step towards this goal.

Having above explanation, let us consider following situation where both c_i, n_i satisfying Jacobi-identity can be constructed by Feynman rule, i.e., the theory can be constructed using cubic vertex with coupling constant $\mathcal{F}_{abc}\widetilde{\mathcal{F}}_{abc}$. We want to know under this assumption, which dual form comes out naturally. The conclusion we found is that the dual DDM (6.8) is more compatible with double-copy formulation.

To see that, let us note that the total amplitude can be constructed recursively as

$$\mathcal{A}(1,2,\ldots,n) = \sum_{i=1}^{n-1}\sum_{Split} \mathcal{F}^{1e_1e_2}\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_2}\frac{\mathcal{A}(e_1,u_1,\ldots,u_i)}{P^2_{u_1,\ldots,u_i}}\frac{\mathcal{A}(e_2,v_1,\ldots,v_{n-2-i},n)}{P^2_{v_1,\ldots,v_{n-2-i},n}}, \tag{6.9}$$

where the second sum is over all possible separations of $(n-1)$ particles into two subsets $\{u\},\{v\}$ with $n_u = i$. Assuming the color-decomposition holds for lower-point amplitude $\mathcal{A}$, we can substitute the lower-point DDM-form into above equation and obtain

$$\begin{aligned}
&\mathcal{A}(1,2,\ldots,n)\\
&= \sum_{i=1}^{n-1}\sum_{Split} \mathcal{F}^{1e_1e_i}\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\times\left(\sum_{\alpha\in perm\{u_1,\ldots,u_{i-1}\}} \mathcal{F}^{e_1\alpha_1e_2}\mathcal{F}^{e_2\alpha_2e_3}\ldots\mathcal{F}^{e_{i-1}\alpha_{i-1}u_i}\frac{\widetilde{A}(e_1,\alpha_1,\ldots,\alpha_{i-1},u_i)}{P^2_{u_1,\ldots,u_i}}\right)\\
&\times\left(\sum_{\beta\in perm\{v_1,\ldots,v_{n-2-i}\}} \mathcal{F}^{e_i\beta_1e_{i+1}}\mathcal{F}^{e_{i+1}\beta_2e_{i+2}}\ldots\mathcal{F}^{e_{n-3}\beta_{n-i-2}n}\frac{\widetilde{A}(e_i,\beta_1,\ldots,\beta_{n-i-2},n)}{P^2_{v_1,\ldots,v_{n-i-2},n}}\right).\\
&= \sum_{i=1}^{n-1}\sum_{Split} \mathcal{F}^{1e_1e_i}\mathcal{F}^{e_1\alpha_1e_2}\ldots\mathcal{F}^{e_{i-1}\alpha_{i-1}u_i}\mathcal{F}^{e_i\beta_1e_{i+1}}\ldots\mathcal{F}^{e_{n-3}\beta_{n-i-2}n}\\
&\left[\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\frac{\widetilde{A}(e_1,\alpha_1,\ldots,\alpha_{i-1},u_i)}{P^2_{u_1,\ldots,u_i}}\frac{\widetilde{A}(e_i,\beta_1,\ldots,\beta_{n-i-2},n)}{P^2_{v_1,\ldots,v_{n-i-2},n}}\right]
\end{aligned} \tag{6.10}$$

where for given permutations $\alpha_1,\ldots,\alpha_i$ of us and $\beta_1,\ldots,\beta_{n-i-2}$ of vs, the contraction of $\mathcal{F}$s has the structure at the left handed side of figure 6. After applying Jacobi identity,

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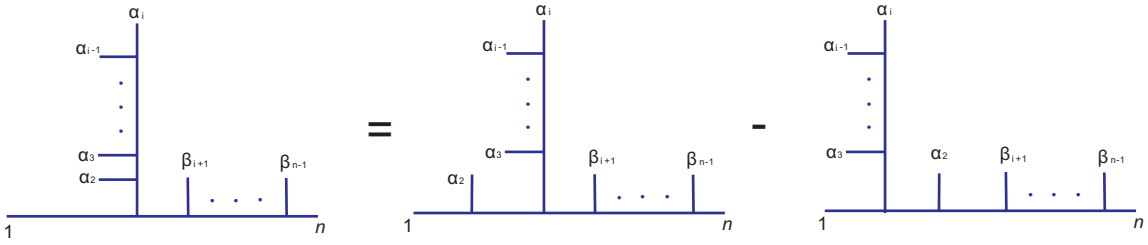

Figure 6. We can use Jacobi identity to reduce the contraction of $\mathcal{F}$s.

Figure 7. A DDM chain with contractions of structure constants $\mathcal{F}^{12e_1}\mathcal{F}^{e_13e_2}\dots\mathcal{F}^{e_{n-3},n-1,n}$

$\mathcal{F}^{1e_1e_i}\mathcal{F}^{e_1\alpha_1e_2}$ becomes

$$\mathcal{F}^{1e_1e_i}\mathcal{F}^{e_1\alpha_1e_2}=\mathcal{F}^{1\alpha_1e_1}\mathcal{F}^{e_1e_2e_i}-\mathcal{F}^{1e_2e_1}\mathcal{F}^{e_1\alpha_1e_i}, \tag{6.11}$$

i.e., the right handed side of figure 6. Iterating this procedure like the one did in [27], we get a sum of 2^{i-1} DDM chains (e.g., figure 7) where the ordered set $O\{\alpha_1,\dots,\alpha_{i-1}\}$ is split into two ordered sets $O\{\sigma\}$ and $O\{\rho\}$ and the form is given by $(-1)^s\mathcal{F}^{1\sigma_1e}\dots\mathcal{F}^{e\sigma_te}\mathcal{F}^{eu_ie}\mathcal{F}^{e\rho_se}\dots\mathcal{F}^{e\rho_1e}\mathcal{F}^{e\beta_1e}\dots\mathcal{F}^{e_{n-3},\beta_{n-2-i},n}$. All these forms are multiplied by $\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\widetilde{A}(e_1,\alpha_1,\dots,\alpha_{i-1},u_i)\widetilde{A}(e_i,\beta_1,\dots,\beta_{n-i-2},n)$. Doing same things to other permutations of $u_1,\dots,u_{i-1}$s and collecting all terms having same DDM chain structure, we get

$$\begin{aligned}
&\mathcal{F}^{1\sigma_1e_1}\dots\mathcal{F}^{e\sigma_te}\mathcal{F}^{eu_ie}\mathcal{F}^{e\rho_se}\dots\mathcal{F}^{e\rho_1e}\mathcal{F}^{e\beta_1e}\dots\mathcal{F}^{e_{n-3},\beta_{n-2-i},n}\\
&\times\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\frac{1}{P^2_{u_1,\dots,u_i}}\left[\sum_{\gamma\in OP(\{\sigma\}\bigcup\{\rho\})}\widetilde{A}(e_1,\gamma_1,\dots,\gamma_{i-1},u_i)\right]\\
&\frac{1}{P^2_{v_1,\dots,v_{n-2-i},n}}\widetilde{A}(e_i,\beta_1,\dots,\beta_{n-i-2},n)\\
=\;&\mathcal{F}^{1\sigma_1e_1}\dots\mathcal{F}^{e\sigma_te}\mathcal{F}^{eu_ie}\mathcal{F}^{e\rho_se}\dots\mathcal{F}^{e\rho_1e}\mathcal{F}^{e\beta_1e}\dots\mathcal{F}^{e_{n-3},\beta_{n-2-i},n}\\
&\times\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\frac{1}{P^2_{u_1,\dots,u_i}}\widetilde{A}(e_1,\sigma_1,\dots,\sigma_t,u_i,\rho_s,\dots,\rho_1)\frac{1}{P^2_{v_1,\dots,v_{n-2-i},n}}\widetilde{A}(e_i,\beta_1,\dots,\beta_{n-i-2},n).
\end{aligned} \tag{6.12}$$

where the KK-relation has been used for the sum in square bracket.

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Putting this result back to recursion relation we reach our final claim

$$
\begin{aligned}
&\mathcal{A}(1,2,\ldots,n)\\
&=\sum_{i=1}^{n-1}\sum_{Split}\sum_{\alpha\in S\{u\}}\sum_{\beta\in S\{v_1\}}\Big[\mathcal{F}^{1\sigma_1 e_1}\ldots\mathcal{F}^{e\sigma_t e}\mathcal{F}^{eu_i e}\mathcal{F}^{e\rho_s e}\ldots\mathcal{F}^{e\rho_1 e}\mathcal{F}^{e\beta_1 e}\ldots\mathcal{F}^{e_{n-3},\beta_{n-2-i},n}\\
&\quad\times\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\frac{1}{P^2_{u_1,\ldots,u_i}}\widetilde{A}(e_1,\sigma_1,\ldots,\sigma_t,u_i,\rho_s,\ldots,\rho_1)\frac{1}{P^2_{v_1,\ldots,v_{n-2-i},n}}\widetilde{A}(e_i,\beta_1,\ldots,\beta_{n-i-2},n)\Big]\\
&=\sum_{\sigma\in S_{n-2}}\mathcal{F}^{a_1 a_{\sigma_2} e_1}\ldots\mathcal{F}^{e_{n-3}a_{\sigma_{n-1}}a_n}\\
&\quad\times\sum_{i=1}^{n-1}\widetilde{\mathcal{F}}^{\bar{1}\bar{e}_1\bar{e}_i}\frac{\widetilde{A}(e_1,\sigma_1,\ldots,\sigma_t,u_i,\rho_s,\ldots,\rho_1)}{P^2_{u_1,\ldots,u_i}}\frac{\widetilde{A}(e_i,\beta_1,\ldots,\beta_{n-i-2},n)}{P^2_{v_1,\ldots,v_{n-2-i},n}}\\
&=\sum_{\sigma\in S_{n-2}}\mathcal{F}^{a_1 a_{\sigma_2} e_1}\ldots\mathcal{F}^{e_{n-3}a_{\sigma_{n-1}}a_n}\widetilde{A}(1\sigma(2\ldots n-1)n)
\end{aligned}
\tag{6.13}
$$

where at the last step we have used the recursion relation for color ordered amplitudes.

7 Conclusion

In this paper we have presented an algorithm which allows systematic construction of the BCJ numerators as well as the kinematic-dual to the DDM formulation. We have shown that assuming gauge symmetry provides enough degrees of freedom, we can express tree-level amplitudes as linear combinations of cubic graph contributions, where Jacobi-like relations between kinematic numerators can be made manifest.

Although our construction is systematically, it is a little bit hard to use practically. In other words, our results is just a small step toward the simple construction of BCJ numerators, which can have important applications for loop calculations of gravity amplitudes.

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A Off-shell KK relation from Berends-Giele recursion

The KK-relation was first written down in [28] without proof. With our knowledge, a proof can be found in [27]. Since the off-shell tensors can be constructed by Berends-Giele

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recursion relation [32], it is natural to prove the off-shell KK relation by this recursion relation and in this appendix we provide a proof for reader's convenience.

The off-shell KK relation is given as

$$J(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\{\alpha\}\bigcup\{\beta^T\})} J(1,\sigma,n), \tag{A.1}$$

where $J(1,2,\ldots,n)$ is an off-shell tensor. After contracting J with on-shell polarization vectors of external legs, it becomes a color-ordered amplitude $A(1,2,\ldots,n)$, and thus the off-shell KK relation becomes the on-shell KK relation.

According to Berends-Giele recursion relation, for a given tensor, we can pick out a leg, for example, the leg 1, to construct whole tensor recursively. In the formula, the leg 1 can be connected to either a three-point vertex or a four point vertex, i.e., we can separate the tensor into $J(1,2,\ldots,n) = J^{(3)}(1,2,\ldots,n) + J^{(4)}(1,2,\ldots,n)$. We will do the same separation at both sides of (A.1) and show the matching for each part.

Connecting to 3-point vertex: in this case, the R.H.S. of KK relation (A.1) can be expressed by

$$\begin{aligned}
&(-)^{n_\beta} \sum_{\substack{\alpha\to\alpha_A,\alpha_B \\ \beta\to\beta_A,\beta_B}} \sum_{\substack{\sigma_A\in OP(\{\alpha_A\}\bigcup\{\beta_B\}^T) \\ \sigma_B\in OP(\{\alpha_B\}\bigcup\{\beta_A\}^T)}} V_{(3)}^{1e_1e_2} \frac{1}{P^2_{\alpha_A,\beta_B}} J(e_1,\sigma_A) \frac{1}{P^2_{\alpha_B,\beta_A}} J(e_2,\sigma_B,n), \\
&= (-)^{n_\beta} \sum_{\alpha\to\alpha_A,\alpha_B;\beta\to\beta_A,\beta_B} V_{(3)}^{1e_1e_2} \frac{1}{P^2_{\alpha_A,\beta_B}} \left(\sum_{\sigma_A\in OP(\{\alpha_A\}\bigcup\{\beta_B\}^T)} J(e_1,\sigma_A) \right) \\
&\quad \times \frac{1}{P^2_{\alpha_B,\beta_A}} \left(\sum_{\sigma_B\in OP(\{\alpha_B\}\bigcup\{\beta_A\}^T)} J(e_2,\sigma_B,n) \right)
\end{aligned} \tag{A.2}$$

where the first sum is over all possible splitting of set α,β into two subsets (including the case, for example, $\alpha_A = \emptyset$) and the second sum is over all possible relative ordering between subsets α_i,β_j. Now we consider the sum in (A.2) for different splitting:

(i) *If both α_A and β_B sets are nonempty*, we can use lower-point generalized U(1)-decoupling identity (4.18)

$$\sum_{\sigma_A\in OP(\{\alpha_A\}\bigcup\{\beta_B\}^T)} J(e_1,\sigma_A) = 0. \tag{A.3}$$

Thus this case does not have nonzero contribution.

(ii) *If β_B set is empty*, we have

$$\sum_{\alpha\to\alpha_A,\alpha_B} V^{1e_1e_2} \frac{1}{P^2_{\alpha_A}} J(e_1,\alpha_A) \times \frac{1}{P^2_{\alpha_B,\beta}} J(e_2,\alpha_B,n,\beta), \tag{A.4}$$

where we have used lower-point KK relation to sum up the last line in (A.2).

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(iii) *If α_A is empty*, we have

$$\sum_{\beta\to\beta_A,\beta_B} V^{1e_2e_1}\frac{1}{P^2_{\alpha,\beta_A}}J(e_2,\alpha,n,\beta_A)\times\frac{1}{P^2_{\beta_B}}J(e_1,\beta_B), \tag{A.5}$$

where we have used lower-point KK relations for the second bracket, the color-order reversed relation for the first brackets as well as the antisymmetry of three-point vertex $V^{1,2,3}=(-1)V^{1,3,2}$ (so the overall factor $(-)^{n_\beta}$ disappears).

The sum of contributions from (ii) and (iii) is just the recursive expansion of $J^{(3)}(1,\alpha,n,\beta)$.

Connecting to 4-point vertex: in this case, the R.H.S. of KK relation (A.1) is given as

$$\begin{aligned}&(-)^{n_\beta}\sum_{\alpha\to\alpha_A,\alpha_B,\alpha_C;\beta\to\beta_A,\beta_B,\beta_C} V_{(4)}^{1e_1e_2e_3}\frac{1}{P^2_{\alpha_A,\beta_A}}\left(\sum_{\sigma_A\in OP(\{\alpha_A\}\bigcup\{\beta_A^T\})}J(e_1,\sigma_A)\right)\\&\times\frac{1}{P^2_{\alpha_B,\beta_B}}\left(\sum_{\sigma_B\in OP(\{\alpha_B\}\bigcup\{\beta_B^T\})}J(e_2,\sigma_B)\right)\times\frac{1}{P^2_{\alpha_C,\beta_C}}\left(\sum_{\sigma_C\in OP(\{\alpha_C\}\bigcup\{\beta_C^T\})}J(e_3,\sigma_C,n)\right).\end{aligned} \tag{A.6}$$

where the sum is over all possible splitting of sets α,β into three subsets (with possible empty subset). For given splittings $\alpha\to\alpha_A,\alpha_B,\alpha_C$, $\beta\to\beta_A,\beta_B,\beta_C$, there are several cases:

(i) *If both $\{\alpha_A\}$ and $\{\beta_A\}$ are nonempty or both $\{\alpha_B\}$ and $\{\beta_B\}$ are nonempty*, we can use lower-point generalized U(1)-decoupling identity (A.3) and the sum is zero for the first or the second brackets in (A.6).

(ii) *If $\sigma_A=OP(\{\alpha_A\})$, $\sigma_B=OP(\{\alpha_B\})$, $\sigma_C\in OP(\{\alpha_C\}\bigcup\{\beta^T\})$* , we have nonzero contribution

$$\sum_{\alpha\to\alpha_A,\alpha_B,\alpha_C} V_{(4)}^{1e_1e_2e_3}\quad\frac{1}{P^2_{\alpha_A}}J(e_1,\alpha_A)\frac{1}{P^2_{\alpha_B}}J(e_2,\alpha_B)\frac{1}{P^2_{\alpha_C,\beta}}J(e_3,\alpha_C,n,\beta), \tag{A.7}$$

where we have used lower-point KK relation to sum up the last bracket.

(iii) *If $\sigma_A=OP(\{\beta_C^T\})$,$\sigma_B=OP(\{\beta_B^T\})$, $\sigma_C\in OP(\{\alpha\}\bigcup\{\beta_A^T\})$*, we have nonzero contribution

$$\sum_{\beta\to\beta_A,\beta_B,\beta_C} V_{(4)}^{1e_1e_2e_3}\quad\frac{1}{P^2_{\alpha,\beta_A}}J(e_1,\alpha,n,\beta_A)\frac{1}{P^2_{\beta_B}}J(e_2,\beta_B)\frac{1}{P^2_{\beta_C}}J(e_3,\beta_C), \tag{A.8}$$

where we have used lower-point KK relation for the third bracket and the color-order reversed relation for the first and second brackets as well as the symmetry of four-vertex $V_{(4)}^{1234}=V_{(4)}^{1432}$.

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(iv) *If* $\sigma_A = OP(\{\alpha_A\})$, $\sigma_B = OP(\{\beta_B^T\})$, $\sigma_C \in OP(\{\alpha_B\}\bigcup\{\beta_A^T\})$, the nonzero contribution is given as

$$(-)^{n_\beta} \sum_{\alpha\to\alpha_A,\alpha_B;\beta\to\beta_A,\beta_B} V_{(4)}^{1e_1e_2e_3} \frac{1}{P^2_{\alpha_A}} J(e_1,\alpha_A) \frac{1}{P^2_{\beta_B}} J(e_2,\beta_B^T) \times \frac{1}{P^2_{\alpha_B,\beta_A}} \left(\sum_{\sigma\in OP(\{\alpha_B\}\bigcup\{\beta_A^T\})} J(e_3,\sigma,n) \right). \tag{A.9}$$

Similarly, *If* $\sigma_A = OP(\{\beta_B^T\})$, $\sigma_B = OP(\{\alpha_A\})$, $\sigma_C \in OP(\{\alpha_B\}\bigcup\{\beta_A^T\})$, we have

$$(-)^{n_\beta} \sum_{\alpha\to\alpha_A,\alpha_B;\beta\to\beta_A,\beta_B} V_{(4)}^{1e_2e_1e_3} \frac{1}{P^2_{\beta_B}} J(e_2,\beta_B^T) \frac{1}{P^2_{\alpha_A}} J(e_1,\alpha_A) \times \frac{1}{P^2_{\alpha_B,\beta_A}} \left(\sum_{\sigma\in OP(\{\alpha_B\}\bigcup\{\beta_A^T\})} J(e_3,\sigma,n) \right). \tag{A.10}$$

where it is worth to notice that the 4-point vertex is written as $V_{(4)}^{1e_2e_1e_3}$. The reason doing so is because the 4-point vertex is

$$V^{1234} = i\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{i}{2}(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} + \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}). \tag{A.11}$$

so we have following identity

$$V^{1234} + V^{1324} = -V^{1243}. \tag{A.12}$$

Using this identity and lower-point KK relation for the third brackets and the color order reversed relation for the first or the second brackets (thus the factor $(-)^{n_\beta}$ disappears), the sum of above two contributions becomes

$$\sum_{\alpha\to\alpha_A,\alpha_B;\beta\to\beta_A,\beta_B} V_{(4)}^{1e_1e_3e_2} \frac{1}{P^2_{\alpha_A}} J(e_1,\alpha_A) \frac{1}{P^2_{\alpha_B,\beta_A}} J(e_3,\alpha_B,n,\beta_A) \frac{1}{P^2_{\beta_B}} J(e_2,\beta_B). \tag{A.13}$$

The sum of (ii), (iii), (iv) is just $J^{(4)}(1,\alpha,n,\beta)$.

Having shown both 3-point vertex part and 4-point vertex part have KK-relation, we have shown the whole off-shell tensor $J(1,\alpha,n,\beta)$ has the KK-relation. In the proof, we have used the antisymmetry of three-point vertex under exchanging a pair of indices as well as the identity between 4-point vertex. This proof shows that if a tensor is constructed only by three-point vertices, it obeys KK relation when the three-point vertex is antisymmetry under exchanging a pair of indices.

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