



The wavelet transform method applied to a coupled chaotic system with asymmetric coupling schemes



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ABSTRACT

The wavelet transform method originated by Wei et al. (2002) [19] is an effective tool for enhancing the transverse stability of the synchronous manifold of a coupled chaotic system. Much of the theoretical study on this matter is centered on networks that are symmetrically coupled. However, in real applications, the coupling topology of a network is often asymmetric; see Belykh et al. (2006) [23,24], Chavez et al. (2005) [25], Hwang et al. (2005) [26], Juang et al. (2007) [17], and Wu (2003) [13]. In this work, a certain type of asymmetric sparse connection topology for networks of coupled chaotic systems is presented. Moreover, our work here represents the first step in understanding how to actually control the stability of global synchronization from dynamical chaos for asymmetrically connected networks of coupled chaotic systems via the wavelet transform method. In particular, we obtain the following results. First, it is shown that the lower bound for achieving synchrony of the coupled chaotic system with the wavelet transform method is independent of the number of nodes. Second, we demonstrate that the wavelet transform method as applied to networks of coupled chaotic systems is even more effective and controllable for asymmetric coupling schemes as compared to the symmetric cases.

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1. Introduction

Network synchronization is one of the most important research fields for the simulation of natural phenomena. Such interesting phenomena are ubiquitous in engineering [1], physics [2], chemistry [3], biology [4], etc. Indeed, many results [5–13] give analytical criteria for determining the range of coupling strength for acquiring locally or even globally stable synchronization. As a matter of fact, in networks of oscillators that are smooth and continuous in time, the synchronous solution becomes stable when the strength of coupling between oscillators exceeds a critical value. General approaches to the local synchronization of networks of coupled chaotic systems have been proposed, including the criteria based on the master stability function (MSF) [6,10,14] originated by Pecora and Carroll [10], and the matrix measures approach [15]. Recently, global synchronization of networks of coupled chaotic systems with asymmetric coupling has also been intensively studied. The methods include Lyapunov function-based criteria [6], the partial contraction approach [16], and the matrix measure approach [15,17,18]. All the works mentioned above, in proving global synchronization, use explicitly or implicitly the fact that the larger the coupling strength is, the more easily the corresponding system can be synchronized. This, in turn, suggests that only the second-largest eigenvalue λ_2 of the coupling matrix has a determining effect on the synchrony of the system. However, as the number of nodes of the system increases, the corresponding λ_2 tends to 0. Hence, to synchronize the system, a greater strength of coupling is needed. The wavelet transform method originated in [19] is an effective tool

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for enhancing the transverse stability of the synchronous manifold of a coupled chaotic system. Much of the theoretical study [20–22] on this matter is centered on networks that are symmetrically coupled. However, in real applications, the coupling topology of a network is often asymmetric [13,17,23–26].

Typically, we assume that the coupled system has N nodes (oscillators) and \mathbf{u}_i is the m -dimensional vector of dynamical variables of the i th node. Let the isolated (uncoupling) dynamics be $\dot{\mathbf{u}}_i = f(\mathbf{u}_i)$ for each node. Here, \mathbf{u}_i has a chaotic dynamics in the sense that its largest Lyapunov exponent is positive. Let $H : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a linear vectorial function describing the coupling within the components of each node. Thus, the dynamics of the i th node are given as follows:

$$\dot{\mathbf{u}}_i = f(\mathbf{u}_i) + \epsilon \sum_{j=1}^N a_{ij}H(\mathbf{u}_j), \quad i = 1, 2, \dots, N, \tag{1.1a}$$

where ϵ is a coupling strength and $A = (a_{ij})_{N \times N}$ is the coupling matrix with conditions $\sum_{j=1}^N a_{ij} = 0$ for $1 \leq i \leq N$. Furthermore, let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N)^T$, and $F(\mathbf{u}) = (f(\mathbf{u}_1), f(\mathbf{u}_2), \dots, f(\mathbf{u}_N))^T$. We may write (1.1a) in the following vector–matrix form:

$$\dot{\mathbf{u}} = F(\mathbf{u}) + \epsilon(A \otimes H)\mathbf{u}, \tag{1.1b}$$

where \otimes is the Kronecker product. We mention that there are various coupling schemes contained in Eq. (1.1b). Next, we briefly introduce the concept of the wavelet transform method [19] to reconstruct the network topology of (1.1). For more details, see [20–22]. Let $N = 2n$ and $n \in \mathbb{N}$, the natural numbers. Write A as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}_{n \times n},$$

where the dimension of each block matrix A_{kl} is 2×2 . By means of an i -scale wavelet operator W [27], the matrix A is transformed into $W(A)$, of the form

$$W(A) = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{n1} & \cdots & \tilde{A}_{nn} \end{pmatrix}_{n \times n},$$

where each entry of \tilde{A}_{kl} is the average of the entries of A_{kl} , $1 \leq k, l \leq n$. After reconstruction, the coupling matrix A becomes $A + \alpha W(A)$. Here α is a wavelet constant.

In Eq. (1.1b), the coupling matrix A gives the topological connectivity of nodes. In this work, we consider a generalized nearest neighbor coupling matrix $A(\beta, \vec{v})$ with mixed boundary conditions. In particular, $A(\beta, \vec{v})$ has the form

$$A(\beta, \vec{v}) =: \begin{pmatrix} a + (1 - \beta)c & b & 0 & \cdots & 0 & \beta c \\ c & a & b & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c & a & b \\ \beta b & 0 & \cdots & 0 & c & a + (1 - \beta)b \end{pmatrix} \\ =: \begin{pmatrix} A_1(\beta, \vec{v}) & A_2(1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & A_3(\beta, \vec{v}) \\ A_3(1, \vec{v}) & A_1(1, \vec{v}) & A_2(1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & A_3(1, \vec{v}) & A_1(1, \vec{v}) & A_2(1, \vec{v}) \\ A_2(\beta, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & A_3(1, \vec{v}) & A_4(\beta, \vec{v}) \end{pmatrix} \tag{1.2a}$$

where $\vec{v} := (a, b, c)$ is an undetermined vector with three real numbers a, b, c which satisfy $b, c \geq 0, a + b + c = 0, N = 2n \geq 4, n \in \mathbb{N}$, and

$$A_1(\beta, \vec{v}) = \begin{pmatrix} a + (1 - \beta)c & b \\ c & a \end{pmatrix}, \quad A_2(\beta, \vec{v}) = \begin{pmatrix} 0 & 0 \\ \beta b & 0 \end{pmatrix}, \quad A_3(\beta, \vec{v}) = \begin{pmatrix} 0 & \beta c \\ 0 & 0 \end{pmatrix}, \tag{1.2b}$$

$$A_4(\beta, \vec{v}) = \begin{pmatrix} a & b \\ c & a + (1 - \beta)b \end{pmatrix}, \quad \text{and} \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It should be noted that the choice of $\beta, 0 \leq \beta \leq 1$, gives what type of boundary conditions the coupling configuration has. In particular, $\beta = 0, \beta = 1$ and $0 < \beta < 1$ give the periodic, Neumann and mixed boundary conditions, respectively.

The coupling configuration of the networks in Eq. (1.2a) is quite general, and includes asymmetric connections between nodes and/or some competitive ($a_{ij} < 0, i \neq j$) couplings between cells \mathbf{u}_i and \mathbf{u}_j , and partial-state coupling with nonzero off-diagonal connections.

Applying the wavelet transform method to $A(\beta, \vec{v})$, we obtain that the matrix $W(A(\beta, \vec{v}))$ is of the following form:

$$W(A(\beta, \vec{v})) =: \begin{pmatrix} \tilde{A}_1(\beta, \vec{v}) & \tilde{A}_2(1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & \tilde{A}_3(\beta, \vec{v}) \\ \tilde{A}_3(1, \vec{v}) & \tilde{A}_1(1, \vec{v}) & \tilde{A}_2(1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \tilde{A}_3(1, \vec{v}) & \tilde{A}_1(1, \vec{v}) & \tilde{A}_2(1, \vec{v}) \\ \tilde{A}_2(\beta, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & \tilde{A}_3(1, \vec{v}) & \tilde{A}_4(\beta, \vec{v}) \end{pmatrix} \quad (1.3a)$$

where

$$\begin{aligned} \tilde{A}_1(\beta, \vec{v}) &= -\frac{(b + \beta c)}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \tilde{A}_2(\beta, \vec{v}) &= \frac{\beta b}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \tilde{A}_3(\beta, \vec{v}) &= \frac{\beta c}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \tilde{A}_4(\beta, \vec{v}) &= -\frac{(c + \beta b)}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{and } \mathbf{0} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (1.3b)$$

Thus, the newly reconstructed coupling matrix is given by

$$C(\alpha, \beta, \vec{v}) := A(\beta, \vec{v}) + \alpha W(A(\beta, \vec{v}))$$

$$=: \begin{pmatrix} C_1(\alpha, \beta, \vec{v}) & C_2(\alpha, 1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & C_3(\alpha, \beta, \vec{v}) \\ C_3(\alpha, 1, \vec{v}) & C_1(\alpha, 1, \vec{v}) & C_2(\alpha, 1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & C_3(\alpha, 1, \vec{v}) & C_1(\alpha, 1, \vec{v}) & C_2(\alpha, 1, \vec{v}) \\ C_2(\alpha, \beta, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & C_3(\alpha, 1, \vec{v}) & C_4(\alpha, \beta, \vec{v}) \end{pmatrix} \quad (1.4a)$$

where

$$C_i(\alpha, \beta, \vec{v}) = A_i(\beta, \vec{v}) + \alpha \tilde{A}_i(\beta, \vec{v}) \quad \text{for } 1 \leq i \leq 4. \quad (1.4b)$$

The eigenvalues of $C(\alpha, \beta, \vec{v})$ are denoted by $\lambda_i(\alpha, \beta, \vec{v})$ which satisfy the following conditions:

$$0 = \Re(\lambda_1(\alpha, \beta, \vec{v})) \geq \Re(\lambda_2(\alpha, \beta, \vec{v})) \geq \cdots \geq \Re(\lambda_i(\alpha, \beta, \vec{v})) \geq \cdots \geq \Re(\lambda_N(\alpha, \beta, \vec{v}))$$

where $\Re(z)$ denotes the real part for complex number z . For fixed $0 \leq \beta \leq 1$, we mention that the graphs of a mapping of the wavelet parameter α into $\lambda_i(\alpha, \beta, \vec{v})$ are called the (real) eigencurves of $C(\alpha, \beta, \vec{v})$ for $1 \leq i \leq N$.

Global synchronization of coupled chaotic systems has been intensively studied with asymmetric connections [13,17, 23–26]. In particular, Juang et al. [17] reported that the coupling configuration A of the networks includes asymmetric connections, and the second-largest real parts of the eigenvalues, $\Re(\lambda_2)$, can dominate the stability of the global synchronous behavior for the networks of coupled chaotic systems in Eq. (1.1). Mathematically speaking, it was shown that the lower bound ϵ_c on the critical coupling strength is proportional to $-\Re(\lambda_2)$ (see, e.g., Theorems 3.1–3.2 of [17]). Namely,

$$\epsilon_c \propto \frac{1}{-\Re(\lambda_2)}.$$

Consequently, as the number of nodes increases, which in turn makes λ_2 closer to the zero, the coupling strength that is needed to get the system synchronized becomes greater. In this work, we consider a certain type of asymmetric sparse connection topology for networks of coupled chaotic systems. In particular, we obtain the following results. First, it is shown that the lower bound for achieving synchrony of the coupled chaotic system with the wavelet transform method is independent of the number of nodes. Second, we demonstrate that the wavelet transform method as applied to networks of coupled chaotic systems is even more effective and controllable for asymmetric coupling schemes as compared to the symmetric cases.

2. The main results for $C(\alpha, 1, \vec{v})$

In this section, the influence of the wavelet transform method on the nearest neighbor coupling with generalized periodic boundary conditions $A(1, \vec{v})$ is considered. In this case, we obtain that $C_1(\alpha, 1, \vec{v}) = C_4(\alpha, 1, \vec{v})$, and the coupling matrix

$C(\alpha, 1, \vec{v})$ is indicated as follows:

$$C(\alpha, 1, \vec{v}) = \begin{pmatrix} C_1(\alpha, 1, \vec{v}) & C_2(\alpha, 1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & C_3(\alpha, 1, \vec{v}) \\ C_3(\alpha, 1, \vec{v}) & C_1(\alpha, 1, \vec{v}) & C_2(\alpha, 1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & C_3(\alpha, 1, \vec{v}) & C_1(\alpha, 1, \vec{v}) & C_2(\alpha, 1, \vec{v}) \\ C_2(\alpha, 1, \vec{v}) & \mathbf{0} & \cdots & \mathbf{0} & C_3(\alpha, 1, \vec{v}) & C_1(\alpha, 1, \vec{v}) \end{pmatrix} \quad (2.1a)$$

where

$$\begin{aligned} C_1(\alpha, 1, \vec{v}) &= \begin{pmatrix} a & b \\ c & a \end{pmatrix} + \alpha \left[\frac{a}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] = C_4(\alpha, 1, \vec{v}), \\ C_2(\alpha, 1, \vec{v}) &= \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} + \alpha \left[\frac{b}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right], \quad \text{and} \quad C_3(\alpha, 1, \vec{v}) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} + \alpha \left[\frac{c}{4} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right]. \end{aligned} \quad (2.1b)$$

First, we solve the corresponding eigenvalue problem $C(\alpha, 1, \vec{v})\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)^T$ and $\mathbf{b}_i \in \mathbb{C}^2, 1 \leq i \leq n$, in block component form; we obtain

$$C_3(\alpha, 1, \vec{v})\mathbf{b}_{i-1} + C_1(\alpha, 1, \vec{v})\mathbf{b}_i + C_2(\alpha, 1, \vec{v})\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, \quad i = 1, \dots, n.$$

The boundary conditions would yield that

$$\begin{aligned} C_3(\alpha, 1, \vec{v})\mathbf{b}_0 + C_1(\alpha, 1, \vec{v})\mathbf{b}_1 + C_2(\alpha, 1, \vec{v})\mathbf{b}_2 &= \lambda\mathbf{b}_1 = C_1(\alpha, 1, \vec{v})\mathbf{b}_1 + C_2(\alpha, 1, \vec{v})\mathbf{b}_2 + C_3(\alpha, 1, \vec{v})\mathbf{b}_n \\ C_3(\alpha, 1, \vec{v})\mathbf{b}_{n-1} + C_1(\alpha, 1, \vec{v})\mathbf{b}_n + C_2(\alpha, 1, \vec{v})\mathbf{b}_{n+1} &= \lambda\mathbf{b}_n = C_2(\alpha, 1, \vec{v})\mathbf{b}_1 + C_3(\alpha, 1, \vec{v})\mathbf{b}_{n-1} + C_1(\alpha, 1, \vec{v})\mathbf{b}_n \end{aligned}$$

or, equivalently,

$$\mathbf{b}_0 = \mathbf{b}_n, \quad \text{and} \quad \mathbf{b}_1 = \mathbf{b}_{n+1}.$$

To study the block difference equation, we first seek to find the solution \mathbf{b}_i of the form

$$\mathbf{b}_i = \delta^i \begin{pmatrix} 1 \\ v \end{pmatrix} \quad i = 1, \dots, n.$$

Substituting these \mathbf{b}_i into the block difference equation, we have

$$\left[C_3(\alpha, 1, \vec{v})\delta^{-1} + (C_1(\alpha, 1, \vec{v}) - \lambda I) + C_2(\alpha, 1, \vec{v})\delta \right] \begin{pmatrix} 1 \\ v \end{pmatrix} = 0.$$

The existence of a nontrivial solution $(1, v)^T$ guarantees that the following condition holds:

$$\det \left[C_3(\alpha, 1, \vec{v})\delta^{-1} + (C_1(\alpha, 1, \vec{v}) - \lambda I) + C_2(\alpha, 1, \vec{v})\delta \right] = 0.$$

We assume, for the moment, that the equation has four roots, say $\delta_1, \delta_2, \delta_3$, and δ_4 . Hence, the general solution can then be written as

$$\mathbf{b}_i = c_1\delta_1^i \begin{pmatrix} 1 \\ v_1 \end{pmatrix} + c_2\delta_2^i \begin{pmatrix} 1 \\ v_2 \end{pmatrix} + c_3\delta_3^i \begin{pmatrix} 1 \\ v_3 \end{pmatrix} + c_4\delta_4^i \begin{pmatrix} 1 \\ v_4 \end{pmatrix}$$

for $i = 1, \dots, n$. Here, $v_k, k = 1, 2, 3, 4$, are some constants depending on $\delta_k, k = 1, 2, 3, 4$. Applying the boundary conditions to the coupling matrix $C(\alpha, 1, \vec{v})$, we get the matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ v_1 & v_2 & v_3 & v_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ v_1\delta_1 & v_2\delta_2 & v_3\delta_3 & v_4\delta_4 \end{pmatrix} \text{diag}(\delta_1^n - 1, \delta_2^n - 1, \delta_3^n - 1, \delta_4^n - 1) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

Now, if $\text{diag}(\delta_1^n - 1, \delta_2^n - 1, \delta_3^n - 1, \delta_4^n - 1)$ is singular, then the above equation has a nontrivial solution $c := (c_1, c_2, c_3, c_4)^T$. Note that $\text{diag}(\delta_1^n - 1, \delta_2^n - 1, \delta_3^n - 1, \delta_4^n - 1)$ is singular if and only if the $\delta_k, k = 1, 2, 3, 4$, satisfy $\delta^n = 1$. Returning to the eigenvalue problem for $C(\alpha, 1, \vec{v})$, we conclude that the eigenvalue λ satisfies the following conditions:

$$\begin{cases} \det \left[C_3(\alpha, 1, \vec{v})\delta^{-1} + (C_1(\alpha, 1, \vec{v}) - \lambda I) + C_2(\alpha, 1, \vec{v})\delta \right] = 0, \\ \delta^n = 1. \end{cases} \quad (2.2)$$

To solve the system of equations, we first note that the $\Omega_m(\alpha, \vec{v}) := \frac{\alpha}{4}(c\delta_m^{-1} + a + b\delta_m)$, where $\delta_m := e^{i2m\pi/n}$, are the roots of $\delta^n = 1$ for $m = 0, \dots, (n - 1)$. Hence, for $m = 0, \dots, (n - 1)$, the determinant in Eq. (2.2) can be rewritten as follows:

$$\begin{aligned} \det \left[C_3(\alpha, 1, \vec{v})\delta_m^{-1} + (C_1(\alpha, 1, \vec{v}) - \lambda I) + C_2(\alpha, 1, \vec{v})\delta_m \right] &= \det \left[\begin{pmatrix} \Omega_m(\alpha, \vec{v}) - a - \lambda & \Omega_m(\alpha, \vec{v}) + c\delta_m^{-1} + b \\ \Omega_m(\alpha, \vec{v}) + c + b\delta_m & \Omega_m(\alpha, \vec{v}) - a - \lambda \end{pmatrix} \right] \\ &= (\Omega_m(\alpha, \vec{v}) - a - \lambda)^2 - (\Omega_m(\alpha, \vec{v}) + c\delta_m^{-1} + b)(\Omega_m(\alpha, \vec{v}) + c + b\delta_m) = 0. \end{aligned}$$

Consequently, the eigenvalue formula for the eigenvalue problem $C(\alpha, 1, \vec{v})\mathbf{b} = \lambda\mathbf{b}$ can be derived and described as in the following theorem.

Theorem 2.1. Let N be any positive even integer with $N = 2n \geq 4$, $n \in \mathbb{N}$, and let $C(\alpha, 1, \vec{v})$ be the $N \times N$ matrix which is given in Eq. (2.1). Then all the eigenvalues of $C(\alpha, 1, \vec{v})$ are of the following form:

$$\lambda_m^\pm(\alpha, 1, \vec{v}) = (a + \Omega_m(\alpha, \vec{v})) \pm \sqrt{(\Omega_m(\alpha, \vec{v}) + c\delta_m^{-1} + b)(\Omega_m(\alpha, \vec{v}) + c + b\delta_m)} \tag{2.3}$$

where $\Omega_m(\alpha, \vec{v}) = \frac{\alpha}{4}(c\delta_m^{-1} + a + b\delta_m)$ and $\delta_m := e^{i2m\pi/n}$ are the roots of $\delta^n = 1$ for $m = 0, \dots, (n - 1)$. Here, we choose the nonnegative real parts of complex numbers for the last term in Eq. (2.3).

From Theorem 2.1, we find that the eigenvalues $\lambda_m^\pm(\alpha, 1, \vec{v})$ of $C(\alpha, 1, \vec{v})$ have the following properties.

Proposition 2.1. Let $\lambda_m^\pm(\alpha, 1, \vec{v})$ be the eigenvalues of the matrix $C(\alpha, 1, \vec{v})$ which is given in Theorem 2.1, and let the dimension of $C(\alpha, 1, \vec{v})$ be $N \times N$, $N = 2n \geq 4$, $n \in \mathbb{N}$. Then, we obtain that

$$\lim_{\alpha \rightarrow \infty} \Re(\lambda_m^-(\alpha, 1, \vec{v})) = -\infty, \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} \lambda_m^+(\alpha, 1, \vec{v}) = \frac{1}{2}[3a - b\delta_m - c\delta_m^{-1}] =: \Lambda_m(1, \vec{v}),$$

for $m = 1, \dots, (n - 1)$.

Proof. First, we define $\Omega_m(\alpha, \vec{v}) = \frac{\alpha}{4}(c\delta_m^{-1} + a + b\delta_m) =: \alpha\Delta_m(\alpha, \vec{v})$ for $m = 1, \dots, (n - 1)$. By substituting the Euler formula $\delta_m = e^{i\frac{2m\pi}{n}} = \cos \frac{2m\pi}{n} + i \sin \frac{2m\pi}{n}$ and $a + b + c = 0$ into $\Delta_m(\alpha, \vec{v})$, we obtain

$$\Delta_m(\alpha, \vec{v}) = \frac{a}{4} \left(1 - \cos \frac{2m\pi}{n} \right) + i \left(\frac{(b - c)}{4} \sin \frac{2m\pi}{n} \right), \quad m = 1, \dots, (n - 1).$$

Moreover, the limit of $\lambda_m^-(\alpha, 1, \vec{v})$ can be calculated as follows:

$$\lim_{\alpha \rightarrow \infty} \left[\alpha \cdot \left(\left(\Delta_m(\alpha, \vec{v}) + \frac{a}{\alpha} \right) - \sqrt{\left(\Delta_m(\alpha, \vec{v}) + \frac{c\delta_m^{-1} + b}{\alpha} \right) \left(\Delta_m(\alpha, \vec{v}) + \frac{c + b\delta_m}{\alpha} \right)} \right) \right]$$

$$= \left(\lim_{\alpha \rightarrow \infty} \alpha \right) \cdot 2\Delta_m(\alpha, \vec{v}).$$

By using the fact $\Re(\Delta_m(\alpha, \vec{v})) < 0$ for $m = 1, \dots, (n - 1)$, we have

$$\lim_{\alpha \rightarrow \infty} \Re(\lambda_m^-(\alpha, 1, \vec{v})) = \left(\lim_{\alpha \rightarrow \infty} \alpha \right) \cdot 2\Re(\Delta_m(\alpha, \vec{v})) = -\infty, \quad m = 1, \dots, (n - 1).$$

Furthermore, we handle the limit of $\lambda_m^+(\alpha, 1, \vec{v})$ as follows:

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \lambda_m^+(\alpha, 1, \vec{v}) &= \lim_{\alpha \rightarrow \infty} \frac{(\alpha\Delta_m(\alpha, \vec{v}) + a)^2 - (\alpha\Delta_m(\alpha, \vec{v}) + c\delta_m^{-1} + b)(\alpha\Delta_m(\alpha, \vec{v}) + c + b\delta_m)}{(\alpha\Delta_m(\alpha, \vec{v}) + a) - \sqrt{(\alpha\Delta_m(\alpha, \vec{v}) + c\delta_m^{-1} + b)(\alpha\Delta_m(\alpha, \vec{v}) + c + b\delta_m)}} \\ &= \lim_{\alpha \rightarrow \infty} \frac{2\alpha\Delta_m(\alpha, \vec{v}) + a^2 - (c\delta_m^{-1} + b + c + b\delta_m)\alpha\Delta_m(\alpha, \vec{v}) - (c\delta_m^{-1} + b)(c + b\delta_m)}{(\alpha\Delta_m(\alpha, \vec{v}) + a) - \sqrt{(\alpha\Delta_m(\alpha, \vec{v}) + c\delta_m^{-1} + b)(\alpha\Delta_m(\alpha, \vec{v}) + c + b\delta_m)}} \\ &= \frac{2a\Delta_m(\alpha, \vec{v}) - (c\delta_m^{-1} + b + c + b\delta_m)\Delta_m(\alpha, \vec{v})}{2\Delta_m(\alpha, \vec{v})} \\ &= \frac{1}{2}[3a - b\delta_m - c\delta_m^{-1}] =: \Lambda_m(1, \vec{v}), \quad m = 1, \dots, (n - 1). \end{aligned}$$

Hence, the proposition holds true. \square

We are now in a position to describe the main results for $\Re(\lambda_2(\alpha, 1, \vec{v}))$ of $C(\alpha, 1, \vec{v})$.

Theorem 2.2. Let N be any positive even integer with $N = 2n \geq 4$ and $n \in \mathbb{N}$. $C(\alpha, 1, \vec{v})$ is an $N \times N$ matrix which is given in Eq. (2.1), and $\Re(\lambda_2(\alpha, 1, \vec{v}))$ are the second-largest real parts of the eigenvalues of $C(\alpha, 1, \vec{v})$. Then, there exists a critical wavelet constant $\tilde{\alpha}_c > 0$ such that

$$\Re(\lambda_2(\alpha, 1, \vec{v})) = \Re\left(\lambda_{\lfloor \frac{n}{2} \rfloor}^+(\alpha, 1, \vec{v})\right) = \Re\left(\lambda_{\lfloor \frac{n}{2} \rfloor + 1}^+(\alpha, 1, \vec{v})\right)$$

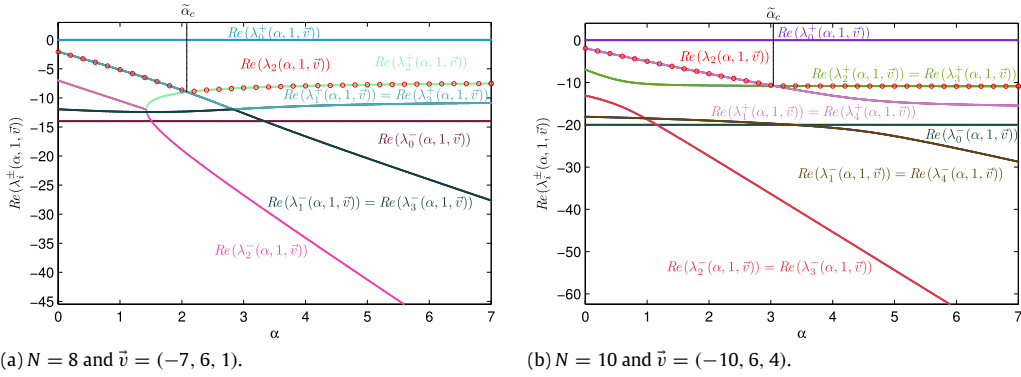


Fig. 2.1. The graphs give all the real parts of the eigencurves of $C(\alpha, 1, \vec{v})$, and the red dotted line is $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$. (a) $N = 8$, which is a multiple of 4, and $\vec{v} = (-7, 6, 1)$. (b) $N = 10$, which is even but not a multiple of 4, and $\vec{v} = (-10, 6, 4)$.

for all $\alpha \geq \tilde{\alpha}_c$. Furthermore,

$$2a \leq \lim_{\alpha \rightarrow \infty} \Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \frac{a}{2} \left(3 + \cos \frac{2 \lfloor \frac{n}{2} \rfloor \pi}{n} \right) \leq a < 0.$$

Proof. Let N be a positive even integer with $N = 2n \geq 4$, $n \in \mathbb{N}$. For $m = 1, \dots, (n - 1)$, using Proposition 2.1(c) and the Euler formula $\delta_m = e^{i \frac{2m\pi}{n}} = \cos \frac{2m\pi}{n} + i \sin \frac{2m\pi}{n}$, we obtain that $\lim_{\alpha \rightarrow \infty} \Re\epsilon(\lambda_m^-(\alpha, 1, \vec{v})) = -\infty$, and

$$\lim_{\alpha \rightarrow \infty} \lambda_m^+(\alpha, 1, \vec{v}) = \frac{1}{2} [3a - b\delta_m - c\delta_m^{-1}] = \frac{a}{2} \left(3 + \cos \frac{2m\pi}{n} \right) + \frac{i}{2} (b - c) \sin \frac{2m\pi}{n} =: A_m(1, \vec{v}).$$

In fact, $2a \leq \Re\epsilon(A_m(1, \vec{v})) \leq a \leq 0$ for $m = 1, \dots, (n - 1)$. Obviously, $\Re\epsilon(A_m(1, \vec{v}))$ attains its maximum at two values $m = \lfloor \frac{n}{2} \rfloor$, and $\lfloor \frac{n}{2} \rfloor + 1$ for $m = 1, \dots, (n - 1)$. Thus, there exists a critical wavelet constant $\tilde{\alpha}_c > 0$ such that

$$\Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \Re\epsilon(\lambda_{\lfloor \frac{n}{2} \rfloor}^+(\alpha, 1, \vec{v})) = \Re\epsilon(\lambda_{\lfloor \frac{n}{2} \rfloor + 1}^+(\alpha, 1, \vec{v}))$$

for all $\alpha \geq \tilde{\alpha}_c$. Moreover, we also obtain that

$$2a \leq \lim_{\alpha \rightarrow \infty} \Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \frac{a}{2} \left(3 + \cos \frac{2 \lfloor \frac{n}{2} \rfloor \pi}{n} \right) \leq a \leq 0.$$

The proof of the theorem is just completed. \square

As shown in the proof of Theorem 2.2, if we let N be a multiple of 4, then $\Re\epsilon(A_m(1, \vec{v}))$ attains its maximum at only one value $m = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. In this case, $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \Re\epsilon(\lambda_{\frac{n}{2}}^+(\alpha, 1, \vec{v}))$ is independent of the choice of the number of nodes (oscillators) N for all $\alpha \geq \tilde{\alpha}_c$. Hence, we describe these results as the following corollary.

Corollary 2.1. Let N be any positive even integer with $N = 2n \geq 4$, and $n \in \mathbb{N}$. If, in addition, N is multiple of 4, $C(\alpha, 1, \vec{v})$ is an $N \times N$ matrix which is given in Eq. (2.1), and $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$ are the second-largest real parts of the eigenvalues of $C(\alpha, 1, \vec{v})$. Then, there exists a critical wavelet constant $\tilde{\alpha}_c > 0$ such that

$$\Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \Re\epsilon \left(\frac{a}{2} (2 + \alpha) + \frac{1}{2} \sqrt{\alpha^2 a^2 - 4(a + 2b)^2} \right)$$

for all $\alpha \geq \tilde{\alpha}_c$. In this case, $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$ is independent of the choice of the number of nodes (oscillators) N . Furthermore,

$$\lim_{\alpha \rightarrow \infty} \Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = a < 0.$$

Remark 2.1. In Theorem 2.2, we see that the limit of the second-largest real parts of the eigenvalues $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$ is in between the values $2a$ and a as α tends to infinity. Using the continuity of $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$ in $\alpha \geq \tilde{\alpha}_c$, we can suitably choose $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$ such that $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$ is independent of the choice of the number of nodes (oscillators) N .

Remark 2.2. In Fig. 2.1(a), (rep. Fig. 2.1(b)), we have, respectively, that $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \Re\epsilon(\lambda_1^+(\alpha, 1, \vec{v})) = \Re\epsilon(\lambda_3^+(\alpha, 1, \vec{v}))$ if $\alpha \leq \alpha_c$ and $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v})) = \Re\epsilon(\lambda_2^+(\alpha, 1, \vec{v}))$ if $\alpha \geq \alpha_c$. From Fig. 2.2, we see that, for fixed a , the greater the imbalance between b and c , the lower their corresponding $\Re\epsilon(\lambda_2(\alpha, 1, \vec{v}))$. As a result, we may conclude numerically that

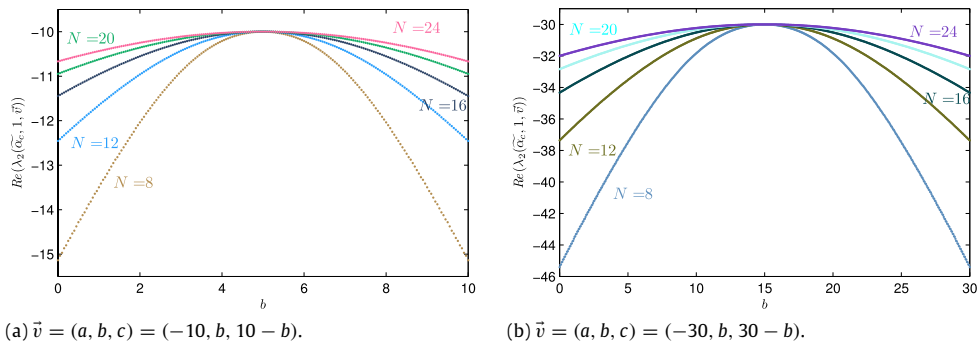


Fig. 2.2. The parts (a) and (b) are mappings of the parameter b into $\Re(\lambda_2(\alpha, 1, \vec{v}))$ with $\vec{v} = (-10, b, 10 - b)$ and $\vec{v} = (-30, b, 30 - b)$ for $0 \leq b \leq 10$ and $0 \leq b \leq 30$, respectively. For each \vec{v} , the corresponding wavelet constant α is chosen such that $\Re(\lambda_2(\alpha, 1, \vec{v}))$ is a minimum, as seen in Fig. 2.1.

the asymmetric coupling of the coupled systems is easier to control via the wavelet transform method than its symmetric counterpart.

Conclusion. By adjusting the wavelet constant of the wavelet transform method [19], it was obtained that the method can greatly reduce the coupling strength needed to synchronize a coupled chaotic system. In this work, a certain type of asymmetric sparse connection topology for coupled chaotic systems with the wavelet transform method was studied analytically. It is proved that on applying the wavelet transform method to the coupled chaotic systems, the lower bound d_c on the coupling strength d for achieving synchrony is independent of the number of nodes N of the systems. Furthermore, we show that asymmetric coupling schemes for networks of coupled chaotic systems obtained with the wavelet transform method are even more effective and controllable as compared to those for the symmetric cases. We conclude our work by suggesting some possible future work. It would be of interest to develop new techniques for finding the explicit eigenvalue formula for $C(\alpha, \beta, \vec{v})$, where $\beta \neq 1$. Note that our techniques given in [21] fail to carry over to Robin boundary conditions. It is also very much worthwhile to pursue the cases where the coupling matrix has the form $C(\alpha, \beta, \vec{v})$, $0 \leq \beta \leq 1$.

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