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Convergence for elliptic equations in periodic perforated domains

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ABSTRACT

Convergence for the solutions of elliptic equations in periodic perforated domains is concerned. Let ϵ denote the size ratio of the holes of a periodic perforated domain to the whole domain. It is known that, by energy method, the gradient of the solutions of elliptic equations is bounded uniformly in ϵ in L^2 space. Also, when ϵ approaches 0, the elliptic solutions converge to a solution of some simple homogenized elliptic equation. In this work, above results are extended to general $W^{1,p}$ space for $p > 1$. More precisely, a uniform $W^{1,p}$ estimate in ϵ for $p \in (1, \infty]$ and a $W^{1,p}$ convergence result for $p \in (\frac{n}{n-2}, \infty]$ for the elliptic solutions in periodic perforated domains are derived. Here n is the dimension of the space domain. One also notes that the L^p norm of the second order derivatives of the elliptic solutions in general cannot be bounded uniformly in ϵ.

1. Introduction

Convergence for the solutions of elliptic equations in periodic perforated domains is presented. Let $Y \equiv (0,1)^n$ for $n \geqslant 3$ consist of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y_m})$, $\epsilon \in (0,1]$, $\Omega_m^\epsilon \equiv \{x \mid x \in \epsilon(Y_m - j)$ for some $j \in \mathbb{Z}^n\}$ with boundary $\partial\Omega_m^\epsilon$, and $\Omega_f^\epsilon \equiv \mathbb{R}^n \setminus \overline{\Omega_m^\epsilon}$ be a connected region. The problem that we consider is

$$\begin{cases} -\nabla \cdot (\nabla U_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } \Omega_f^\epsilon, \\ (\nabla U_\epsilon + Q_\epsilon) \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ |U_\epsilon|(x) = o(1) & \text{for large } |x|, \end{cases} \tag{1.1}$$

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where Q_ϵ, F_ϵ are given functions and $\vec{\mathbf{n}}^\epsilon$ is a unit normal vector on $\partial\Omega_m^\epsilon$. $|U_\epsilon|(x) = o(1)$ for large $|x|$ means $\lim_{|x|\to\infty} |U_\epsilon|(x) = 0$. If Q_ϵ, F_ϵ both are bounded with compact support, by energy method, a solution of (1.1) in Hilbert space $\mathcal{D}^{1,2}(\Omega_f^\epsilon)$ (see definition in Section 2) exists uniquely for each ϵ. The L^2 norm of the gradient of the solution of (1.1) in Ω_f^ϵ is bounded uniformly in ϵ. If, in addition, $Q_\epsilon = 0$ and $F_\epsilon = F$ in (1.1), by compactness principle [3], there exists a function $U_0 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that the solution U_ϵ of (1.1) satisfies

$$\nabla U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \to \mathcal{K}\nabla U_0 \quad \text{in } L^2(\mathbb{R}^n) \text{ weakly as } \epsilon \to 0, \tag{1.2}$$

where $\mathcal{X}_{\Omega_f^\epsilon}$ is the characteristic function on Ω_f^ϵ and $\mathcal{K}$ is a positive definite matrix depending on Y_f. Moreover, U_0 in (1.2) satisfies

$$\begin{cases} -\nabla \cdot (\mathcal{K}\nabla U_0) = |Y_f| F & \text{in } \mathbb{R}^n, \\ |U_0|(x) = o(1) & \text{for large } |x|, \end{cases} \tag{1.3}$$

where $|Y_f|$ is the volume of Y_f. It is interesting to know whether a uniform bound of the solution of (1.1) in ϵ in Ω_f^ϵ can be derived in L^p space and whether a convergence rate of U_ϵ can be obtained in L^p space for any $p \in (1, \infty]$ or not. One example in *Remark* 2.1 below shows that the L^p estimate of the second derivatives of the solution of (1.1) may not be bounded uniformly in ϵ in general.

There are some literatures related to this work. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [13,17]. Uniform Hölder, L^p, and Lipschitz estimates in ϵ for uniform elliptic equations with periodic smooth oscillatory coefficients were proved in [4,5]. Uniform Lipschitz estimate in ϵ for the Laplace equation in periodic perforated domains was considered in [22]. Uniform Hölder and Lipschitz estimates in ϵ for non-uniform elliptic equations with periodic discontinuous oscillatory coefficients were shown in [24]. By homogenization theory, the solutions of elliptic equations in periodic perforated domains in general converge to a solution of some homogenized elliptic equation with convergence rate ϵ in L^2 norm and with convergence rate $\epsilon^{1/2}$ in H^1 norm as ϵ closes to 0 (see [6,12,20] and references therein). Higher order asymptotic expansion for the solutions of elliptic equations in perforated domains could be found in [7,14]. Rigorous proof of higher order convergence rate for the solution of (1.1) in Hilbert spaces was considered in [6,8,20]. In this work, we shall derive a uniform $W^{1,p}$ estimate in ϵ for the solution of (1.1) in Ω_f^ϵ and prove a $W^{1,p}$ convergence result with convergence rate ϵ for the solution of (1.1) for $p > 1$ case. The approaches to derive above results are similar to those in [4,5] and are based on the following steps: First we prove the existence of the Green's function of the Laplace operator in perforated domains. Next we find approximations of the Green's function of the Laplace operator in perforated domains. Then we derive a uniform $W^{1,p}$ estimate in ϵ for elliptic equations in perforated domains. Finally an asymptotic expansion technique is used to derive the $W^{1,p}$ convergence rate for the solutions of (1.1) and (1.3). Concerning the approximation of the Green's function of the Laplace operator in bounded perforated domains, some uniform approximations in ϵ in L^∞ space for the Green's function can be found in the review paper [18] and references therein. However, they are not enough for our purpose. Uniform approximations in ϵ in L^∞ space for the zero, the first, and the second order derivatives of the Green's function are needed here, and they are derived by an approach different from [18].

The rest of this manuscript is organized as follows: Notation and main results are stated in Section 2. In Section 3, we present uniform Hölder and uniform Lipschitz estimates in ϵ for the solutions of elliptic equations in periodic perforated domains; prove an L^∞ convergence result for an elliptic equation in perforated domains; derive a priori estimates for some interface problems; show the existence of the Green's function of the Laplace operator in perforated domains; and give some estimates for the zero order and the first order derivatives of the Green's function. Approximation of the second order derivatives of the Green's function of the Laplace operator in perforated domains is derived in Section 4. Main results (that is, a uniform $W^{1,p}$ estimate in ϵ and a $W^{1,p}$ convergence result for the solution of (1.1)) are shown in Section 5. Finally, uniform Hölder and uniform Lipschitz estimates in ϵ for the solutions of elliptic equations in perforated domains claimed in Section 3 are proved in Section 6.

2. Notation and main result

$W^{s,p}(\mathbb{D})$ denotes a Sobolev space with norm $\|\cdot\|_{W^{s,p}(\mathbb{D})}$, $C^{k,\alpha}(\mathbb{D})$ is a Hölder space with norm $\|\cdot\|_{C^{k,\alpha}(\mathbb{D})}$, $W^{s,p}_{loc}(\mathbb{D}) \equiv \{\zeta \mid \zeta \in W^{s,p}(D)$ for any compact subset D of $\mathbb{D}\}$, $C^{k,\alpha}_{loc}(\mathbb{D}) \equiv \{\zeta \mid \zeta \in C^{k,\alpha}(D)$ for any compact subset D of $\mathbb{D}\}$, and $[\zeta]_{C^{0,\alpha}}$ is the Hölder semi-norm of ζ, where $s \geqslant -1$, $p \in [1,\infty]$, $k \geqslant 0$, $\alpha \in [0,1]$ (see [2,11]). $H^s(\mathbb{D}) \equiv W^{s,2}(\mathbb{D})$, $L^p(\mathbb{D}) \equiv W^{0,p}(\mathbb{D})$, $H^s_{loc}(\mathbb{D}) \equiv W^{s,2}_{loc}(\mathbb{D})$, $C(\mathbb{D}) \equiv C^{0,0}(\mathbb{D})$. $C^\infty(\mathbb{D})$ is a space of infinitely differentiable functions in $\mathbb{D}$, $C^\infty_0(\mathbb{D})$ is a subset of $C^\infty(\mathbb{D})$ with compact support in $\mathbb{D}$, and $C^\infty_{per}(\mathbb{R}^n)$ is a subset of $C^\infty(\mathbb{R}^n)$ of $(0,1)^n$-periodic functions. $H^s_{per}(\mathbb{D})$ (resp. $C^{k,\alpha}_{per}(\mathbb{D})$) is the closure of $C^\infty_{per}(\mathbb{R}^n)$ under the H^s (resp. $C^{k,\alpha}$) norm and $\|\zeta\|_{H^s_{per}(\mathbb{D})} \equiv \|\zeta\|_{H^s(\mathbb{D}\cap(0,1)^n)}$ (resp. $\|\zeta\|_{C^{k,\alpha}_{per}(\mathbb{D})} \equiv \|\zeta\|_{C^{k,\alpha}(\mathbb{D}\cap(0,1)^n)}$) for $s \geqslant 1$, $k \geqslant 0$, $\alpha \in [0,1]$. $W^{s,p}_0(\mathbb{D})$ is the closure of $C^\infty_0(\mathbb{D})$ under the $W^{s,p}$ norm for $s \geqslant 1$, $p \in (1,\infty)$. $supp(\zeta)$ is the support of ζ. For Banach spaces $\mathbf{B}_1$ and $\mathbf{B}_2$, define $\|\zeta_1,\dots,\zeta_k\|_{\mathbf{B}_1} \equiv \|\zeta_1\|_{\mathbf{B}_1} + \cdots + \|\zeta_k\|_{\mathbf{B}_1}$ and $\|\zeta\|_{\mathbf{B}_1\cap\mathbf{B}_2} \equiv \|\zeta\|_{\mathbf{B}_1} + \|\zeta\|_{\mathbf{B}_2}$. Let $B_r(x)$ denote a ball centered at x with radius $r > 0$. For any set $\mathbb{D} \subset \mathbb{R}^n$, $|\mathbb{D}|$ is the volume of $\mathbb{D}$, $\overline{\mathbb{D}}$ is the closure of $\mathbb{D}$, $\mathcal{X}_{\mathbb{D}}$ is the characteristic function on $\mathbb{D}$, $\mathbb{D}/r \equiv \{x \mid rx \in \mathbb{D}\}$ for $r > 0$, $dist(x, \partial\mathbb{D})$ is the distance between x and $\partial\mathbb{D}$, and

$$\fint_{\mathbb{D}} \zeta(x)\,dx \equiv \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \zeta(x)\,dx \quad \text{if } \zeta \in L^1(\mathbb{D}).$$

If $\zeta \in L^1(\mathbb{D})$, $(\zeta)_{x,r} \equiv \fint_{B_r(x)\cap\mathbb{D}} \zeta(y)\,dy$. Let $\Omega_m \equiv \{x \mid x \in Y_m - j$ for some $j \in \mathbb{Z}^n\}$, $\Omega_f \equiv \mathbb{R}^n \setminus \overline{\Omega_m}$, $\Omega^\omega_m \equiv \omega\Omega_m$, $\Omega^\omega_f \equiv \omega\Omega_f$, and $\partial\Omega^\omega_m$ be the boundary of Ω^ω_m for any $\omega \in (0,\infty)$. $\mathcal{D}^{1,2}(\mathbb{R}^n) \equiv \{\zeta \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla\zeta \in L^2(\mathbb{R}^n)\}$ under the norm $\|\zeta\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \equiv \|\nabla\zeta\|_{L^2(\mathbb{R}^n)}$ is a Hilbert space (see p. 168 [15]). $\mathcal{D}^{1,2}(\Omega^\epsilon_f) \equiv \{\zeta|_{\Omega^\epsilon_f} \mid$ for $\zeta \in \mathcal{D}^{1,2}(\mathbb{R}^n)\}$ for any $\epsilon \in (0,1]$ with norm $\|\zeta\|_{\mathcal{D}^{1,2}(\Omega^\epsilon_f)} \equiv \|\nabla\zeta\|_{L^2(\Omega^\epsilon_f)}$ and $\mathcal{D}^{1,2}(\mathbb{R}^n \setminus \overline{Y_m}) \equiv \{\zeta|_{\mathbb{R}^n\setminus\overline{Y_m}} \mid$ for any $\zeta \in \mathcal{D}^{1,2}(\mathbb{R}^n)\}$ with norm $\|\zeta\|_{\mathcal{D}^{1,2}(\mathbb{R}^n\setminus\overline{Y_m})} \equiv \|\nabla\zeta\|_{L^2(\mathbb{R}^n\setminus\overline{Y_m})}$. It is easy to see that both $\mathcal{D}^{1,2}(\Omega^\epsilon_f)$, $\mathcal{D}^{1,2}(\mathbb{R}^n \setminus \overline{Y_m})$ are also Hilbert spaces. Denote by x_i for $i \in \{1,\dots,n\}$ the i-th component of $x \in \mathbb{R}^n$, $\mathbb{R}^n_+ \equiv \{x \mid x_n > 0\}$, and $\partial\mathbb{R}^n_+ \equiv \{x \mid x_n = 0\}$.

Next we present our main results.

Theorem 2.1. *Suppose*

A1. *Y_m is a smooth simply-connected sub-domain of $(0,1)^n$ for $n \geqslant 3$,*
A2. *$\epsilon \in (0,1]$, $p \in (1,\infty)$, $Q_\epsilon \in L^p(\Omega^\epsilon_f)$ with support in $B_t(0)$ for some $t > 0$,*

then a $W^{1,p}_{loc}(\Omega^\epsilon_f)$ solution of

$$\begin{cases} -\nabla\cdot(\nabla U_\epsilon + Q_\epsilon) = 0 & \text{in } \Omega^\epsilon_f, \\ (\nabla U_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega^\epsilon_m, \\ |U_\epsilon|(x) = o(1) & \text{for large } |x| \end{cases}$$

exists uniquely. The solution U_ϵ satisfies

$$\begin{cases} \|\nabla U_\epsilon\|_{L^p(\Omega_f^\epsilon)} \leqslant c\|Q_\epsilon\|_{L^p(\Omega_f^\epsilon)} & \text{for } p \in (1,\infty), \\ \|U_\epsilon\|_{L^{\frac{np}{n-p}}(\Omega_f^\epsilon)} \leqslant c\|Q_\epsilon\|_{L^p(\Omega_f^\epsilon)} & \text{for } p \in (1,n), \\ \|U_\epsilon\|_{L^p(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_{t,s}\|Q_\epsilon\|_{L^p(\Omega_f^\epsilon)} & \text{for } p \in (1,\infty), \end{cases}$$

where $s > 0$; c is a positive constant independent of ϵ, Q_ϵ; and $c_{t,s}$ is a positive constant independent of ϵ but dependent on t, s.

Theorem 2.2. *Suppose $A1$ and*

A3. *$\epsilon \in (0,1]$, $p \in (1,\infty)$, $F_\epsilon \in W^{-1,p}(\mathbb{R}^n)$ with support in $B_t(0)$ for $t > 0$,*

then a $W^{1,p}_{loc}(\Omega_f^\epsilon)$ solution of

$$\begin{cases} -\Delta U_\epsilon = F_\epsilon & \text{in } \Omega_f^\epsilon, \\ \nabla U_\epsilon \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ |U_\epsilon|(x) = o(1) & \text{for large } |x| \end{cases} \tag{2.1}$$

exists uniquely and satisfies

$$\|U_\epsilon\|_{W^{1,p}(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_{t,s}\|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)}, \tag{2.2}$$

where $s > 0$ and $c_{t,s}$ is a constant independent of ϵ but dependent on t, s. If, in addition, $F_\epsilon \in L^p(\mathbb{R}^n)$ has compact support, the solution of (2.1) satisfies

$$\begin{cases} \|\nabla U_\epsilon\|_{L^{\frac{np}{n-p}}(\Omega_f^\epsilon)} \leqslant c\|F_\epsilon\|_{L^p(\Omega_f^\epsilon)} & \text{for } p \in (1,n), \\ \|U_\epsilon\|_{L^{\frac{np}{n-2p}}(\Omega_f^\epsilon)} \leqslant c\|F_\epsilon\|_{L^p(\Omega_f^\epsilon)} & \text{for } p \in (1,n/2), \end{cases} \tag{2.3}$$

where c is a constant independent of ϵ, F_ϵ.

Theorem 2.1 and Theorem 2.2 imply that

Corollary 2.1. *Suppose $A1$, $\epsilon \in (0,1]$, $p \in (\frac{n}{n-2},\infty)$, and*

A4. *$Q_\epsilon \in L^p(\Omega_f^\epsilon)$, $F_\epsilon \in L^{\frac{np}{n+p}}(\mathbb{R}^n)$ have compact support,*

then a $W^{1,p}(\Omega_f^\epsilon)$ solution of (1.1) exists uniquely and satisfies

$$\|U_\epsilon\|_{W^{1,p}(\Omega_f^\epsilon)} \leqslant c\big(\|Q_\epsilon\|_{L^p(\Omega_f^\epsilon)} + \|Q_\epsilon, F_\epsilon\|_{L^{\frac{np}{n+p}}(\Omega_f^\epsilon)} + \|F_\epsilon\|_{L^{\frac{np}{n+2p}}(\Omega_f^\epsilon)}\big),$$

where c is a constant independent of ϵ, Q_ϵ, F_ϵ.

Theorem 2.3. *If $A1$ holds, $\epsilon \in (0,1]$, and both $Q_\epsilon, F_\epsilon \in L^{n+\delta}(\mathbb{R}^n)$ for any $\delta > 0$ have compact support, then a $\mathcal{D}^{1,2}(\Omega_f^\epsilon)$ solution of*

$$\begin{cases} -\nabla \cdot (\nabla U_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } \Omega_f^\epsilon, \\ (\nabla U_\epsilon + Q_\epsilon) \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon \end{cases} \tag{2.4}$$

exists uniquely and satisfies

$$\|U_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\|Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(\mathbb{R}^n)}, \tag{2.5}$$

where c is a constant independent of ϵ, Q_ϵ, F_ϵ. If, in addition, $Q_\epsilon = 0$ in $\mathbb{R}^n$, then

$$\|\nabla U_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\|F_\epsilon\|_{L^{n+\delta}(\mathbb{R}^n)}, \tag{2.6}$$

where c is a constant independent of ϵ, F_ϵ.

Let $\mathbb{X}^{(i)}(y) \in H^1_{per}(\Omega_f)$ for $i \in \{1, \ldots, n\}$ satisfy, in the cell Y_f,

$$\begin{cases} -\nabla \cdot \left(\nabla \mathbb{X}^{(i)} + \vec{e}_i\right) = 0 & \text{in } Y_f, \\ \left(\nabla \mathbb{X}^{(i)} + \vec{e}_i\right) \cdot \vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m, \\ \displaystyle\int_{Y_f} \mathbb{X}^{(i)}\, dy = 0, \end{cases} \tag{2.7}$$

where $\vec{\mathbf{n}}_y$ denotes a unit normal vector on ∂Y_m and $\vec{e}_i$ is the unit vector in the i-th coordinate direction in $\mathbb{R}^n$. Define

$$\begin{cases} \mathbb{X}_\epsilon^{(i)}(y) \equiv \epsilon \mathbb{X}^{(i)}\left(\dfrac{y}{\epsilon}\right) & \text{for } i \in \{1, \ldots, n\}, \\ \mathbb{X}_\epsilon \equiv \left(\mathbb{X}_\epsilon^{(1)}, \ldots, \mathbb{X}_\epsilon^{(n)}\right), \\ \mathbb{X} \equiv \left(\mathbb{X}^{(1)}, \ldots, \mathbb{X}^{(n)}\right). \end{cases} \tag{2.8}$$

Denote by Ξ an $n \times n$ matrix function whose (i, j)-component is $\partial_i \mathbb{X}^{(j)}$ and define

$$\mathcal{K} \equiv \int_{Y_f} \left(I + \Xi(y)\right) dy, \tag{2.9}$$

where I is the identity matrix. By [3], $\mathcal{K}$ is a constant symmetric positive definite matrix. For $i_1, i_2 \in \{1, \ldots, n\}$, find $\mathbb{S}^{(i_1,i_2)}(y) \in H^1_{per}(\Omega_f)$ satisfying, in the cell Y_f,

$$\begin{cases} \Delta \mathbb{S}^{(i_1,i_2)} + \partial_{i_1} \mathbb{X}^{(i_2)} = -\delta_{i_1,i_2} - \partial_{i_1} \mathbb{X}^{(i_2)} + \dfrac{\mathcal{K}_{i_1,i_2}}{|Y_f|} & \text{in } Y_f, \\ \nabla \mathbb{S}^{(i_1,i_2)} \cdot \vec{\mathbf{n}}_y + \mathbb{X}^{(i_2)} \mathbf{n}_{y_{i_1}} = 0 & \text{on } \partial Y_m, \\ \displaystyle\int_{Y_f} \mathbb{S}^{(i_1,i_2)}\, dy = 0, \end{cases} \tag{2.10}$$

where

$$\delta_{i_1,i_2} \equiv \begin{cases} 1 & \text{if } i_1 = i_2, \\ 0 & \text{if } i_1 \neq i_2, \end{cases}$$

$\mathcal{K}_{i_1,i_2}$ is the (i_1, i_2)-component of $\mathcal{K}$ in (2.9), and $\mathbf{n}_{y_{i_1}}$ is the i_1-th component of $\vec{\mathbf{n}}_y \equiv (\mathbf{n}_{y_1}, \dots, \mathbf{n}_{y_n})$. Let $\mathbb{S}(y) \equiv (\mathbb{S}^{(i_1,i_2)}(y))$ be an $n \times n$ matrix function and $\mathbb{S}_\epsilon(y) \equiv \epsilon^2(\mathbb{S}^{(i_1,i_2)}(\frac{y}{\epsilon}))$. By energy method and Lax–Milgram Theorem [11], $\mathbb{X}^{(i_1)}$ in (2.7) and $\mathbb{S}^{(i_1,i_2)}$ in (2.10) for $i_1, i_2 \in \{1, \dots, n\}$ are solvable uniquely in $H^1_{per}(\Omega_f)$. By Theorem 6.30 [11] and tracing the proof of Theorem 8.24 [11], if Y_m is a $C^{2,\alpha}$ domain for some $\alpha \in (0,1)$, then

$$\left\|\mathbb{X}^{(i_1)}\right\|_{C^{2,\alpha}_{per}(\Omega_f)} + \left\|\mathbb{S}^{(i_1,i_2)}\right\|_{C^{2,\alpha}_{per}(\Omega_f)} \leqslant c \quad \text{for } i_1, i_2 \in \{1, \dots, n\}, \tag{2.11}$$

where c is a constant.

Remark 2.1. The $W^{2,p}$ norm of the solution of (1.1) in general is not bounded uniformly in ϵ even if Q_ϵ, F_ϵ have compact supports and $\|Q_\epsilon\|_{W^{1,p}(\Omega^\epsilon_f)}$, $\|F_\epsilon\|_{L^p(\Omega^\epsilon_f)}$ are bounded independent of ϵ. One example is as follows. Suppose η is a bell-shaped smooth function satisfying $\eta \in C^\infty_0(B_1(0))$, $\eta \in [0,1]$, and $\eta(x) = 1$ in $B_{1/2}(0)$. By (2.7) and (2.8), we have the following equation

$$\begin{cases} -\nabla \cdot \left(\nabla\left(\eta \mathbb{X}^{(1)}_\epsilon\right) - \mathbb{X}^{(1)}_\epsilon \nabla\eta + \eta\vec{e}_1\right) = -\left(\nabla\mathbb{X}^{(1)}_\epsilon + \vec{e}_1\right)\nabla\eta & \text{in } \Omega^\epsilon_f, \\ \left(\nabla\left(\eta\mathbb{X}^{(1)}_\epsilon\right) - \mathbb{X}^{(1)}_\epsilon\nabla\eta + \eta\vec{e}_1\right) \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega^\epsilon_m, \\ \left|\eta\mathbb{X}^{(1)}_\epsilon\right|(x) = 0 & \text{for large } |x|. \end{cases}$$

By (2.11), we see

$$\left\|\mathbb{X}^{(1)}_\epsilon \nabla\eta - \eta\vec{e}_1\right\|_{W^{1,p}(\Omega^\epsilon_f)} + \left\|\left(\nabla\mathbb{X}^{(1)}_\epsilon + \vec{e}_1\right)\nabla\eta\right\|_{L^p(\Omega^\epsilon_f)}$$

is bounded uniformly in ϵ, but $\|\eta\mathbb{X}^{(1)}_\epsilon\|_{W^{2,p}(\Omega^\epsilon_f)}$ is not bounded uniformly in ϵ.

Let Π_ϵ for $\epsilon \in (0,1]$ denote an extension operator in Ω^ϵ_f, which maps a function ζ in Ω^ϵ_f to $\Pi_\epsilon\zeta$ in $\mathbb{R}^n$ and still keeps the same regularity of ζ (see [1] for the existence of such an operator). By Theorem 4.31 [2] and extension theorem [1], we know, for any $\zeta \in W^{1,p}(\Omega^\epsilon_f)$, $p \in (1,n)$, and $\epsilon \in (0,1]$,

$$\|\zeta\|_{L^{\frac{np}{n-p}}(\Omega^\epsilon_f)} \leqslant \|\Pi_\epsilon\zeta\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leqslant c\|\nabla\Pi_\epsilon\zeta\|_{L^p(\mathbb{R}^n)} \leqslant c\|\nabla\zeta\|_{L^p(\Omega^\epsilon_f)}, \tag{2.12}$$

where c is independent of ϵ. If $F \in C^\infty_0(\mathbb{R}^n)$, then the $\mathcal{D}^{1,2}(\Omega^\epsilon_f)$ solution of

$$\begin{cases} -\Delta U_\epsilon = F & \text{in } \Omega^\epsilon_f, \\ \nabla U_\epsilon \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega^\epsilon_m \end{cases} \tag{2.13}$$

exists uniquely by Lax–Milgram Theorem [11], extension theorem [1], and (2.12). Moreover,

$$\|U_\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega^\epsilon_f)} + \|\nabla U_\epsilon\|_{L^2(\Omega^\epsilon_f)} \leqslant c\|F\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},$$

where c is independent of ϵ, F. By compactness principle [3], there is a subsequence of the solution of (2.13) satisfying, for any fixed $r > 0$,

$$\begin{cases} \Pi_\epsilon U_\epsilon \to U_0 & \text{in } L^2\big(B_r(0)\big) \text{ strongly} \\ \nabla U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \to \mathcal{K}\nabla U_0 & \text{in } L^2\big(B_r(0)\big) \text{ weakly} \end{cases} \quad \text{as } \epsilon \to 0,$$

where Π_ϵ is the extension operator used in (2.12), $\mathcal{X}_{\Omega_f^\epsilon}$ is the characteristic function on Ω_f^ϵ, and $\mathcal{K}$ is the one in (2.9). The limit function $U_0 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfies

$$-\nabla \cdot (\mathcal{K}\nabla U_0) = |Y_f| F \quad \text{in } \mathbb{R}^n \tag{2.14}$$

where $|Y_f|$ is the volume of Y_f. Moreover, we have the following results:

Theorem 2.4. *Assume $A1$ and $\epsilon \in (0, 1]$.*

(1) *If $p \in (\frac{n}{n-2}, \infty)$ and $F \in W^{1,p}(\mathbb{R}^n)$ with compact support, the solutions of (2.13) and (2.14) satisfy*

$$\begin{cases} \|U_\epsilon - U_0\|_{L^p(\Omega_f^\epsilon)} \leqslant c\epsilon \|F\|_{W^{1,\frac{np}{n+p}}(\mathbb{R}^n) \cap W^{1,\frac{np}{n+2p}}(\mathbb{R}^n)}, \\ \big\|\nabla U_\epsilon - (I + \nabla \mathbb{X}_\epsilon)\nabla U_0\big\|_{L^p(\Omega_f^\epsilon)} \leqslant c\epsilon \|F\|_{W^{1,p}(\mathbb{R}^n) \cap W^{1,\frac{np}{n+p}}(\mathbb{R}^n)}, \end{cases}$$

where c is a constant independent of ϵ, F.

(2) *If $\delta > 0$ and $F \in W^{1,n+\delta}(\mathbb{R}^n)$ with compact support, the solutions of (2.13) and (2.14) satisfy*

$$\|U_\epsilon - U_0\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\epsilon \|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n) \cap W^{1,n+\delta}(\mathbb{R}^n)},$$

where c is a constant independent of ϵ, F.

(3) *If $\delta > 0$ and $F \in W^{2,n+\delta}(\mathbb{R}^n)$ with compact support, the solutions of (2.13) and (2.14) satisfy*

$$\big\|\nabla U_\epsilon - (I + \nabla \mathbb{X}_\epsilon)\nabla U_0\big\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\epsilon \|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n) \cap W^{2,n+\delta}(\mathbb{R}^n)},$$

where c is a constant independent of ϵ, F. Here I is the identity matrix and $\mathbb{X}_\epsilon$ is defined in (2.8).

3. Existence of the Green's function

This section consists of three subsections. The first subsection is to present uniform Hölder and uniform Lipschitz estimates as well as to show a convergence result for elliptic equations in perforated domains Ω_f^ϵ. We also give a local maximum norm estimate for a non-uniform elliptic equation. In the second and the third subsections, we prove the existence and derive some estimates for the Green's function of the Laplace equation in the perforated domain Ω_f and in $\mathbb{R}^n \setminus \overline{Y_m}$ respectively.

3.1. Some auxiliary results

Let d_0 be a positive constant and let $\mathbf{D}_0$, $\mathbf{D}_1$, $\mathbf{D}$, $\mathcal{A}$, $\mathbf{D}_2$ be smooth domains satisfying

$$\begin{cases} Y_m \subset \mathbf{D}_0 \subset \mathbf{D}_1 \subset \mathbf{D} \subset \mathcal{A} \subset Y \subset \mathbf{D}_2, \\ dist(Y_m, \partial\mathbf{D}_0), dist(\mathbf{D}_0, \partial\mathbf{D}_1), dist(\mathbf{D}_1, \partial\mathbf{D}) \geqslant d_0 > 0, \\ dist(\mathbf{D}, \partial\mathcal{A}), dist(\mathcal{A}, \partial Y), dist(Y, \partial\mathbf{D}_2), dist(\partial\mathbf{D}_2, \partial\Omega_m) \geqslant d_0 > 0. \end{cases} \tag{3.1}$$

Next we give a uniform Hölder estimate for elliptic equations in perforated domains.

Lemma 3.1. *For any $\delta > 0$ and $\epsilon \in (0, 1]$, there is a $\mu \in (0, \frac{\delta}{n+\delta})$ such that any solution of*

$$\begin{cases} -\nabla \cdot (\nabla U_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } B_1(x) \cap \Omega_f^\epsilon, \\ (\nabla U_\epsilon + Q_\epsilon) \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } B_1(x) \cap \partial \Omega_m^\epsilon \end{cases} \tag{3.2}$$

satisfies

$$[U_\epsilon]_{C^{0,\mu}(\overline{B_{1/2}(x)\cap\Omega_f^\epsilon})} \leqslant c\big(\|U_\epsilon\|_{L^2(B_1(x)\cap\Omega_f^\epsilon)} + \|Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(B_1(x)\cap\Omega_f^\epsilon)}\big),$$

where $x \in \mathbb{R}^n$ and c is a constant independent of ϵ, x.

Proof of Lemma 3.1 will be given in Section 6.1. Next we give a pointwise estimate for the solution of (2.13) and the solution of (2.14).

Lemma 3.2. *If $\delta > 0$, $\epsilon \in (0, 1]$, and $F \in W^{1,n+\delta}(\mathbb{R}^n)$ with compact support, the solution of (2.13) and the solution of (2.14) satisfy*

$$\|U_\epsilon - U_0\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\epsilon \|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)},$$

where c is a constant independent of ϵ, F.

Proof. By Lemma 4.4 [11], (2.14) in [11], and Theorem 3 on p. 39 [23], the solution of (2.14) satisfies

$$\big\|\nabla^2 U_0\big\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)} \leqslant c\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)}, \tag{3.3}$$

where c is a constant dependent on n, δ, $\mathcal{K}$ but independent of F. Define

$$\varphi_\epsilon(x) \equiv U_\epsilon(x) - U_0(x) - \mathbb{X}_\epsilon(x)\nabla U_0(x) - \mathbb{S}_\epsilon(x)\nabla^2 U_0(x) \quad \text{in } \Omega_f^\epsilon,$$

where U_ϵ, U_0 are the solutions of (2.13), (2.14) respectively, $\mathbb{X}_\epsilon$ is defined in (2.8), and $\mathbb{S}_\epsilon$ is defined in (2.10). Then function $\varphi_\epsilon \in \mathcal{D}^{1,2}(\Omega_f^\epsilon)$ satisfies

$$\begin{cases} -\nabla \cdot \big(\nabla\varphi_\epsilon + \mathbb{S}_\epsilon \nabla^3 U_0\big) = \mathbb{X}_\epsilon \nabla\Delta U_0 + \nabla\mathbb{S}_\epsilon \nabla^3 U_0 & \text{in } \Omega_f^\epsilon, \\ \big(\nabla\varphi_\epsilon + \mathbb{S}_\epsilon \nabla^3 U_0\big) \cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon. \end{cases} \tag{3.4}$$

By energy method, (2.11), (2.12), and (3.3), the solution of (3.4) satisfies

$$\|\varphi_\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega_f^\epsilon)} + \|\nabla\varphi_\epsilon\|_{L^2(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)}, \tag{3.5}$$

where c is independent of ϵ, F. By Lemma 3.1, (2.11), and (3.5), for any $\epsilon \in (0, 1]$ and $x \in \mathbb{R}^n$,

$$[\varphi_\epsilon]_{C^{0,\mu}(\overline{B_1(x)\cap\Omega_f^\epsilon})} \leqslant c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)}, \tag{3.6}$$

where $\mu \in (0, \frac{\delta}{n+\delta})$ and c is independent of ϵ, x, F. By Hölder inequality and (3.5)–(3.6),

$$\begin{aligned}|\varphi_\epsilon(x)| &\leqslant \left|\varphi_\epsilon(x) - \fint_{B_1(x)\cap\Omega_f^\epsilon} \varphi_\epsilon(y)\,dy\right| + \left|\fint_{B_1(x)\cap\Omega_f^\epsilon} \varphi_\epsilon(y)\,dy\right| \\ &\leqslant c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)},\end{aligned}$$

which implies the lemma. □

Lemma 3.3. *For any $p \in (1,\infty)$, $s \in (n,\infty)$, and $\alpha \in (0,1)$, there is a constant c such that any solution of*

$$\begin{cases} -\nabla\cdot(\nabla\varphi + Q) = F & \text{in } \mathbf{D}_2 \setminus \overline{Y_m}, \\ (\nabla\varphi + Q)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m \end{cases} \tag{3.7}$$

satisfies

$$\begin{cases} \|\varphi\|_{W^{1,p}(Y_f)} \leqslant c\big(\|\varphi\|_{L^p(\mathbf{D}_2\setminus\bar{Y})} + \|Q\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \|F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\|_{W^{-1,p}(\mathbf{D}_2)}\big), \\ \|\nabla\varphi\|_{L^\infty(Y_f)} \leqslant c\big(\|\varphi\|_{L^\infty(\mathbf{D}_2\setminus\overline{Y_m})} + \|Q\|_{C^{0,\alpha}(\overline{\mathbf{D}_2\setminus Y_m})} + \|F\|_{L^s(\mathbf{D}_2\setminus\overline{Y_m})}\big), \end{cases} \tag{3.8}$$

where $\vec{\mathbf{n}}$ is the unit outward normal vector on ∂Y_m and $\partial\mathbf{D}_2$. See (3.1) for $\mathbf{D}_2$.

Proof. Let c denote a constant.

Step 1: Claim if $Q \in C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ and $F \in L^p(\mathbf{D}_2)$ for $p \in (1,\infty)$, any solution of

$$\begin{cases} -\nabla\cdot(\nabla\psi + Q) = F & \text{in } \mathbf{D}_2 \setminus \overline{Y_m}, \\ (\nabla\psi + Q)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \\ \psi = 0 & \text{on } \partial\mathbf{D}_2 \end{cases} \tag{3.9}$$

satisfies

$$\|\psi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\big(\|\psi\|_{L^p(\mathbf{D}_2\setminus\bar{Y})} + \|Q\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \|F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\|_{W^{-1,p}(\mathbf{D}_2)}\big). \tag{3.10}$$

First we assume $\|\psi\|_{L^p(\mathbf{D}_2\setminus\bar{Y})} + \|Q\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \|F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\|_{W^{-1,p}(\mathbf{D}_2)} \leqslant 1$ and let $\hat{\psi}$ be a solution of

$$\begin{cases} -\nabla\cdot(\nabla\hat{\psi} + Q) = F & \text{in } \mathbf{D}_2 \setminus \overline{Y_m}, \\ \hat{\psi} = 0 & \text{on } \partial Y_m \cup \partial\mathbf{D}_2. \end{cases} \tag{3.11}$$

By [19], there exists a unique function $\hat{\psi} \in W_0^{1,p}(\mathbf{D}_2 \setminus \overline{Y_m})$ satisfying (3.11) and

$$\|\hat{\psi}\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c \quad \text{for } p \in (1,\infty). \tag{3.12}$$

Set $\check{\psi} \equiv \psi - \hat{\psi}$. (3.9), (3.11), and $Q \in C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ imply

$$\begin{cases} -\Delta\check{\psi} = 0 & \text{in } \mathbf{D}_2 \setminus \overline{Y_m}, \\ \nabla\check{\psi}\cdot\vec{\mathbf{n}} = -\nabla\hat{\psi}\cdot\vec{\mathbf{n}} & \text{on } \partial Y_m, \\ \check{\psi} = 0 & \text{on } \partial\mathbf{D}_2. \end{cases} \tag{3.13}$$

By the Green's formula, (3.13), and Theorem 6.5.1 [9], we see that

$$\check{\psi}/2 - \mathcal{T}_{\partial Y_m}(\check{\psi}) = \mathcal{E}_{\partial Y_m}(\nabla\hat{\psi}\cdot\vec{\mathbf{n}}|_{\partial Y_m}) + \mathcal{E}_{\partial \mathbf{D}_2}(\nabla\check{\psi}\cdot\vec{\mathbf{n}}|_{\partial \mathbf{D}_2}) \quad \text{on } \partial Y_m, \tag{3.14}$$

where $\mathcal{E}_{\partial Y_m}$, $\mathcal{E}_{\partial \mathbf{D}_2}$ (resp. $\mathcal{T}_{\partial Y_m}$) are the single-layer potentials (resp. double-layer potential) (see (4.1) in [25] for definition). By (3.12), (3.14), and Lemma 4.1 [25],

$$\|\check{\psi}\|_{W^{1-1/p,p}(\partial Y_m)} \leqslant c\big(1 + \|\nabla\psi\cdot\vec{\mathbf{n}}\|_{W^{-1/p,p}(\partial \mathbf{D}_2)}\big) \quad \text{for } p \in (1,\infty). \tag{3.15}$$

Let η be a smooth function satisfying $\eta \in C^\infty(\mathbf{D}_2)$, $\eta \in [0,1]$, $\eta(x) = 1$ if $dist(x, \partial\mathbf{D}_2) < d_0/2$, $\eta(x) = 0$ if $x \in Y$, $\|\nabla\eta\|_{W^{1,\infty}(\mathbf{D}_2)} \leqslant c$. See (3.1) for d_0. For the existence of such a function η, we refer the reader to Lemma 7.2, Chapter 1 [14]. Multiply (3.9) by η to see

$$\begin{cases} -\nabla\cdot\big(\nabla(\eta\psi) - \psi\nabla\eta + \eta Q\big) = \eta F - \nabla\eta(\nabla\psi + Q) & \text{in } \mathbf{D}_2 \setminus \overline{Y_m}, \\ \eta\psi = 0 & \text{on } \partial Y_m \cup \partial\mathbf{D}_2. \end{cases} \tag{3.16}$$

By [19], the solution of (3.16) satisfies

$$\|\nabla\psi\cdot\vec{\mathbf{n}}\|_{W^{-1/p,p}(\partial\mathbf{D}_2)} \leqslant c\|\eta\psi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c. \tag{3.17}$$

(3.13), (3.15), and (3.17) imply

$$\|\check{\psi}\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c.$$

Together with (3.12), we obtain (3.10).

Step 2: Claim if $Q \in C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ and $F \in L^p(\mathbf{D}_2)$ for $p \in (1,\infty)$, any solution of (3.9) satisfies

$$\|\psi\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\big(\|Q\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \|F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\|_{W^{-1,p}(\mathbf{D}_2)}\big). \tag{3.18}$$

For any $\zeta \in C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ and $r \in [2,\infty)$, let ρ be the solution of

$$\begin{cases} -\Delta\rho = \zeta & \text{in } \mathbf{D}_2 \setminus \overline{Y_m}, \\ \nabla\rho\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \\ \rho = 0 & \text{on } \partial\mathbf{D}_2. \end{cases} \tag{3.19}$$

By Lax–Milgram Theorem and Theorem 7.26 [11], the H^1 solution of (3.19) exists uniquely and

$$\|\rho\|_{L^{\frac{2n}{n-2}}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\rho\|_{H^1(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^2(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})}.$$

So, if $r \in [2, \frac{2n}{n-2}]$, then

$$\|\rho\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})}. \tag{3.20}$$

By (3.10) and (3.20), if $r \in [2, \frac{2n}{n-2}]$.

$$\|\rho\|_{W^{1,r}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})}, \tag{3.21}$$

If $r > \frac{2n}{n-2}$, by (3.21),

$$\|\rho\|_{W^{1,\frac{2n}{n-2}}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^{\frac{2n}{n-2}}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})}. \tag{3.22}$$

Theorem 7.26 [11] and (3.22) then imply

$$\begin{cases} \|\rho\|_{L^{\frac{2n}{n-4}}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\rho\|_{W^{1,\frac{2n}{n-2}}(\mathbf{D}_2\setminus\overline{Y_m})} & \text{if } n>4 \text{ and } r>\dfrac{2n}{n-4}, \\ \|\rho\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\rho\|_{W^{1,\frac{2n}{n-2}}(\mathbf{D}_2\setminus\overline{Y_m})} & \text{if } \begin{cases} n\leqslant 4 \text{ and } r>\dfrac{2n}{n-2}, \text{ or} \\ n>4 \text{ and } r\leqslant \dfrac{2n}{n-4}. \end{cases} \end{cases} \tag{3.23}$$

So if $n \leqslant 4$ and $r > \frac{2n}{n-2}$ or if $n > 4$ and $r \in [2, \frac{2n}{n-4}]$, by (3.10) and (3.22)–(3.23),

$$\|\rho\|_{W^{1,r}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})}. \tag{3.24}$$

Repeating above argument, we see that the solution of (3.19) satisfies (3.24) for all $r \geqslant 2$. Since $C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ is dense in $L^r(\mathbf{D}_2 \setminus \overline{Y_m})$ for all $r \geqslant 2$, we see (3.24) holds for any $\zeta \in L^r(\mathbf{D}_2 \setminus \overline{Y_m})$.

Let $\zeta \in L^r(\mathbf{D}_2 \setminus \overline{Y_m})$ for $r \in [2,\infty)$, multiply (3.9) by ρ obtained from (3.19), and use extension theorem [1] (or see remark above (2.12)), Green Theorem, and Hölder inequality to obtain

$$\begin{aligned} -\int_{\mathbf{D}_2\setminus\overline{Y_m}} \psi\zeta\,dx &= \int_{\mathbf{D}_2\setminus\overline{Y_m}} \psi\Delta\rho\,dx = \int_{\mathbf{D}_2\setminus\overline{Y_m}} (Q\nabla\rho - F\rho)\,dx \\ &= \int_{\mathbf{D}_2\setminus\overline{Y_m}} (Q\nabla\rho - F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\Pi_1\rho)\,dx \\ &\leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})}\big(\|Q\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \|F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\|_{W^{-1,p}(\mathbf{D}_2)}\big), \end{aligned} \tag{3.25}$$

for $p \in (1,2]$, $1/p + 1/r = 1$. So

$$\|\psi\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\big(\|Q\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \|F\mathcal{X}_{\mathbf{D}_2\setminus\overline{Y_m}}\|_{W^{-1,p}(\mathbf{D}_2)}\big) \quad \text{for } p \in (1,2]. \tag{3.26}$$

Eqs. (3.10) and (3.26) imply (3.18) for $p \in (1,2]$. Similarly take ρ to be the solution of (3.19) with $\zeta \in L^r(\mathbf{D}_2 \setminus \overline{Y_m})$ for $r \in (1,2]$. Since (3.18) holds for $p \in (1,2]$, the solution of (3.19) satisfies

$$\|\rho\|_{W^{1,r}(\mathbf{D}_2\setminus\overline{Y_m})} \leqslant c\|\zeta\|_{L^r(\mathbf{D}_2\setminus\overline{Y_m})} \quad \text{for } r \in (1,2]. \tag{3.27}$$

Again we multiply (3.9) by ρ obtained from (3.19) with $\zeta \in L^r(\mathbf{D}_2 \setminus \overline{Y_m})$ and $r \in (1,2]$ as well as argue as (3.25) to see that (3.18) holds for $p \in [2,\infty)$. So (3.18) holds for $p \in (1,\infty)$.

Step 3: By Lax–Milgram Theorem [11] and maximal principle (see Theorem 3.1 and Lemma 3.4 [11]), we see that the $W^{1,p}$ solution of (3.9) exists uniquely and satisfies (3.18) for any $Q \in C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ and $F \in L^p(\mathbf{D}_2)$ for $p \in (1,\infty)$. Since $C_0^\infty(\mathbf{D}_2 \setminus \overline{Y_m})$ is dense in $L^p(\mathbf{D}_2 \setminus \overline{Y_m})$ and $L^p(\mathbf{D}_2)$ is dense in $W^{-1,p}(\mathbf{D}_2)$ for $p \in (1,\infty)$, the solution of (3.9) exists uniquely and satisfies (3.18) for any $Q \in L^p(\mathbf{D}_2 \setminus \overline{Y_m})$, $F \in W^{-1,p}(\mathbf{D}_2)$ by using a limiting argument.

Step 4: Let $\hat{\eta}$ be another smooth function satisfying $\hat{\eta} \in C_0^\infty(\mathbf{D}_2)$, $\hat{\eta} \in [0,1]$, $\hat{\eta} = 1$ in Y, $\|\nabla\hat{\eta}\|_{W^{1,\infty}(\mathbf{D}_2)} \leqslant c$. Multiply (3.7) by $\hat{\eta}$ to see

$$\begin{cases} -\nabla\cdot\big(\nabla(\hat\eta\varphi)-\varphi\nabla\hat\eta+\hat\eta Q\big)=\hat\eta F-\nabla\hat\eta(\nabla\varphi+Q) & \text{in } \mathbf{D}_2\setminus\overline{Y_m},\\ \big(\nabla(\hat\eta\varphi)-\varphi\nabla\hat\eta+\hat\eta Q\big)\cdot\vec{\mathbf{n}}=0 & \text{on } \partial Y_m,\\ \hat\eta\varphi=0 & \text{on } \partial\mathbf{D}_2. \end{cases}$$

Let $\psi=\hat\eta\varphi$ and use (3.18) to obtain $(3.8)_1$.

Step 5: One can modify the argument from Step 1 to Step 4 and employ Lemma 3.2 [24] and $(3.8)_1$ to obtain $(3.8)_2$. So we skip it. □

We also have a uniform Lipschitz estimate for (3.2).

Lemma 3.4. *If $\delta>0$, $\epsilon\in(0,1]$, and $\alpha\in(0,1)$, any solution U_ϵ of (3.2) satisfies*

$$\|\nabla U_\epsilon\|_{L^\infty(B_{1/2}(x)\cap\Omega_f^\epsilon)}\leqslant c\Big(\|U_\epsilon\|_{L^\infty(B_1(x)\cap\Omega_f^\epsilon)}+\epsilon^{-\mu/2}\|\widetilde{Q}_\epsilon\|_{C^{0,\alpha}(\overline{\Omega_f^\epsilon}/\epsilon)} + \big\|\epsilon^{-1+\mu/2}Q_\epsilon, F_\epsilon\big\|_{L^{n+\delta}(B_1(x)\cap\Omega_f^\epsilon)}\Big), \tag{3.28}$$

where $\widetilde{Q}_\epsilon(x)=Q_\epsilon(\epsilon x)$, $\mu\equiv\frac{\delta}{n+\delta}$, $x\in\mathbb{R}^n$, and c is a constant independent of ϵ, x.

Proof of Lemma 3.4 is in Section 6.2. For any $\nu\in(0,1]$ and $\omega\in(0,\infty)$, we define

$$\mathbf{E}^{\nu,\omega}(y)\equiv\begin{cases}1 & \text{if } y\in\Omega_f^\omega,\\ \nu^2 & \text{if } y\in\Omega_m^\omega.\end{cases}$$

Lemma 3.5. *If $\nu\in(0,1]$, $\omega\in[1,\infty)$, and $0\in\partial\Omega_m^\omega$, any solution of*

$$\nabla\cdot\big(\mathbf{E}^{\nu,\omega}\nabla\varphi\big)=0 \quad \text{in } B_1(0) \tag{3.29}$$

satisfies

$$\|\varphi\|_{H^k(B_{1/2}(0)\cap\Omega_f^\omega)}+\nu\|\varphi\|_{H^k(B_{1/2}(0)\cap\Omega_m^\omega)}\leqslant c\big(\|\varphi\|_{L^2(B_1(0)\cap\Omega_f^\omega)}+\nu\|\varphi\|_{L^2(B_1(0)\cap\Omega_m^\omega)}\big), \tag{3.30}$$

where $k\in\mathbb{N}$ and c is a constant independent of ν, ω.

Proof. Let ∂_i denote the partial derivative in the x_i direction, ∂_i^t be t times of partial derivative ∂_i, and $\partial_{i_1,\dots,i_k}=\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}$ for $i,i_1,\dots,i_k\in\{1,\dots,n\}$. We consider the following problem:

$$\begin{cases} -\nabla\cdot(A\nabla\Phi)+Q\,\nabla\Phi=0 & \text{in } B_1(0)\cap\{x\mid x_n<0\},\\ -\nu^2\nabla\cdot(A\nabla\phi)+\nu^2 Q\,\nabla\phi=0 & \text{in } B_1(0)\cap\{x\mid x_n>0\},\\ A\nabla\Phi\cdot\vec{\mathbf{n}}=\nu^2A\nabla\phi\cdot\vec{\mathbf{n}} & \text{on } B_1(0)\cap\{x\mid x_n=0\},\\ \Phi=\phi & \text{on } B_1(0)\cap\{x\mid x_n=0\}, \end{cases} \tag{3.31}$$

where A is a positive definite matrix, both A, Q are smooth functions, and $\vec{\mathbf{n}}$ is a normal vector on the plane $\{x\mid x_n=0\}$.

We claim that there is a constant c independent of ν such that, for any $k\in\mathbb{N}$,

$$\|\Phi\|_{H^k(B_{1/2}(0)\cap\{x_n<0\})}+\nu\|\phi\|_{H^k(B_{1/2}(0)\cap\{x_n>0\})} \leqslant c\big(\|\Phi\|_{L^2(B_1(0)\cap\{x_n<0\})}+\nu\|\phi\|_{L^2(B_1(0)\cap\{x_n>0\})}\big). \tag{3.32}$$

Proof of the claim: This is done by induction on k. Let η be a bell-shaped function satisfying $\eta \in C_0^\infty(B_1(0))$, $\eta \in [0,1]$, and $\eta = 1$ in $B_{3/4}(0)$. Multiply (3.31) by $\eta(\Phi\mathcal{X}_{\{x_n \leqslant 0\}} + \phi\mathcal{X}_{\{x_n \geqslant 0\}})$ and do integration by parts to obtain

$$\begin{aligned}&\|\Phi\|_{H^1(B_{3/4}(0)\cap\{x_n<0\})} + \nu\|\phi\|_{H^1(B_{3/4}(0)\cap\{x_n>0\})}\\&\quad\leqslant c\big(\|\Phi\|_{L^2(B_1(0)\cap\{x_n<0\})} + \nu\|\phi\|_{L^2(B_1(0)\cap\{x_n>0\})}\big),\end{aligned}$$

where c is independent of ν. So we prove (3.32) for $k=1$ case. Let us assume (3.32) for some $k \in \mathbb{N}$ and $k \geqslant 1$. Let $\hat\eta$ be a bell-shaped function satisfying $\hat\eta \in C_0^\infty(B_{3/4}(0))$, $\hat\eta \in [0,1]$, and $\hat\eta = 1$ in $B_{1/2}(0)$. Take partial derivative $\partial_{i_1,\ldots,i_k}$ of (3.31) for $i_1,\ldots,i_k \in \{1,\ldots,n-1\}$, multiply the resulting equations by $\hat\eta(\partial_{i_1,\ldots,i_k}\Phi\mathcal{X}_{\{x_n\leqslant 0\}} + \partial_{i_1,\ldots,i_k}\phi\mathcal{X}_{\{x_n\geqslant 0\}})$, and do integration by parts to obtain

$$\begin{aligned}&\|\partial_{i_1,\ldots,i_k}\Phi\|_{H^1(B_{1/2}(0)\cap\{x_n<0\})} + \nu\|\partial_{i_1,\ldots,i_k}\phi\|_{H^1(B_{1/2}(0)\cap\{x_n>0\})}\\&\quad\leqslant c\big(\|\Phi\|_{L^2(B_1(0)\cap\{x_n<0\})} + \nu\|\phi\|_{L^2(B_1(0)\cap\{x_n>0\})}\big),\end{aligned}\tag{3.33}$$

where c is independent of ν. If we take partial derivative $\partial_n^\ell\partial_{i_1,\ldots,i_{k-1-\ell}}$ of (3.31) for $i_1,\ldots,i_{k-1-\ell} \in \{1,\ldots,n-1\}$, $\ell \in \{0,\ldots,k-1\}$, and $k \geqslant 1$, then

$$\begin{cases}\partial_n^{2+\ell}\partial_{i_1,\ldots,i_{k-1-\ell}}\Phi\\ \quad = \displaystyle\sum_{s=0}^{1+\ell}\sum_{t_1+\cdots+t_{n-1}=0}^{k+1-s} C_{s,t_1,\ldots,t_{n-1}}\partial_n^s\partial_1^{t_1}\cdots\partial_{n-1}^{t_{n-1}}\Phi \quad \text{in } B_1(0)\cap\{x_n<0\},\\ \partial_n^{2+\ell}\partial_{i_1,\ldots,i_{k-1-\ell}}\phi\\ \quad = \displaystyle\sum_{s=0}^{1+\ell}\sum_{t_1+\cdots+t_{n-1}=0}^{k+1-s} C_{s,t_1,\ldots,t_{n-1}}\partial_n^s\partial_1^{t_1}\cdots\partial_{n-1}^{t_{n-1}}\phi \quad \text{in } B_1(0)\cap\{x_n>0\},\end{cases}\tag{3.34}$$

where $C_{s,t_1,\ldots,t_{n-1}}$ is smooth in $\{x_n<0\}\cup\{x_n>0\}$. By (3.33)–(3.34), we obtain (3.32) for $k+1$ case and we prove the claim.

Because of $A1$, we can find an open set $\mathcal{O}_\omega$ and a smooth diffeomorphism τ_ω with positive Jacobian determinant for each $\omega \geqslant 1$ such that $0 \in \mathcal{O}_\omega$, $\tau_\omega(\mathcal{O}_\omega) \to B_1(0)$, $\tau_\omega(\mathcal{O}_\omega \cap \partial\Omega_m^\omega) \subset \{x_n = 0\}$, $\tau_\omega(\mathcal{O}_\omega \cap \Omega_m^\omega) \subset \{x_n > 0\}$, and $\tau_\omega(\mathcal{O}_\omega \cap \Omega_f^\omega) \subset \{x_n < 0\}$. After transform by the mapping τ_ω, Eq. (3.29) can be written as (3.31) in the new coordinate system. (3.30) follows from (3.32). $\square$

Lemma 3.6. *If $\nu, \epsilon \in (0,1]$ and $r > 0$, any solution of*

$$\nabla\cdot\big(\mathbf{E}^{\nu,\epsilon}\nabla\varphi\big) = 0 \quad \text{in } B_r(x)\tag{3.35}$$

satisfies

$$\big|\varphi(x)\mathcal{X}_{\Omega_f^\epsilon} + \nu\varphi(x)\mathcal{X}_{\Omega_m^\epsilon}\big| \leqslant c\left|\;\fint_{B_r(x)}\varphi^2(y)\mathcal{X}_{\Omega_f^\epsilon} + \nu^2\varphi^2(y)\mathcal{X}_{\Omega_m^\epsilon}\,dy\right|^{1/2},\tag{3.36}$$

where c is a constant independent of x, ν, ϵ, r.

Proof. Assume $x=0\in\Omega_f^\epsilon$ and define $\widetilde{\varphi}(y)\equiv\varphi(ry)$, $\mathbf{E}^{\nu,\epsilon/r}(y)\equiv\mathbf{E}^{\nu,\epsilon}(ry)$. Then (3.35) implies

$$\nabla\cdot\big(\mathbf{E}^{\nu,\epsilon/r}\nabla\widetilde{\varphi}\big)=0\quad\text{in } B_1(0). \tag{3.37}$$

If $1\leqslant\epsilon/r$ (resp. $\epsilon/r<1$), Theorems 7.26, 8.24 [11] and Lemma 3.5 for $k>n/2$ (resp. Lemma 4.3 [24]) imply

$$[\widetilde{\varphi}]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega_f^\epsilon/r})}\leqslant c\big(\|\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega_f^\epsilon/r)}+\|\nu\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega_m^\epsilon/r)}\big),$$

where $\mu>0$ and c is a constant independent of ν, ϵ, r. So

$$\begin{aligned}
\big|\varphi(0)\big|=\big|\tilde{\varphi}(0)\big|&\leqslant\bigg|\widetilde{\varphi}(0)-\fint_{B_{1/2}(0)\cap\Omega_f^\epsilon/r}\widetilde{\varphi}(y)\,dy\bigg|+\bigg|\fint_{B_{1/2}(0)\cap\Omega_f^\epsilon/r}\widetilde{\varphi}(y)\,dy\bigg|\\
&\leqslant[\widetilde{\varphi}]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega_f^\epsilon/r})}\fint_{B_{1/2}(0)\cap\Omega_f^\epsilon/r}|y|^\mu\,dy+\bigg|\fint_{B_{1/2}(0)\cap\Omega_f^\epsilon/r}\widetilde{\varphi}(y)\,dy\bigg|\\
&\leqslant c\bigg|\fint_{B_1(0)\cap\Omega_f^\epsilon/r}\big|\widetilde{\varphi}(y)\big|^2\,dy+\fint_{B_1(0)\cap\Omega_m^\epsilon/r}\big|\nu\widetilde{\varphi}(y)\big|^2\,dy\bigg|^{1/2}\\
&=c\bigg|\fint_{B_r(0)\cap\Omega_f^\epsilon}\big|\varphi(y)\big|^2\,dy+\fint_{B_r(0)\cap\Omega_m^\epsilon}\big|\nu\varphi(y)\big|^2\,dy\bigg|^{1/2}.
\end{aligned}\tag{3.38}$$

If $0\neq x\in\Omega_f^\epsilon$, we shift x to 0 and repeat the above argument to see that (3.38) with 0 replaced by x still holds. So (3.36) is proved for $x\in\Omega_f^\epsilon$.

Next we assume $x=0\in\epsilon Y_m\subset\Omega_m^\epsilon$ and define $\widetilde{\varphi}(y)=\varphi(ry)$. So (3.37) still holds. If $\epsilon/r\leqslant1$, by maximal principle [11] and $0\in\frac{\epsilon}{r}Y_m$, we know that maximal value of $|\widetilde{\varphi}|$ in the region $\frac{\epsilon}{r}Y_m$ is bounded by the maximal value of $|\widetilde{\varphi}|$ on the boundary of $\frac{\epsilon}{r}Y_m$. Since (3.36) holds in Ω_f^ϵ,

$$\begin{aligned}
\big|\varphi(0)\big|=\big|\widetilde{\varphi}(0)\big|&\leqslant\max_{z\in\partial\frac{\epsilon}{r}Y_m}\big|\widetilde{\varphi}(z)\big|\\
&\leqslant\max_{z\in\partial\frac{\epsilon}{r}Y_m}c\bigg|\fint_{B_{d_0}(z)}\big|\widetilde{\varphi}(y)\big|^2\mathcal{X}_{\Omega_f^\epsilon/r}(y)+\nu^2\big|\widetilde{\varphi}(y)\big|^2\mathcal{X}_{\Omega_m^\epsilon/r}(y)\,dy\bigg|^{1/2}\\
&\leqslant c\bigg|\fint_{B_1(0)}\big|\widetilde{\varphi}(y)\big|^2\mathcal{X}_{\Omega_f^\epsilon/r}(y)+\nu^2\big|\widetilde{\varphi}(y)\big|^2\mathcal{X}_{\Omega_m^\epsilon/r}(y)\,dy\bigg|^{1/2}\\
&\leqslant c\bigg|\fint_{B_r(0)}\big|\varphi(y)\big|^2\mathcal{X}_{\Omega_f^\epsilon}(y)+\nu^2\big|\varphi(y)\big|^2\mathcal{X}_{\Omega_m^\epsilon}(y)\,dy\bigg|^{1/2},
\end{aligned}$$

where d_0 is defined in (3.1) and c is independent of ν, ϵ/r. If $\epsilon/r>1$, Theorems 7.26, 8.24 [11] and Lemma 3.5 with $k>\frac{n}{2}$ imply

$$\nu[\widetilde{\varphi}]_{C^{0,\mu}(\overline{B_{1/4}(0)\cap\Omega^\epsilon_m/r})} \leqslant c\big(\|\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega^\epsilon_f/r)} + \nu\|\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega^\epsilon_m/r)}\big),$$

where μ, c are positive constants independent of ν, ϵ/r.

$$\begin{aligned}
\big|\varphi(0)\big| = \big|\widetilde{\varphi}(0)\big| &\leqslant \bigg|\widetilde{\varphi}(0) - \fint_{B_{1/4}(0)\cap\Omega^\epsilon_m/r} \widetilde{\varphi}(y)\,dy\bigg| + \bigg|\fint_{B_{1/4}(0)\cap\Omega^\epsilon_m/r} \widetilde{\varphi}(y)\,dy\bigg| \\
&\leqslant c\big([\widetilde{\varphi}]_{C^{0,\mu}(\overline{B_{1/4}(0)\cap\Omega^\epsilon_m/r})} + \|\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega^\epsilon_m/r)}\big) \\
&\leqslant c\nu^{-1}\big(\|\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega^\epsilon_f/r)} + \nu\|\widetilde{\varphi}\|_{L^2(B_1(0)\cap\Omega^\epsilon_m/r)}\big) \\
&\leqslant c\nu^{-1}\bigg|\fint_{B_r(0)} \big|\varphi(y)\big|^2\mathcal{X}_{\Omega^\epsilon_f}(y) + \nu^2\big|\varphi(y)\big|^2\mathcal{X}_{\Omega^\epsilon_m}(y)\,dy\bigg|^{1/2},
\end{aligned}$$

where c is independent of ν, ϵ/r. So (3.36) holds for $x = 0 \in \epsilon Y_m \subset \Omega^\epsilon_m$ case. If $x \in \Omega^\epsilon_m$ and $x \neq 0$, we shift the coordinate system such that the origin of the coordinate system is located at x. Then we see that (3.36) holds for $x \in \Omega^\epsilon_m$. □

3.2. The Green's function in Ω_f

Let $\mathcal{G}_{\nu,\epsilon}$ for $\nu, \epsilon \in (0, 1]$ denote the Green's function of

$$\begin{cases} -\nabla_y \cdot \big(\mathbf{E}^{\nu,\epsilon}(y)\nabla_y\mathcal{G}_{\nu,\epsilon}(x, y)\big) = \delta(x - y) \quad \text{in } \mathbb{R}^n, \\ \mathcal{G}_{\nu,\epsilon}(x, y) \to 0 \quad \text{as } |x - y| \to \infty. \end{cases} \tag{3.39}$$

By Theorem 5.4 [16] and remark on pp. 62, 67 [16], $\mathcal{G}_{\nu,\epsilon}(x,\cdot) \in H^1_{loc}(\mathbb{R}^n \setminus \{x\}) \cap W^{1,1}_{loc}(\mathbb{R}^n)$ exists uniquely when $n \geqslant 3$ and

$$\mathcal{G}_{\nu,\epsilon}(x, y) = \mathcal{G}_{\nu,\epsilon}(y, x) \quad \text{for } x \neq y. \tag{3.40}$$

Lemma 3.7. *There is a constant c independent of $\nu, \epsilon \in (0, 1]$ such that*

$$\big|\mathcal{G}_{\nu,\epsilon}(x, y)\big| \leqslant \begin{cases} c|x-y|^{2-n} & \text{if } x, y \in \Omega^\epsilon_f, \\ c\nu^{-1}|x-y|^{2-n} & \text{if } x \in \Omega^\epsilon_f,\ y \in \Omega^\epsilon_m \text{ or if } x \in \Omega^\epsilon_m,\ y \in \Omega^\epsilon_f, \\ c\nu^{-2}|x-y|^{2-n} & \text{if } x, y \in \Omega^\epsilon_m, \\ c|x-y|^{2-n} & \text{if } |x-y| > \epsilon, \end{cases} \tag{3.41}$$

$$\big|\nabla_y\mathcal{G}_{\nu,\epsilon}(x, y)\big| + \big|\nabla_x\mathcal{G}_{\nu,\epsilon}(x, y)\big| \leqslant c|x-y|^{1-n} \quad \text{if } x, y \in \Omega^\epsilon_f. \tag{3.42}$$

For any $x \in \Omega_m$, there is a number $d > 0$ such that if $0 < |x - y| < d$, then

$$\big|\nabla_y\mathcal{G}_{\nu,1}(x, y)\big| \leqslant c\nu^{-2}|x-y|^{1-n}, \tag{3.43}$$

where c is a constant independent of ν.

Proof. Proof of (3.41)$_1$ (that is, for $x, y \in \Omega_f^\epsilon$ case). Set $r \equiv |x-y|$ for $x, y \in \Omega_f^\epsilon$. Let $F \in C_0^\infty(B_{r/3}(y))$ and find $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfying

$$-\nabla \cdot \left(\mathbf{E}^{\nu,\epsilon} \nabla \varphi\right) = F \mathcal{X}_{\Omega_f^\epsilon} + \nu F \mathcal{X}_{\Omega_m^\epsilon}.$$

By Lax–Milgram Theorem [11], extension theorem [1], and (2.12) for $p = 2$, φ is solvable uniquely in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. By Definition 5.1 and p. 67 [16] and Lemma 3.6, we see, for $x \in \Omega_f^\epsilon$,

$$\begin{cases} \varphi(x) = \displaystyle\int\limits_{B_{r/3}(y)} \mathcal{G}_{\nu,\epsilon}(x,z) F(z) \mathcal{X}_{\Omega_f^\epsilon}(z) + \mathcal{G}_{\nu,\epsilon}(x,z) \nu F(z) \mathcal{X}_{\Omega_m^\epsilon}(z)\, dz, \\ \left|\varphi(x)\right| \leqslant c \left| \displaystyle\fint\limits_{B_{r/3}(x)} \varphi^2(z) \mathcal{X}_{\Omega_f^\epsilon}(z) + \nu^2 \varphi^2(z) \mathcal{X}_{\Omega_m^\epsilon}(z)\, dz \right|^{1/2}, \end{cases} \tag{3.44}$$

where c is independent of x, y, ν, ϵ, r. (2.12), (3.44), extension theorem [1], and Hölder inequality [11] imply

$$\begin{aligned} &\left| \int\limits_{B_{r/3}(y)} \mathcal{G}_{\nu,\epsilon}(x,z) F(z) \mathcal{X}_{\Omega_f^\epsilon}(z) + \mathcal{G}_{\nu,\epsilon}(x,z) \nu F(z) \mathcal{X}_{\Omega_m^\epsilon}(z)\, dz \right| \\ &\quad \leqslant c \left| \fint\limits_{B_{r/3}(x)} \varphi^2(z) \mathcal{X}_{\Omega_f^\epsilon}(z) + \nu^2 \varphi^2(z) \mathcal{X}_{\Omega_m^\epsilon}(z)\, dz \right|^{1/2} \\ &\quad \leqslant c \left| \fint\limits_{B_{r/3}(x)} \left|\varphi(z) \mathcal{X}_{\Omega_f^\epsilon}(z) + \nu \varphi(z) \mathcal{X}_{\Omega_m^\epsilon}(z)\right|^{\frac{2n}{n-2}} dz \right|^{\frac{n-2}{2n}} \\ &\quad \leqslant c \frac{\|\nabla \varphi \mathcal{X}_{\Omega_f^\epsilon} + \nu \nabla \varphi \mathcal{X}_{\Omega_m^\epsilon}\|_{L^2(\mathbb{R}^n)}}{r^{\frac{n}{2}-1}} \leqslant c \frac{\|F\|_{L^2(B_{r/3}(y))}}{r^{\frac{n}{2}-2}}, \end{aligned} \tag{3.45}$$

where c is independent of x, y, ν, ϵ, r. Multiply (3.45) by r^{-n} to obtain

$$\begin{aligned} &\left| \fint\limits_{B_{r/3}(y)} \mathcal{G}_{\nu,\epsilon}(x,z) F(z) \mathcal{X}_{\Omega_f^\epsilon}(z) + \mathcal{G}_{\nu,\epsilon}(x,z) \nu F(z) \mathcal{X}_{\Omega_m^\epsilon}(z)\, dz \right| \\ &\quad \leqslant \frac{c}{r^{n-2}} \left| \fint\limits_{B_{r/3}(y)} F^2(z)\, dz \right|^{1/2}. \end{aligned} \tag{3.46}$$

Since $y \in \Omega_f^\epsilon$, Eqs. (3.39), (3.46) and Lemma 3.6 imply

$$\left|\mathcal{G}_{\nu,\epsilon}(x,y)\right| \leqslant c \left| \fint\limits_{B_{r/3}(y)} \left|\mathcal{G}_{\nu,\epsilon}(x,z)\right|^2 \mathcal{X}_{\Omega_f^\epsilon}(z) + \nu^2 \left|\mathcal{G}_{\nu,\epsilon}(x,z)\right|^2 \mathcal{X}_{\Omega_m^\epsilon}(z)\, dz \right|^{1/2} \leqslant \frac{c}{r^{n-2}}.$$

So (3.41)$_1$ holds. (3.41)$_{2,3}$ are proved exactly by following the argument of (3.41)$_1$. (3.41)$_4$ (that is, $|x-y| > \epsilon$ case) follows from Theorem 3.1 [11], (3.41)$_1$, and (3.1).

Proof of (3.42). Assume $x, y \in \Omega_f^\epsilon$, $x \neq y$, and $y = 0$. We define $r \equiv |x|$, then

$$-\nabla_z \cdot \big(\mathbf{E}^{\nu,\epsilon}(z)\nabla_z \mathcal{G}_{\nu,\epsilon}(x,z)\big) = 0 \quad \text{in } B_{r/2}(0).$$

If $\varphi(z) \equiv \mathcal{G}_{\nu,\epsilon}(x, \frac{r}{2}z)$, then

$$-\nabla \cdot \big(\mathbf{E}^{\nu,2\epsilon/r}\nabla\varphi\big)(z) = 0 \quad \text{in } B_1(0).$$

Suppose $2\epsilon/r > 1$, by Theorem 2.10 [11], Lemma 3.5 with $k > \frac{n}{2} + 1$, and (3.41), we get

$$\begin{aligned}\big|\nabla_y \mathcal{G}_{\nu,\epsilon}(x,0)\big| &= \frac{2}{r}\big|\nabla\varphi(0)\big| \\ &\leqslant \frac{c}{r}\big(\|\varphi\|_{L^2(B_1(0)\cap\Omega_f^{2\epsilon/r})} + \nu\|\varphi\|_{L^2(B_1(0)\cap\Omega_m^{2\epsilon/r})}\big) \leqslant c|x|^{1-n},\end{aligned} \tag{3.47}$$

where c is independent of x, y, ν, ϵ/r. Suppose $2\epsilon/r \leqslant 1$, by Lemma 5.3 [24] and (3.41), we also get (3.47). Assume $x, y \in \Omega_f^\epsilon$ and $x \neq y$. One can shift y to 0, repeat the above process, and obtain $|\nabla_y \mathcal{G}_{\nu,\epsilon}(x,y)| \leqslant c|x-y|^{1-n}$, where c is independent of ν, ϵ. By (3.40), we also obtain $|\nabla_x \mathcal{G}_{\nu,\epsilon}(x,y)| \leqslant c|x-y|^{1-n}$, where c is independent of ν, ϵ. So (3.42) is proved. (3.43) can be proved in a similar way as (3.42), so we skip it. □

Lemma 3.8. *There is a unique function $G(x,y)$ in $\Omega_f \times \Omega_f$ satisfying, for any $x \in \Omega_f$,*

$$\begin{cases} -\Delta_y G(x,y) = \delta(x-y) & \text{in } \Omega_f, \\ \nabla_y G(x,y)\cdot \vec{\mathbf{n}}_y = 0 & \text{on } \partial\Omega_m, \\ \big|G(x,y)\big| \leqslant c|x-y|^{2-n} & \text{if } x \neq y, \\ \big|\nabla_y G(x,y)\big| + \big|\nabla_x G(x,y)\big| \leqslant c|x-y|^{1-n} & \text{if } x \neq y, \\ G(x,y) = G(y,x) & \text{if } x \neq y, \end{cases} \tag{3.48}$$

where $\vec{\mathbf{n}}_y$ is a unit vector normal to $\partial\Omega_m$ and c is a constant.

For any $x \in \Omega_f$, there is a number $d > 0$ such that if $0 < |x-y| < d$, then

$$\big|\nabla_x\nabla_y G(x,y)\big| \leqslant c|x-y|^{-n}, \tag{3.49}$$

where c is a constant independent of x, y.

Proof. If $x \in \Omega_f$, $dist(x, \partial\Omega_m) > 0$. Assume $t_i \to 0$ and $s_i \to \infty$ as $i \to \infty$. For any $s_i > dist(x, \partial\Omega_m) > t_i > 0$, we define $\mathbb{D}_{t_i,s_i}(x) \equiv \{y \in \mathbb{R}^n |\ 0 < t_i < |x-y| < s_i\}$. From (3.39),

$$-\nabla_y \cdot \big(\mathbf{E}^{\nu,1}\nabla_y \mathcal{G}_{\nu,1}(x,y)\big) = 0 \quad \text{in } \mathbb{D}_{t_i,s_i}(x). \tag{3.50}$$

Since $x \in \Omega_f$, by maximal principle (see Theorem 3.1 and Lemma 3.4 [11]) and $(3.41)_1$, $\|\mathcal{G}_{\nu,1}(x,\cdot)\|_{L^\infty(\mathbb{D}_{t_i,s_i}(x))}$ with fixed t_i, s_i is bounded independent of ν. If $j \in \mathbb{Z}^n$ and the closure $\overline{Y} - j \subset \mathbb{D}_{t_i,s_i}(x)$, by Lemma 3.3 [24], we see

$$\big\|\mathcal{G}_{\nu,1}(x,\cdot)\big\|_{C^{1,\alpha}(\overline{\mathbf{D}\setminus Y_m} - j)} + \big\|\mathcal{G}_{\nu,1}(x,\cdot)\big\|_{C^{1,\alpha}(\overline{Y_m} - j)} \leqslant c\big\|\mathcal{G}_{\nu,1}(x,\cdot)\big\|_{L^\infty(\mathbb{D}_{t_i,s_i}(x))},$$

where $Y_m \subset \mathbf{D} \subset Y$ (see (3.1)), $\alpha \in (0,1)$, and c is independent of ν. So there are $\widetilde{t}_i$, $\widetilde{s}_i$ satisfying (1) $0 < t_i < \widetilde{t}_i < \widetilde{s}_i < s_i$, (2) $\widetilde{t}_i \to 0$, $\widetilde{s}_i \to \infty$ as $i \to \infty$, and (3)

$$\left\| \mathcal{G}_{\nu,1}(x,\cdot) \right\|_{C^{1,\alpha}(\overline{\Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x)})} + \left\| \mathcal{G}_{\nu,1}(x,\cdot) \right\|_{C^{1,\alpha}(\overline{\Omega_m \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x)})} \leqslant c, \tag{3.51}$$

where $\alpha \in (0,1)$ and c is bounded independent of ν but dependent on t_i. Therefore for each fixed $\widetilde{t}_i$, $\widetilde{s}_i$, there is a convergent subsequence of $\mathcal{G}_{\nu,1}(x,\cdot)$ in $C^{1,\widetilde{\alpha}}(\overline{\Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x)})$ for some $\widetilde{\alpha} < \alpha < 1$ as $\nu \to 0$. By a diagonal process, we can even extract a subsequence of $\mathcal{G}_{\nu,1}(x,\cdot)$ (same notation for subsequence) such that, for all $\widetilde{t}_i$, $\widetilde{s}_i$ with $0 < \widetilde{t}_i < \widetilde{s}_i < \infty$,

$$\mathcal{G}_{\nu,1}(x,\cdot) \text{ converges to } G(x,\cdot) \text{ in } C^{1,\widetilde{\alpha}}\left(\overline{\Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x)}\right) \text{ as } \nu \to 0. \tag{3.52}$$

If $\zeta \in C_0^\infty(\mathbb{R}^n)$ and $x \notin supp(\zeta)$ (that is, the support of ζ), there are $\widetilde{t}_i$, $\widetilde{s}_i$ such that $supp(\zeta) \subset \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x)$. Multiply (3.50) by ζ to see, by (3.51)–(3.52),

$$\begin{aligned} 0 &= \lim_{\nu \to 0} \int \nabla_y \mathcal{G}_{\nu,1}(x,y) \nabla\zeta(y) \mathcal{X}_{\Omega_f} + \nu^2 \nabla_y \mathcal{G}_{\nu,1}(x,y) \nabla\zeta(y) \mathcal{X}_{\Omega_m}\, dy \\ &= \int_{\Omega_f} \nabla_y G(x,y) \nabla\zeta(y)\, dy. \end{aligned} \tag{3.53}$$

(3.40), (3.52)–(3.53), and Lemma 3.7 imply, for all $\widetilde{t}_i$, $\widetilde{s}_i$ and for any $x \in \Omega_f$,

$$\begin{cases} -\Delta_y G(x,y) = 0 & \text{in } \Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x), \\ \nabla_y G(x,y) \cdot \vec{\mathbf{n}}_y = 0 & \text{on } \partial\Omega_m \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x), \\ \left|G(x,y)\right| \leqslant c|x-y|^{2-n} & \text{for } y \in \Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x), \\ \left|\nabla_y G(x,y)\right| \leqslant c|x-y|^{1-n} & \text{for } y \in \Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x), \\ G(x,y) = G(y,x) & \text{for } y \in \Omega_f \cap \mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x), \end{cases} \tag{3.54}$$

where c is a constant independent of $\widetilde{t}_i$, $\widetilde{s}_i$.

For any $F, Q \in C(\mathbb{R}^n)$ with compact support, there is a unique $\varphi_{\nu,1} \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfying, by Lax–Milgram Theorem [11], extension theorem [1], and (2.12) for $p = 2$,

$$-\nabla \cdot \left(\mathbf{E}^{\nu,1} \nabla\varphi_{\nu,1} + Q\mathcal{X}_{\Omega_f}\right) = F\mathcal{X}_{\Omega_f} \quad \text{in } \mathbb{R}^n.$$

By energy method, (2.12), and Lemma 3.3, we see that there is a subsequence of $\varphi_{\nu,1}$ (same notation for subsequence) such that (1) $\varphi_{\nu,1}$ converges to φ in $\mathcal{D}^{1,2}(\Omega_f) \cap W^{1,p}(B_r(0) \cap \Omega_f)$ weakly for any $r > 0$, $p \in (n,\infty)$ as $\nu \to 0$ and (2) φ satisfies

$$\begin{cases} -\nabla \cdot (\nabla\varphi + Q) = F & \text{in } \Omega_f, \\ (\nabla\varphi + Q) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega_m, \end{cases} \tag{3.55}$$

where $\vec{\mathbf{n}}$ is a unit vector normal to $\partial\Omega_m$. By Definition 5.1 and p. 67 [16], (3.42), and (3.43),

$$\varphi_{\nu,1}(x) = \int_{\Omega_f} \mathcal{G}_{\nu,1}(x,y) F(y)\, dy - \int_{\Omega_f} \nabla_y \mathcal{G}_{\nu,1}(x,y) Q(y)\, dy \quad \text{for } x \in \mathbb{R}^n. \tag{3.56}$$

By Lemma 3.7, (3.52), and (3.56), for any $x \in \Omega_f$ and $d < \frac{dist(x,\partial\Omega_m)}{2}$,

$$\begin{aligned}
&\left|\varphi(x) - \int\limits_{\Omega_f \setminus B_d(x)} \big(G(x,y)F(y) - \nabla_y G(x,y)Q(y)\big)\,dy\right| \\
&= \lim_{\nu\to 0}\left|\varphi_{\nu,1}(x) - \int\limits_{\Omega_f \setminus B_d(x)} \big(\mathcal{G}_{\nu,1}(x,y)F(y) - \nabla_y \mathcal{G}_{\nu,1}(x,y)Q(y)\big)\,dy\right| \\
&= \lim_{\nu\to 0}\left|\int\limits_{B_d(x)} \big(\mathcal{G}_{\nu,1}(x,y)F(y) - \nabla_y \mathcal{G}_{\nu,1}(x,y)Q(y)\big)\,dy\right| \\
&\leqslant cd\|Q,F\|_{L^\infty(B_d(x))},
\end{aligned}$$

where c is independent of d. So, for any $x \in \Omega_f$ and $\delta > 0$, there is a $\widetilde{d} < dist(x,\partial\Omega_m)$ such that if $d < \widetilde{d}$, then $cd\|Q,F\|_{L^\infty(B_d(x))} < \delta$. So we obtain

$$\varphi(x) = \int\limits_{\Omega_f} G(x,y)F(y)\,dy - \int\limits_{\Omega_f} \nabla_y G(x,y)Q(y)\,dy. \tag{3.57}$$

(3.54), (3.55), and (3.57) imply the existence of $G(x,y)$ and (3.48). The uniqueness of $G(x,\cdot)$ for any $x \in \Omega_f$ is due to $(3.48)_3$ and maximal principle [11].

Let $x \in \Omega_f$, $d < dist(x,\partial\Omega_m)$, $y \in B_{d/4}(x) \setminus \{x\}$, and $r \equiv |x-y|$. By (3.48),

$$-\Delta_z \partial_{y_i} G(z,y) = 0 \quad \text{in } B_{r/2}(x),$$

where ∂_{y_i} is the partial derivative with respect to y_i for $i \in \{1,\dots,n\}$. By Theorem 2.10 [11] and $(3.48)_4$,

$$\left|\nabla_x \partial_{y_i} G(x,y)\right| \leqslant \frac{c}{r}\left\|\partial_{y_i} G(\cdot,y)\right\|_{L^\infty(B_{r/2}(x))} \leqslant c|x-y|^{-n},$$

where c is a constant independent of x, y. So we prove (3.49). □

Remark 3.1. From (3.55) and (3.57) in the proof of Lemma 3.8, we know that for any $F, Q \in C(\mathbb{R}^n)$ with compact support, the $\mathcal{D}^{1,2}(\Omega_f)$ solution of

$$\begin{cases} -\nabla\cdot(\nabla\varphi + Q) = F & \text{in } \Omega_f, \\ (\nabla\varphi + Q)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial\Omega_m \end{cases} \tag{3.58}$$

exists uniquely and satisfies

$$\varphi(x) = \int\limits_{\Omega_f} G(x,y)F(y)\,dy - \int\limits_{\Omega_f} \nabla_y G(x,y)Q(y)\,dy \quad \text{for } x \in \Omega_f. \tag{3.59}$$

Following the argument of Lemma 4.1 [11] and employing $(3.48)_4$ and (3.59), if $F \in C(\mathbb{R}^n)$ with compact support, the $\mathcal{D}^{1,2}(\Omega_f)$ solution of

$$\begin{cases} -\Delta\varphi = F & \text{in } \Omega_f, \\ \nabla\varphi \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega_m \end{cases}$$

satisfies

$$\nabla\varphi(x) = \int_{\Omega_f} \nabla_x G(x,y) F(y)\,dy \quad \text{for } x \in \Omega_f.$$

Tracing the proof of Lemma 4.2 [11] as well as employing (3.49), we have

Lemma 3.9. *If $Q \in C_0^\infty(\mathbb{R}^n)$ with support in $B_r(0)$ for some $r > 0$, then the $\mathcal{D}^{1,2}(\Omega_f)$ solution of*

$$\begin{cases} -\nabla \cdot (\nabla\varphi + Q) = 0 & \text{in } \Omega_f, \\ (\nabla\varphi + Q) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega_m \end{cases}$$

satisfies

$$\partial_{x_j}\varphi(x) = -\int_{B_s(0)\cap\Omega_f} \partial_{x_j}\nabla_y G(x,y) Q(y)\,dy + Q(x)\int_{B_s(0)\cap\Omega_f} \partial_{x_j}\nabla_y \Gamma(x,y)\,dy$$
$$+ Q(x)\int_{\partial(B_s(0)\cap\Omega_f)} \nabla_y \Gamma(x,y)\mathbf{n}_j\,d\sigma_y \quad \text{for any } x \in \Omega_f, \tag{3.60}$$

where $s > 1 + r$, Γ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$, $j \in \{1, \dots, n\}$, and $\mathbf{n}_j$ is the j-th component of $\vec{\mathbf{n}} = (\mathbf{n}_1, \dots, \mathbf{n}_n)$ which is a unit outward normal vector on $\partial(B_s(0) \cap \Omega_f)$.

Moreover, there is a constant c independent of r such that

$$\left|\nabla\varphi(x) + \int_{\Omega_f} \nabla_x\nabla_y G(x,y) Q(y)\,dy\right| \leqslant c\big|Q(x)\big| \quad \text{for any } x \in \Omega_f. \tag{3.61}$$

Proof. Define, for any $x \in \Omega_f$ and $\delta < dist(x, \partial\Omega_m)/2 < 1$,

$$\varphi_\delta(x) \equiv -\int_{B_s(0)\cap\Omega_f} \nabla_y G(x,y)\eta_\delta\big(|x-y|\big) Q(y)\,dy,$$

where $\eta_\delta(x) = \eta(x/\delta)$ and $\eta \in C_0^\infty(\mathbb{R})$ is an even function satisfying $\eta \in [0,1]$, $\eta(x) = 0$ in $|x| \leqslant 1/2$, $\eta(x) = 1$ in $|x| \geqslant 1$, and $\eta'(x) \geqslant 0$ for $x \geqslant 0$. By (3.48), $\varphi_\delta \in C^1(\Omega_f)$ and φ_δ converges to

$$\varphi(x) \equiv -\int_{B_s(0)\cap\Omega_f} \nabla_y G(x,y) Q(y)\,dy$$

in $L^\infty(\Omega_f)$ as $\delta \to 0$. Define $\rho(x,\cdot)$ for $x \in \Omega_f$ as

$$\rho(x,y) \equiv G(x,y) - \Gamma(x,y) \quad \text{for } y \in \Omega_f.$$

Then $\Delta_y \rho(x,\cdot) = 0$ in Ω_f and, by Theorem 6.30 [11] and Lemma 3.8, $\|\rho(x,\cdot)\|_{C^2(\overline{\Omega_f})}$ and $\|\nabla_x \nabla_y \rho\|_{L^\infty(\Omega_f\times\Omega_f)}$ are bounded by a constant depending on $dist(x,\partial\Omega_m)$. If $x=(x_1,\dots,x_n)\in\Omega_f$, for any $j\in\{1,\dots,n\}$,

$$\begin{aligned}
\partial_{x_j}\varphi_\delta(x) &= -\int\limits_{B_s(0)\cap\Omega_f} \partial_{x_j}\big(\nabla_y G(x,y)\eta_\delta\big(|x-y|\big)\big) Q(y)\,dy \\
&= -\int\limits_{B_s(0)\cap\Omega_f} \partial_{x_j}\big(\nabla_y G(x,y)\eta_\delta\big(|x-y|\big)\big)\big(Q(y)-Q(x)\big)\,dy \\
&\quad - Q(x)\int\limits_{B_s(0)\cap\Omega_f} \big(\partial_{x_j}\big(\nabla_y \Gamma(x,y)\eta_\delta\big(|x-y|\big)\big) + \partial_{x_j}\big(\nabla_y \rho(x,y)\eta_\delta\big(|x-y|\big)\big)\big)\,dy \\
&= -\int\limits_{B_s(0)\cap\Omega_f} \partial_{x_j}\big(\nabla_y G(x,y)\eta_\delta\big(|x-y|\big)\big)\big(Q(y)-Q(x)\big)\,dy \\
&\quad + Q(x)\int\limits_{\partial(B_s(0)\cap\Omega_f)} \nabla_y \Gamma(x,y)\mathbf{n}_j\,d\sigma_y - Q(x)\int\limits_{B_s(0)\cap\Omega_f} \partial_{x_j}\big(\nabla_y \rho(x,y)\eta_\delta\big(|x-y|\big)\big)\,dy.
\end{aligned}$$

By Lemma 3.8 and arguing as the proof of Lemma 4.2 [11], if $\delta\to 0$, then $\partial_{x_j}\varphi_\delta(x)$ converges to the right-hand side of (3.60) for any $x\in\Omega_f$, $s>1+r$, and $j\in\{1,\dots,n\}$. So we prove (3.60).

From (2.13) in [11],

$$\int\limits_{\partial B_t(x)} \partial_{y_i}\partial_{y_j}\Gamma(x,y)\,d\sigma_y = 0 \quad \text{for any } t>0 \text{ and } i,j\in\{1,\dots,n\}. \tag{3.62}$$

If $x\notin B_r(0)\cap\Omega_f$, (3.61) is obvious from (3.60). If $x\in B_r(0)\cap\Omega_f$ and $s>1+r$, Divergence Theorem [11] and (3.62) imply

$$\begin{aligned}
&\left|\int\limits_{B_s(0)\cap\Omega_f} \partial_{x_j}\nabla_y\Gamma(x,y)\,dy + \int\limits_{\partial(B_s(0)\cap\Omega_f)} \nabla_y\Gamma(x,y)\mathbf{n}_j\,d\sigma_y\right| \\
&\quad = \left|-\int\limits_{B_s(0)\cap\Omega_f} \partial_{y_j}\nabla_y\Gamma(x,y)\,dy + \int\limits_{\partial(B_s(0)\cap\Omega_f)} \nabla_y\Gamma(x,y)\mathbf{n}_j\,d\sigma_y\right| \\
&\quad \leqslant \int\limits_{B_{s+|x|+1}(x)\setminus B_{s-|x|-1}(x)} \big|\nabla_y^2\Gamma(x,y)\big|\,dy + c \\
&\quad \leqslant c\int\limits_{B_{s+|x|+1}(x)\setminus B_{s-|x|-1}(x)} |x-y|^{-n}\,dy + c \leqslant c\ln\frac{s+r+1}{s-r-1} + c,
\end{aligned} \tag{3.63}$$

where c is a constant independent of r, s. We note that $(\mathbf{n}_1,\dots,\mathbf{n}_n)$ is a unit outward normal vector on $\partial(B_s(0)\cap\Omega_f)$ in (3.63). If s is much larger than r, the right-hand side of (3.63) is bounded by a constant independent of r. Together with (3.60), we obtain (3.61). □

If we define

$$G_\omega(x, y) \equiv \omega^{2-n} G(x/\omega, y/\omega) \quad \text{for any } \omega \in (0, \infty), \tag{3.64}$$

then $G_\omega(x, y)$ satisfies, by Lemma 3.8,

$$\begin{cases} -\Delta_y G_\omega(x, y) = \delta(x - y) & \text{in } \Omega_f^\omega, \\ \nabla_y G_\omega(x, y) \cdot \vec{\mathbf{n}}^\omega = 0 & \text{on } \partial\Omega_m^\omega, \\ \left|G_\omega(x, y)\right| \leqslant c|x - y|^{2-n} & \text{for } x \neq y, \\ \left|\nabla_y G_\omega(x, y)\right| + \left|\nabla_x G_\omega(x, y)\right| \leqslant c|x - y|^{1-n} & \text{for } x \neq y, \\ G_\omega(x, y) = G_\omega(y, x) & \text{for } x \neq y, \end{cases} \tag{3.65}$$

where $\vec{\mathbf{n}}^\omega$ is a unit vector normal to $\partial\Omega_m^\omega$ and c is a constant independent of ω.

3.3. *The Green's function in $\mathbb{R}^n \setminus \overline{Y_m}$*

For $\nu \in (0, 1]$ and $\omega \in (0, \infty)$, let us define

$$\widetilde{\mathbf{E}}^{\nu,\omega}(y) \equiv \begin{cases} 1 & \text{if } y \in \mathbb{R}^n \setminus \omega\overline{Y_m}, \\ \nu^2 & \text{if } y \in \omega Y_m. \end{cases}$$

Let $\mathcal{G}^*_{\nu,\epsilon}$ for $\nu, \epsilon \in (0, 1]$ denote the Green's function of

$$\begin{cases} -\nabla_y \cdot \left(\widetilde{\mathbf{E}}^{\nu,\epsilon}(y) \nabla_y \mathcal{G}^*_{\nu,\epsilon}(x, y)\right) = \delta(x - y) & \text{in } \mathbb{R}^n, \\ \mathcal{G}^*_{\nu,\epsilon}(x, y) \to 0 \quad \text{as } |x - y| \to \infty. \end{cases} \tag{3.66}$$

By Theorem 5.4 [16] and remark on pp. 62, 67 [16], we see that $\mathcal{G}^*_{\nu,\epsilon}(x, \cdot) \in H^1_{loc}(\mathbb{R}^n \setminus \{x\}) \cap W^{1,1}_{loc}(\mathbb{R}^n)$ exists uniquely when $n \geqslant 3$ and

$$\mathcal{G}^*_{\nu,\epsilon}(x, y) = \mathcal{G}^*_{\nu,\epsilon}(y, x) \quad \text{for } x \neq y. \tag{3.67}$$

A modification of the proof of Lemma 4.3 [24], we have:

If $\delta > 0$ and $\nu, \epsilon \in (0, 1]$, any solution φ of

$$-\nabla \cdot \left(\widetilde{\mathbf{E}}^{\nu,\epsilon} \nabla\varphi\right) = 0 \quad \text{in } B_1(x)$$

satisfies

$$[\varphi]_{C^{0,\mu}(\overline{B_{1/2}(x) \setminus \epsilon Y_m})} \leqslant c\left(\|\varphi\|_{L^2(B_1(x) \setminus \epsilon\overline{Y_m})} + \nu\|\varphi\|_{L^2(B_1(x) \cap \epsilon Y_m)}\right), \tag{3.68}$$

where $\mu \equiv \frac{\delta}{n+\delta}$, $x \in \mathbb{R}^n$, and c is independent of ν, ϵ, x.

Employing (3.66)–(3.68) and following the proofs of Lemmas 3.6, 3.7, 3.8, 3.9, we see that there is a subsequence of $\mathcal{G}^*_{\nu,1}(x, \cdot)$ converging to $\mathbb{G}^*_0(x, \cdot)$ in $C^{1,\alpha}(\overline{\mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x) \setminus Y_m})$ as ν closes to 0 for all $0 < \widetilde{t}_i < \widetilde{s}_i < \infty$ and $x \in \mathbb{R}^n \setminus \overline{Y_m}$, where $\alpha \in (0, 1)$. Here $\widetilde{t}_i$, $\widetilde{s}_i$, $\mathbb{D}_{\widetilde{t}_i,\widetilde{s}_i}(x)$ are defined in a same way as those in the proof of Lemma 3.8. Also following the arguments of Lemma 3.8 and Remark 3.1, we know:

$\mathbb{G}_0^*(x,y)$ defined in $(\mathbb{R}^n \setminus \overline{Y_m}) \times (\mathbb{R}^n \setminus \overline{Y_m})$ satisfies, for any $x \in \mathbb{R}^n \setminus \overline{Y_m}$,

$$\begin{cases} -\Delta_y \mathbb{G}_0^*(x,y) = \delta(x-y) & \text{in } \mathbb{R}^n \setminus \overline{Y_m}, \\ \nabla_y \mathbb{G}_0^*(x,y) \cdot \vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m, \\ \left|\mathbb{G}_0^*(x,y)\right| \leqslant c|x-y|^{2-n} & \text{for } x \neq y, \\ \left|\nabla_y \mathbb{G}_0^*(x,y)\right| \leqslant c|x-y|^{1-n} & \text{for } x \neq y, \\ \mathbb{G}_0^*(x,y) = \mathbb{G}_0^*(y,x) & \text{for } x \neq y, \end{cases} \tag{3.69}$$

where $\vec{\mathbf{n}}_y$ is a unit vector normal to ∂Y_m and c is a constant.

If $F, Q \in C_0^\infty(\mathbb{R}^n)$ with support in $B_r(0)$ for some $r > 0$, the $\mathcal{D}^{1,2}(\mathbb{R}^n \setminus \overline{Y_m})$ solution of

$$\begin{cases} -\nabla \cdot (\nabla \varphi + Q) = F & \text{in } \mathbb{R}^n \setminus \overline{Y_m}, \\ (\nabla \varphi + Q) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m \end{cases}$$

exists uniquely and satisfies, for $x \in \mathbb{R}^n \setminus \overline{Y_m}$,

$$\begin{cases} \varphi(x) = \displaystyle\int_{\mathbb{R}^n \setminus \overline{Y_m}} \mathbb{G}_0^*(x,y) F(y)\,dy - \int_{\mathbb{R}^n \setminus \overline{Y_m}} \nabla_y \mathbb{G}_0^*(x,y) Q(y)\,dy, \\ \partial_{x_j}\varphi(x) = \displaystyle\int_{\mathbb{R}^n \setminus \overline{Y_m}} \partial_{x_j}\mathbb{G}_0^*(x,y) F(y)\,dy - \int_{B_s(0) \setminus \overline{Y_m}} \partial_{x_j}\nabla_y \mathbb{G}_0^*(x,y) Q(y)\,dy \\ \qquad + Q(x) \displaystyle\int_{\partial(B_s(0) \setminus \overline{Y_m})} \nabla_y \Gamma(x,y) \mathbf{n}_j\,d\sigma_y + Q(x) \int_{B_s(0) \setminus \overline{Y_m}} \partial_{x_j}\nabla_y \Gamma(x,y)\,dy, \\ \left|\partial_{x_j}\varphi(x) + \displaystyle\int_{\mathbb{R}^n \setminus \overline{Y_m}} \big(\partial_{x_j}\nabla_y \mathbb{G}_0^*(x,y) Q(y) - \partial_{x_j}\mathbb{G}_0^*(x,y) F(y)\big)\,dy\right| \leqslant c\left|Q(x)\right|, \end{cases} \tag{3.70}$$

where $s > 1 + r$, c is a constant independent of r, Γ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$, $j \in \{1, \ldots, n\}$, and $\mathbf{n}_j$ is the j-th component of $\vec{\mathbf{n}} = (\mathbf{n}_1, \ldots, \mathbf{n}_n)$ which is a unit outward normal vector on $\partial(B_s(0) \setminus \overline{Y_m})$.

4. The second derivatives of the Green's function $G(x,y)$

From Lemma 3.8, we know some estimates for the zero order and the first order derivatives of the Green's function $G(x,y)$. This section is to find an approximation for the second order derivatives of $G(x,y)$ and it consists of two subsections. The first subsection is the approximation of $G(x,y)$ for $|x-y| \geqslant 1$. The second subsection is the approximation of $G(x,y)$ for $|x-y| \leqslant n+1$.

4.1. Approximation of $G(x,y)$ for $|x-y| \geqslant 1$

Let $G_0(x,y)$ denote the Green's function of

$$\begin{cases} -\nabla_y \cdot \big(\mathcal{K}\nabla_y G_0(x,y)\big) = \delta(x-y) & \text{in } \mathbb{R}^n, \\ G_0(x,y) \to 0 \quad \text{as } |x-y| \to \infty, \end{cases} \tag{4.1}$$

where $\mathcal{K}$ is the constant symmetric positive definite matrix in (2.9). By change of variable, the Green's function G_0 of (4.1) can be transformed to the fundamental solution of the Laplace equation in some new coordinate system. Together with the results on p. 17 [11], we see that there is a constant c such that

$$\begin{cases} |G_0(x,y)| \leqslant c|x-y|^{2-n} & \text{if } x \neq y, \\ |\nabla_y G_0(x,y)| \leqslant c|x-y|^{1-n} & \text{if } x \neq y, \\ G_0(x,y) = \omega^{2-n} G_0(x/\omega, y/\omega) & \text{for any } \omega > 0. \end{cases} \tag{4.2}$$

Lemma 4.1. *If $\omega \in (0,1]$, $|x-y| \geqslant \frac{1}{2}$, and $x, y \in \Omega_f^\omega$, then*

$$|G_\omega(x,y) - G_0(x,y)| \leqslant c\omega^\sigma, \tag{4.3}$$

where $c, \sigma > 0$ are constants independent of ω. See (3.64) for $G_\omega(x,y)$.

Proof. We fix $x \in \Omega_f^\omega$ for $\omega \in (0,1]$ and define

$$c_1 \equiv \sup_{\substack{1/5 \leqslant |x-y| \\ y \in \Omega_f^\omega}} |G_\omega(x,y) - G_0(x,y)|.$$

By $(3.65)_3$ and $(4.2)_1$, c_1 is a constant independent of ω, x. $(3.65)_4$ and $(4.2)_2$ imply that $G_\omega(x,y)$ (resp. $G_0(x,y)$) is a uniformly Lipschitz continuous function (independent of ω) of y in $\Omega_f^\omega \setminus B_{1/4}(x)$ (resp. $\mathbb{R}^n \setminus B_{1/4}(x)$). So there is a positive constant c_2 independent of ω, x such that

$$|\nabla_y G_\omega(x,z)| + |\nabla_y G_0(x,y)| \leqslant c_2 \quad \text{for } \begin{cases} z \in \Omega_f^\omega \setminus B_{1/4}(x), \\ y \in \mathbb{R}^n \setminus B_{1/4}(x). \end{cases} \tag{4.4}$$

Now we define

$$\theta_{\omega,x} \equiv \sup_{\substack{1/2 \leqslant |x-y| \\ y \in \Omega_f^\omega}} |G_\omega(x,y) - G_0(x,y)|.$$

By $(3.65)_3$ and $(4.2)_1$, there is a $y_{\omega,x} \in \overline{\Omega_f^\omega}$ satisfying $|x - y_{\omega,x}| \geqslant \frac{1}{2}$ and

$$\theta_{\omega,x} = |G_\omega(x, y_{\omega,x}) - G_0(x, y_{\omega,x})| \leqslant c_1.$$

Pick a number $\beta > n$ so that $\rho_{\omega,x} \equiv \frac{\theta_{\omega,x}}{\beta c_2} \leqslant 1/4$. By (4.4),

$$|G_\omega(x,z) - G_0(x,y)| \geqslant \theta_{\omega,x}/5 \quad \text{for } \begin{cases} z \in B_{\rho_{\omega,x}}(y_{\omega,x}) \cap \Omega_f^\omega, \\ y \in B_{\rho_{\omega,x}}(y_{\omega,x}). \end{cases} \tag{4.5}$$

Let $F \in C_0^\infty(B_{\rho_{\omega,x}}(y_{\omega,x}))$ satisfy

$$\begin{cases} F \in [0,1], \\ F \equiv 1 \quad \text{on } B_{\rho_{\omega,x}/2}(y_{\omega,x}), \\ \|\nabla F\|_{L^\infty(B_{\rho_{\omega,x}}(y_{\omega,x}))} \leqslant 4/\rho_{\omega,x}. \end{cases}$$

Put F in the right-hand side of (2.13) with $\epsilon = \omega$ and (2.14) to obtain U_ω, U_0. Since $\omega \in (0,1]$, there is a constant c_3 independent of ω satisfying, by Lemma 3.2,

$$\|U_\omega - U_0\|_{L^\infty(\Omega_f^\omega)} \leqslant c_3 \omega |\rho_{\omega,x}|^{\frac{-\delta}{n+\delta}}. \tag{4.6}$$

We also note, by (2.13), (2.14), (4.4), (4.5), Remark 3.1, and Taylor expansion,

$$\begin{aligned} \big|U_\omega(x) - U_0(x)\big| &= \left| \int_{\Omega_f^\omega} G_\omega(x,y)F(y)\,dy - \int_{\mathbb{R}^n} G_0(x,y)|Y_f|F(y)\,dy \right| \\ &= \left| \int_{\Omega_f^\omega} (G_\omega - G_0)(x,y)F(y)\,dy + \int_{\Omega_f^\omega} G_0(x,y)F(y)\,dy - \int_{\mathbb{R}^n} G_0(x,y)|Y_f|F(y)\,dy \right| \\ &\geqslant c_4 \rho_{\omega,x}^{n+1} - \sum_{j\in\mathbb{Z}^n} \omega^n |Y_f| \big|G_0(x,y_j)F(y_j) - G_0(x,z_j)F(z_j)\big| \quad \text{for } y_j, z_j \in \omega(Y - j) \\ &\geqslant c_4 \rho_{\omega,x}^{n+1}\big(1 - c_5 \omega \rho_{\omega,x}^{-2}\big), \end{aligned} \tag{4.7}$$

where c_4, c_5 are independent of ω, x. If $1 - c_5\omega\rho_{\omega,x}^{-2} > 1/2$, Eqs. (4.6)–(4.7) imply $\rho_{\omega,x}^{n+1} \leqslant c\omega\rho_{\omega,x}^{\frac{-\delta}{n+\delta}}$. So (4.3) holds. If $1 - c_5\omega\rho_{\omega,x}^{-2} \leqslant 1/2$, then $\rho_{\omega,x} \leqslant c\sqrt{\omega}$, which also implies (4.3). So this lemma holds. □

Lemma 4.2. *If $\omega \in (0,1]$, $|x-y| > 2/3$, and $x, y \in \Omega_f^\omega$, then*

$$\big|\nabla_y G_\omega(x,y) - \big(I + \nabla\mathbb{X}(y/\omega)\big)\nabla_y G_0(x,y)\big| \leqslant c\omega^\sigma, \tag{4.8}$$

where I is the identity matrix, $\mathbb{X}$ is defined in (2.8), and the constants $c, \sigma > 0$ are independent of ω.

Proof. For $i_1, i_2 \in \{1,\ldots,n\}$, find $\mathbb{T}^{(i_1,i_2)}(y) \in H^1_{per}(\Omega_f)$ satisfying, in the cell Y_f,

$$\begin{cases} \Delta\mathbb{T}^{(i_1,i_2)} + \partial_{i_1}\mathbb{X}^{(i_2)} = -\delta_{i_1,i_2} - \partial_{i_1}\mathbb{X}^{(i_2)} & \text{in } Y_f, \\ \nabla\mathbb{T}^{(i_1,i_2)} \cdot \vec{\mathbf{n}}_y + \mathbb{X}^{(i_2)}\mathbf{n}_{y_{i_1}} = 0 & \text{on } \partial Y_m, \\ \displaystyle\int_{Y_f} \mathbb{T}^{(i_1,i_2)}\,dy = 0. \end{cases}$$

See (2.7) for $\mathbb{X}^{(i)}$ and see (2.10) for δ_{i_1,i_2}, $\mathbf{n}_{y_{i_1}}$. Let $\mathbb{T}(y) \equiv (\mathbb{T}^{(i_1,i_2)}(y))$ be an $n \times n$ matrix function and $\mathbb{T}_\omega(y) \equiv \omega^2(\mathbb{T}^{(i_1,i_2)}(\frac{y}{\omega}))$. Note $\mathbb{T}^{(i_1,i_2)}$ is a special case of (2.10). As (2.11),

$$\|\mathbb{T}\|_{C^{2,\alpha}_{per}(\Omega_f)} \leqslant c \quad \text{for } \alpha \in (0,1), \tag{4.9}$$

where c is a constant. Define φ_ω in $\Omega_f^\omega \setminus B_{1/2}(x)$ as, for any fixed $x \in \Omega_f^\omega$,

$$\varphi_\omega(y) \equiv G_\omega(x,y) - G_0(x,y) - \mathbb{X}_\omega(y)\nabla_y G_0(x,y) - \mathbb{T}_\omega(y)\nabla_y^2 G_0(x,y),$$

where $\mathbb{X}_\omega$ is defined in (2.8). Then function φ_ω satisfies

$$\begin{cases} -\nabla\cdot\big(\nabla\varphi_\omega + \mathbb{T}_\omega\nabla_y^3 G_0\big) = \mathbb{X}_\omega\nabla_y\Delta_y G_0 + \nabla\mathbb{T}_\omega\nabla_y^3 G_0 & \text{in } \Omega_f^\omega\setminus B_{1/2}(x), \\ \big(\nabla\varphi_\omega + \mathbb{T}_\omega\nabla_y^3 G_0\big)\cdot\vec{\mathbf{n}}^\omega = 0 & \text{on } \partial\Omega_m^\omega\setminus B_{1/2}(x). \end{cases}$$

By Lemma 3.4, Lemma 4.1, (2.11), and (4.9), $\|\nabla\varphi_\omega\|_{L^\infty(\Omega_f^\omega\setminus B_{2/3}(x))}\leqslant c\omega^\sigma$. So (4.8) holds. □

For any fixed $y\in\Omega_f^\omega$, we define

$$\begin{cases} \zeta_\omega(x) \equiv \nabla_y G_\omega(x,y) \\ \zeta_0(x) \equiv \big(I+\nabla\mathbb{X}(y/\omega)\big)\nabla_y G_0(x,y) \end{cases} \quad \text{for } x\in\Omega_f^\omega\setminus B_{2/3}(y).$$

By (3.65), we see

$$\begin{cases} -\Delta\zeta_\omega = 0 & \text{in } \Omega_f^\omega\setminus B_{2/3}(y), \\ \nabla\zeta_\omega\cdot\vec{\mathbf{n}}^\omega = 0 & \text{on } \partial\Omega_m^\omega\setminus B_{2/3}(y). \end{cases}$$

By Lemma 4.2, $\|\zeta_\omega-\zeta_0\|_{L^\infty(\Omega_f^\omega\setminus B_{2/3}(y))}\leqslant c\omega^\sigma$. Tracing the argument from (2.13) to (2.14), we see that ζ_0 satisfies

$$-\nabla\cdot(\mathcal{K}\nabla\zeta_0) = 0 \quad \text{in } \mathbb{R}^n\setminus B_{2/3}(y),$$

where $\mathcal{K}$ is the matrix in (2.9). Following the proof of Lemma 4.2, we obtain

Lemma 4.3. *If $\omega\in(0,1]$, $|x-y|\geqslant 3/4$, and $x,y\in\Omega_f^\omega$, then*

$$\big|\nabla_x\nabla_y G_\omega(x,y) - \big(I+\nabla\mathbb{X}(x/\omega)\big)\big(I+\nabla\mathbb{X}(y/\omega)\big)\nabla_x\nabla_y G_0(x,y)\big| \leqslant c\omega^\sigma$$

where $c,\sigma>0$ are constants independent of ω.

Lemma 4.3 then implies

Corollary 4.1. *There are constants $c,\sigma>0$ so that, for $x,y\in\Omega_f$ and $1\leqslant|x-y|$,*

$$\big|\nabla_x\nabla_y G(x,y) - \big(I+\nabla\mathbb{X}(x)\big)\big(I+\nabla\mathbb{X}(y)\big)\nabla_x\nabla_y G_0(x,y)\big| \leqslant \frac{c}{|x-y|^{n+\sigma}}. \tag{4.10}$$

Proof. $G_\omega(x,y) = \omega^{2-n}G(\frac{x}{\omega},\frac{y}{\omega})$ and $G_0(x,y)=\omega^{2-n}G_0(\frac{x}{\omega},\frac{y}{\omega})$ for $\omega\in(0,1]$ by (3.64) and (4.2). Hence if $\frac{\xi}{\omega}=x$, $\frac{\eta}{\omega}=y$, and $|\xi-\eta|=1$, then, by Lemma 4.3,

$$\begin{aligned} &\big|\nabla_x\nabla_y G(x,y) - \big(I+\nabla\mathbb{X}(x)\big)\big(I+\nabla\mathbb{X}(y)\big)\nabla_x\nabla_y G_0(x,y)\big| \\ &\quad = \big|\nabla_x\nabla_y G(\xi/\omega,\eta/\omega) - \big(I+\nabla\mathbb{X}(\xi/\omega)\big)\big(I+\nabla\mathbb{X}(\eta/\omega)\big)\nabla_x\nabla_y G_0(\xi/\omega,\eta/\omega)\big| \\ &\quad = \big|\nabla_\xi\nabla_\eta G_\omega(\xi,\eta) - \big(I+\nabla\mathbb{X}(\xi/\omega)\big)\big(I+\nabla\mathbb{X}(\eta/\omega)\big)\nabla_\xi\nabla_\eta G_0(\xi,\eta)\big|\omega^n \leqslant c\omega^{n+\sigma}. \end{aligned}$$

If we take $\omega=\frac{1}{|x-y|}$, then $\omega\in(0,1]$ implies $|x-y|\geqslant 1$. So we prove (4.10). □

4.2. Approximation of $G(x, y)$ for $|x-y| \leqslant n+1$

Define $\gamma(x, y) \equiv \mathbb{G}_0^*(x, y) - \Gamma(x, y)$, where $\mathbb{G}_0^*$ is the Green's function in (3.69) and Γ is the fundamental solution of the Laplace operator in $\mathbb{R}^n$. By (3.69), γ is a function in $(\mathbb{R}^n \setminus \overline{Y_m}) \times (\mathbb{R}^n \setminus \overline{Y_m})$ satisfying, for any $x \in \mathbb{R}^n \setminus \overline{Y_m}$,

$$\begin{cases} -\Delta_y \gamma(x, y) = 0 & \text{in } \mathbb{R}^n \setminus \overline{Y_m}, \\ \nabla_y \gamma(x, y) \cdot \vec{\mathbf{n}}_y = -\nabla_y \Gamma(x, y) \cdot \vec{\mathbf{n}}_y & \text{on } \partial Y_m, \\ \left|\gamma(x, y)\right| \leqslant c|x-y|^{2-n} & \text{for } x \neq y, \\ \gamma(x, y) = \gamma(y, x) & \text{for } x \neq y, \end{cases} \tag{4.11}$$

where c is a constant.

Also note, for each $j \in \mathbb{Z}^n$, $\mathbb{G}_j^*(x, y) \equiv \Gamma(x, y) + \gamma(x+j, y+j)$ is defined in $(\mathbb{R}^n \setminus \overline{Y_m} - j) \times (\mathbb{R}^n \setminus \overline{Y_m} - j)$ and satisfies

$$\begin{cases} -\Delta_y \mathbb{G}_j^*(x, y) = \delta(x-y) & \text{in } \mathbb{R}^n \setminus \overline{Y_m} - j, \\ \nabla_y \mathbb{G}_j^*(x, y) \cdot \vec{\mathbf{n}}_y = 0 & \text{on } \partial Y_m - j, \\ \left|\mathbb{G}_j^*(x, y)\right| \leqslant c|x-y|^{2-n} & \text{for } x \neq y, \\ \mathbb{G}_j^*(x, y) = \mathbb{G}_j^*(y, x) & \text{for } x \neq y. \end{cases} \tag{4.12}$$

Let us use notation in (3.1). $(4.11)_3$ implies

$$\begin{cases} \left\|\gamma(x, \cdot)\right\|_{L^\infty(\partial Y_m)} + \left\|\Gamma(x, \cdot)\right\|_{C^{2,\alpha}(\partial Y_m)} \leqslant c & \text{if } x \in \mathbb{R}^n \setminus \mathbf{D}_0, \\ \left\|\nabla_y \Gamma(\cdot, y)\right\|_{C^{2,\alpha}(\partial Y_m)} \leqslant c & \text{if } y \in \mathbb{R}^n \setminus \mathbf{D}_0, \end{cases} \tag{4.13}$$

where $\alpha \in (0,1)$ and c is a constant independent of x, y. By (4.11), $(4.13)_1$, and Theorem 3.1 [11],

$$\begin{cases} \left\|\gamma(x, \cdot)\right\|_{L^\infty(\mathbb{R}^n \setminus \overline{Y_m})} \leqslant c & \text{if } x \in \mathbb{R}^n \setminus \mathbf{D}_0, \\ \left\|\gamma(\cdot, y)\right\|_{L^\infty(\mathbb{R}^n \setminus \overline{Y_m})} \leqslant c & \text{if } y \in \mathbb{R}^n \setminus \mathbf{D}_0, \end{cases} \tag{4.14}$$

where c is independent of x, y. By (4.11), $(4.14)_1$, and Corollary 6.3 and Theorem 6.30 [11], we obtain

$$\left\|\gamma(x, \cdot)\right\|_{C^{2,\alpha}(\mathbb{R}^n \setminus \overline{Y_m})} \leqslant c \quad \text{if } x \in \mathbb{R}^n \setminus \mathbf{D}_0, \tag{4.15}$$

where $\alpha \in (0,1)$ and c is independent of x. By (4.11), $(4.14)_2$, and Corollary 6.3 [11],

$$\left\|\gamma(x, \cdot)\right\|_{C^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{D})} \leqslant c \quad \text{if } x \in \mathbf{D}_0 \setminus \overline{Y_m}, \tag{4.16}$$

where c is a constant independent of x. By (4.11), for any $y = (y_1, \ldots, y_n) \in \mathbb{R}^n \setminus \overline{Y_m}$,

$$-\Delta_x \partial_{y_i} \gamma(\cdot, y) = 0 \quad \text{in } \mathbb{R}^n \setminus \mathbf{D}_0, \tag{4.17}$$

where ∂_{y_i} is the partial derivative with respect to y_i for $i \in \{1, \ldots, n\}$. (4.15), (4.17), and Corollary 6.3 [11] imply

$$\left\| \partial_{y_i} \gamma(\cdot, y) \right\|_{C^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{D})} \leqslant c \quad \text{if } y \in \mathbb{R}^n \setminus \overline{Y_m}, \tag{4.18}$$

where $\alpha \in (0, 1)$ and c is independent of $y = (y_1, \ldots, y_n)$.

Let $\breve{\phi} \in C_0^\infty(\mathbb{R}^n)$ be a bell-shaped smooth function and satisfy $\breve{\phi} \in [0, 1]$, $\breve{\phi}(x) = 1$ in $x \in \mathbf{D}$, *supp*$(\breve{\phi}) \subset \mathcal{A}$. Then $\breve{\mathcal{X}}_{\mathcal{A}}(x, y) \equiv \breve{\phi}(x)\breve{\phi}(y)$ is a smooth function in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\begin{cases} \breve{\mathcal{X}}_{\mathcal{A}}(x, y) = \breve{\mathcal{X}}_{\mathcal{A}}(y, x) \in [0, 1], \\ \breve{\mathcal{X}}_{\mathcal{A}}(x, y) = \begin{cases} 1 & \text{if } x, y \in \mathbf{D}, \\ 0 & \text{if } x \notin \mathcal{A} \text{ or } y \notin \mathcal{A}. \end{cases} \end{cases} \tag{4.19}$$

For any fixed $x \in \Omega_f$, we define, for $y \in \Omega_f \setminus \{x\}$,

$$\begin{aligned} G^*(x, y) &\equiv \Gamma(x, y) + \sum_{j \in \mathbb{Z}^n} \gamma(x + j, y + j)\breve{\mathcal{X}}_{\mathcal{A}}(x + j, y + j) \\ &= \Gamma(x, y)\left(1 - \sum_{j \in \mathbb{Z}^n} \breve{\mathcal{X}}_{\mathcal{A}}(x + j, y + j)\right) + \sum_{j \in \mathbb{Z}^n} \mathbb{G}_j^*(x, y)\breve{\mathcal{X}}_{\mathcal{A}}(x + j, y + j). \end{aligned} \tag{4.20}$$

Because of (4.12)$_4$ and (4.19)$_1$,

$$G^*(x, y) = G^*(y, x) \quad \text{in } \Omega_f \times \Omega_f \setminus \{x = y\}. \tag{4.21}$$

Define, in $\Omega_f^\omega \times \Omega_f^\omega$ for $\omega \in [\frac{1}{n+1}, \infty)$,

$$G_\omega^*(x, y) \equiv \omega^{2-n} G^*\left(\frac{x}{\omega}, \frac{y}{\omega}\right). \tag{4.22}$$

Lemma 4.4.

(1) *There is a constant c such that*

$$\left|\left(1 - \breve{\mathcal{X}}_{\mathcal{A}}(x, y)\right)\gamma(x, y)\right| \leqslant c \quad \text{for } x, y \in \mathbb{R}^n \setminus \overline{Y_m}. \tag{4.23}$$

(2) *For $\omega \in [\frac{1}{n+1}, \infty)$, there is a constant c independent of ω such that*

$$\left|G_\omega^*(x, y)\right| \leqslant c|x - y|^{2-n} \quad \text{for } x, y \in \Omega_f^\omega \text{ and } x \neq y. \tag{4.24}$$

Proof. Let c denote a constant independent of $\omega \in [\frac{1}{n+1}, \infty)$. By (4.14), $|\gamma(x, y)| \leqslant c$ if $x \notin \mathbf{D}$ or $y \notin \mathbf{D}$. If $x, y \in \mathbf{D}$, (4.19)$_2$ implies $1 - \mathcal{X}_{\mathcal{A}}^\dagger(x, y) = 0$. So we see that (4.23) holds. (4.24) is from the definition of G_ω^* and (4.11). $\square$

Lemma 4.5. *If $\omega \in [\frac{1}{n+1}, \infty)$, $|x - y| \geqslant \frac{1}{2}$, and $x, y \in \Omega_f^\omega$, then*

$$\left|G_\omega(x, y) - G_\omega^*(x, y)\right| \leqslant c\omega^{2-n}, \tag{4.25}$$

where c is a constant independent of ω.

Proof. Fix $x \in \Omega_f^\omega$ and define

$$c_1 \equiv \sup_{\substack{y \in \Omega_f^\omega \\ 1/5 \leqslant |x-y|}} \left| G_\omega(x, y) - G_\omega^*(x, y) \right|.$$

By (3.65)$_3$ and (4.24), we see that c_1 is independent of ω, x. (3.65), (3.69), and (4.22) imply that $G_\omega(x, y)$ and $G_\omega^*(x, y)$ are uniformly Lipschitz continuous functions (independent of ω) of y in $\Omega_f^\omega \setminus B_{1/4}(x)$. So there is a positive constant c_2 (independent of ω, x) such that

$$\left| \nabla_y G_\omega(x, y) \right| + \left| \nabla_y G_\omega^*(x, y) \right| \leqslant c_2 \quad \text{for } y \in \Omega_f^\omega \setminus B_{1/4}(x). \tag{4.26}$$

Let us define

$$\theta_{\omega,x} \equiv \sup_{\substack{y \in \Omega_f^\omega \\ 1/2 \leqslant |x-y|}} \left| G_\omega(x, y) - G_\omega^*(x, y) \right|.$$

Clearly, $\theta_{\omega,x} \leqslant c_1$. By (3.65)$_3$ and (4.24), there is a $y_{\omega,x} \in \overline{\Omega_f^\omega}$ such that

$$1/2 \leqslant |x - y_{\omega,x}| \quad \text{and} \quad \theta_{\omega,x} = \left| G_\omega(x, y_{\omega,x}) - G_\omega^*(x, y_{\omega,x}) \right|.$$

Take a number $\beta > n$ so that

$$\rho_{\omega,x} \equiv \frac{\theta_{\omega,x}}{\beta c_2} \leqslant \min\{1/4, d_0\}. \tag{4.27}$$

See (3.1) for d_0. Then we see, by (4.26),

$$\left| G_\omega(x, y) - G_\omega^*(x, y) \right| \geqslant \theta_{\omega,x}/5 \quad \text{for } y \in B_{\rho_{\omega,x}}(y_{\omega,x}) \cap \Omega_f^\omega. \tag{4.28}$$

Because of $A1$, one can take $F_{\omega,x} \in C_0^\infty(B_{\rho_{\omega,x}}(y_{\omega,x}) \cap \Omega_f^\omega)$ such that

$$\begin{cases} F_{\omega,x} \in [0, 1], \\ F_{\omega,x} = 1 \qquad \text{on } B_{\rho_{\omega,x}/\beta}(y_{\omega,x} + \hat{y}_{\omega,x}). \end{cases} \tag{4.29}$$

The point $y_{\omega,x} + \hat{y}_{\omega,x}$ is chosen so that $B_{\rho_{\omega,x}/\beta}(y_{\omega,x} + \hat{y}_{\omega,x}) \subset B_{\rho_{\omega,x}}(y_{\omega,x}) \cap \Omega_f^\omega$.

If $y_{\omega,x} \in \omega(\overline{Y_f} - j)$ for some $j \in \mathbb{Z}^n$, then we consider the following problems:

$$\begin{cases} -\Delta U_\omega = F_{\omega,x} & \text{in } \Omega_f^\omega, \\ \nabla U_\omega \cdot \vec{\mathbf{n}}^\omega = 0 & \text{on } \partial\Omega_m^\omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta \widetilde{U}_\omega = F_{\omega,x} & \text{in } \mathbb{R}^n \setminus \omega(\overline{Y_m} - j), \\ \nabla \widetilde{U}_\omega \cdot \vec{\mathbf{n}}^\omega = 0 & \text{on } \omega(\partial Y_m - j). \end{cases} \tag{4.30}$$

By Lax–Milgram Theorem [11], extension theorem [1], and (2.12), both U_ω, $\widetilde{U}_\omega$ are solvable uniquely in $\mathcal{D}^{1,2}$ space, and

$$\begin{cases} -\Delta(U_\omega - \widetilde{U}_\omega) = 0 & \text{in } \Omega_f^\omega, \\ \nabla(U_\omega - \widetilde{U}_\omega) \cdot \vec{\mathbf{n}}^\omega = \begin{cases} 0 \\ -\nabla \widetilde{U}_\omega \cdot \vec{\mathbf{n}}^\omega \end{cases} & \begin{matrix} \text{on } \omega(\partial Y_m - j), \\ \text{on } \partial\Omega_m^\omega \setminus \omega(\partial Y_m - j). \end{matrix} \end{cases} \tag{4.31}$$

We claim that, for any $\omega \in [\frac{1}{n+1}, \infty)$,

$$\|U_\omega - \widetilde{U}_\omega\|_{L^\infty(\Omega_f^\omega)} \leqslant c\omega^{2-n}\|F_{\omega,x}\|_{L^1(\mathbb{R}^n)} \leqslant c\omega^{2-n}\rho_{\omega,x}^n, \tag{4.32}$$

where c is independent of ω, x.

Proof of the claim: Because of $y_{\omega,x} \in \omega(\overline{Y_f} - j)$, there is a smooth domain $\mathbf{B}_\omega$ such that, for $\omega \in [\frac{1}{n+1}, \infty)$,

(S1) $\mathbf{B}_\omega \supset \overline{\omega(Y_m - j) \cup supp(F_{\omega,x})}$, $\mathbf{B}_\omega \cap \overline{\omega(Y_m - i)} = \emptyset$ for $i \neq j$,
(S2) $dist(\partial\mathbf{B}_\omega, supp(F_{\omega,x}))$ is greater than ωd_0 (see (3.1) for d_0).

By (S1) as well as Theorem 3.1 and Lemma 3.4 in [11], we see, from (4.31),

$$\|U_\omega - \widetilde{U}_\omega\|_{L^\infty(\mathbf{B}_\omega \cap \Omega_f^\omega)} \leqslant \|U_\omega - \widetilde{U}_\omega\|_{L^\infty(\partial\mathbf{B}_\omega)}. \tag{4.33}$$

By Remark 3.1, (3.65), $(3.70)_1$, (4.12), and (4.30), we have, for $z \in \Omega_f^\omega \setminus \mathbf{B}_\omega$,

$$\begin{cases} \left|U_\omega(z)\right| = \left|\displaystyle\int_{\Omega_f^\omega} G_\omega(z, y)F_{\omega,x}(y)\,dy\right| \leqslant \dfrac{c\|F_{\omega,x}\|_{L^1(\mathbb{R}^n)}}{dist(z, supp(F_{\omega,x}))^{n-2}}, \\ \left|\widetilde{U}_\omega(z)\right| = \left|\displaystyle\int_{\mathbb{R}^n\setminus\omega(\overline{Y_m}-j)} \omega^{2-n}\mathbb{G}_j^*\left(\frac{z}{\omega}, \frac{y}{\omega}\right)F_{\omega,x}(y)\,dy\right| \leqslant \dfrac{c\|F_{\omega,x}\|_{L^1(\mathbb{R}^n)}}{dist(z, supp(F_{\omega,x}))^{n-2}}, \end{cases}$$

where c is independent of ω, x, and $F_{\omega,x}$. By (S2),

$$\|U_\omega - \widetilde{U}_\omega\|_{L^\infty(\Omega_f^\omega\setminus\mathbf{B}_\omega)} \leqslant c\omega^{2-n}\|F_{\omega,x}\|_{L^1(\mathbb{R}^n)} \leqslant c\omega^{2-n}\rho_{\omega,x}^n, \tag{4.34}$$

where c is independent of ω, x. (4.33)–(4.34) imply (4.32).

By (4.28)–(4.30) and Remark 3.1, we see that, for some $j \in \mathbb{Z}^n$,

$$\begin{aligned} \rho_{\omega,x}^{n+1} &\leqslant c\int_{B_{\rho_{\omega,x}/\beta}(y_{\omega,x}+\hat{y}_{\omega,x})} \frac{\theta_{\omega,x}}{5}\,dz \leqslant \left|\int_{\Omega_f^\omega}\big(G_\omega(x,y) - G_\omega^*(x,y)\big)F_{\omega,x}(y)\,dy\right| \\ &\leqslant \left|U_\omega(x) - \widetilde{U}_\omega(x)\right| \\ &\quad + c\left|\int_{\Omega_f^\omega} \omega^{2-n}\gamma\left(\frac{x}{\omega}+j, \frac{y}{\omega}+j\right)\left(1 - \check{\mathcal{X}}_{\mathcal{A}}\left(\frac{x}{\omega}+j, \frac{y}{\omega}+j\right)\right)F_{\omega,x}(y)\,dy\right|, \end{aligned} \tag{4.35}$$

where c is independent of ω, x. By (4.23), (4.32), and (4.35),

$$\rho_{\omega,x}^{n+1} \leqslant c\omega^{2-n}\rho_{\omega,x}^n.$$

So (4.25) holds and we prove the lemma. □

Lemma 4.6. *If $\omega \in [\frac{1}{n+1}, \infty)$, $|x-y| > 2/3$, and $x, y \in \Omega_f^\omega$, then*

$$\left|\nabla_y G_\omega(x,y) - \nabla_y G_\omega^*(x,y)\right| \leqslant c\omega^{2-n}, \tag{4.36}$$

where c is a constant independent of ω.

Proof. If $x \in \Omega_f^\omega \setminus \sum_{j\in\mathbb{Z}^n} \omega(\mathcal{A}-j)$, $G_\omega^*(x,\cdot) = \Gamma(x,\cdot)$ in $\Omega_f^\omega \setminus \{x\}$ by (4.19), (4.20). So

$$\begin{cases} -\Delta_y(G_\omega - G_\omega^*) = 0 & \text{in } \Omega_f^\omega \setminus \{x\}, \\ \nabla_y(G_\omega - G_\omega^*)\cdot \vec{\mathbf{n}}^\omega = -\nabla_y \Gamma(x,\cdot)\cdot \vec{\mathbf{n}}^\omega & \text{on } \partial\Omega_m^\omega. \end{cases}$$

Since the distance from x to $\partial\Omega_m^\omega$ is of order ω (see (3.1)),

$$\left\|\nabla_y \Gamma(x,\cdot)\right\|_{C^{1,\alpha}(\partial\Omega_m^\omega)} \leqslant c\omega^{1-n},$$

where $\alpha \in (0,1)$ and c is independent of x, ω. By Theorem 6.30 [11] and Lemma 4.5, we obtain (4.36).

If $x \in \omega(\mathcal{A}\setminus \overline{Y_m} - j)$ for some $j \in \mathbb{Z}^n$, then

$$G_\omega^*(x,y) = \omega^{2-n}\mathbb{G}_0^*(x/\omega + j, y/\omega + j) + \omega^{2-n}\gamma(x/\omega + j, y/\omega + j)\big(\check{\mathcal{X}}_{\mathcal{A}}(x/\omega + j, y/\omega + j) - 1\big),$$

for $y \in \Omega_f^\omega \setminus \{x\}$. By (4.11),

$$\begin{cases} -\Delta_y\big(G_\omega - G_\omega^*\big) = \begin{cases} 0 & \text{in } \begin{cases} \Omega_f^\omega \setminus \omega(\mathcal{A}-j), \text{ or} \\ \omega(\mathbf{D}\setminus \overline{Y_m} - j), \end{cases} \\ 2\omega^{-n}(\nabla_y\gamma\nabla_y\check{\mathcal{X}}_{\mathcal{A}})\left(\dfrac{x}{\omega}+j, \dfrac{\cdot}{\omega}+j\right) & \\ \quad + \omega^{-n}(\gamma\Delta_y\check{\mathcal{X}}_{\mathcal{A}})\left(\dfrac{x}{\omega}+j, \dfrac{\cdot}{\omega}+j\right) & \text{in } \omega(\mathcal{A}\setminus \mathbf{D} - j), \end{cases} \\ \nabla_y\big(G_\omega - G_\omega^*\big)\cdot \vec{\mathbf{n}}^\omega = \begin{cases} -\left(1 - \check{\phi}\left(\dfrac{x}{\omega}+j\right)\right)\nabla_y\Gamma(x,\cdot)\cdot\vec{\mathbf{n}}^\omega & \text{on } \omega(\partial Y_m - j), \\ -\nabla_y\Gamma(x,\cdot)\cdot\vec{\mathbf{n}}^\omega & \text{on } \partial\Omega_m^\omega \setminus \omega(\partial Y_m - j). \end{cases} \end{cases}$$

Because the distance from x to $\partial\Omega_m^\omega \setminus \omega(\partial Y_m - j)$ is of order ω (see (3.1)), by $(4.13)_1$, (4.15), and (4.16),

$$\begin{cases} \left\|\omega^{-n}(\nabla_y\gamma\nabla_y\check{\mathcal{X}}_{\mathcal{A}})\left(\dfrac{x}{\omega}+j, \dfrac{\cdot}{\omega}+j\right)\right\|_{C^{0,\alpha}(\omega(\mathcal{A}\setminus\mathbf{D}-j))} \\ \quad + \left\|\omega^{-n}(\gamma\Delta_y\check{\mathcal{X}}_{\mathcal{A}})\left(\dfrac{x}{\omega}+j, \dfrac{\cdot}{\omega}+j\right)\right\|_{C^{0,\alpha}(\omega(\mathcal{A}\setminus\mathbf{D}-j))} \leqslant c\omega^{-n}, \\ \sup\limits_{\substack{i\in\mathbb{Z}^n \\ i\neq j}} \left\|\nabla_y\Gamma(x,\cdot)\right\|_{C^{1,\alpha}(\omega(\partial Y_m - i))} \leqslant c\omega^{1-n}, \\ \left\|\left(1 - \check{\phi}\left(\dfrac{x}{\omega}+j\right)\right)\nabla_y\Gamma(x,\cdot)\right\|_{C^{1,\alpha}(\omega(\partial Y_m - j))} \leqslant c\omega^{1-n}, \end{cases}$$

where $\alpha \in (0,1)$ and c is independent of x, ω. We get (4.36) by Theorem 6.30 [11] and Lemma 4.5. $\square$

Lemma 4.7. *If $\omega \in [\frac{1}{n+1}, \infty)$, $|x-y| > 3/4$, and $x, y \in \Omega_f^\omega$, then*

$$\left|\nabla_x \nabla_y G_\omega(x,y) - \nabla_x \nabla_y G_\omega^*(x,y)\right| \leqslant c\omega^{2-n}, \tag{4.37}$$

where the constant c is independent of ω.

Proof. To show (4.37), we modify the argument for (4.36) and use the facts $G_\omega(x,y) = G_\omega(y,x)$ and $G_\omega^*(x,y) = G_\omega^*(y,x)$ (see $(3.65)_5$ and (4.21)). Define $\varphi_\omega(x) \equiv \partial_{y_k} G_\omega(x,y)$ and $\widetilde{\varphi}_\omega(x) \equiv \partial_{y_k} G_\omega^*(x,y)$ for $k \in \{1,\dots,n\}$. If $y \in \Omega_f^\omega \setminus \sum_{j\in\mathbb{Z}^n} \omega(\mathcal{A} - j)$, then $G_\omega^*(\cdot, y) = \Gamma(\cdot, y)$ in $\Omega_f^\omega \setminus \{y\}$ and

$$\begin{cases} -\Delta(\varphi_\omega - \widetilde{\varphi}_\omega) = 0 & \text{in } \Omega_f^\omega \setminus B_{2/3}(y), \\ \nabla(\varphi_\omega - \widetilde{\varphi}_\omega) \cdot \vec{\mathbf{n}}^\omega = -\nabla_x \partial_{y_k} \Gamma(\cdot, y) \cdot \vec{\mathbf{n}}^\omega & \text{on } \partial\Omega_m^\omega \setminus B_{2/3}(y). \end{cases}$$

Because the distance from y to $\partial\Omega_m^\omega$ is of order ω (see (3.1)),

$$\left\|\nabla_x \partial_{y_k} \Gamma(\cdot, y)\right\|_{C^{1,\alpha}(\partial\Omega_m^\omega)} \leqslant c\omega^{-n},$$

where $\alpha \in (0,1)$ and c is independent of y, ω, k. By (4.36) and Theorem 6.30 [11], we have (4.37).

If $y \in \omega(\mathcal{A} \setminus \overline{Y_m} - j)$ for some $j \in \mathbb{Z}^n$, then

$$\begin{aligned} G_\omega^*(x,y) = {} & \omega^{2-n} \mathbb{G}_0^*(x/\omega + j, y/\omega + j) \\ & + \omega^{2-n} \gamma(x/\omega + j, y/\omega + j)\big(\check{\mathcal{X}}_{\mathcal{A}}(x/\omega + j, y/\omega + j) - 1\big), \end{aligned}$$

for $x \in \Omega_f^\omega \setminus \{y\}$. Note $(4.11)_{1,4}$ imply $\Delta_x \gamma(\frac{x}{\omega} + j, \frac{y}{\omega} + j) = 0$ for $x, y \in \Omega_f^\omega$. So

$$\begin{cases} -\Delta(\varphi_\omega - \widetilde{\varphi}_\omega) = \begin{cases} 0 & \text{in } \begin{cases} \Omega_f^\omega \setminus (\omega(\mathcal{A} - j) \cup B_{2/3}(y)), \\ \text{or} \\ \omega(\mathbf{D} \setminus \overline{Y_m} - j) \setminus B_{2/3}(y), \end{cases} \\ 2\omega^{-n-1} \partial_{y_k}(\nabla_x \gamma \nabla_x \check{\mathcal{X}}_{\mathcal{A}})\left(\dfrac{\cdot}{\omega} + j, \dfrac{y}{\omega} + j\right) & \\ \quad + \omega^{-n-1} \partial_{y_k}(\gamma \Delta_x \check{\mathcal{X}}_{\mathcal{A}})\left(\dfrac{\cdot}{\omega} + j, \dfrac{y}{\omega} + j\right) & \text{in } \omega(\mathcal{A} \setminus \mathbf{D} - j) \setminus B_{2/3}(y), \end{cases} \\ \nabla(\varphi_\omega - \widetilde{\varphi}_\omega) \cdot \vec{\mathbf{n}}^\omega = \begin{cases} -\left(1 - \check{\phi}\left(\dfrac{y}{\omega} + j\right)\right) \nabla_x \partial_{y_k} \Gamma(\cdot, y) \cdot \vec{\mathbf{n}}^\omega & \\ \quad + \omega^{-1} \check{\phi}'\left(\dfrac{y}{\omega} + j\right) \nabla_x \Gamma(\cdot, y) \cdot \vec{\mathbf{n}}^\omega & \text{on } \omega(\partial Y_m - j) \setminus B_{2/3}(y), \\ -\nabla_x \partial_{y_k} \Gamma(\cdot, y) \cdot \vec{\mathbf{n}}^\omega & \text{on } \partial\Omega_m^\omega \setminus \omega(\partial Y_m - j). \end{cases} \end{cases}$$

Since the distance from y to $\partial\Omega_m^\omega \setminus \omega(\partial Y_m - j)$ is of order ω (see (3.1)), by $(4.13)_2$ and (4.18),

$$\begin{cases} \left\| \omega^{-n-1} \partial_{y_k} (\nabla_x \gamma \nabla_x \check{\mathcal{X}}_{\mathcal{A}}) \left(\frac{\cdot}{\omega} + j, \frac{y}{\omega} + j \right) \right\|_{C^{0,\alpha}(\omega(\mathcal{A} \backslash \mathbf{D} - j))} \\ \quad + \left\| \omega^{-n-1} \partial_{y_k} (\gamma \Delta_x \check{\mathcal{X}}_{\mathcal{A}}) \left(\frac{\cdot}{\omega} + j, \frac{y}{\omega} + j \right) \right\|_{C^{0,\alpha}(\omega(\mathcal{A} \backslash \mathbf{D} - j))} \leqslant c\omega^{-n-1}, \\ \sup\limits_{\substack{i \in \mathbb{Z}^n \\ i \neq j}} \|\nabla_x \partial_{y_k} \Gamma(\cdot, y)\|_{C^{1,\alpha}(\omega(\partial Y_m - i))} \leqslant c\omega^{-n}, \\ \left\| \left(1 - \check{\phi}\left(\frac{y}{\omega} + j \right) \right) \nabla_x \partial_{y_k} \Gamma(\cdot, y) \right\|_{C^{1,\alpha}(\omega(\partial Y_m - j))} \\ \quad + \left\| \omega^{-1} \check{\phi}\left(\frac{y}{\omega} + j \right) \nabla_x \Gamma(\cdot, y) \right\|_{C^{1,\alpha}(\omega(\partial Y_m - j))} \leqslant c\omega^{-n}, \end{cases}$$

where $\alpha \in (0, 1)$ and c is independent of y, ω, k. We get (4.37) by (4.36) and Theorem 6.30 [11]. □

By Lemma 4.7 and tracing the argument of Corollary 4.1, we have

Corollary 4.2. *There is a constant $c > 0$ so that, for $x, y \in \Omega_f$ and $|x - y| \leqslant n + 1$,*

$$\left|\nabla_x \nabla_y G(x, y) - \nabla_x \nabla_y G^*(x, y)\right| \leqslant c\, |x - y|^{-2}.$$

5. Proof of main results

In this section, we prove the main results. Theorem 2.1 is in Section 5.1, Theorem 2.2 is proved in Section 5.2, Theorem 2.3 is proved in Section 5.3, and Theorem 2.4 is in Section 5.4.

5.1. Proof of Theorem 2.1

Lemma 5.1. *If $Q \in [C_0^\infty(\mathbb{R}^n)]^n$, then*

$$\left\| \int_{\Omega_f} \nabla_x \nabla_y G(x, y) Q(y)\, dy \right\|_{L^p(\Omega_f)} \leqslant c \|Q\|_{L^p(\Omega_f)} \quad \text{for any } p \in (1, \infty),$$

where c is a constant.

Proof. Let η be a bell-shaped function in $C_0^\infty(\mathbb{R})$ satisfying $\eta \in [0, 1]$, $\eta(z) = 1$ for $|z| < n$ and $\eta(z) = 0$ for $|z| \geqslant n + 1$, and set $\nabla_x \nabla_y G(x, y) = \mathcal{L}^0(x, y) + \mathcal{L}^c(x, y)$, where

$$\mathcal{L}^0(x, y) \equiv \eta\big(|x - y|\big) \nabla_x \nabla_y G^* + \big(1 - \eta\big(|x - y|\big)\big)\big(I + \nabla \mathbb{X}(x)\big)\big(I + \nabla \mathbb{X}(y)\big) \nabla_x \nabla_y G_0.$$

By (4.20), we define

$$\int_{\Omega_f} \nabla_x \nabla_y G(x, y) Q(y)\, dy = \sum_{i=1}^{4} \mathcal{F}_i(Q)(x), \tag{5.1}$$

where

$$\mathcal{F}_1(Q)(x) \equiv \int_{\Omega_f} \eta(|x-y|)Q(y)\nabla_x\nabla_y\Big(\sum_{k\in\mathbb{Z}^n} \mathbb{G}_0^*(x+k,y+k)\check{\mathcal{X}}_{\mathcal{A}}(x+k,y+k)\Big)\,dy,$$

$$\mathcal{F}_2(Q)(x) \equiv \int_{\Omega_f} \eta(|x-y|)Q(y)\nabla_x\nabla_y\Big(\Gamma(x,y)\Big(1-\sum_{k\in\mathbb{Z}^n}\check{\mathcal{X}}_{\mathcal{A}}(x+k,y+k)\Big)\Big)\,dy,$$

$$\mathcal{F}_3(Q)(x) \equiv \int_{\Omega_f} \big(1-\eta(|x-y|)\big)Q(y)\big(I+\nabla\mathbb{X}(x)\big)\big(I+\nabla\mathbb{X}(y)\big)\nabla_x\nabla_y G_0(x,y)\,dy,$$

$$\mathcal{F}_4(Q)(x) \equiv \int_{\Omega_f} \mathcal{L}^c(x,y)Q(y)\,dy.$$

From (4.19), we know $\check{\mathcal{X}}_{\mathcal{A}}(x+k,y+k)\neq 0$ for any $k\in\mathbb{Z}^n$ only if $|x-y|<\sqrt{n}$ and $x+k, y+k\in Y$. If $x\in Y_f - j$ for some $j\in\mathbb{Z}^n$, by change of variable and the definition of η,

$$\begin{aligned}
\mathcal{F}_1(Q)(x) &= \int_{Y_f - j} Q(y)\nabla_x\nabla_y\big(\mathbb{G}_0^*(x+j,y+j)\check{\phi}(x+j)\check{\phi}(y+j)\big)\,dy \\
&= \nabla\check{\phi}(x+j)\int_{Y_f} Q(y-j)\big(\nabla_y\mathbb{G}_0^*(x+j,y)\check{\phi}(y)+\mathbb{G}_0^*(x+j,y)\nabla\check{\phi}(y)\big)\,dy \\
&\quad + \check{\phi}(x+j)\int_{Y_f} Q(y-j)\check{\phi}(y)\nabla_x\nabla_y\mathbb{G}_0^*(x+j,y)\,dy \\
&\quad + \check{\phi}(x+j)\int_{Y_f} Q(y-j)\nabla_x\mathbb{G}_0^*(x+j,y)\nabla\check{\phi}(y)\,dy.
\end{aligned} \tag{5.2}$$

Define $V_j^{(1)}(z) \equiv \int_{Y_f}\mathbb{G}_0^*(z,y)Q(y-j)\nabla\check{\phi}(y)\,dy$ for $z\in\mathbb{R}^n\setminus\overline{Y_m}$. By $(3.70)_1$, $V_j^{(1)}$ is the unique $\mathcal{D}^{1,2}(\mathbb{R}^n\setminus\overline{Y_m})$ solution of

$$\begin{cases} -\Delta V_j^{(1)} = Q(\cdot - j)\nabla\check{\phi} & \text{in } \mathbb{R}^n\setminus\overline{Y_m}, \\ \nabla V_j^{(1)}\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m. \end{cases}$$

Define $V_j^{(2)}(z) \equiv \int_{Y_f}\nabla_y\mathbb{G}_0^*(z,y)Q(y-j)\check{\phi}(y)\,dy$ for $z\in\mathbb{R}^n\setminus\overline{Y_m}$. By $(3.70)_1$, $V_j^{(2)}$ is the unique $\mathcal{D}^{1,2}(\mathbb{R}^n\setminus\overline{Y_m})$ solution of

$$\begin{cases} -\nabla\cdot\big(\nabla V_j^{(2)} - Q(\cdot - j)\nabla\check{\phi}\big) = 0 & \text{in } \mathbb{R}^n\setminus\overline{Y_m}, \\ \big(\nabla V_j^{(2)} - Q(\cdot - j)\nabla\check{\phi}\big)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m. \end{cases}$$

By Lemma 3.3,

$$\big\|V_j^{(1)}, V_j^{(2)}\big\|_{W^{1,p}(Y_f)} \leqslant c\big(\big\|V_j^{(1)}, V_j^{(2)}\big\|_{L^p(\mathbf{D}_2\setminus\bar{Y})} + \big\|Q(\cdot - j)\big\|_{L^p(Y_f)}\big), \tag{5.3}$$

where $p\in(1,\infty)$ and c is a constant independent of j. See (3.1) for $\mathbf{D}_2$. By (3.69),

$$\left|V_j^{(1)}(z)\right| + \left|V_j^{(2)}(z)\right| \leqslant c \int\limits_{Y_f} \left|Q(y-j)\right|\left(|z-y|^{2-n} + |z-y|^{1-n}\right) dy \quad \text{for } z \in \mathbf{D}_2 \setminus \bar{Y},$$

where c is independent of j. Tracing the proof of Lemma 7.12 [11], we see

$$\left\| V_j^{(1)}, V_j^{(2)} \right\|_{L^p(\mathbf{D}_2 \setminus \bar{Y})} \leqslant c \left\| Q(\cdot - j) \right\|_{L^p(Y_f)} \quad \text{for } p \in (1,\infty), \tag{5.4}$$

where c is a constant independent of j. By $(3.70)_3$,

$$\left\| \int\limits_{Y_f} \nabla_z \nabla_y \mathbb{G}_0^*(\cdot, y) Q(y-j) \check{\phi}(y)\, dy \right\|_{L^p(Y_f)} \leqslant \left\| \nabla V_j^{(2)}, Q(\cdot - j) \right\|_{L^p(Y_f)}, \tag{5.5}$$

where $p \in (1,\infty)$ and c is independent of j. Since $supp(\check{\phi}(x+j)) \cap supp(\check{\phi}(x+i)) = \emptyset$ if $i \neq j$, (5.2)–(5.5) imply

$$\begin{aligned} \left\| \mathcal{F}_1(Q) \right\|_{L^p(\Omega_f)}^p &\leqslant c \sum_{j \in \mathbb{Z}^n} \left(\left\| V_j^{(1)}, V_j^{(2)} \right\|_{W^{1,p}(Y_f)}^p + \left\| Q(\cdot - j) \right\|_{L^p(Y_f)}^p \right) \\ &\leqslant c \| Q \|_{L^p(\Omega_f)}^p, \end{aligned} \tag{5.6}$$

where $p \in (1,\infty)$ and c is a constant. So there is a constant c such that

$$\left\| \mathcal{F}_1(Q) \right\|_{L^p(\Omega_f)} \leqslant c \| Q \|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty). \tag{5.7}$$

By (4.19), $\mathcal{F}_2(Q)$ can be written as $\mathcal{F}_2(Q) = \mathcal{F}_{21}(Q) - \mathcal{F}_{22}(Q) - \mathcal{F}_{23}(Q)$, where

$$\begin{aligned} \mathcal{F}_{21}(Q)(x) &= \int\limits_{\Omega_f} Q(y) \nabla_x \nabla_y \Gamma(x,y)\, dy, \\ \mathcal{F}_{22}(Q)(x) &= \int\limits_{\Omega_f} \left(1 - \eta\left(|x-y|\right)\right) Q(y) \nabla_x \nabla_y \Gamma(x,y)\, dy, \\ \mathcal{F}_{23}(Q)(x) &= \int\limits_{\Omega_f} \eta\left(|x-y|\right) Q(y) \nabla_x \nabla_y \left(\Gamma(x,y) \sum_{k \in \mathbb{Z}^n} \check{\mathcal{X}}_{\mathcal{A}}(x+k, y+k) \right) dy. \end{aligned}$$

It is not difficult to see, by Theorem 3 on p. 39 [23] and (2.13) in [11],

$$\left\| \mathcal{F}_{21}(Q) \right\|_{L^p(\mathbb{R}^n)} + \left\| \mathcal{F}_{22}(Q) \right\|_{L^p(\mathbb{R}^n)} \leqslant c \| Q \|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty),$$

where c is a constant. By Lemma 7.12 and Lemma 4.2 [11] and repeating the argument from (5.2) to (5.6), we obtain

$$\left\| \mathcal{F}_{23}(Q) \right\|_{L^p(\Omega_f)} \leqslant c \| Q \|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty),$$

where c is a constant. Therefore,

$$\left\|\mathcal{F}_2(Q)\right\|_{L^p(\Omega_f)} \leqslant c\|Q\|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty), \tag{5.8}$$

where c is a constant. Since $\mathcal{K}$ is a constant symmetric positive definite matrix (see (2.9)), we do change of variable as the remark before Lemma 4.1 as well as use both Theorem 3 on p. 39 [23] and (2.13) in [11] to obtain

$$\left\|\mathcal{F}_3(Q)\right\|_{L^p(\Omega_f)} \leqslant c\|Q\|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty), \tag{5.9}$$

where c is a constant. Finally,

$$\mathcal{F}_4(Q)(x) \equiv \int\limits_{\Omega_f} \eta\big(|x-y|\big)\mathcal{L}_1^c(x,y)Q(y) + \big(1-\eta\big(|x-y|\big)\big)\mathcal{L}_2^c(x,y)Q(y)\,dy,$$

where

$$\mathcal{L}_1^c(x,y) = \nabla_x\nabla_y G(x,y) - \nabla_x\nabla_y G^*(x,y),$$
$$\mathcal{L}_2^c(x,y) = \nabla_x\nabla_y G(x,y) - \big(I+\nabla\mathbb{X}(x)\big)\big(I+\nabla\mathbb{X}(y)\big)\nabla_x\nabla_y G_0(x,y).$$

Define

$$\mathcal{P}_1(Q)(x) \equiv \begin{cases} \int_{\Omega_f} \eta(|x-y|)\mathcal{L}_1^c(x,y)Q(y)\,dy & \text{if } x \in \Omega_f, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_f. \end{cases} \tag{5.10}$$

Clearly, $\mathcal{P}_1$ is a linear map. For any $\delta > 0$, by Fubini Theorem [21], Corollary 4.2, and change of variable,

$$\begin{aligned} &\delta\left|\left\{x \in \mathbb{R}^n \,\middle|\, \left|\mathcal{P}_1(Q)(x)\right| > \delta\right\}\right| \\ &\quad \leqslant \int\limits_{\Omega_f} \left|\mathcal{P}_1(Q)\right| dx \leqslant \int\limits_{\Omega_f}\int\limits_{\Omega_f} \eta\big(|x-y|\big)\left|\mathcal{L}_1^c(x,y)Q(y)\right| dy\,dx \\ &\quad \leqslant \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\eta(|x-y|)}{|x-y|^2}\mathcal{X}_{\Omega_f}(x)\left|Q(y)\right|\mathcal{X}_{\Omega_f}(y)\,dx\,dy \leqslant c\|Q\|_{L^1(\Omega_f)}, \end{aligned} \tag{5.11}$$

where c is a constant. By (5.10) and Corollary 4.2, if $x \in \Omega_f$,

$$\left|\mathcal{P}_1(Q)(x)\right| \leqslant \int\limits_{\Omega_f} \eta\big(|x-y|\big)\left|\mathcal{L}_1^c(x,y)Q(y)\right| dy \leqslant c\|Q\|_{L^\infty(\Omega_f)}, \tag{5.12}$$

where c is a constant. By (5.11)–(5.12) and Theorem 5 on p. 21 [23], we see

$$\left\|\mathcal{P}_1(Q)\right\|_{L^p(\mathbb{R}^n)} \leqslant c\|Q\|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty), \tag{5.13}$$

where c is a constant. Similarly, if we define

$$\mathcal{P}_2(Q)(x) \equiv \begin{cases} \int_{\Omega_f} (1-\eta(|x-y|))\mathcal{L}_2^c(x,y)Q(y)\,dy & \text{if } x \in \Omega_f, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_f, \end{cases}$$

as well as employ Theorem 5 on p. 21 [23] and Corollary 4.1, then

$$\|\mathcal{P}_2(Q)\|_{L^p(\mathbb{R}^n)} \leqslant c\|Q\|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty),$$

where c is a constant. Together with (5.13), we have

$$\|\mathcal{F}_4(Q)\|_{L^p(\Omega_f)} \leqslant c\|Q\|_{L^p(\Omega_f)} \quad \text{for } p \in (1,\infty), \tag{5.14}$$

where c is a constant.

The lemma follows from (5.1), (5.7), (5.8), (5.9), and (5.14). □

Lemma 5.2. *If $p \in (1,\infty)$ and $Q \in [L^p(\mathbb{R}^n)]^n$ with support in $B_t(0)$ for some $t > 0$, then a $W^{1,p}_{loc}(\Omega_f)$ solution of*

$$\begin{cases} -\nabla \cdot (\nabla U + Q) = 0 & \text{in } \Omega_f, \\ (\nabla U + Q)\cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega_m, \\ |U|(x) = o(1) & \text{for large } |x| \end{cases} \tag{5.15}$$

exists uniquely. The solution of (5.15) satisfies

$$\begin{cases} \|\nabla U\|_{L^p(\Omega_f)} \leqslant c\|Q\|_{L^p(\Omega_f)} & \text{if } p \in (1,\infty), \\ \|U\|_{L^p(B_s(0)\cap\Omega_f)} \leqslant c_{t,s}\|Q\|_{L^p(\Omega_f)} & \text{if } p \in (1,\infty), \\ \|U\|_{L^{\frac{np}{n-p}}(\Omega_f)} \leqslant c\|Q\|_{L^p(\Omega_f)} & \text{if } p \in (1,n), \end{cases} \tag{5.16}$$

where $s > 0$, $\vec{\mathbf{n}}$ is a unit vector normal to $\partial\Omega_m$, c is a constant independent of t, and $c_{t,s}$ is a constant dependent on t, s.

Proof. By Remark 3.1, we know that the $\mathcal{D}^{1,2}$ solution of (5.15) exists uniquely if $Q \in [C_0^\infty(\mathbb{R}^n)]^n$. Lemma 5.1 and (3.61) imply $(5.16)_1$. By Remark 3.1 and Lemma 3.8, the solution of (5.15) satisfies

$$|U|(x) = \left|\int_{\Omega_f} \nabla_y G(x,y)Q(y)\,dy\right| \leqslant c\int_{\Omega_f} |x-y|^{1-n}|Q|(y)\,dy \quad \text{for } x \in \Omega_f. \tag{5.17}$$

By Lemma 7.12 [11] and (5.17), we have $(5.16)_2$. So a $W^{1,p}_{loc}$ solution of (5.15) for $p \in (1,\infty)$ exists. By Theorem 1 on p. 119 [23] and (5.17), we obtain $(5.16)_3$. The uniqueness of the $W^{1,p}_{loc}$ solution of (5.15) for $p \in (1,\infty)$ is due to the maximal principle (see Theorem 3.1 and Lemma 3.4 in [11]). So we prove Lemma 5.2 for $Q \in [C_0^\infty(\mathbb{R}^n)]^n$ case. If $Q \in [L^p(\mathbb{R}^n)]^n$ with compact support, the lemma can be proved by a limiting argument.

Theorem 2.1 is a direct result of Lemma 5.2. □

5.2. Proof of Theorem 2.2

Assume $F_\epsilon \in C_0^\infty(B_t(0))$ for some $t > 0$. By Remark 3.1 and (2.12), the $\mathcal{D}^{1,2}(\Omega_f^\epsilon)$ solution of (2.1) exists uniquely and

$$\|U_\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega_f^\epsilon)} \leqslant c_t \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{H^{-1}(\mathbb{R}^n)}, \tag{5.18}$$

where c_t is independent of ϵ but dependent on t. Let $\xi \in C_0^\infty(B_s(0))$ and $s > 0$. By Remark 3.1, (3.64), and Fubini Theorem [21],

$$\begin{aligned} \int_{B_s(0)\cap\Omega_f^\epsilon} \nabla U_\epsilon\, \xi\, dx &= \int_{B_s(0)\cap\Omega_f^\epsilon} \int_{B_t(0)\cap\Omega_f^\epsilon} \nabla_x G_\epsilon(x,y) F_\epsilon(y)\xi(x)\, dy\, dx \\ &= \int_{B_t(0)\cap\Omega_f^\epsilon} \int_{B_s(0)\cap\Omega_f^\epsilon} \nabla_x G_\epsilon(x,y)\xi(x)\, dx\, F_\epsilon(y)\, dy. \end{aligned} \tag{5.19}$$

If we define

$$\varphi_\epsilon(y) \equiv \int_{B_s(0)\cap\Omega_f^\epsilon} \nabla_x G_\epsilon(x,y)\xi(x)\, dx,$$

then, by Remark 3.1 and (3.64)–(3.65),

$$\begin{cases} -\nabla\cdot(\nabla\varphi_\epsilon - \xi) = 0 & \text{in } \Omega_f^\epsilon, \\ (\nabla\varphi_\epsilon - \xi)\cdot \vec{\mathbf{n}}^\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ |\varphi_\epsilon|(y) = o(1) & \text{for large } |y|. \end{cases}$$

By Theorem 2.1, we see

$$\|\varphi_\epsilon\|_{W^{1,r}(B_\ell(0)\cap\Omega_f^\epsilon)} \leqslant c_{\ell,s}\|\xi\|_{L^r(\Omega_f^\epsilon)} \quad \text{for any } r \in (1,\infty), \tag{5.20}$$

where $\ell > 0$ and $c_{\ell,s}$ is independent of ϵ but dependent on ℓ, s. By (5.20), extension theorem [1], and Hölder inequality,

$$\begin{aligned} \int_{B_s(0)\cap\Omega_f^\epsilon} \nabla U_\epsilon\, \xi\, dx &= \int_{B_t(0)\cap\Omega_f^\epsilon} \varphi_\epsilon F_\epsilon\, dy = \int_{\mathbb{R}^n} \Pi_\epsilon\varphi_\epsilon F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\, dy \\ &\leqslant \|\Pi_\epsilon\varphi_\epsilon\|_{W^{1,r}(B_{t+1}(0))} \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)} \\ &\leqslant c_{t,s}\|\xi\|_{L^r(\Omega_f^\epsilon)} \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)}, \end{aligned} \tag{5.21}$$

where $\frac{1}{r} + \frac{1}{p} = 1$ and $c_{t,s}$ is independent of ϵ but dependent on t, s. Since $C_0^\infty(B_s(0))$ is dense in $L^r(B_s(0))$ for $r \in (1,\infty)$ and $s > 0$, we know

$$\|\nabla U_\epsilon\|_{L^p(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_{t,s}\|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)} \quad \text{for any } p \in (1,\infty). \tag{5.22}$$

If $p \in [2, \infty)$, then

$$\|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{H^{-1}(\mathbb{R}^n)} \leqslant c_t \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)}, \tag{5.23}$$

where c_t depends on t. (5.18), (5.22), and (5.23) imply that a $W^{1,p}_{loc}(\Omega_f^\epsilon)$ solution of (2.1) for $p \in [2, \frac{2n}{n-2}]$ exists and (2.2) holds. That is,

$$\|U_\epsilon\|_{W^{1,p}(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_{t,s} \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)}, \tag{5.24}$$

where $c_{t,s}$ is a constant independent of ϵ but dependent on t, s. If $p \in (\frac{2n}{n-2}, \frac{2n}{n-4}]$, we know

$$\|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,\frac{2n}{n-2}}(\mathbb{R}^n)} \leqslant c_t \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)},$$

where c_t depends on t. By Theorem 7.26 [11], (5.24), and a similar argument as (2.12), if $\frac{2n}{n-2} < n$ and $p \in (\frac{2n}{n-2}, \frac{2n}{n-4}]$, then

$$\|U_\epsilon\|_{L^{\frac{2n}{n-4}}(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_s \|U_\epsilon\|_{W^{1,\frac{2n}{n-2}}(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_{t,s} \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)}, \tag{5.25}$$

where $c_{t,s}$ is independent of ϵ but dependent on t, s. Theorem 7.26 [11], (5.22), and (5.25) imply that, if $\frac{2n}{n-2} \geqslant n$ and $p \in (\frac{2n}{n-2}, \infty)$ or if $\frac{2n}{n-2} < n$ and $p \in (\frac{2n}{n-2}, \frac{2n}{n-4}]$, then a $W^{1,p}_{loc}(\Omega_f^\epsilon)$ solution of (2.1) exists and (2.2) holds. Repeating the same process, we see that a $W^{1,p}_{loc}(\Omega_f^\epsilon)$ solution of (2.1) exists and (2.2) holds for $p \in [2, \infty)$.

Since (2.2) holds for $p \in [2, \infty)$, a modification of the argument from (5.19) to (5.21) shows

$$\|U_\epsilon\|_{L^p(B_s(0)\cap\Omega_f^\epsilon)} \leqslant c_{t,s} \|F_\epsilon \mathcal{X}_{B_t(0)\cap\Omega_f^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)} \quad \text{for } p \in (1, 2], \tag{5.26}$$

where $c_{t,s}$ is a positive constant independent of ϵ but dependent on t, s. (5.22) and (5.26) imply that a $W^{1,p}_{loc}(\Omega_f^\epsilon)$ solution of (2.1) exists and (2.2) holds for $p \in (1, 2]$. So (2.2) holds for $p \in (1, \infty)$.

Remark 3.1 and (3.65) imply

$$\begin{cases} |U_\epsilon|(x) \leqslant c \displaystyle\int_{\Omega_f^\epsilon} |x-y|^{2-n} |F_\epsilon|(y)\,dy \\ |\nabla U_\epsilon|(x) \leqslant c \displaystyle\int_{\Omega_f^\epsilon} |x-y|^{1-n} |F_\epsilon|(y)\,dy \end{cases} \quad \text{for } x \in \Omega_f^\epsilon,$$

where c is a constant independent of ϵ, F_ϵ. By Theorem 1 on p. 119 [23], we obtain (2.3). Uniqueness of the $W^{1,p}_{loc}(\Omega_f^\epsilon)$ solution of (2.1) for $p \in (1, \infty)$ is due to Theorem 3.1 and Lemma 3.4 [11]. So Theorem 2.2 holds for $F_\epsilon \in C_0^\infty(\mathbb{R}^n)$ case. If $F_\epsilon \in W^{-1,p}(\mathbb{R}^n)$ with compact support, Theorem 2.2 can be proved by a limiting argument.

5.3. Proof of Theorem 2.3

By Lax–Milgram Theorem [11], extension theorem [1], and (2.12), we know that the $\mathcal{D}^{1,2}(\Omega_f^\epsilon)$ solution of (2.4) exists uniquely and satisfies

$$\|U_\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega_f^\epsilon)} \leqslant c\big(\|Q_\epsilon\|_{L^2(\mathbb{R}^n)} + \|F_\epsilon\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\big), \tag{5.27}$$

where c is independent of ϵ. For any $x \in \Omega_f^\epsilon$, by Lemma 3.1 and (5.27),

$$\begin{aligned}
\big|U_\epsilon(x)\big| &\leqslant \left|U_\epsilon(x) - \fint_{B_{1/2}(x)\cap\Omega_f^\epsilon} U_\epsilon(y)\,dy\right| + \left|\fint_{B_{1/2}(x)\cap\Omega_f^\epsilon} U_\epsilon(y)\,dy\right| \\
&\leqslant [U_\epsilon]_{C^{0,\mu}(\overline{B_{1/2}(x)\cap\Omega_f^\epsilon})} \fint_{B_{1/2}(x)\cap\Omega_f^\epsilon} |y|^\mu\,dy + \left|\fint_{B_{1/2}(x)\cap\Omega_f^\epsilon} U_\epsilon(y)\,dy\right| \\
&\leqslant c\|Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(\mathbb{R}^n)},
\end{aligned}$$

where $\mu, \delta > 0$ and c is independent of ϵ, x. So we prove (2.5). (2.6) follows from (2.5) and Lemma 3.4.

5.4. Proof of Theorem 2.4

Lemma 5.3. *Suppose*

$$\begin{cases} Q \in L^p(\mathbb{R}^n), \qquad F \in L^{\frac{np}{n+p}}(\mathbb{R}^n) \quad \text{for } p \in \left(\dfrac{n}{n-2}, \infty\right), \\ |Q|(x) = O\big(|x|^{-n-1}\big), \qquad |F|(x) = O\big(|x|^{-n-1}\big), \end{cases} \tag{5.28}$$

then a $W^{1,p}(\Omega_f)$ solution of

$$\begin{cases} -\nabla\cdot(\nabla\varphi + Q) = F & \text{in } \Omega_f, \\ (\nabla\varphi + Q)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial\Omega_m, \\ |\varphi|(x) = o(1) \end{cases} \tag{5.29}$$

exists uniquely and satisfies

$$\begin{cases} \|\varphi\|_{L^p(\Omega_f)} \leqslant c\big(\|Q\|_{L^{\frac{np}{n+p}}(\Omega_f)} + \|F\|_{L^{\frac{np}{n+2p}}(\Omega_f)}\big), \\ \|\nabla\varphi\|_{L^p(\Omega_f)} \leqslant c\big(\|Q\|_{L^p(\Omega_f)} + \|F\|_{L^{\frac{np}{n+p}}(\Omega_f)}\big), \end{cases} \tag{5.30}$$

where c is a positive constant independent of Q, F. $|\zeta|(x) = O(|x|^{-n-1})$ for $\zeta \in \{Q, F\}$ means that when $|x|$ is large, $|\zeta|(x) \leqslant c|x|^{-n-1}$ where c is a constant.

Proof. The uniqueness of the solution of (5.29) is from maximal principle (see Theorem 3.1 and Lemma 3.4 [11]). If $Q, F \in C_0^\infty(\mathbb{R}^n)$, by Remark 3.1 and Lemma 3.9, the solution of (5.29) exists uniquely in $\mathcal{D}^{1,2}(\Omega_f)$ and satisfies, for $x \in \Omega_f$,

$$\varphi(x)=\int_{\Omega_f} G(x,y)F(y)\,dy-\int_{\Omega_f}\nabla_y G(x,y)Q(y)\,dy,\tag{5.31}$$

$$\left|\nabla\varphi(x)-\int_{\Omega_f}\nabla_x G(x,y)F(y)\,dy+\int_{\Omega_f}\nabla_x\nabla_y G(x,y)Q(y)\,dy\right|\leqslant c\left|Q(x)\right|,\tag{5.32}$$

where c is a constant independent of Q, F. By Theorem 1 on p. 119 [23], (3.48), and (5.31), we get $(5.30)_1$. By Theorem 1 on p. 119 [23] and (3.48),

$$\left\|\int_{\Omega_f}\nabla_x G(x,y)F(y)\,dy\right\|_{L^p(\Omega_f)}\leqslant c\|F\|_{L^{\frac{np}{n+p}}(\Omega_f)},$$

where c is a constant. Together with Lemma 5.1 and (5.32), we obtain $(5.30)_2$ for $Q, F\in C_0^\infty(\mathbb{R}^n)$ case.

For general case, (5.28) implies $Q, F\in L^1(\mathbb{R}^n)$. We can find $Q_m, F_m\in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} Q_m\to Q & \text{in } L^p(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)\\ F_m\to F & \text{in } L^{\frac{np}{n+p}}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)\end{cases}\quad \text{as } m\to\infty.\tag{5.33}$$

Let φ_m be the solution of (5.29) corresponding to Q_m and F_m. Then (5.30) implies

$$\begin{cases} \|\varphi_m\|_{L^p(\Omega_f)}\leqslant c\left(\|Q_m\|_{L^{\frac{np}{n+p}}(\Omega_f)}+\|F_m\|_{L^{\frac{np}{n+2p}}(\Omega_f)}\right),\\ \|\nabla\varphi_m\|_{L^p(\Omega_f)}\leqslant c\left(\|Q_m\|_{L^p(\Omega_f)}+\|F_m\|_{L^{\frac{np}{n+p}}(\Omega_f)}\right),\end{cases}\tag{5.34}$$

where c is independent of Q_m, F_m. There is a subsequence $\widetilde{\varphi}_m$ of φ_m such that, by Theorem 7.26 [11] and (5.33)–(5.34),

$$\widetilde{\varphi}_m\to\varphi\quad \text{weakly in } W^{1,p}(\Omega_f) \text{ and pointwise in } \Omega_f \text{ as } m\to\infty,\tag{5.35}$$

and φ satisfies $(5.29)_{1,2}$ and (5.30).

Next we claim φ, Q, F satisfy (5.31) almost everywhere in Ω_f. For any fixed $x\in\Omega_f$,

$$\begin{aligned}&\left|\int_{\Omega_f} G(x,y)(F_m-F)(y)\,dy\right|\\ &\quad\leqslant\left|\int_{\Omega_f\setminus B_1(x)} G(x,y)(F_m-F)(y)\,dy\right|+\left|\int_{\Omega_f\cap B_1(x)} G(x,y)(F_m-F)(y)\,dy\right|.\end{aligned}\tag{5.36}$$

By (3.48) and (5.33),

$$\lim_{m\to\infty}\left|\int_{\Omega_f\setminus B_1(x)} G(x,y)(F_m-F)(y)\,dy\right|\leqslant\lim_{m\to\infty} c\|F_m-F\|_{L^1(\Omega_f)}=0.\tag{5.37}$$

By Fubini Theorem [21] and (3.48),

$$\lim_{m\to\infty}\left|\int\limits_{\Omega_f}\int\limits_{\Omega_f\cap B_1(x)} G(x,y)(F_m-F)(y)\,dy\,dx\right|$$
$$\leqslant \lim_{m\to\infty} c\int\limits_{\mathbb{R}^n}\int\limits_{B_1(x)} |x-y|^{2-n}|F_m-F|(y)\,dy\,dx$$
$$=\lim_{m\to\infty} c\int\limits_{\mathbb{R}^n}\int\limits_{B_1(0)} |z|^{2-n}|F_m-F|(x-z)\,dz\,dx \leqslant \lim_{m\to\infty} c\|F_m-F\|_{L^1(\Omega_f)}=0.$$

This means that there is a subsequence of

$$\int\limits_{\Omega_f\cap B_1(x)} G(x,y)F_m(y)\,dy \quad \text{converging to} \quad \int\limits_{\Omega_f\cap B_1(x)} G(x,y)F(y)\,dy \tag{5.38}$$

pointwise almost everywhere in Ω_f. By (5.36)–(5.38), we prove $\int_{\Omega_f} G(x,y)F_m(y)\,dy$ converges to $\int_{\Omega_f} G(x,y)F(y)\,dy$ pointwise almost everywhere in Ω_f. By a similar argument, we also have $\int_{\Omega_f} \nabla_y G(x,y)Q_m(y)\,dy$ converging to $\int_{\Omega_f} \nabla_y G(x,y)Q(y)\,dy$ pointwise almost everywhere in Ω_f. Together with (5.35), we prove the claim. That is, functions φ, Q, F satisfy (5.31) almost everywhere in Ω_f.

Because $(5.28)_2$ holds and because φ, Q, F satisfy (5.31), it is not difficult to see that φ satisfies $(5.29)_3$. Then uniqueness of the solution implies the lemma. □

If U_0 solves (2.14) and F in (2.14) is in $W^{1,p}(\mathbb{R}^n)$ with compact support for $p\in(\frac{n}{n-2},\infty)$, then, by Definition 5.1 and p. 67 [16],

$$U_0(x)=\int\limits_{\mathbb{R}^n} G_0(x,y)|Y_f|F(y)\,dy \quad \text{for } x\in\mathbb{R}^n,$$

where G_0 is the Green's function in (4.1). By change of variable, G_0 can be transformed to the fundamental solution of the Laplace equation in some new coordinate system. By Lemma 4.4 [11], (2.14) in [11], and Theorem 3 on p. 39 [23], we know

$$\begin{cases} \left\|\nabla^2 U_0\right\|_{W^{i,p}(\mathbb{R}^n)} \leqslant c\|F\|_{W^{i,p}(\mathbb{R}^n)} & \text{for } p\in\left(\dfrac{n}{n-2},\infty\right) \text{ and } i\in\{1,2\},\\ \left|\nabla^i U_0\right|(x)=O\left(|x|^{2-n-i}\right) & \text{for } i\in\{0,1,2,3\}, \end{cases} \tag{5.39}$$

where c is a constant dependent on n, p, $\mathcal{K}$. Define

$$\varphi_\epsilon(x)\equiv U_\epsilon(x)-U_0(x)-\mathbb{X}_\epsilon(x)\nabla U_0(x)-\mathbb{S}_\epsilon(x)\nabla^2 U_0(x) \quad \text{for } x\in\Omega_f^\epsilon,$$

where U_ϵ, U_0 are the solutions of (2.13) and (2.14), $\mathbb{X}_\epsilon$ is defined in (2.8), and $\mathbb{S}_\epsilon$ is defined in (2.10). As in the proof of Lemma 3.2,

$$\begin{cases} -\nabla\cdot\left(\nabla\varphi_\epsilon+\mathbb{S}_\epsilon\nabla^3 U_0\right)=\mathbb{X}_\epsilon\nabla\Delta U_0+\nabla\mathbb{S}_\epsilon\nabla^3 U_0 & \text{in } \Omega_f^\epsilon,\\ \left(\nabla\varphi_\epsilon+\mathbb{S}_\epsilon\nabla^3 U_0\right)\cdot\vec{\mathbf{n}}^\epsilon=0 & \text{on } \partial\Omega_m^\epsilon,\\ |\varphi_\epsilon|(x)=o(1). \end{cases} \tag{5.40}$$

$(5.40)_3$ is due to (2.11), Remark 3.1, (3.48), and $(5.39)_2$. By Lemma 5.3, (2.11), and (5.39), the solution of (5.40) satisfies

$$\begin{cases} \|\varphi_\epsilon\|_{L^p(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,\frac{np}{n+p}}(\mathbb{R}^n)\cap W^{1,\frac{np}{n+2p}}(\mathbb{R}^n)} \\ \|\nabla\varphi_\epsilon\|_{L^p(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,p}(\mathbb{R}^n)\cap W^{1,\frac{np}{n+p}}(\mathbb{R}^n)} \end{cases} \quad \text{for } p \in \left(\frac{n}{n-2}, \infty\right),$$

where c is independent of ϵ. So if $p \in (\frac{n}{n-2}, \infty)$, then

$$\begin{cases} \|U_\epsilon - U_0\|_{L^p(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,\frac{np}{n+p}}(\mathbb{R}^n)\cap W^{1,\frac{np}{n+2p}}(\mathbb{R}^n)}, \\ \left\|\nabla U_\epsilon - (I + \nabla\mathbb{X}_\epsilon)\nabla U_0\right\|_{L^p(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,p}(\mathbb{R}^n)\cap W^{1,\frac{np}{n+p}}(\mathbb{R}^n)}, \end{cases} \tag{5.41}$$

where c is independent of ϵ.

Similarly, by (2.11), (5.39)–(5.40), Lemma 3.2, and Lemma 3.4, we have

$$\begin{cases} \|U_\epsilon - U_0\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{1,n+\delta}(\mathbb{R}^n)}, \\ \left\|\nabla U_\epsilon - (I + \nabla\mathbb{X}_\epsilon)\nabla U_0\right\|_{L^\infty(\Omega_f^\epsilon)} \leqslant c\epsilon\|F\|_{W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)\cap W^{2,n+\delta}(\mathbb{R}^n)}, \end{cases} \tag{5.42}$$

where c is independent of ϵ. By (5.41) and (5.42), we prove Theorem 2.4.

6. Proof of Lemmas 3.1, 3.4

From Remark 2.1, we know the $W^{2,p}$ norm of the solutions of (3.2) in $B_{1/2}(0) \cap \Omega_f^\epsilon$ in general is not bounded uniformly in ϵ even if $\|Q_\epsilon\|_{W^{1,p}(\Omega_f^\epsilon)}$, $\|F_\epsilon\|_{L^p(\Omega_f^\epsilon)}$ are bounded independent of ϵ. But Lemma 3.1 and Lemma 3.4 prove that the Hölder norm and the Lipschitz norm of the solutions of (3.2) in $B_{1/2}(0) \cap \Omega_f^\epsilon$ are bounded uniformly in ϵ.

6.1. Proof of Lemma 3.1

Lemma 6.1. *For any $\delta > 0$, there are $\theta_1, \theta_2 \in (0, 1)$ (dependent on δ, Y_f) with $\theta_1 < \theta_2^2$ and $\epsilon_0 \in (0, 1)$ (dependent on δ, θ_1, θ_2) such that if U_λ, Q_λ, F_λ satisfy*

$$\begin{cases} -\nabla\cdot(\nabla U_\lambda + Q_\lambda) = F_\lambda & \text{in } B_1(0) \cap \Omega_f^\lambda, \\ (\nabla U_\lambda + Q_\lambda)\cdot\vec{\mathbf{n}}^\lambda = 0 & \text{on } B_1(0) \cap \partial\Omega_m^\lambda, \end{cases} \tag{6.1}$$

and

$$\max\left\{\|U_\lambda\|_{L^2(B_1(0)\cap\Omega_f^\lambda)}, \epsilon_0^{-1}\|Q_\lambda, F_\lambda\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\lambda)}\right\} \leqslant 1,$$

then, for any $0 < \lambda \leqslant \epsilon_0$ and $\theta \in [\theta_1, \theta_2]$,

$$\fint_{B_\theta(0)} \left|\Pi_\lambda U_\lambda - (\Pi_\lambda U_\lambda)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu} \tag{6.2}$$

where $\mu \equiv \frac{\delta}{n+\delta}$, $\vec{\mathbf{n}}^\lambda$ is the unit vector normal to $\partial\Omega_m^\lambda$, and Π_λ is the extension operator in [1] (or see (2.12)).

Proof. Consider the following problem $LU \equiv -\nabla\cdot(\mathcal{K}\nabla U) = 0$, where $\mathcal{K}$ is the constant matrix in (2.9). Then U satisfies, by Theorem 9.11 [11],

$$\|U\|_{W^{1,r}(B_{1/2}(0))} \leqslant c\|U\|_{L^2(B_{3/4}(0))} \quad \text{for any } r > n,$$

where c depends on $\mathcal{K}$, r. If μ' satisfies $\mu < \mu' < 1$, then

$$\fint_{B_\theta(0)} \left|U - (U)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu'} \fint_{B_{3/4}(0)} U^2\, dx \tag{6.3}$$

for θ sufficiently small (see p. 70 [10]). Fix $\theta_1, \theta_2 \in (0, 1/2)$ with $\theta_1 < \theta_2^2$ such that (6.3) holds for $\theta \in [\theta_1, \theta_2]$.

We claim (6.2). If not, there is a sequence $\{\theta_\lambda, U_\lambda, Q_\lambda, F_\lambda\}$ satisfying (6.1) and

$$\begin{cases} \theta_\lambda \in [\theta_1, \theta_2], \\ \|U_\lambda\|_{L^2(B_1(0)\cap\Omega_f^\lambda)} \leqslant 1, \\ \lim\limits_{\lambda\to 0} \|Q_\lambda, F_\lambda\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\lambda)} = 0, \\ \fint_{B_{\theta_\lambda}(0)} \left|\Pi_\lambda U_\lambda - (\Pi_\lambda U_\lambda)_{0,\theta_\lambda}\right|^2 dx > \theta_\lambda^{2\mu}. \end{cases} \tag{6.4}$$

By compactness principle [3], we can extract a subsequence (same notation for subsequence) such that

$$\begin{cases} \theta_\lambda \to \theta_* & \\ \Pi_\lambda U_\lambda \to U & \text{in } L^2(B_{3/4}(0)) \text{ strongly} \\ \mathcal{X}_{\Omega_f^\lambda}\nabla U_\lambda \to \mathcal{K}\nabla U & \text{in } L^2(B_{3/4}(0)) \text{ weakly} \end{cases} \quad \text{as } \lambda \to 0. \tag{6.5}$$

Here $\mathcal{X}_{\Omega_f^\lambda}$ is the characteristic function on Ω_f^λ and $\mathcal{K}$ is the constant positive definite matrix in (2.9). Moreover, U satisfies $LU = 0$ in $B_{3/4}(0)$. If θ_2 is small enough (dependent on δ, Y_f), Eqs. (6.3)–(6.5) then imply

$$\begin{aligned} \theta_*^{2\mu} &= \lim_{\lambda\to 0} \theta_\lambda^{2\mu} \leqslant \lim_{\lambda\to 0} \fint_{B_{\theta_\lambda}(0)} \left|\Pi_\lambda U_\lambda - (\Pi_\lambda U_\lambda)_{0,\theta_\lambda}\right|^2 dx \\ &= \fint_{B_{\theta_*}(0)} U^2\, dx - \left|\fint_{B_{\theta_*}(0)} U\, dx\right|^2 = \fint_{B_{\theta_*}(0)} \left|U - (U)_{0,\theta_*}\right|^2 dx < \theta_*^{2\mu}. \end{aligned}$$

So we get $\theta_*^{2\mu} < \theta_*^{2\mu}$, which is impossible. Therefore we prove (6.2). □

Lemma 6.2. *Let δ, ϵ_0, θ_1, θ_2, μ be same as those in Lemma 6.1. For any $\epsilon \leqslant \epsilon_0$, $\theta \in [\theta_1, \theta_2]$, and k with $\epsilon/\theta^k \leqslant \epsilon_0$, if U_ϵ, Q_ϵ, F_ϵ satisfy*

$$\begin{cases} -\nabla\cdot(\nabla U_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } B_1(0)\cap\Omega_f^\epsilon, \\ (\nabla U_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon = 0 & \text{on } B_1(0)\cap\partial\Omega_m^\epsilon, \end{cases} \tag{6.6}$$

then

$$\fint_{B_{\theta^k}(0)} \left|\Pi_\epsilon U_\epsilon - (\Pi_\epsilon U_\epsilon)_{0,\theta^k}\right|^2 dx \leqslant \theta^{2k\mu} J_\epsilon^2, \tag{6.7}$$

where $J_\epsilon \equiv \|U_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)} + \epsilon_0^{-1}\|Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)}$.

Proof. We only consider $J_\epsilon < \infty$ case and this is done by induction on k. For $k=1$, we define $\hat{U}_\epsilon \equiv \frac{U_\epsilon}{J_\epsilon}$, $\hat{Q}_\epsilon \equiv \frac{Q_\epsilon}{J_\epsilon}$, $\hat{F}_\epsilon \equiv \frac{F_\epsilon}{J_\epsilon}$. Then $\hat{U}_\epsilon$, $\hat{Q}_\epsilon$, $\hat{F}_\epsilon$ satisfy (6.6) and

$$\max\left\{\|\hat{U}_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)}, \epsilon_0^{-1}\|\hat{Q}_\epsilon, \hat{F}_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)}\right\} \leqslant 1.$$

By Lemma 6.1 (in this case $\lambda = \epsilon$),

$$\fint_{B_\theta(0)} \left|\Pi_\epsilon \hat{U}_\epsilon - (\Pi_\epsilon \hat{U}_\epsilon)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}.$$

This implies (6.7) for $k=1$ case. Suppose (6.7) holds for some k satisfying $\epsilon/\theta^k \leqslant \epsilon_0$, we define

$$\begin{cases} \hat{U}_\epsilon \equiv J_\epsilon^{-1}\theta^{-k\mu}\left(U_\epsilon\left(\theta^k x\right) - (\Pi_\epsilon U_\epsilon)_{0,\theta^k}\right) \\ \hat{Q}_\epsilon \equiv J_\epsilon^{-1}\theta^{k(1-\mu)} Q_\epsilon\left(\theta^k x\right) & \text{in } B_1(0)\cap\Omega_f^\epsilon/\theta^k. \\ \hat{F}_\epsilon \equiv J_\epsilon^{-1}\theta^{k(2-\mu)} F_\epsilon\left(\theta^k x\right) \end{cases}$$

Then they satisfy

$$\begin{cases} -\nabla\cdot(\nabla\hat{U}_\epsilon + \hat{Q}_\epsilon) = \hat{F}_\epsilon & \text{in } B_1(0)\cap\Omega_f^\epsilon/\theta^k, \\ (\nabla\hat{U}_\epsilon + \hat{Q}_\epsilon)\cdot\vec{\mathbf{n}}^{\epsilon/\theta^k} = 0 & \text{on } B_1(0)\cap\partial\Omega_m^\epsilon/\theta^k, \end{cases} \tag{6.8}$$

where $\vec{\mathbf{n}}^{\epsilon/\theta^k}$ is the unit vector normal to $\partial\Omega_m^\epsilon/\theta^k$. By induction,

$$\max\left\{\|\hat{U}_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon/\theta^k)}, \epsilon_0^{-1}\|\hat{Q}_\epsilon, \hat{F}_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\theta^k)}\right\} \leqslant 1.$$

Since $\epsilon/\theta^k \leqslant \epsilon_0$, by Lemma 6.1 (in this case $\lambda = \epsilon/\theta^k$), we obtain

$$\fint_{B_\theta(0)} \left|\Pi_{\epsilon/\theta^k}\hat{U}_\epsilon - (\Pi_{\epsilon/\theta^k}\hat{U}_\epsilon)_{0,\theta}\right|^2 dx \leqslant \theta^{2\mu}. \tag{6.9}$$

Note, by [1],

$$\fint_{B_\theta(0)} \left|\Pi_{\epsilon/\theta^k}\hat{U}_\epsilon - (\Pi_{\epsilon/\theta^k}\hat{U}_\epsilon)_{0,\theta}\right|^2 dx = \fint_{B_{\theta^{k+1}}(0)} \frac{|\Pi_\epsilon U_\epsilon - (\Pi_\epsilon U_\epsilon)_{0,\theta^{k+1}}|^2}{J_\epsilon^2\,\theta^{2k\mu}}\, dx. \tag{6.10}$$

Eqs. (6.9)–(6.10) imply the inequality (6.7) for $k+1$ case. $\square$

Lemma 6.3. *For any $\delta > 0$, there is a $\mu_* \in (0, \mu)$ such that the solutions of (6.6) satisfy, for any $\epsilon \in (0, 1]$,*

$$[U_\epsilon]_{C^{0,\mu_*}(B_{1/2}(0)\cap\overline{\Omega^\epsilon_f})} \leqslant c J_\epsilon,$$

where c is a constant independent of ϵ. μ, J_ϵ are same as those in Lemma 6.2.

Proof. Let ϵ_0, θ_1, θ_2 be same as those in Lemma 6.2, define $\epsilon_* \equiv \epsilon_0\theta_2/2$, and let $\epsilon \leqslant \epsilon_*$. Denote by c a constant independent of ϵ. Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there exist $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ such that $r = \theta^k$. Lemma 6.2 implies that the solutions of (6.6) satisfy, for any $r \in [\epsilon/\epsilon_0, \theta_2]$,

$$\fint_{B_r(0)} \left|\Pi_\epsilon U_\epsilon - (\Pi_\epsilon U_\epsilon)_{0,r}\right|^2 dx \leqslant cr^{2\mu} J_\epsilon^2. \tag{6.11}$$

Since $2\epsilon/\epsilon_0 \in [\epsilon/\epsilon_0, \theta_2]$, we define, for any $\widetilde{\mu} \in (0, \mu)$,

$$\begin{cases} \hat{U}_\epsilon(x) \equiv J_\epsilon^{-1}\epsilon^{-\widetilde{\mu}}\big(U_\epsilon(\epsilon x) - (\Pi_\epsilon U_\epsilon)_{0,2\epsilon/\epsilon_0}\big) \\ \hat{Q}_\epsilon(x) \equiv J_\epsilon^{-1}\epsilon^{1-\widetilde{\mu}} Q_\epsilon(\epsilon x) \\ \hat{F}_\epsilon(x) \equiv J_\epsilon^{-1}\epsilon^{2-\widetilde{\mu}} F_\epsilon(\epsilon x) \end{cases} \quad \text{in } B_{2/\epsilon_0}(0) \cap \Omega^\epsilon_f/\epsilon. \tag{6.12}$$

Then they satisfy

$$\begin{cases} -\nabla\cdot(\nabla\hat{U}_\epsilon + \hat{Q}_\epsilon) = \hat{F}_\epsilon & \text{in } B_{2/\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon, \\ (\nabla\hat{U}_\epsilon + \hat{Q}_\epsilon)\cdot\vec{\mathbf{n}}^{\epsilon/\epsilon} = 0 & \text{on } B_{2/\epsilon_0}(0)\cap\partial\Omega^\epsilon_m/\epsilon, \end{cases} \tag{6.13}$$

where $\vec{\mathbf{n}}^{\epsilon/\epsilon}$ is the unit vector normal to $\partial\Omega^\epsilon_m/\epsilon$. Take $r = 2\epsilon/\epsilon_0$ in (6.11). We see

$$\|\hat{U}_\epsilon\|_{L^2(B_{2/\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon)} + \|\hat{Q}_\epsilon, \hat{F}_\epsilon\|_{L^{n+\delta}(B_{2/\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon)} \leqslant c,$$

where c is independent of $\widetilde{\mu}$. Tracing the proof of Theorem 8.24 [11], there is a $\mu_* \in (0, \mu)$ such that

$$[\hat{U}_\epsilon]_{C^{0,\mu_*}(B_{1/\epsilon_0}(0)\cap\overline{\Omega^\epsilon_f}/\epsilon)} \leqslant c, \tag{6.14}$$

where c is independent of $\widetilde{\mu}$. If the $\widetilde{\mu}$ in (6.12) is taken to be the μ_* in (6.14), by Theorem 1.2 on p. 70 [10], we see that (6.11) with μ replaced by μ_* also holds for $r \leqslant \epsilon/\epsilon_0$. So (6.11) holds for $r \in (0, \theta_2]$. We then shift the origin to any point $x \in B_{1/2}(0)$, repeat above argument, and see that (6.11) with 0 (resp. μ) replaced by x (resp. μ_*) also holds for $r > 0$. By Theorem 1.2 on p. 70 [10], we see:

For any $\delta > 0$, there is a $\mu_* \in (0, \mu)$ and an $\epsilon_* \in (0, 1)$ (dependent on δ, Y_f) such that if $\epsilon \in (0, \epsilon_*)$, then the solutions of (6.6) satisfy

$$[U_\epsilon]_{C^{0,\mu_*}(B_{1/2}(0)\cap\overline{\Omega^\epsilon_f})} \leqslant c J_\epsilon. \tag{6.15}$$

From the proof of Theorem 8.24 [11], we also see:

For any $\delta > 0$, there is a $\mu_* \in (0, \mu)$ such that if $\epsilon \in [\epsilon_*, 1]$, then the solutions of (6.6) satisfy

$$[U_\epsilon]_{C^{0,\mu_*}(B_{1/2}(0)\cap\overline{\Omega^\epsilon_f})} \leqslant c J_\epsilon. \tag{6.16}$$

Combining (6.15) and (6.16), we prove this lemma. □

By shifting the coordinate, we see that Lemma 3.1 is a direct consequence of Lemma 6.3.

6.2. *Proof of Lemma 3.4*

For convenience, let us assume $0 \in \Omega_f^\epsilon$.

Lemma 6.4. *For any $\delta > 0$, there exist a constant $\theta \in (0,1)$ (dependent on δ, Y_f) and a constant $\epsilon_0 \in (0,1)$ (dependent on θ, δ) so that if*

$$\max\left\{\|U_\lambda\|_{L^\infty(B_1(0)\cap\Omega_f^\lambda)}, \epsilon_0^{-1}\|Q_\lambda, F_\lambda\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\lambda)}\right\} \leqslant 1,$$

then the solutions of (6.1) satisfy, for any $0 < \lambda \leqslant \epsilon_0$,

$$\sup_{x\in B_\theta(0)} \left|\Pi_\lambda U_\lambda(x) - \Pi_\lambda U_\lambda(0) - \big(x + \Pi_\lambda \mathbb{X}_\lambda(x)\big)\mathbf{b}_\lambda\right| \leqslant \theta^{1+\mu/2}, \tag{6.17}$$

where $\mu \equiv \frac{\delta}{n+\delta}$, $\mathbf{b}_\lambda \equiv \frac{\mathcal{K}^{-1}}{|B_\theta(0)|}\int_{B_\theta(0)\cap\Omega_f^\lambda} \nabla U_\lambda\, dx$, $\mathcal{K}^{-1}$ is the inverse matrix of the positive definite matrix $\mathcal{K}$ in (2.9), and $\mathbb{X}_\lambda$ is defined in (2.8).

Proof. If U is a solution of $-\nabla\cdot(\mathcal{K}\nabla U) = 0$ in $B_{3/4}(0)$, then, by Theorem 6.2 [11],

$$\|U\|_{C^{2+\mu}(\overline{B_{1/2}(0)})} \leqslant c\|U\|_{L^\infty(B_{3/4}(0))}.$$

By Taylor's expansion, if μ' satisfies $\mu < \mu' < 1$, then, for θ small enough,

$$\sup_{x\in B_\theta(0)} \left|U(x) - U(0) - x(\nabla U)_{0,\theta}\right| \leqslant \theta^{1+\frac{\mu'}{2}}\|U\|_{L^\infty(B_{3/4}(0))}. \tag{6.18}$$

Fix a value θ such that (6.18) holds. We claim (6.17). If not, there is a sequence $\{U_\lambda, Q_\lambda, F_\lambda\}$ satisfying (6.1) and

$$\begin{cases} \|U_\lambda\|_{L^\infty(B_1(0)\cap\Omega_f^\lambda)} \leqslant 1, \\ \lim\limits_{\lambda\to 0}\|Q_\lambda, F_\lambda\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\lambda)} = 0, \\ \sup\limits_{x\in B_\theta(0)} \left|\Pi_\lambda U_\lambda(x) - \Pi_\lambda U_\lambda(0) - \big(x + \Pi_\lambda\mathbb{X}_\lambda(x)\big)\mathbf{b}_\lambda\right| > \theta^{1+\mu/2}. \end{cases} \tag{6.19}$$

After extraction of a subsequence (same notation for subsequence), we have, by [1] and Lemma 3.1,

$$\begin{cases} \Pi_\lambda U_\lambda \to U & \text{in } L^\infty\big(B_{3/4}(0)\big) \text{ strongly} \\ \mathcal{X}_{\Omega_f^\lambda}\nabla U_\lambda \to \mathcal{K}\nabla U & \text{in } L^2\big(B_{3/4}(0)\big) \text{ weakly} \end{cases} \quad \text{as } \lambda \to 0. \tag{6.20}$$

Here $\mathcal{X}_{\Omega_f^\lambda}$ is the characteristic function on Ω_f^λ. Eqs. (6.19) and (6.20) imply that U satisfies $-\nabla\cdot(\mathcal{K}\nabla U) = 0$ in $B_{3/4}(0)$. By (6.18), $(6.19)_3$, and (6.20), we get

$$\theta^{1+\mu/2} \leqslant \lim_{\lambda\to 0}\sup_{B_\theta(0)}\left|\Pi_\lambda U_\lambda(x) - \Pi_\lambda U_\lambda(0) - \left(x + \Pi_\lambda \mathbb{X}_\lambda(x)\right)\mathbf{b}_\lambda\right|$$

$$= \sup_{B_\theta(0)}\left|U(x) - U(0) - x(\nabla U)_{0,\theta}\right| \leqslant \theta^{1+\frac{\mu'}{2}}\|U\|_{L^\infty(B_{3/4}(0))}.$$

If θ is small enough, the right-hand side of above inequality is less than $\theta^{1+\frac{\mu''}{2}}$ for some $\mu'' \in (\mu, \mu')$. Then we get contradiction. So (6.17) holds. $\square$

Lemma 6.5. *Let δ, θ, ϵ_0, μ be same as those in Lemma 6.4 and let U_ϵ, Q_ϵ, F_ϵ satisfy (6.6). For any $\epsilon \leqslant \epsilon_0$ and k with $\epsilon/\theta^k \leqslant \epsilon_0$, there are $\mathbf{a}_k^\epsilon$, $\mathbf{b}_k^\epsilon$ such that*

$$\begin{cases} \left|\mathbf{a}_k^\epsilon\right| + \left|\mathbf{b}_k^\epsilon\right| \leqslant c\mathcal{J}_\epsilon, \\ \sup\limits_{x\in B_{\theta^k}(0)}\left|\Pi_\epsilon U_\epsilon(x) - \Pi_\epsilon U_\epsilon(0) - \epsilon\mathbf{a}_k^\epsilon - \left(x + \Pi_\epsilon\mathbb{X}_\epsilon(x)\right)\mathbf{b}_k^\epsilon\right| \leqslant \theta^{k(1+\frac{\mu}{2})}\mathcal{J}_\epsilon, \end{cases} \tag{6.21}$$

where $\mathcal{J}_\epsilon \equiv \|U_\epsilon\|_{L^\infty(B_1(0)\cap\Omega_f^\epsilon)} + \epsilon_0^{-1}\|\epsilon^{-1+\mu/2}Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)}$ and c is a constant independent of ϵ.

Proof. This is done by induction on k. Take $\mathbf{a}_1^\epsilon = 0$, $\mathbf{b}_1^\epsilon = \frac{\mathcal{K}^{-1}}{|B_\theta(0)|}\int_{B_\theta(0)\cap\Omega_f^\epsilon}\nabla U_\epsilon\,dx$ and define $\hat{U}_\epsilon \equiv \frac{U_\epsilon}{\mathcal{J}_\epsilon}$, $\hat{Q}_\epsilon \equiv \frac{Q_\epsilon}{\mathcal{J}_\epsilon}$, $\hat{F}_\epsilon \equiv \frac{F_\epsilon}{\mathcal{J}_\epsilon}$. Then $\hat{U}_\epsilon$, $\hat{Q}_\epsilon$, $\hat{F}_\epsilon$ satisfy (6.1) and

$$\max\left\{\|\hat{U}_\epsilon\|_{L^\infty(B_1(0)\cap\Omega_f^\epsilon)}, \epsilon_0^{-1}\|\hat{Q}_\epsilon, \hat{F}_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)}\right\} \leqslant 1.$$

By Lemma 6.4 ($\lambda = \epsilon$ in this case), we obtain (6.21) for $k = 1$. If (6.21) holds for some k satisfying $\epsilon/\theta^k \leqslant \epsilon_0$, then we define, in $B_1(0) \cap \Omega_f^\epsilon/\theta^k$,

$$\begin{cases} \hat{U}_\epsilon(x) \equiv \theta^{-k(1+\mu/2)}\mathcal{J}_\epsilon^{-1}\left(U_\epsilon\left(\theta^k x\right) - \Pi_\epsilon U_\epsilon(0) - \epsilon\mathbf{a}_k^\epsilon - \left(\theta^k x + \mathbb{X}_\epsilon(\theta^k x)\right)\mathbf{b}_k^\epsilon\right), \\ \hat{Q}_\epsilon(x) \equiv \theta^{-k\mu/2}\mathcal{J}_\epsilon^{-1}Q_\epsilon\left(\theta^k x\right), \\ \hat{F}_\epsilon(x) \equiv \theta^{k(1-\mu/2)}\mathcal{J}_\epsilon^{-1}F_\epsilon\left(\theta^k x\right). \end{cases}$$

Then $\hat{U}_\epsilon$, $\hat{Q}_\epsilon$, $\hat{F}_\epsilon$ satisfy (6.1) in $B_1(0) \cap \Omega_f^\epsilon/\theta^k$ (or see (6.8)) by (2.7). By induction,

$$\max\left\{\|\hat{U}_\epsilon\|_{L^\infty(B_1(0)\cap\Omega_f^\epsilon/\theta^k)}, \epsilon_0^{-1}\|\hat{Q}_\epsilon, \hat{F}_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\theta^k)}\right\} \leqslant 1. \tag{6.22}$$

By Lemma 6.4 ($\lambda = \epsilon/\theta^k \leqslant \epsilon_0$ in this case), we have

$$\sup_{B_\theta(0)}\left|\Pi_{\epsilon/\theta^k}\hat{U}_\epsilon(x) - \Pi_{\epsilon/\theta^k}\hat{U}_\epsilon(0) - \left(x + \Pi_{\epsilon/\theta^k}\mathbb{X}_{\epsilon/\theta^k}(x)\right)\mathbf{b}_{\epsilon/\theta^k}\right| \leqslant \theta^{1+\frac{\mu}{2}}, \tag{6.23}$$

where $\mathbf{b}_{\epsilon/\theta^k} \equiv \frac{\mathcal{K}^{-1}}{|B_\theta(0)|}\int_{B_\theta(0)\cap\Omega_f^\epsilon/\theta^k}\nabla\hat{U}_\epsilon\,dx$. Rewrite (6.23) in terms of U_ϵ to obtain, by [1],

$$\begin{aligned} &\sup_{B_\theta(0)}\Big|\Pi_\epsilon U_\epsilon\left(\theta^k x\right) - \Pi_\epsilon U_\epsilon(0) + \epsilon\Pi_1\mathbb{X}(0)\mathbf{b}_k^\epsilon - \left(\theta^k x + \Pi_\epsilon\mathbb{X}_\epsilon\left(\theta^k x\right)\right)\mathbf{b}_k^\epsilon \\ &\quad - \mathcal{J}_\epsilon\theta^{k(1+\mu/2)}\left(x + \theta^{-k}\Pi_\epsilon\mathbb{X}_\epsilon\left(\theta^k x\right)\right)\mathbf{b}_{\epsilon/\theta^k}\Big| \leqslant \mathcal{J}_\epsilon\theta^{(k+1)(1+\mu/2)}. \end{aligned} \tag{6.24}$$

Define

$$\mathbf{a}^{\epsilon}_{k+1} \equiv -\Pi_1 \mathbb{X}(0)\mathbf{b}^{\epsilon}_k \quad \text{and} \quad \mathbf{b}^{\epsilon}_{k+1} \equiv \mathbf{b}^{\epsilon}_k + \mathcal{J}_\epsilon \theta^{k\mu/2}\mathbf{b}_{\epsilon/\theta^k}. \tag{6.25}$$

By (6.22) and (6.23), $|\mathbf{b}_{\epsilon/\theta^k}|$ is bounded uniformly in ϵ, k. So we get $(6.21)_1$. Substituting (6.25) into (6.24) and making the change of variables $\theta^k x$ to x, we obtain $(6.21)_2$. □

Lemma 6.6. *Let ϵ_0, μ, $\mathcal{J}_\epsilon$ be same as those in Lemma 6.5. For any $\epsilon \leqslant \epsilon_0$, the solutions of (6.6) satisfy*

$$\|\nabla U_\epsilon\|_{L^\infty(B_{1/2}(0)\cap\Omega^\epsilon_f)} \leqslant c\mathcal{J}_{*,\epsilon} \equiv c\big(\mathcal{J}_\epsilon + \epsilon^{-\mu/2}\|Q_\epsilon(\epsilon x)\|_{C^{0,\alpha}(\overline{\Omega^\epsilon_f}/\epsilon)}\big),$$

where $\alpha > 0$ and the constant c is independent of ϵ.

Proof. Let c be a constant independent of ϵ. Let $k \in \mathbb{N}$ satisfy $\epsilon/\theta^k \leqslant \epsilon_0 < \epsilon/\theta^{k+1}$, where θ is same as that in Lemma 6.5. By Lemma 6.5,

$$\sup_{B_{\epsilon/\epsilon_0}(0)} \big|\Pi_\epsilon U_\epsilon(x) - \Pi_\epsilon U_\epsilon(0) - \epsilon \mathbf{a}^{\epsilon}_k - \big(x + \Pi_\epsilon \mathbb{X}_\epsilon(x)\big)\mathbf{b}^{\epsilon}_k\big| \leqslant c\left|\frac{\epsilon}{\epsilon_0}\right|^{1+\frac{\mu}{2}} \mathcal{J}_{*,\epsilon}. \tag{6.26}$$

Define, in $B_{1/\epsilon_0}(0) \cap \Omega^\epsilon_f/\epsilon$,

$$\begin{cases} \hat{U}_\epsilon(x) \equiv \epsilon^{-(1+\mu/2)}\mathcal{J}^{-1}_{*,\epsilon}\big(U_\epsilon(\epsilon x) - \Pi_\epsilon U_\epsilon(0) - \epsilon\mathbf{a}^{\epsilon}_k - \big(\epsilon x + \mathbb{X}_\epsilon(\epsilon x)\big)\mathbf{b}^{\epsilon}_k\big), \\ \hat{Q}_\epsilon(x) \equiv \epsilon^{-\mu/2}\mathcal{J}^{-1}_{*,\epsilon} Q_\epsilon(\epsilon x), \\ \hat{F}_\epsilon(x) \equiv \epsilon^{1-\mu/2}\mathcal{J}^{-1}_{*,\epsilon} F_\epsilon(\epsilon x). \end{cases}$$

Then $\hat{U}_\epsilon$, $\hat{Q}_\epsilon$, $\hat{F}_\epsilon$ satisfy (6.1) in $B_{1/\epsilon_0}(0) \cap \Omega^\epsilon_f/\epsilon$ (or see (6.13)) by (2.7). By (6.26),

$$\|\hat{U}_\epsilon\|_{L^\infty(B_{1/\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon)} + \|\hat{F}_\epsilon\|_{L^{n+\delta}(B_{1/\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon)} + \|\hat{Q}_\epsilon\|_{C^{0,\alpha}(\overline{B_{1/\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon})} \leqslant c.$$

By Theorem 6.30 and Theorem 9.19 [11],

$$\|\hat{U}_\epsilon\|_{W^{1,\infty}(B_{1/2\epsilon_0}(0)\cap\Omega^\epsilon_f/\epsilon)} \leqslant c.$$

Since $\nabla\hat{U}_\epsilon(0) = \frac{\nabla U_\epsilon(0) - (I+\nabla\mathbb{X}(0))\mathbf{b}^{\epsilon}_k}{\epsilon^{\mu/2}\mathcal{J}_{*,\epsilon}}$,

$$\big|\nabla U_\epsilon(0)\big| = \big|\big(I + \nabla\mathbb{X}(0)\big)\mathbf{b}^{\epsilon}_k + \epsilon^{\mu/2}\mathcal{J}_{*,\epsilon}\nabla\hat{U}_\epsilon(0)\big| \leqslant c\mathcal{J}_{*,\epsilon}. \tag{6.27}$$

Since one can shift the origin to any point in Ω^ϵ_f and gets the same result as (6.27), we conclude $\|\nabla U_\epsilon\|_{L^\infty(B_{1/2}(0)\cap\Omega^\epsilon_f)} \leqslant c\mathcal{J}_{*,\epsilon}$. □

By Lemma 6.6, we know that for any $\delta > 0$, there is a constant $\epsilon_0 \in (0,1)$ such that, for $\epsilon \in (0,\epsilon_0)$, any solution of (3.2) satisfies (3.28). By $(3.8)_2$, we also know that for any $\delta > 0$ and $\epsilon \in [\epsilon_0, 1]$, any solution of (3.2) satisfies (3.28). So we prove Lemma 3.4.

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