

New Constructions of Zero-Correlation Zone Sequences

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Abstract—In this paper, we propose three classes of systematic approaches for constructing zero-correlation zone (ZCZ) sequence families. In most cases, these approaches are capable of generating sequence families that achieve the upper bounds on the family size (K) and the ZCZ width (T) for a given sequence period (N). Our approaches can produce various binary and polyphase ZCZ families with desired parameters (N, K, T) and alphabet size. They also provide additional tradeoffs amongst the above four system parameters and are less constrained by the alphabet size. Furthermore, the constructed families have nested-like property that can be either decomposed or combined to constitute smaller or larger ZCZ sequence sets. We make detailed comparisons with related works and present some extended properties. For each approach, we provide examples to numerically illustrate the proposed construction procedure.

Index Terms—Hadamard matrix, mutually orthogonal complementary set of sequences, periodic correlation, upsampling, zero-correlation zone (ZCZ) sequence.

I. INTRODUCTION

FAMILIES of sequences with some desired periodic or aperiodic autocorrelation (AC) and cross-correlation (CC) properties are useful in communication and radar systems [1]–[11] for applications in identification, synchronization, ranging, or/and interference mitigation. For example, to minimize the multiple access interference (MAI) and self-interference (e.g., intersymbol interference) in a multiuser, multipath environment or to avoid inter-antenna interference in a multiple-input, multiple-output system, one would like to have an *ideal sequence set* whose periodic AC functions are nonzero only at the zeroth correlation lag ($\tau = 0$) and whose pairwise periodic CC values are identically zero at any τ for all pairs of sequences. Similar aperiodic properties are called for in designing pulse compressed radar signal or 2-D array waveforms to have an impulse-like ambiguity function satisfying the resolution requirements.

Unfortunately, the ideal sequence set does not exist, i.e., it is impossible to have impulse-like AC functions and zero CC functions simultaneously in a sequence set. In fact, bounds on

the magnitude of CC and AC values derived in [2] and [3] suggest that the design of sequence sets involves the tradeoff between AC and CC values. An alternate compromise is to require that the ideal AC and CC properties be maintained only at correlation lags within a window called zero-correlation zone (ZCZ) [1]. Sequences with such properties are known as ZCZ sequences. Little or no system performance degradation results if the correlation values outside the ZCZ are immaterial to the application of concern. For example, if the maximum channel delay spread T_m and the maximum distance between a base station and co-channel users D_m are known, a direct sequence spread spectrum-based multiple access system using a family of ZCZ sequences with ZCZ width $|\tau| \leq T_m + 2D_m/c$, where c is the speed of light, will be able to suppress MAI and multipath self interference.

Other than the restrictions on the magnitude of correlation values, practical implementation concerns prefer that the choice of the sequence period be flexible and the family size be as large as possible while keeping the desired AC and CC properties intact. One also hope that the elements of the sequences be drawn from an alphabet set as small as possible. ZCZ sequences with these desired properties have found applications in code division multiple access (CDMA) [1], code-hopping CDMA [4], orthogonal frequency division multiplexing [5], multiantenna [6], free-space optical [7], underwater acoustic [8], ultrawideband [9], inverse synthetic aperture radar (ISAR) imaging [10], and radar sensing [11] systems. They are used for spectrum-spreading [4], [8], [9], [12] and pulse compression [11], and as pilot sequences for channel estimation [5], timing synchronization [6], ranging [13], and frequency offset estimation [14].

Numerous ZCZ sequence generation methods have been proposed. The methods presented in [15]–[17] are based on complementary sequence sets, while in [18]–[20] some novel interleaving techniques were shown to be effective. They can be generalized to construct 2-D ZCZ arrays [21], [22] as well. Sets of ZCZ sequences derived from manipulating perfect sequences were suggested, among others, in [23]–[25]. Park *et al.* [12] construct sequences that has nonzero AC only at subperiodic correlation lags and zero CC across all lags. By requiring the transform domain sequences to satisfy some special properties, the authors in [14] and [26] present methods that generate sequences having zero CC across all lags. Some ZCZ sequence sets can be partitioned into smaller subsets so that the zero-CC zone of any two sequences drawn from different subsets is wider than that among intrasubset sequences. Ternary or polyphase sequences with such a property have been constructed via interleaving techniques [27], [28], and in [29] and [30], quadrature amplitude-modulated sequences are shown to be derivable from binary or ternary sequences.

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In this paper, we present three systematic approaches for generating families of sequences whose periodic AC and CC functions satisfy a variety of ZCZ requirements. While some known ZCZ sequence construction methods employ Hadamard matrices in time domain (e.g., [20] and [25]), our first approach uses such matrices to meet the desired transform domain properties of a ZCZ sequence set instead. Sequence sets generated from this approach are, by construction, optimal in the sense that the upper bounds for family sizes and ZCZ widths are achieved. We further employ a filtering operation to convert sequences of nonconstant modulus symbols into polyphase ones without changing the correlation properties.

Based upon a basic binary sequence (to be defined in Section IV) whose AC function satisfies the ZCZ requirement, the second approach generates ZCZ sequence families by a special nonuniform upsampling on unitary matrices. The construction of basic sequences seems trivial and straightforward, but from these simple sequences we are able to synthesize desired polyphase ZCZ sequences through some refining steps that include nonuniform upsampling and filtering.

Our third approach invokes the notion of complementary set of sequences [31], [32]. It bears the flavor of the second approach and makes use of a basic binary sequence which meets the ZCZ constraint as well as a collection of mutually orthogonal complementary sets. While this method is capable of generating binary sequences with sequence parameters identical to those given in [15] and [17], it can also produce polyphase sequence sets that are unobtainable by the conventional complementary set-based approaches.

The rest of this paper is organized as follows. We introduce basic definitions and properties related to our investigation in the next section. Section III begins with a brief summary of important transform domain properties, followed by the analysis and synthesis of the proposed transform domain approach. We then show some ZCZ sequence sets generated by the transform domain method in Section III-D. A class of direct synthesis methods is presented in Section IV and construction examples are given in Section IV-E. In Section V, a complementary sequence set-based extension of the second approach is proposed, followed by numerical construction examples given in Section V-D. For each proposed approach, we tabulate the parametric constraint comparisons with related methods. More detailed comparisons and discussions are given in the form of remarks. Finally, some concluding notes are provided in Section VI.

II. DEFINITIONS AND FUNDAMENTAL PROPERTIES

Definition 1: An (N, K) sequence set \mathbf{X} is a set of K sequences of period N .

Definition 2: The periodic CC function of two period- N sequences $u \equiv \{u(n)\}$ and $v \equiv \{v(n)\}$ is defined as

$$\theta_{uv}(\tau) = \sum_{n=0}^{N-1} u(n)v^*(n-\tau) = u(\tau) \circledast v^*(-\tau) \quad (1)$$

where \circledast denotes the circular convolution.

Thus, the periodic AC function of sequence u is simply $\theta_{uu}(\tau)$. Since these CC and AC functions are also of period N , to simplify the discussion, we shall, throughout in this paper, limit the representations and examples of sequences or sequence sets to a single period ($0 \leq \tau \leq N-1$) unless necessary.

Definition 3: A sequence $\{u(n)\}$ that has an impulse-like (or ideal) AC function, i.e., $\theta_{uu}(\tau) = \theta_{uu}(0)\delta(\tau)$, is called a *perfect sequence*, where

$$\delta(\tau) \stackrel{\text{def}}{=} \begin{cases} 1, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases} \quad (2)$$

is the Dirac delta function.

Definition 4: A sequence $\{u_v(n)\}$ is said to be obtained from *filtering* the sequence $u = \{u(n)\}$ by the sequence $v = \{v(n)\}$ of the same period if

$$u_v(n) \stackrel{\text{def}}{=} u(n) \circ v(n) \stackrel{\text{def}}{=} u(n) \circledast v^*(-n) \equiv \theta_{uv}(n). \quad (3)$$

Definition 5: An (N, K) sequence set, $\mathbf{C} = \{C_0, C_1, \dots, C_{K-1}\}$ is called an (N, K, T) ZCZ sequence family (or set) if $\forall C_i, C_j \in \mathbf{C}, i \neq j, \theta_{C_i C_j}(\tau) = 0$ and $\theta_{C_i C_i}(\tau) = \theta_{C_i C_i}(0)\delta(\tau), |\tau|_N \leq T < N$ where T is the ZCZ width and $|k|_N \stackrel{\text{def}}{=} k \bmod N$.

In [33], the following is proved.

Lemma 1: The sequence period N , cardinality K , and ZCZ width T of an (N, K, T) ZCZ family must satisfy the inequality

$$K(T+1) \leq N. \quad (4)$$

For ± 1 -valued binary sequence set, the bound becomes tighter [18]

$$KT \leq \frac{N}{2}, \quad K > 1. \quad (5)$$

This lemma describes the fundamental tradeoff among the sequence period, family size, and ZCZ width. For a fixed N , increasing the family size must be achieved at the cost of reduced ZCZ width and vice versa. Note that for a set with a single perfect sequence, (4) is automatically satisfied because $K = 1$ and $T = N - 1$.

Definition 6: An $N \times N$ matrix \mathbf{U} is called a *Hadamard matrix* of order N if and only if it satisfies two conditions.

- (i) **Unimodularity:** the components of \mathbf{U} are of the same magnitude \sqrt{N} .
- (ii) **Orthogonality:** $\mathbf{U}\mathbf{U}^H = N\mathbf{I}_N$ where \mathbf{I}_N is the $N \times N$ identity matrix and $(\cdot)^H$ denotes the conjugate transpose of the enclosed matrix.

Definition 7: The matrix

$$\mathbf{F}_M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)} \\ 1 & W_M^{-2} & W_M^{-4} & \cdots & W_M^{-2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)^2} \end{bmatrix} \quad (6)$$

is called the M -discrete Fourier transform (M -DFT) matrix, where $W_M^k = e^{j2\pi k/M}$, and its Hermitian $\mathbf{F}_M^H = \mathbf{F}_M^{-1}$ is called the inverse M -DFT (M -IDFT) matrix. The set of complex M th roots of unity, $\{W_M^k : k = 0, 1, \dots, M-1\}$, is called the M -ary phase-shift keying (M -PSK) set and a sequence with elements from the M -PSK constellation is called an M -PSK sequence or a polyphase sequence in general.

Note that DFT matrices form a subclass of the so-called Butson Hadamard matrices [34].

Definition 8: The k th Kronecker power of matrix \mathbf{U} , denoted by $\otimes^k \mathbf{U}$, is defined as

$$\otimes^k \mathbf{U} = \underbrace{\mathbf{U} \otimes \mathbf{U} \otimes \dots \otimes \mathbf{U}}_{\mathbf{U} \text{ appears } k \text{ times}} \quad (7)$$

where \otimes denotes the Kronecker product.

Definition 9: The matrices

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (8)$$

and

$$\mathbf{H}_{2^n} = \otimes^n \mathbf{H}_2 = \begin{bmatrix} \mathbf{H}_{2^{n-1}} & \mathbf{H}_{2^{n-1}} \\ \mathbf{H}_{2^{n-1}} & -\mathbf{H}_{2^{n-1}} \end{bmatrix}, \quad n = 2, 3, \dots \quad (9)$$

are called *Sylvester Hadamard matrices*.

The following lemma is essential to derive our construction methods in Section III.

Lemma 2 [34]: The Kronecker (tensor) product of any two Hadamard matrices is a Hadamard matrix.

III. TRANSFORM DOMAIN CONSTRUCTION METHODS

We first review some transform domain properties of sequences and their correlation functions. A class of ZCZ sequence construction approaches based on transform domain properties is then presented. Detailed comparisons with two related proposals are made and a few construction examples are provided.

A. Useful Transform Domain Properties

Denote by $\text{DFT}\{u(n)\}$, the DFT of a periodic sequence $\{u(n)\}$ and by $\text{IDFT}\{U(k)\}$, the inverse DFT (IDFT) of a periodic transform domain sequence $\{U(k)\}$. We then immediately have

Lemma 3 [14]: The DFT of the CC function $\theta_{uv}(\tau)$ of two period- N sequences, $\{u(n)\}$ and $\{v(n)\}$, is equal to $U(k)V^*(k)$, where $\{U(k)\} = \text{DFT}\{u(n)\}$ and $\{V(k)\} = \text{DFT}\{v(n)\}$.

Since the AC function of $\{u(n)\}$ can be expressed as $\theta_{uu}(n) = u(n) \otimes u^*(-n)$, its DFT is given by $\Theta_{uu}(k) = |U(k)|^2$. Therefore, it is straightforward to show the following.

Corollary 1 [14]: Sequence $\{u(n)\}$ is a perfect sequence if and only if $|U(k)|^2$ is constant for all k .

Based on the aforementioned properties, we can easily prove the following.

Lemma 4: The AC and CC functions of a set of sequences are invariant (up to a scaling factor) to filtering if the filtering sequence v is a perfect sequence.

It will become clear in subsequent sections that this lemma makes the filtering operator very useful in transforming a sequence set into one with entries of the sequences taken from a desired constellation while maintaining the correlation properties.

B. Basic Constructions

Definition 10: A sequence $\{u(n)\}$ in an (N, K) sequence set is said to have a *subperiod* of J , where $J|N$, if it is also periodic with period $J < N$, i.e., $u(n) = u(\ell J + n)$, for $0 \leq \ell < N/J$ and $0 \leq n < J$.

Now note that Lemma 3 implies

$$\theta_{uv}(\tau) = \sum_{k=0}^{N-1} \Theta_{uv}(k) e^{j2\pi\tau k} = \sum_{k=0}^{N-1} U(k)V^*(k) e^{j2\pi\tau k} \quad (10)$$

where $\Theta_{uv}(k) = \text{DFT}\{\theta_{uv}(\tau)\}$. When $\{U(k)\}$ and $\{V(k)\}$, regarded as N -dimensional vectors, are orthogonal, we have

$$\theta_{uv}(0) = \sum_{k=0}^{N-1} \Theta_{uv}(k) = \sum_{k=0}^{N-1} U(k)V^*(k) = 0. \quad (11)$$

If the sequence $\{\Theta_{uv}(k)\}$ has a subperiod of $J = \frac{N}{m}$, then

$$\begin{aligned} \theta_{uv}(\tau) &= \sum_{k=0}^{J-1} \Theta_{uv}(k) e^{j2\pi\tau k} + \sum_{k=J}^{2J-1} \Theta_{uv}(k) e^{j2\pi\tau k} + \dots \\ &+ \sum_{k=(m-1)J}^{N-1} \Theta_{uv}(k) e^{j2\pi\tau k} \\ &= \sum_{k=0}^{J-1} \Theta_{uv}(k) e^{j2\pi\tau k} \left(1 + e^{j2\pi\tau \frac{J}{m}} + \dots + e^{j2\pi\tau \frac{(m-1)J}{m}} \right). \end{aligned}$$

The identity

$$1 + \alpha + \alpha^2 + \dots + \alpha^{m-1} = 0, \quad \forall \alpha = W_m^{\tau}, \quad |\tau|_m \neq 0 \quad (12)$$

then gives the following.

Lemma 5: The CC function $\theta_{uv}(\tau)$ of two period- N sequences $\{u(n)\}$ and $\{v(n)\}$ is identical zero $\forall |\tau|_N \leq T$ if the associated DFT vectors $\{U(k)\}$ and $\{V(k)\}$ are orthogonal and their Hadamard product, $\{U(k)V^*(k)\}$, has a subperiod of $J = N/(T+1)$, where T is a positive integer.

The recursive Kronecker construction of the Sylvester Hadamard matrices (9) gives at least two sets of row vectors (i.e., upper and lower half parts of \mathbf{H}_{2^n}) that satisfy both the orthogonality and subperiodicity requirements. This property still holds when we replace Sylvester Hadamard matrices by other classes of Hadamard matrices produced by a recursive Kronecker construction similar to (9). Furthermore, as elements of a Hadamard matrix have constant modulus, the AC of all sequences derived by taking IDFT on rows of a Hadamard matrix is 0 for all nonzero correlation lags by Corollary 1. These two observations suggest that ZCZ families can be obtained by using proper subsets of row vectors from a Hadamard matrix. To have a precise definition of ‘‘proper subsets,’’ we need the following.

Definition 11: A regular p th-order M -partition on an $N \times N$ matrix \mathbf{H} , where $N = M^n$, is the set of $m = N/K = M^p K \times$

N submatrices, each formed by nonoverlapping $K = M^{n-p}$ consecutive rows of \mathbf{H} .

Proper subsets of row vectors that generate ZCZ families are obtained by performing p th-order M -partition on the n th Kronecker power of a Hadamard matrix.

Lemma 6: Let \mathbf{U} be a Hadamard matrix of order M and \mathbf{H} be the Hadamard matrix of order N generated by the n th Kronecker power of \mathbf{U} , i.e.,

$$\mathbf{H} = [\mathbf{h}_0^T, \mathbf{h}_1^T, \dots, \mathbf{h}_{N-1}^T]^T = \otimes^n \mathbf{U} \quad (13)$$

where $N = M^n$, $n \geq 2$, and \mathbf{h}_ℓ is the ℓ th row¹ of \mathbf{H} . We perform a regular p th-order M -partition on \mathbf{H} to obtain the $m = M^p$ submatrices

$$\tilde{\mathbf{H}}_i = [\mathbf{h}_{iK}^T, \dots, \mathbf{h}_{(i+1)K-1}^T]^T, \quad i = 0, 1, \dots, m-1. \quad (14)$$

Then, for each i , the set of K length- N sequences $\mathbf{A}_i \stackrel{def}{=} \{A_{i,0}, A_{i,1}, \dots, A_{i,K-1}\}$, where $A_{i,j} = \text{IDFT}\{\mathbf{h}_{iK+j}\}$, is an $(N, K, m-1)$ ZCZ sequence family that achieves the upper bound (4). Furthermore, all member sequences in the family are perfect sequences.

Proof: The matrix \mathbf{H} can be expressed in the stacked form, $\mathbf{H} = [\tilde{\mathbf{H}}_0^T, \tilde{\mathbf{H}}_1^T, \dots, \tilde{\mathbf{H}}_{m-1}^T]^T$, where the submatrix $\tilde{\mathbf{H}}_i$ is of the form

$$[a_{i,0}\mathbf{B}, a_{i,1}\mathbf{B}, \dots, a_{i,m-1}\mathbf{B}]$$

where $a_{i,j}$'s have unit magnitudes and $\mathbf{B} = \otimes^{n-p} \mathbf{U}$. It follows immediately that the Hadamard products of two distinct rows of $\tilde{\mathbf{H}}_i$ has a period of $M^{n-p} = K$. ■

The aforementioned construction gives ZCZ sequences of length M^n , $n \geq 2$. That the upper bound (4) is achieved is a result of our partition method described by Definition 11. The sequence length constraint can be relaxed by using Kronecker construction of Hadamard matrices of different orders. Using Lemma 2 and an argument similar to that in deriving the aforementioned lemma, we obtain

Theorem 1: Let \mathbf{H} be the $N \times N$ Hadamard matrix

$$\mathbf{H} = [\mathbf{h}_0^T, \mathbf{h}_1^T, \dots, \mathbf{h}_{N-1}^T]^T \stackrel{def}{=} \mathbf{U}_{n-1} \otimes \dots \otimes \mathbf{U}_0 \quad (15)$$

where \mathbf{U}_k , $k = 0, 1, \dots, n-1$, are $M_k \times M_k$ (not necessarily distinct) Hadamard matrices and $N = \prod_{k=0}^{n-1} M_k$, $n \geq 2$. Partition \mathbf{H} into $m = \frac{N}{K}$ submatrices of size $K \times N$

$$\tilde{\mathbf{H}}_i = [\mathbf{h}_{iK}^T, \dots, \mathbf{h}_{(i+1)K-1}^T]^T, \quad i = 0, 1, \dots, m-1 \quad (16)$$

each formed by nonoverlapping $K = \prod_{k=0}^{n-p-1} M_k$ consecutive rows of \mathbf{H} with $p > 0$. Then, for each i , the set of K period- N sequences $\mathbf{A}_i \stackrel{def}{=} \{A_{i,0}, A_{i,1}, \dots, A_{i,K-1}\}$, where $A_{i,j} = \text{IDFT}\{\mathbf{h}_{iK+j}\}$, is an $(N, K, m-1)$ ZCZ sequence family that achieves the upper bound (4).²

¹For convenience, all the column, row, and vector elements' indices start with 0 instead of 1.

²Technically, the theorem is also valid for $p = 0$, as the resulting set has a ZCZ width 0. We will implicitly ignore this trivial case and assume $p > 0$ in the subsequent discussion.

Note that the recursive generation of Hadamard matrices defined by (9) and (13) is a special case of (15), i.e., the aforementioned theorem generalize Theorems 1 and 2 of [35].

C. Polyphase ZCZ Sequences

The ZCZ sequences generated by Lemma 6 and Theorem 1 are not necessary of constant modulus but can be converted into polyphase sequences without altering the desired AC and CC properties by a proper filtering process; see Definition 4 and Lemma 4. To find the filtering perfect sequences, we need the following two properties.

Lemma 7 [36]: Let U be a length- N polyphase perfect sequence with entries drawn from the N -PSK constellation. Then, both $\text{IDFT}\{U\}$ and $\text{DFT}\{U\}$ are polyphase perfect sequences.

Lemma 8 [37]: Let L be a natural number and $N = L^2$. Define the length- N polyphase sequence $\{u(k)\}$ by

$$u(k_1L + k_2) = W_L^{\beta(k_2)k_1 + r(k_2)}, \quad 0 \leq k_1, k_2 < L \quad (17)$$

where $\{\beta(k_2) : k_2 = 0, 1, \dots, L-1\}$ is a permutation of $\{0, 1, \dots, L-1\}$, and $r(k_2)$ is a rational number depending on k_2 . Then, the sequences $\{u(k)\}$,

$$\{e^{j\theta_{k_2}} u(k_1L + k_2) : 0 \leq \theta_{k_2} < 2\pi, 0 \leq k_1, k_2 < L\} \quad (18)$$

and

$$\{W_L^{\ell k_1} u(k_1L + k_2) : 0 \leq k_1, k_2 < L\} \text{ for any integer } \ell \quad (19)$$

are all polyphase perfect sequences.

Based on the aforementioned results, we propose a transform domain construction of polyphase ZCZ sequences as follows.

Corollary 2: Let \mathbf{u} be a length- N perfect sequence of the form (17), $N = \prod_{k=0}^{n-1} M_k = L^2$ for some L , and $\tilde{\mathbf{H}}_i$ be the i th submatrix defined by (15) and (16) using M_k -DFT or M_k -IDFT matrices \mathbf{U}_k 's. Then, $\mathbf{C}_i = \{\text{IDFT}\{\mathbf{h}_{iK+n}\} \circ \text{IDFT}\{\mathbf{u}\} : 0 \leq n \leq K-1\}$ is an $(N, K, \frac{N}{K}-1)$ bound-achieving polyphase ZCZ sequence set.

Proof: Since the entries in the n th row of $\tilde{\mathbf{H}}_i$ render the general expression

$$[\mathbf{H}]_{iK+n, k_1L+k_2} \stackrel{def}{=} h_{iK+n}(k_1L+k_2) = e^{j\theta_{k_2}(n)} W_L^{\ell(n)k_1}$$

for $0 \leq k_1, k_2 < L$, where $\ell(n) \in \mathbb{Z}$ (integers) and $0 \leq \theta_{k_2}(n) < 2\pi$, the products $h_{iK+n}(k)u^*(k)$ are of the forms (17) and (19) and are integer powers of W_N . Lemmas 7 and 8 imply that the sequence

$$\begin{aligned} C_{i,n}(k) &= \text{IDFT}\{h_{iK+n}(k)\} \circ \text{IDFT}\{u(k)\} \\ &= \text{IDFT}\{h_{iK+n}(k)u^*(k)\} \end{aligned}$$

has polyphase entries. Invoking Theorem 1 and Lemma 4, we conclude that $\{C_{i,n} : 0 \leq n < K\}$ is an $(N, K, \frac{N}{K}-1)$ polyphase ZCZ family. ■

Remark 1 (Polyphase Constraint and Sequence Length Selection): Theorem 1 provides a general transform domain approach using Hadamard matrices to construct bound-achieving sets of arbitrary nonprime length ZCZ sequences. In contrast,

TABLE I
TRANSFORM DOMAIN-BASED POLYPHASE ZCZ SEQUENCE SETS

	Tsai [14]	Brodzik [26]		Corollary 2
Sequence length N	$n_1 n_2$	L^3		$\prod_{k=0}^{n-1} M_k = L^2$
Set size K	n_2	L		$\prod_{k=0}^{n-p-1} M_k$
ZCZ width T	$n_1 - 1$	$L^2 - 1$, L prime	$L - 1$, L nonprime	$\frac{N}{K} - 1$
Upper-bound (4) achieved?	Yes	Yes	No	Yes
Perfect sequence used	Length- n_1 , n_p -phase	No explicit use of perfect sequences		Length- N
Alphabet size	$\text{lcm}(N, n_p)$	N		N

Corollary 2 focuses on the generation of polyphase sequences and can be regarded as an extension of a special case of the former. The polyphase requirement is satisfied by invoking an additional filtering operation and the use of special Hadamard matrices; see Example 1 in the ensuing section. As a result, the choice of the sequence length is limited to perfect squares ($N = L^2$).

Remark 2 (Nested Structure): Every $K \times N$ submatrix $\tilde{\mathbf{H}}_i$ can be further partitioned into $K/K' = \prod_{k=n-p}^{n-p-1} M_k$ submatrices of size $K' \times N$, where $p < p' < n$, $K = \prod_{k=0}^{n-p-1} M_k$, and $K' = \prod_{k=0}^{n-p'-1} M_k$ so that each submatrix can be used to construct an $(N, K', \frac{N}{K'} - 1)$ ZCZ sequence set \mathbf{C}_i^j with larger ZCZ width and $\bigcup_{j=0}^{K'-1} \mathbf{C}_i^j = \mathbf{C}_i$. This partition can be done in a nested manner, i.e., each subset can be further decomposed to render even smaller sequence subsets or $\tilde{\mathbf{H}}_i$ can be merged with proper neighboring submatrices to construct a larger set.

Remark 3 (Tradeoff Between AC and CC): The identity (12) actually gives a stronger CC property than what is specified by the ZCZ width; it implies that the CC values are identically zero except at $\tau = s(T+1)$, $s \in \mathbb{Z}$. This is still weaker than the constructions of [14] and [26] which yield perfect (zero) CC at all lags. Perfect CC is achieved by requiring that each transform domain sequence has sparse nonzero elements and support (set of the nonzero coordinates) disjoint from the supports of all other transform domain sequences. Nevertheless, their AC functions are not as good as ours as all the sequences constructed by our approach are perfect sequences.

Remark 4 (Tradeoff Between Sequence Length and Alphabet Size): Tsai's approach [14] is more flexible in the choice of sequence length but requires a very large constellation for elements of the sequences. Our approach, on the other hand, requires the smallest constellation and is more flexible than [26] in selecting the sequence length N .

We summarize various parameter constraints for our approach, [14], and [26] in Table I.

D. ZCZ Sequence Sets Generated by Transform Domain Approach

In this section, we present some construction examples using the proposed transform domain method. All ZCZ sequences obtained are perfect sequences. To minimize the number of notations, we use \mathbf{C}_i and \mathbf{A}_i to denote sequences generated by the methods of Corollary 2 and Theorem 1, respectively. The same

notation may refer to different sequences in different examples when there is no danger of ambiguity.

Example 1 (Use of Three DFT Matrices of Unequal Dimensions): Partitioning the Hadamard matrix $\mathbf{H} = \mathbf{F}_6 \otimes \mathbf{F}_3 \otimes \mathbf{F}_2$ into submatrices $\tilde{\mathbf{H}}_0, \tilde{\mathbf{H}}_1, \dots, \tilde{\mathbf{H}}_{17}$ and performing IDFT on the rows of $\tilde{\mathbf{H}}_{10}$, we obtain two sequences

$$\begin{aligned} \mathbf{A}_0 &= (000W_{12}^{21}00000W_{12}^700000W_{12}^500000W_{12}^{15}00000 \\ &\quad W_{12}^100000W_{12}^{23}00) \\ \mathbf{A}_1 &= (000W_{12}^{15}00000W_{12}^100000W_{12}^{23}00000W_{12}^{21}00000 \\ &\quad W_{12}^700000W_{12}^500) \end{aligned}$$

To convert them into ones with constant moduli, we filter them by the perfect polyphase sequence [14]

$$\begin{aligned} U_{36} &= (W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^5 W_6^4 W_6^3 W_6^2 W_6^1 \\ &\quad W_6^0 W_6^4 W_6^2 W_6^0 W_6^4 W_6^2 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 \\ &\quad W_6^0 W_6^2 W_6^4 W_6^0 W_6^2 W_6^4 W_6^0 W_6^1 W_6^2 W_6^3 W_6^4 W_6^5) \end{aligned} \quad (20)$$

which satisfies (17). The resulting (36, 2, 17) bound-achieving ZCZ sequence set consists of

$$\begin{aligned} \mathbf{C}_0 &= \mathbf{A}_0 \circ U_{36} = (W_{12}^{11} W_{12}^8 W_{12}^9 W_{12}^0 W_{12}^3 W_{12}^2 W_{12}^5 W_{12}^4 W_{12}^7 \\ &\quad W_{12}^0 W_{12}^5 W_{12}^6 W_{12}^{11} W_{12}^{12} W_{12}^5 W_{12}^0 W_{12}^7 W_{12}^{10} \\ &\quad W_{12}^5 W_{12}^8 W_{12}^3 W_{12}^0 W_{12}^9 W_{12}^2 W_{12}^{11} W_{12}^4 W_{12}^1 \\ &\quad W_{12}^0 W_{12}^{11} W_{12}^6 W_{12}^5 W_{12}^{12} W_{12}^{11} W_{12}^0 W_{12}^1 W_{12}^{10}) \\ \mathbf{C}_1 &= \mathbf{A}_1 \circ U_{36} = (W_{12}^5 W_{12}^8 W_{12}^3 W_{12}^0 W_{12}^9 W_{12}^2 W_{12}^{11} W_{12}^4 W_{12}^1 \\ &\quad W_{12}^0 W_{12}^{11} W_{12}^6 W_{12}^5 W_{12}^{12} W_{12}^{11} W_{12}^0 W_{12}^1 W_{12}^{10} \\ &\quad W_{12}^{11} W_{12}^8 W_{12}^9 W_{12}^0 W_{12}^3 W_{12}^2 W_{12}^5 W_{12}^4 W_{12}^7 \\ &\quad W_{12}^0 W_{12}^5 W_{12}^6 W_{12}^{11} W_{12}^{12} W_{12}^5 W_{12}^0 W_{12}^7 W_{12}^{10}). \end{aligned}$$

If instead we take IDFT on the rows of the first submatrix $\tilde{\mathbf{G}}_0$ of $\mathbf{G} = [\tilde{\mathbf{G}}_0^T, \tilde{\mathbf{G}}_1^T, \dots, \tilde{\mathbf{G}}_{11}^T]^T = \mathbf{F}_2 \otimes \mathbf{F}_6 \otimes \mathbf{F}_3$ and filter the resulting sequences $\{\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2\}$ through (20), we obtain the bound-achieving (36, 3, 11) set

$$\begin{aligned} \mathbf{C}_0 &= \mathbf{A}_0 \circ U_{36} = (W_6^0 W_6^1 W_6^2 W_6^3 W_6^4 W_6^5 W_6^0 W_6^2 W_6^4 \\ &\quad W_6^0 W_6^2 W_6^4 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 \\ &\quad W_6^0 W_6^4 W_6^2 W_6^0 W_6^4 W_6^2 W_6^0 W_6^5 W_6^4 \\ &\quad W_6^3 W_6^2 W_6^1 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0) \\ \mathbf{C}_1 &= \mathbf{A}_1 \circ U_{36} = (W_6^0 W_6^5 W_6^4 W_6^3 W_6^2 W_6^1 W_6^0 W_6^0 W_6^0 \\ &\quad W_6^0 W_6^0 W_6^0 W_6^0 W_6^1 W_6^2 W_6^3 W_6^4 W_6^5 \\ &\quad W_6^0 W_6^2 W_6^4 W_6^0 W_6^2 W_6^4 W_6^0 W_6^3 W_6^0 \\ &\quad W_6^3 W_6^0 W_6^3 W_6^0 W_6^4 W_6^2 W_6^0 W_6^4 W_6^2) \\ \mathbf{C}_2 &= \mathbf{A}_2 \circ U_{36} = (W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^4 W_6^2 \\ &\quad W_6^0 W_6^4 W_6^2 W_6^0 W_6^5 W_6^4 W_6^3 W_6^2 W_6^1 \\ &\quad W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^1 W_6^2 \\ &\quad W_6^3 W_6^4 W_6^5 W_6^0 W_6^2 W_6^4 W_6^0 W_6^2 W_6^4). \end{aligned}$$

Example 2 (Construction Based on Kronecker Power of a DFT Matrix): Let $\mathbf{H} = \mathbf{F}_3 \otimes \mathbf{F}_3 \otimes \mathbf{F}_3 \otimes \mathbf{F}_3$ and denote by $\tilde{\mathbf{H}}_0, \tilde{\mathbf{H}}_1, \dots, \tilde{\mathbf{H}}_{26}$ the submatrices obtained by performing regular third-order 3-partition on \mathbf{H} . Choosing $\tilde{\mathbf{H}}_2$ and performing

IDFT on its rows, we obtain sequences $\{A_0, A_1, A_2\}$. Filtering them by polyphase perfect sequence

$$U_{81} = (W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^8 \\ W_9^7 W_9^6 W_9^5 W_9^4 W_9^3 W_9^2 W_9^1 W_9^0 W_9^7 W_9^5 W_9^3 \\ W_9^1 W_9^8 W_9^6 W_9^4 W_9^2 W_9^0 W_9^6 W_9^3 W_9^0 W_9^6 W_9^3 \\ W_9^0 W_9^6 W_9^3 W_9^0 W_9^5 W_9^1 W_9^6 W_9^2 W_9^7 W_9^3 W_9^8 \\ W_9^4 W_9^0 W_9^4 W_9^8 W_9^3 W_9^7 W_9^2 W_9^6 W_9^1 W_9^5 W_9^0 \\ W_9^3 W_9^6 W_9^0 W_9^3 W_9^6 W_9^0 W_9^3 W_9^6 W_9^0 W_9^2 W_9^4 \\ W_9^6 W_9^8 W_9^1 W_9^3 W_9^5 W_9^7 W_9^0 W_9^1 W_9^2 W_9^3 W_9^4 \\ W_9^5 W_9^6 W_9^7 W_9^8)$$

we obtain

$$C_0 = A_0 \circ U_{81} = (W_9^0 W_9^1 W_9^2 W_9^6 W_9^7 W_9^8 W_9^3 W_9^4 W_9^5 W_9^0 W_9^2 \\ W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^3 W_9^6 W_9^3 \\ W_9^6 W_9^0 W_9^6 W_9^0 W_9^3 W_9^0 W_9^4 W_9^8 W_9^6 W_9^1 W_9^5 \\ W_9^3 W_9^7 W_9^2 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 \\ W_9^1 W_9^0 W_9^6 W_9^3 W_9^3 W_9^0 W_9^6 W_9^6 W_9^3 W_9^0 W_9^0 \\ W_9^7 W_9^5 W_9^6 W_9^4 W_9^2 W_9^3 W_9^1 W_9^8 W_9^0 W_9^8 W_9^7 \\ W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^0 W_9^0 W_9^3 W_9^3 \\ W_9^3 W_9^6 W_9^6 W_9^6) \\ C_1 = A_1 \circ U_{81} = (W_9^0 W_9^7 W_9^5 W_9^6 W_9^4 W_9^2 W_9^3 W_9^1 W_9^8 W_9^0 W_9^8 \\ W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^0 W_9^0 W_9^3 \\ W_9^3 W_9^3 W_9^6 W_9^6 W_9^6 W_9^0 W_9^1 W_9^2 W_9^6 W_9^7 W_9^8 \\ W_9^3 W_9^4 W_9^5 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 \\ W_9^4 W_9^0 W_9^3 W_9^6 W_9^3 W_9^6 W_9^0 W_9^6 W_9^0 W_9^3 W_9^0 \\ W_9^4 W_9^8 W_9^6 W_9^1 W_9^5 W_9^3 W_9^7 W_9^2 W_9^0 W_9^5 W_9^1 \\ W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^6 W_9^3 W_9^3 W_9^0 \\ W_9^6 W_9^6 W_9^3 W_9^0) \\ C_2 = A_2 \circ U_{81} = (W_9^0 W_9^4 W_9^8 W_9^6 W_9^1 W_9^5 W_9^3 W_9^7 W_9^2 W_9^0 W_9^5 \\ W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^6 W_9^3 W_9^3 \\ W_9^0 W_9^6 W_9^6 W_9^3 W_9^0 W_9^7 W_9^5 W_9^6 W_9^4 W_9^2 \\ W_9^3 W_9^1 W_9^8 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 \\ W_9^7 W_9^0 W_9^0 W_9^0 W_9^3 W_9^3 W_9^3 W_9^6 W_9^6 W_9^6 W_9^0 \\ W_9^1 W_9^2 W_9^6 W_9^7 W_9^8 W_9^3 W_9^4 W_9^5 W_9^0 W_9^2 W_9^4 \\ W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^3 W_9^6 W_9^3 W_9^6 \\ W_9^0 W_9^6 W_9^0 W_9^3)$$

which form an $(81, 3, 26)$ ZCZ sequence set that satisfies (4).

Example 3 (Quadriphase Sequences Derived From a Sylvester Hadamard Matrix): Partition the Sylvester Hadamard matrix \mathbf{H}_{16} into four submatrices, $\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$, and select the first submatrix, $\mathbf{H}_0 = [\mathbf{h}_0^T, \mathbf{h}_1^T, \mathbf{h}_2^T, \mathbf{h}_3^T]^T$. Filtering the IDFT of \mathbf{h}_i by

$$U_{16} = (W_4^0 W_4^0 W_4^0 W_4^0 W_4^3 W_4^2 W_4^1 \\ W_4^0 W_4^2 W_4^0 W_4^2 W_4^0 W_4^1 W_4^2 W_4^3) \quad (21)$$

for each i , we have

$$C_0 = (W_4^0 W_4^1 W_4^2 W_4^3 W_4^0 W_4^2 W_4^0 W_4^2 \\ W_4^0 W_4^3 W_4^2 W_4^1 W_4^0 W_4^0 W_4^0 W_4^0) \\ C_1 = (W_4^0 W_4^3 W_4^2 W_4^1 W_4^0 W_4^0 W_4^0 W_4^0 \\ W_4^0 W_4^1 W_4^2 W_4^3 W_4^0 W_4^2 W_4^0 W_4^2) \\ C_2 = (W_4^0 W_4^1 W_4^0 W_4^1 W_4^0 W_4^2 W_4^2 W_4^0 \\ W_4^0 W_4^3 W_4^0 W_4^3 W_4^0 W_4^0 W_4^2 W_4^2) \\ C_3 = (W_4^0 W_4^3 W_4^0 W_4^3 W_4^0 W_4^0 W_4^2 W_4^2 \\ W_4^0 W_4^1 W_4^0 W_4^1 W_4^0 W_4^2 W_4^2 W_4^0)$$

a quadriphase $(16, 4, 3)$ ZCZ sequence family that satisfies (4).

Note that if a third-order 2-partition is used instead, we have a set of only two sequences but with a larger ZCZ width, i.e., we obtain a quadriphase $(16, 2, 7)$ ZCZ sequence set consisting of $\{A_0 \circ U_{16}, A_1 \circ U_{16}\}$ or $\{A_2 \circ U_{16}, A_3 \circ U_{16}\}$.

IV. DIRECT SYNTHESIS METHOD

A. Preliminaries

We now present an alternate approach that is capable of generating ZCZ sequences of arbitrary nonprime periods.

Definition 12: A binary (0- and 1-valued) sequence of period N which satisfies the ZCZ width constraint T on its AC function is called a basic (N, T) sequence.

A basic sequence can be obtained by the simple rule given in the following.

Lemma 9: A binary sequence $B = (b_0, b_1, \dots, b_{N-1})$, $b_i \in \{0, 1\}$, is a basic (N, T) sequence if the minimum run length of 0's is T (in the circular sense), where a run refers to a string of identical symbols and T is also called the *minimum spacing* of B .

B. Synthesis Process

Two new operations are needed.

Definition 13: A basic (N, T) sequence B with Hamming weight $w_H(B)$ can be expressed as the sum (via componentwise addition) of M length- N binary sequences, $\{B_i\}_{i=0}^{M-1}$, with disjoint nonempty supports so that $\sum_{i=0}^{M-1} w_H(B_i) = w_H(B)$ and $w_H(B_i) \geq 1$. The sequence set $\{B_i\}_{i=0}^{M-1}$ is said to be an *orthogonal tone decomposition* of B .

It is trivial to see that $\{B_i\}_{i=0}^{M-1}$ is a binary (N, M, T) ZCZ sequence family and each B_i is a basic (N, T_i) sequence with $T_i \geq T$.

Definition 14: Let $V = (v(0), v(1), \dots, v(N-1))$ be a length- N binary sequence with Hamming weight $w_H(V) = k$ and $\mathbf{U} = [u_{ij}]$ be any matrix having k columns and arbitrary number of rows k' . The V -upsampled matrix of \mathbf{U} is the $k' \times N$ matrix $\mathbf{P} = [p_{ij}]$ defined by

$$p_{ij} = \begin{cases} u_{im}, & j = s_V(m), m = 0, 1, \dots, k-1 \\ 0, & \text{otherwise} \end{cases} \quad (22)$$

where $s_V(m)$ is the coordinate of sequence V 's m th nonzero entry. We denote the aforementioned row-wise nonuniform up-sampling operation on \mathbf{U} by $\mathbf{P} = \mathbf{U}\Delta V$.

TABLE II
POLYPHASE ZCZ SEQUENCE SETS WITH SEQUENCE LENGTH $N = n_1 n_2$ USING n_ρ -PSK PERFECT SEQUENCE

	Torii [18]	Tang [19]		Popovic [25]	Park [12]	Theorem 2	
Perfect sequence length	$n_1 = kt, k \leq n_1$	n_1		N	$n_1 = k^2$	n_1	
Set size K	k	n_2		n_2	n_2	n_2	
ZCZ width T	$(n_1 - 2)k^{\ell-1}$	$n_1 - 1$	$n_1 - 2$	$n_1 - 1$	$n_1 - 1$	$n_1 - 1$	$n_1 - 2$
Constraints on n_2	$n_2 = k^\ell, \ell > 1$	$\gcd(n_1, n_2) = 1$	$n_1 n_2$ or $n_2 n_1$	None	$\gcd(n_1, n_2) = 1$	$\gcd(n_1, n_2) = 1$	$\gcd(n_1, n_2) \neq 1$
Upper-bound (4) achieved?	No	Yes	No	Yes	Yes	Yes	No
Alphabet size	$\text{lcm}(k, n_P)$	$\text{lcm}(n_2, n_P)$		$\text{lcm}(n_1, n_2)$ or $\text{lcm}(2n_1, n_2)$	N	$\text{lcm}(n_2, n_P)$	

Obviously, the nonzero entries in all rows of the matrix $\mathbf{P} = \mathbf{U}\Delta V$ are in the same positions. Hence, if V is an (N, T) basic sequence constructed by the procedure described in Lemma 9, then each row has the same minimum spacing T and all CC (including AC) values are zero at $0 < \tau \leq T$. Values of all CC functions at $\tau = 0$ are zero when \mathbf{U} is unitary in which case rows of \mathbf{P} all have ZCZ width T . Invoking Lemma 4, we have the following.

Lemma 10: Let B be a basic (N, T) sequence with $w_H(B) = K$, $\mathbf{B} \stackrel{\text{def}}{=} \{B_i\}_{i=0}^{M-1}$ be an orthogonal tone decomposition of B , $w_H(B_i) = K_i$, and $\mathbf{U}_i, 0 \leq i < M$ be $K_i \times K_i$ unitary matrices (not necessarily distinct). Then, for each i , the rows of nonuniform upsampled matrix $\mathbf{P}_i = \mathbf{U}_i \Delta B_i$ constitute an (N, K_i, T_i) ZCZ sequence family, where $T_i \geq T$ is the minimum spacing of B_i . Moreover, the rows of all \mathbf{P}_i 's constitute an (N, K, T) ZCZ sequence set.

C. Polyphase ZCZ Sequences

Lemma 10 does not guarantee a constant modulus constellation for the entries of the generated sequences. We need a special class of basic sequences and a suitable perfect sequence to generate polyphase sequence families.

Theorem 2: Let $A' = \{a'_n\}$ be a length- N' perfect $N_{A'}$ -PSK sequence, where $2 \leq N_{A'} \leq 2N'$ and A be the perfect sequence of length $N = N_r N'$ derived from N_r -fold upsampling on A' . An $(N, N_r, N' - 1)$ or $(N, N_r, N' - 2)$ ZCZ ℓ -PSK sequence family, where $\ell = \text{lcm}(N_{A'}, N_r)$, can be obtained by filtering the rows of $\mathbf{P} = \mathbf{F}_{N_r} \Delta B$ by A , where $B = (b_0, b_1, \dots, b_{N-1})$ is the weight- N_r basic sequence defined by

$$b_i = \begin{cases} 1, & i = kN', k = 0, 1, \dots, N_r - 1 \\ 0, & \text{otherwise} \end{cases} \quad (23)$$

if N_r and N' are relatively prime, or by

$$b_i = \begin{cases} 1, & i = kN', k = 0, 1, \dots, \frac{L_0}{N'} - 1, \text{ or} \\ & i = \ell L_0 + \left(\frac{N}{L_0} - \ell\right) + kN', \text{ where} \\ & \ell = 1, 2, \dots, \frac{N}{L_0} - 1 \\ & k = 0, 1, \dots, \frac{L_0}{N'} - 1 \\ 0, & \text{otherwise} \end{cases} \quad (24)$$

if $\gcd(N_r, N') \neq 1$, where $L_0 = \text{lcm}(N_r, N')$.

Proof: See the Appendix. ■

D. Properties, Constraints, and Comparisons

The following three properties about the approach described above are easily verifiable.

Remark 5 (Parameter Relations): For a fixed N and $K = N_r$, ZCZ sequence families generated by (23) achieve the upper bound (4) and those generated from (24) satisfy the relation $K(T + 1) = N - N_r$.

Remark 6 (Nested-Like and Interset Properties): The construction described in Lemma 10 results in a nested-like structure similar to that of Remark 2. Instead of decomposing a Hadamard matrix, we decompose a basic sequence of minimum spacing S into several basic sequences of minimum spacing $S' \geq S$ and use the latter basic sequences to construct sequence sets whose union constitutes a larger ZCZ set with a ZCZ width smaller than that of individual subset; see the second part of Example 7.

The construction of Theorem 2 needs a special choice of the Hadamard matrix and basic sequence used because of the polyphase requirement. But as a special case of Lemma 10, it still preserves the nested-like structure. In fact, the basic sequences defined by (23) and (24) can be cyclically shifted to generate distinct polyphase ZCZ sequence families with the same (N, K, T) . The zero CC zone width between a sequence from the set based on B and one from the set based on a circularly shifted version of B is determined by the CC function of the two basic sequences used. If, for instance, $\mathbf{C}_0 = \{C_{0,0}, C_{0,1}, \dots, C_{0,K-1}\}$ and $\mathbf{C}_1 = \{C_{1,0}, C_{1,1}, \dots, C_{1,K-1}\}$ are derived from basic sequence $B(n)$ and $B'(n) = B(|n - n'|_{N'})$, respectively, then $\theta_{C_0, C_1, \ell}(\tau) = 0, \forall i, \ell$ and $|\tau|_N \leq T'$, where $T' < T$ is the zero-CC zone width of $\theta_{B, B'}(\tau)$. As a result, the set $\mathbf{C}_0 \cup \mathbf{C}_1$ has the ZCZ width $T' < T$; see Example 4 in Section IV-E.

Remark 7 (Binary Sequences): To generate binary ZCZ sequences, one has to use binary Hadamard matrices, which exist for $N_r = 2^\ell, 12 \times 2^\ell$, or $20 \times 2^\ell$ [34], to replace the N_r -DFT matrix, \mathbf{F}_{N_r} , in constructing \mathbf{P} and reduce the required alphabet size to just $\text{lcm}(N_{A'}, 2) = 2$; see Examples 9 and 10.

The parameter selection constraints and related properties for our and some related existing methods are given in Table II. We provide more comparisons in the following remarks.

Remark 8: Theorem 2 does not explicitly mention any restriction on the alphabet size. As these constructions need to use a length- N' perfect sequence and $N_r \times N_r$ Hadamard matrices, which do not always exist for all lengths (N'), matrix dimension (N_r), and all constellation sizes ($N_{A'}$), the ZCZ width, sequence length, and family size are thus implicitly constrained by the alphabet size.

Remark 9: Tang and Mow [19] classify the ZCZ sequences construction methods into two major categories, i.e., 1) those based on complementary sets and 2) those derived from perfect sequences. Our approach belongs to the latter category and generates sequences with length $N = n_1 n_2$, where n_1 is the length of a perfect sequences. The constructions proposed in [18]–[25] have similar constraints on the sequence length N and those mentioned in the next three remarks.

Remark 10: In [18], an $(N, k, (n_1 - 2)k^{\ell-1})$ set is constructed by using a length- n_1 perfect sequence, where $n_1 = kt$, $k \leq n_1$, but n_2 must be of the form k^ℓ , $\ell > 1$. The interleaving scheme [19] requires that either 1) $\gcd(n_1, n_2) = 1$ or 2) $n_1 | n_2$ or $n_2 | n_1$ to generate an $(N, n_2, n_1 - 1)$ or $(N, n_2, n_1 - 2)$ ZCZ family. The length constraints in 1) is similar to that for the construction (23) while 2) leads to ZCZ families of the same parameters as those by the construction (24) except that the latter is only constrained by $\gcd(n_1, n_2) \neq 1$.

Remark 11: A length- N ($N = n_1 n_2$) Frank–Chu perfect sequence is used in [25] to generate an $(N, n_2, n_1 - 1)$ family. This method also calls for the use of an $n_2 \times n_2$ DFT or binary Hadamard matrix. However, for the case when n_1 is a perfect square and a DFT (or binary Hadamard) matrix is used, our approach needs an alphabet of size $\text{lcm}(n_2, \sqrt{n_1})$ or $\text{lcm}(2, \sqrt{n_1})$ instead of $\text{lcm}(n_2, n_1)$, $\text{lcm}(n_2, 2n_1)$, or $\text{lcm}(2, n_1)$ required by [25]. Moreover, as [25] is primarily interested in polyphase (nonbinary) sequences, their approach is not applicable for binary set since it requires $n_1 = 2$. Our constructions, on the other hand, can be applied to generate both binary and nonbinary families.

Remark 12: The construction based on (23) generates sequences that possess the same correlation properties as those of the so-called PS sequences [12]. These sequences are bound-achieving; they have nonzero AC values only on subperiodic correlation lags at $\tau = m(T + 1)$, $m \in \mathbb{Z}$, and zero CC for all lags. While the PS sequences require that $\gcd(n_1, n_2) = 1$, where n_1 is a perfect square, to construct an $(N, n_2, n_1 - 1)$ family, our method does not impose any constraint on n_1 . Moreover, when n_1 is a perfect square, our approach can generate sequences, which, for the convenience of reference, are called *PS-like sequences* that require a constellation of size $\text{lcm}(n_2, \sqrt{n_1}) = N/\sqrt{n_1}$ as opposed to $\text{lcm}(n_1, n_2) = N$ required by the PS approach [12]. Similarly, we refer to those families derived from (23) using nonperfect square n_1 as *generalized PS sequences* for these sequences cannot be generated by the PS method. Some PS-like and generalized PS sequence sets are given in Section IV-E.

E. Examples of Direct Synthesized Sequence Sets

Example 4 (PS-Like Sequences): Following the procedure described in Theorem 2 with $N_r = 2$, $N' = 9$, $B = (10000000100000000)$ and \mathbf{U} being the Sylvester

Hadamard matrix \mathbf{H}_2 , we obtain $\mathbf{P} = \mathbf{U}\Delta B = [P_0^T, P_1^T]^T$, where

$$\begin{aligned} P_0 &= (1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0) \\ P_1 &= (1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0). \end{aligned}$$

Filtering them by the upsampled perfect sequence $A = (W_3^0 W_3^0 W_3^0 W_3^0 W_3^0 W_3^0 W_3^1 W_3^1 W_3^1 W_3^1 W_3^2 W_3^2 W_3^2 W_3^2 W_3^2 W_3^2 W_3^2 W_3^2)$, we have

$$\begin{aligned} C_{0,0} &= P_0 \circ A = (W_6^0 W_6^2 W_6^2 W_6^0 W_6^4 W_6^0 W_6^0 W_6^4 \\ &\quad W_6^0 W_6^2 W_6^2 W_6^0 W_6^4 W_6^0 W_6^0 W_6^4 W_6^4) \\ C_{0,1} &= P_1 \circ A = (W_6^0 W_6^5 W_6^2 W_6^3 W_6^4 W_6^3 W_6^0 W_6^3 W_6^4 \\ &\quad W_6^3 W_6^2 W_6^5 W_6^0 W_6^1 W_6^0 W_6^3 W_6^0 W_6^1). \end{aligned} \quad (25)$$

It can be shown that

$$\theta_{C_{0,0}C_{0,1}}(\tau) = 0, \quad |\theta_{C_{0,0}C_{0,0}}(\tau)| = |\theta_{C_{0,1}C_{0,1}}(\tau)| = 18\delta(|\tau|_9)$$

and $\mathbf{C}_0 = \{C_{0,0}, C_{0,1}\}$ is an $(18, 2, 8)$ bound-achieving ZCZ sequence family.

Using cyclically shifted basic sequences $B'(n) = B(|n - 3|_{18})$ and $B''(n) = B(|n - 6|_{18})$, we obtain two new $(18, 2, 8)$ ZCZ sequence sets $\mathbf{C}_1 = \{C_{1,0}, C_{1,1}\}$ and $\mathbf{C}_2 = \{C_{2,0}, C_{2,1}\}$ whose members are

$$\begin{aligned} C_{1,0} &= (W_6^0 W_6^0 W_6^4 W_6^0 W_6^2 W_6^2 W_6^0 W_6^4 W_6^0 \\ &\quad W_6^0 W_6^0 W_6^4 W_6^0 W_6^2 W_6^2 W_6^0 W_6^4 W_6^0) \\ C_{1,1} &= (W_6^3 W_6^0 W_6^1 W_6^0 W_6^5 W_6^2 W_6^3 W_6^4 W_6^3 \\ &\quad W_6^0 W_6^3 W_6^4 W_6^3 W_6^2 W_6^5 W_6^1 W_6^0 W_6^0) \\ C_{2,0} &= (W_6^0 W_6^4 W_6^0 W_6^0 W_6^0 W_6^4 W_6^0 W_6^2 W_6^2 \\ &\quad W_6^0 W_6^4 W_6^0 W_6^0 W_6^0 W_6^4 W_6^0 W_6^2 W_6^2) \\ C_{2,1} &= (W_6^0 W_6^1 W_6^0 W_6^3 W_6^0 W_6^1 W_6^0 W_6^5 W_6^2 \\ &\quad W_6^3 W_6^4 W_6^3 W_6^0 W_6^3 W_6^4 W_6^3 W_6^2 W_6^5). \end{aligned}$$

It can be shown that $\theta_{BB'}(\tau) = \theta_{BB''}(\tau) = \theta_{B'B''}(\tau) = 0$, $\forall |\tau| \leq T' = 2$, and thus, the interset zero-CC zone width is 2. Moreover, the set $\mathbf{C} \stackrel{\text{def}}{=} \bigcup_{i=0}^2 \mathbf{C}_i$ is a bound-achieving $(18, 6, 2)$ ZCZ sequence set.

Example 5 (Length-12 PS-Like Sequences): The set of three PS-like sequences

$$\begin{aligned} P_0 &= (W_3^0 000 W_3^0 000 W_3^0 000) \\ P_1 &= (W_3^0 000 W_3^1 000 W_3^2 000) \\ P_2 &= (W_3^0 000 W_3^2 000 W_3^1 000) \end{aligned}$$

is generated by using $N_r = 3$, $N' = 4$, $B = (100010001000)$, and IDFT matrix $\mathbf{U} = \mathbf{F}_3^H$. Filtering them by $A = (1, 0, 0, 1, 0, 0, 1, 0, 0, -1, 0, 0)$, we obtain the ZCZ sequences

$$\begin{aligned} C_0 &= P_0 \circ A \\ &= (W_6^0 W_6^0 W_6^0 W_6^3 W_6^0 W_6^0 W_6^0 W_6^3 W_6^0 W_6^0 W_6^0 W_6^3) \\ C_1 &= P_1 \circ A \\ &= (W_6^0 W_6^2 W_6^4 W_6^3 W_6^2 W_6^4 W_6^0 W_6^5 W_6^4 W_6^0 W_6^2 W_6^1) \\ C_2 &= P_2 \circ A \\ &= (W_6^0 W_6^4 W_6^2 W_6^3 W_6^4 W_6^2 W_6^0 W_6^1 W_6^2 W_6^0 W_6^4 W_6^5). \end{aligned} \quad (26)$$

It is verifiable that $\forall i, j, i \neq j$,

$$\theta_{C_i C_j}(\tau) = 0, \quad |\theta_{C_i C_i}(\tau)| = 12\delta(|\tau|_4) \quad (27)$$

i.e., $\mathbf{C} = \{C_0, C_1, C_2\}$ is a $(12, 3, 3)$ bound-achieving ZCZ sequence set. This set also possesses the same PS sequence correlation properties [12]. Moreover, both (25) and (26) require only $1/3$ and $1/2$ of the alphabet size required by the original PS construction under the same sequence period constraint.

Example 6 (Generalized PS Sequences): Using the method of Theorem 2 with $N_r = 5$, $N' = 3$, the IDFT matrix $\mathbf{U} = \mathbf{F}_5^H$, $B = (100100100100100)$, and $A = (W_3^0 0000 W_3^2 0000 W_3^4 0000)$, we obtain

$$\begin{aligned} C_0 &= (W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^0 W_{15}^5 \\ &\quad W_{15}^0 W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^0 W_{15}^5 W_{15}^0) \\ C_1 &= (W_{15}^0 W_{15}^{11} W_{15}^{12} W_{15}^3 W_{15}^{14} \\ &\quad W_{15}^0 W_{15}^6 W_{15}^2 W_{15}^3 W_{15}^9 W_{15}^5 W_{15}^6 W_{15}^{12} W_{15}^8 W_{15}^9) \\ C_2 &= (W_{15}^0 W_{15}^2 W_{15}^9 W_{15}^6 W_{15}^8 \\ &\quad W_{15}^0 W_{15}^{12} W_{15}^{14} W_{15}^6 W_{15}^3 W_{15}^5 W_{15}^{12} W_{15}^9 W_{15}^{11} W_{15}^3) \\ C_3 &= (W_{15}^0 W_{15}^8 W_{15}^6 W_{15}^9 W_{15}^2 \\ &\quad W_{15}^0 W_{15}^3 W_{15}^{11} W_{15}^9 W_{15}^{12} W_{15}^5 W_{15}^3 W_{15}^6 W_{15}^{14} W_{15}^{12}) \\ C_4 &= (W_{15}^0 W_{15}^{14} W_{15}^3 W_{15}^{12} W_{15}^{11} \\ &\quad W_{15}^0 W_{15}^9 W_{15}^8 W_{15}^{12} W_{15}^6 W_{15}^5 W_{15}^9 W_{15}^3 W_{15}^2 W_{15}^6) \end{aligned}$$

which constitute a set of $(15, 5, 2)$ bound-achieving generalized PS sequences that have the same correlation properties as the original PS sequences, i.e., $\forall i, j, i \neq j$

$$\theta_{C_i C_j}(\tau) = 0, \quad |\theta_{C_i C_i}(\tau)| = 15\delta(|\tau|_3). \quad (28)$$

As mentioned previously, the PS method [12] cannot produce ZCZ sequences of length $N = 15$.

Previous examples are constructed by using coprime N_r and N' ; we show a set using the construction (24).

Example 7 (Sets Based Noncoprime Parameters and Nested-Like Sets Using Orthogonal Tone Decomposition): By choosing $N_r = 4$, $N' = 6$, and upsampling the Sylvester Hadamard \mathbf{H}_4 by $B = (100000100000010000010000)$, we obtain a $(24, 4, 4)$ ZCZ sequence family by filtering each row of $\mathbf{P} = \mathbf{H}_4 \Delta B$ through $A = (W_{12}^0 000 W_{12}^1 000 W_{12}^4 000 W_{12}^9 000 W_{12}^4 000 W_{12}^1 000)$

$$\begin{aligned} C_0 &= P_0 \circ A \\ &= (W_{12}^0 W_{12}^3 W_{12}^{11} W_{12}^8 W_{12}^{11} W_{12}^8 W_{12}^0 W_{12}^3 W_{12}^8 W_{12}^{11} W_{12}^{11} W_{12}^8 \\ &\quad W_{12}^3 W_{12}^0 W_{12}^8 W_{12}^{11} W_{12}^8 W_{12}^{11} W_{12}^3 W_{12}^0 W_{12}^{11} W_{12}^8 W_{12}^8 W_{12}^{11}) \\ C_1 &= P_1 \circ A \\ &= (W_{12}^0 W_{12}^3 W_{12}^5 W_{12}^2 W_{12}^{11} W_{12}^8 W_{12}^6 W_{12}^9 W_{12}^8 W_{12}^{11} W_{12}^5 W_{12}^2 \\ &\quad W_{12}^3 W_{12}^0 W_{12}^2 W_{12}^5 W_{12}^8 W_{12}^{11} W_{12}^9 W_{12}^6 W_{12}^{11} W_{12}^8 W_{12}^2 W_{12}^5) \\ C_2 &= P_2 \circ A \\ &= (W_{12}^0 W_{12}^9 W_{12}^{11} W_{12}^2 W_{12}^{11} W_{12}^2 W_{12}^0 W_{12}^9 W_{12}^8 W_{12}^5 W_{12}^{11} W_{12}^2 \\ &\quad W_{12}^3 W_{12}^6 W_{12}^8 W_{12}^5 W_{12}^8 W_{12}^5 W_{12}^3 W_{12}^6 W_{12}^{11} W_{12}^2 W_{12}^8 W_{12}^5) \\ C_3 &= P_3 \circ A \\ &= (W_{12}^0 W_{12}^9 W_{12}^5 W_{12}^8 W_{12}^{11} W_{12}^2 W_{12}^6 W_{12}^3 W_{12}^8 W_{12}^5 W_{12}^5 W_{12}^8 \\ &\quad W_{12}^3 W_{12}^6 W_{12}^2 W_{12}^{11} W_{12}^8 W_{12}^5 W_{12}^9 W_{12}^0 W_{12}^{11} W_{12}^2 W_{12}^2 W_{12}^{11}). \end{aligned}$$

Alternatively, we can perform orthogonal tone decomposition on B to obtain two weight-2 basic sequences of same minimum spacing 10

$$\begin{aligned} B_0 &= (100000000000010000000000) \\ B_1 &= (000000100000000000010000). \end{aligned}$$

With $\mathbf{U}_0 = \mathbf{H}_2$ and

$$\mathbf{U}_1 = \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}$$

we filter rows of $\mathbf{U}_0 \Delta B_0$ and $\mathbf{U}_1 \Delta B_1$ by [37]

$$\begin{aligned} A &= (W_6^0 W_6^0 W_6^3 W_6^2 W_6^4 W_6^2 W_6^0 \\ &\quad W_6^3 W_6^6 W_6^6 W_6^0 W_6^1 W_6^2 W_6^0) \end{aligned}$$

to obtain two smaller polyphase sets of larger ZCZ width, $\mathbf{C}_0 = \{C_{0,0}, C_{0,1}\}$ and $\mathbf{C}_1 = \{C_{1,0}, C_{1,1}\}$, where

$$\begin{aligned} C_{0,0} &= (W_6^0 W_6^3 W_6^4 W_6^4 W_6^5 W_6^2 W_6^4 W_6^4 \\ &\quad W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^4 W_6^4 \\ &\quad W_6^2 W_6^5 W_6^4 W_6^4 W_6^3 W_6^0 W_6^0 W_6^0) \\ C_{0,1} &= (W_6^0 W_6^0 W_6^4 W_6^1 W_6^5 W_6^5 W_6^4 W_6^1 \\ &\quad W_6^0 W_6^0 W_6^0 W_6^3 W_6^3 W_6^3 W_6^4 W_6^1 \\ &\quad W_6^2 W_6^2 W_6^4 W_6^1 W_6^3 W_6^3 W_6^0 W_6^3) \\ C_{1,0} &= (W_{12}^8 W_{12}^{11} W_{12}^6 W_{12}^3 W_{12}^0 W_{12}^3 W_{12}^0 W_{12}^9 \\ &\quad W_{12}^8 W_{12}^{11} W_{12}^{10} W_{12}^7 W_{12}^8 W_{12}^{11} W_{12}^0 W_{12}^9 \\ &\quad W_{12}^0 W_{12}^3 W_{12}^6 W_{12}^3 W_{12}^8 W_{12}^{11} W_{12}^4 W_{12}^1) \\ C_{1,1} &= (W_{12}^{11} W_{12}^8 W_{12}^9 W_{12}^0 W_{12}^3 W_{12}^0 W_{12}^3 W_{12}^6 \\ &\quad W_{12}^{11} W_{12}^8 W_{12}^1 W_{12}^4 W_{12}^{11} W_{12}^8 W_{12}^3 W_{12}^6 \\ &\quad W_{12}^3 W_{12}^0 W_{12}^9 W_{12}^0 W_{12}^{11} W_{12}^8 W_{12}^7 W_{12}^{10}). \end{aligned}$$

Both sets are $(24, 2, 10)$ ZCZ sequence sets and together they form another $(24, 4, 4)$ set.

Example 8 (Generalized PS Sequence Set): We can also derive a smaller *generalized PS sequence* set having the same period but a larger ZCZ width. For example, if we choose $N' = 8$, $N_r = 3$ in (23) and use the QPSK perfect sequence [38]

$$A' = (W_4^1 W_4^1 W_4^2 W_4^1 W_4^1 W_4^3 W_4^2 W_4^3)$$

then the three sequences

$$\begin{aligned} C_0 &= (W_{12}^9 W_{12}^3 W_{12}^6 W_{12}^3 W_{12}^9 W_{12}^9 W_{12}^6 W_{12}^9 W_{12}^9 W_{12}^3 W_{12}^6 W_{12}^3 \\ &\quad W_{12}^9 W_{12}^9 W_{12}^6 W_{12}^9 W_{12}^9 W_{12}^3 W_{12}^6 W_{12}^3 W_{12}^9 W_{12}^9 W_{12}^6 W_{12}^9) \\ C_1 &= (W_{12}^9 W_{12}^7 W_{12}^2 W_{12}^3 W_{12}^1 W_{12}^5 W_{12}^6 W_{12}^1 W_{12}^5 W_{12}^3 W_{12}^{10} W_{12}^{11} \\ &\quad W_{12}^9 W_{12}^1 W_{12}^2 W_{12}^9 W_{12}^1 W_{12}^{11} W_{12}^6 W_{12}^7 W_{12}^5 W_{12}^9 W_{12}^{10} W_{12}^5) \\ C_2 &= (W_{12}^9 W_{12}^{11} W_{12}^{10} W_{12}^3 W_{12}^5 W_{12}^1 W_{12}^6 W_{12}^5 W_{12}^1 W_{12}^3 W_{12}^2 W_{12}^7 \\ &\quad W_{12}^9 W_{12}^5 W_{12}^{10} W_{12}^9 W_{12}^5 W_{12}^7 W_{12}^6 W_{12}^{11} W_{12}^1 W_{12}^9 W_{12}^2 W_{12}^1) \end{aligned}$$

constitute an $(N, N_r, N' - 1) = (24, 3, 7)$ bound-achieving ZCZ family. A family with such ZCZ parameter values cannot be generated by the method suggested in [12].

Example 9 (Length-16 Ternary and Binary Sequences): Using the basic sequence $B = (1000000100100100)$, the

Sylvester Hadamard matrix \mathbf{H}_4 as \mathbf{U} , and the perfect sequence $A = (+ 000 + 000 + 000 - 000)$, we obtain

$$\begin{aligned} P_0 &= (+ 000000 + 00 + 00 + 00) \\ P_1 &= (+ 000000 - 00 + 00 - 00) \\ P_2 &= (+ 000000 + 00 - 00 - 00) \\ P_3 &= (+ 000000 - 00 - 00 + 00) \end{aligned}$$

where $+$ and $-$ denote $+1$ and -1 , respectively. Time-domain sequences with zero entries are often undesirable as they require on-off switching. Filtering $\{P_i\}$ by A , we obtain the binary $(16, 4, 2)$ ZCZ sequence family consisting of

$$\begin{aligned} C_0 &= P_0 \circ A = (+ - + + - + + + + + - + + - +) \\ C_1 &= P_1 \circ A = (+ + + - - - + - + - + + + - - -) \\ C_2 &= P_2 \circ A = (+ + - + - - - + + - - - + - + +) \\ C_3 &= P_3 \circ A = (+ - - - - + - - + + - + + + + -). \end{aligned}$$

Example 10 (Length-32 Binary Sequence Set): Let $N_r = 8$ and $N' = 4$. With $B = (10001000000100010010001000100)$, $A = (+ 0000000 + 0000000 + 0000000 - 0000000)$, and $\mathbf{U} = \mathbf{H}_8$, we obtain the binary $(32, 8, 2)$ ZCZ sequence set

$$\begin{aligned} C_0 &= (+ - + + + - + + - + + + - + + + \\ &\quad + + + - + + + - + + - + + + - +) \\ C_1 &= (+ - + + - + - - - + + + + - - - \\ &\quad + + + - - - - + + + - + - - + -) \\ C_2 &= (+ + + - + + + - - - + - - - + - \\ &\quad + - + + + - + + + - - - + - - -) \\ C_3 &= (+ + + - - - - + - - + - + + - + \\ &\quad + - + + - + - - - + - - - + + +) \\ C_4 &= (+ + - + + + - + - - - + - - - + \\ &\quad + - - - + - - - + - + + + - + +) \\ C_5 &= (+ + - + - - - + - - - - + + + + - \\ &\quad + - - - - + + + + - + + - + - -) \\ C_6 &= (+ - - - - + - - - - + - - - + - - \\ &\quad + + - + + + - + + + - + + + -) \\ C_7 &= (+ - - - - + + + - + - - + - + + \\ &\quad + + - + - - - + - + + + - - - +). \end{aligned}$$

The ZCZ families shown in the aforementioned two examples achieve (5), the bound for binary ($N_{A'} = 2$) sequences, but their ZCZ widths are limited by the facts that there exists only one binary perfect sequence (whose length $N' = 4$) and binary Hadamard matrices only exists for certain N_r ; see Remark 7. To increase the ZCZ width and have greater flexibility in choosing the ZCZ parameters, we can use higher order constellations ($N_r > 2$). For example, quadriphase perfect sequences of length $N' = 2, 4, 8$, or 16 do exist [37], [38]. We introduce in Section V an alternate method that offers more choices for the ZCZ width.

V. SEQUENCES DERIVED FROM COMPLEMENTARY SETS OF SEQUENCES

In this section, we generalize the above basic sequence-based approach by replacing rows of an unitary matrix with concatenated sequences. The following definitions can be found in [32].

A. Basic Definitions

Definition 15: The aperiodic CC function of two length- L sequences $u \equiv \{u(n)\}$ and $v \equiv \{v(n)\}$ is defined as

$$\psi_{uv}(\tau) = \sum_{n=\tau}^{L-1} u(n)v^*(n-\tau). \quad (29)$$

The aperiodic AC function of sequence u is obviously $\psi_{uu}(\tau)$.

Definition 16: A set of Q equal-length sequences, $\mathbf{E} = \{E_0, E_1, \dots, E_{Q-1}\}$, forms a *complementary set of sequences (CSS)* if and only if $\forall \tau \neq 0$

$$\sum_{i=0}^{Q-1} \psi_{E_i E_i}(\tau) = 0. \quad (30)$$

Definition 17: A CSS, $\mathbf{F} = \{F_0, F_1, \dots, F_{Q-1}\}$, is said to be a *mate* of the CSS, $\mathbf{E} = \{E_0, E_1, \dots, E_{Q-1}\}$ if

- the lengths of all members in \mathbf{E} and \mathbf{F} are the same;
- for all τ

$$\sum_{i=0}^{Q-1} \psi_{E_i F_i}(\tau) = 0. \quad (31)$$

Definition 18: A collection of complementary sets of sequences $\{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$, where each set contains the same number of sequences, is said to be *mutually orthogonal* if every two sets in the collection are mates of each other.

It has been proved in [31] the following.

Corollary 3: The number of mutually orthogonal CSS's (MOCSS's) K cannot exceed the cardinality of member CSS, Q , i.e., $K \leq Q$.

B. Synthesis Procedure

We now extend the nonuniform upsampling operation defined in Definition 14.

Definition 19: Let V be a length- N binary sequence with $w_H(V) = Q$ and $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$ be a collection of K MOCSS's in which each CSS \mathbf{E}_i consists of Q length- L sequences, i.e., $\mathbf{E}_i = \{E_{i,0}, E_{i,1}, \dots, E_{i,Q-1}\}$, where $E_{i,j} = (e_{i,j}(0), e_{i,j}(1), \dots, e_{i,j}(L-1))$.

The V -upsampled concatenated sequence based on \mathbf{E}_i , $G_i = \mathbf{E}_i \Delta_c V = (g_i(0), g_i(1), \dots, g_i(N+Q(L-1)-1))$ is defined by

$$g_i(n) = \begin{cases} e_{i,j}(m), & n = j(L-1) + s_V(j) + m \\ 0, & \text{otherwise} \end{cases} \quad (32)$$

where $s_V(j)$ is given in Definition 14.

The operator Δ_c is similar to Δ : the latter operates on rows of a matrix, while the former operates on the sequence formed by

concatenating members of the set \mathbf{E}_i and replaces each nonzero element of a basic sequence by a finite-length sequence.

Lemma 11: Let $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$ be a collection of K MOCSS's in which each set \mathbf{E}_i has Q length- L sequences and B be a basic (N, T) sequence of weight Q . The set $\mathbf{G} = \{\mathbf{E}_i \triangleleft_c B\} \stackrel{\text{def}}{=} \{G_0, G_1, \dots, G_{K-1}\}$ forms an $(N + Q(L - 1), K, T)$ ZCZ sequence family.

Proof: Based on Lemma 9 and Definition 19, we can express G_i as

$$G_i = \underbrace{(0 \cdots 0)}_{s_V(0)} \underbrace{e_{i,0}(0) \cdots e_{i,0}(L-1)}_L \underbrace{0 \cdots 0}_{s_V(1) - s_V(0) - 1} \\ \underbrace{e_{i,1}(0) \cdots e_{i,1}(L-1)}_L \underbrace{0 \cdots 0}_{s_V(2) - s_V(1) - 1} \\ \vdots \\ \underbrace{e_{i,Q-1}(0) \cdots e_{i,Q-1}(L-1)}_L \underbrace{0 \cdots 0}_{N - s_V(Q-1) - 1}$$

where $s_V(j) - s_V(j-1) - 1 \geq T$, $j = 0, 1, \dots, Q-1$, and $s_V(0) - s_V(Q-1) + N - 1 \geq T$. Invoking Definitions 17 and 18, we obtain for all $i \neq k$

$$\theta_{G_i G_k}(\tau) = \sum_{j=0}^{Q-1} \psi_{E_{i,j} E_{k,j}}(\tau) = 0, \quad |\tau|_N \leq T. \quad (33)$$

Analogously, Definition 16 gives, for all i , $\theta_{G_i G_i}(\tau) = \sum_{j=0}^{Q-1} \psi_{E_{i,j} E_{i,j}}(\tau) = 0$, $0 < |\tau|_N \leq T$. Therefore, \mathbf{G} forms an $(N + Q(L - 1), K, T)$ ZCZ sequence set. ■

C. Polyphase ZCZ Sequences

Following the idea described in Section IV-C, we can derive another class of polyphase ZCZ sequence families by using suitable perfect and basic sequences. The proof of the next corollary is similar to that of Theorem 2 and is given in the last two paragraphs of Appendix.

Corollary 4: Let A be the length- LN perfect sequence obtained by LN_r -fold upsampling on a length- N' perfect $N_{A'}$ -PSK sequence, A' , where $N = N_r N'$ and $2 \leq N_{A'} \leq N'$. Denote by $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$ a collection of K MOCSS's, where $K \leq N_r$ and each CSS $\mathbf{E}_i = \{E_{i,0}, E_{i,1}, \dots, E_{i,N_r-1}\}$ contains N_r length- L N_c -PSK sequences. An (LN, K, T) ZCZ M -PSK sequence set, $M = \text{lcm}(N_{A'}, N_c)$, with $T = L(N' - 2)$ if $\text{gcd}(N_r, N') \neq 1$ or $T = L(N' - 1)$ if $\text{gcd}(N_r, N') = 1$ can be obtained by the following steps.

- 1) Generate K length- $(N + N_r(L - 1))$ sequences $G'_i = \mathbf{E}_i \triangleleft_c B$, $i = 0, 1, \dots, K - 1$, where B is the weight- N_r basic sequence of length N defined by (24) if $\text{gcd}(N_r, N') \neq 1$ or by (23) if $\text{gcd}(N_r, N') = 1$.
- 2) Replace each zero in G'_i by a length- L all-zero sequence to obtain the augmented sequence G_i .
- 3) Filter each G_i by A .

We have the following four remarks on the MOCSS-based approach.

Remark 13: Similar to Remark 6, the basic sequence B can be cyclically shifted to generate different polyphase ZCZ families with the same ZCZ parameters and alphabet size. These families can be combined to form a larger family with smaller ZCZ width. Likewise, the zero-CC zone width between B and its shifted version B' determine the interset zero-CC zone width between the associated families or the ZCZ width of the combined set. We can also decompose B into several basic sequences $\{B_i\}$ to generate multiple sets with different ZCZ widths.

Remark 14: As mentioned in Section IV, binary sequence sets constructed by Theorem 2 have less choices in ZCZ width. The construction described in Corollary 4 takes advantage of the fact that the member sequence of an MOCSS exists for many values of L and thus allow the ZCZ width to be chosen from the set $\{T = 2L\}$ with the same basic sequence B , set cardinality N_r , and perfect sequence A' .

Remark 15: The authors in [15] and [16] present MOCSS-based methods for generating binary ZCZ sequences. The approach given in [16] was later generalized by [17]. The ZCZ parameters realizable by these methods can be obtained by using our approach described previously. For example, a method given in [15] needs to use a class of recursively generated families of binary CSS $\{\Delta_n\}$. Expressing a family of Q MOCSS's in matrix form [32]

$$\Delta_1 \stackrel{\text{def}}{=} \begin{bmatrix} E_{0,0} & E_{1,0} & \cdots & E_{Q-1,0} \\ E_{0,1} & E_{1,1} & \cdots & E_{Q-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ E_{0,Q-1} & E_{1,Q-1} & \cdots & E_{Q-1,Q-1} \end{bmatrix} \quad (34)$$

where $E_{i,j}$ are length- L binary sequences and each row is a CSS. Then, for $n \geq 2$

$$\Delta_n = \begin{bmatrix} \Delta_{n-1} \diamond \Delta_{n-1} & -\Delta_{n-1} \diamond \Delta_{n-1} \\ -\Delta_{n-1} \diamond \Delta_{n-1} & \Delta_{n-1} \diamond \Delta_{n-1} \end{bmatrix} \quad (35)$$

where $[\mathbf{A} \diamond \mathbf{B}]_{ij}$, the (i, j) th entry of the submatrix $[\mathbf{A} \diamond \mathbf{B}]$, is obtained by concatenating the two sequences, $[\mathbf{A}]_{ij}$ and $[\mathbf{B}]_{ij}$. The concatenation of rows of Δ_n forms a $(4^{n-1}LQ, 2^{n-1}Q, 2^{n-2}L)$ ZCZ sequence set. On the other hand, by using $A' = (1, 1, 1, -1)$, the basic sequence defined by (24) and the family of MOCSS Δ_n with $N_r = 2^{n-1}Q$ and elements of Δ_1 being length $L/4$ sequences, we obtain binary ZCZ sequence sets with the parameters $(4^{n-1}4(L/4)Q, 2^{n-1}Q, 2^{n-1}(L/4) \cdot 2) = (4^{n-1}LQ, 2^{n-1}Q, 2^{n-2}L)$ via Corollary 4.

Remark 16: Our approach offers more choices in parameter values and thus produce sets which are not derivable from the methods of [15] and [17]. More importantly, we can generate not only binary but also nonbinary sequences and the ZCZ parameters for the nonbinary class can be flexibly controlled via N' , which can be any integer and is not affected by the MOCSS chosen.

In Table III, we list key parameters for our and some other MOCSS-based binary ZCZ sequence set constructions.

TABLE III
ZCZ SEQUENCE SETS (EACH USES A COLLECTION OF
 M MOCSS'S OF Q LENGTH- L SEQUENCES)

	Deng [15] and Tang [17], $n \geq 0$			Corollary 4
Sequence length N	$4^n LQ$	$2^{2n-1} LQ$	$2LQ$	$4LQ$
Set size K	$2^n M$	$2^n M$	M	M
ZCZ width T	$2^{n-1} L$	$2^{n-2} L$	L	$2L$
Upper-bound (5) achieved with $M = Q$?	Yes	Yes	Yes	Yes
Alphabet Size	Binary			Binary and polyphase

D. Examples of CSS-Based Polyphase ZCZ Sequence Sets

Two ZCZ sequence construction examples based on CSS are given in this section.

Example 11 ($\gcd(N_r, N') \neq 1$): Let $N = 16$, $N' = N_r = K = L = 4$, $A' = (+ + + -)$, and $B = (100000100100100)$ and choose a collection of mutually orthogonal complementary sets $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ from [32], where

$$\begin{aligned}
 \mathbf{E}_0 &= \{(++++) , (- - + +) , (- + - +) , (+ - - +)\} \\
 \mathbf{E}_1 &= \{(++--) , (- - - -) , (- + + -) , (+ - + -)\} \\
 \mathbf{E}_2 &= \{(- + - +) , (+ - - +) , (+ + + +) , (- - + +)\} \\
 \mathbf{E}_3 &= \{(- + + -) , (+ - + -) , (+ + - -) , (- - - -)\}.
 \end{aligned} \tag{36}$$

Following the procedure of Corollary 4, we obtain the bound-achieving binary (64, 4, 8) ZCZ sequence set

$$\begin{aligned}
 C_0 &= (+ + + + - + + - - + - - + - - + + \\
 &\quad - - - - + - - + - + - + - - + + \\
 &\quad + + + + + - - + - + - + + + - - \\
 &\quad + + + + + - - + + - + - - - + +) \\
 C_1 &= (+ + - - - + - + - + + - - - - - \\
 &\quad - - + + + - + - - + + - - - - - \\
 &\quad + + - - + - + - - + + - + + + + \\
 &\quad + + - - + - + - + - - + - - - -) \\
 C_2 &= (- + - + + + - - + + + + + - - + \\
 &\quad + - + - - - + + + + + + - - + \\
 &\quad - + - + - - + + + + + - + + - \\
 &\quad - + - + - - + + - - - - + - - +) \\
 C_3 &= (- + + - + + + + + - - + - + - \\
 &\quad + - - + - - - - + + - - + - + - \\
 &\quad - + + - - - - - + + - - - + - + \\
 &\quad - + + - - - - - - - + + + - + -).
 \end{aligned}$$

With the same B , A' , and N_r as those used in Example 9, this set extends the ZCZ width without changing the set cardinality.

Example 12 ($\gcd(N_r, N') = 1$): Using the construction (23) with (36), $A' = (W_3^0 W_3^2 W_3^0)$, $N' = 3$, $N_r = 4$, and $L = 4$, we

can obtain a ZCZ sequence set \mathbf{C} of the same (or larger) T with a shorter sequence period LN and slightly larger constellation

$$\begin{aligned}
 C_0 &= (W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^5 W_6^2 W_6^5 W_6^2 \\
 &\quad W_6^3 W_6^3 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^2 W_6^5 W_6^5 W_6^2 \\
 &\quad W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^2 W_6^2 W_6^2 W_6^2 \\
 &\quad W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^5 W_6^5 W_6^2 W_6^2) \\
 C_1 &= (W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^5 W_6^2 W_6^2 W_6^5 \\
 &\quad W_6^3 W_6^3 W_6^3 W_6^0 W_6^0 W_6^0 W_6^3 W_6^3 W_6^2 W_6^5 W_6^2 W_6^5 \\
 &\quad W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^3 W_6^3 W_6^2 W_6^2 W_6^5 W_6^5 \\
 &\quad W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^5 W_6^5 W_6^5 W_6^5) \\
 C_2 &= (W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^2 W_6^2 W_6^2 W_6^2 \\
 &\quad W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^5 W_6^5 W_6^2 W_6^2 \\
 &\quad W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^5 W_6^2 W_6^5 W_6^2 \\
 &\quad W_6^3 W_6^3 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^2 W_6^5 W_6^5 W_6^2 W_6^2) \\
 C_3 &= (W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^3 W_6^3 W_6^2 W_6^2 W_6^5 W_6^5 \\
 &\quad W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^5 W_6^5 W_6^5 W_6^5 \\
 &\quad W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^5 W_6^2 W_6^2 W_6^5 \\
 &\quad W_6^3 W_6^3 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^2 W_6^5 W_6^2 W_6^5).
 \end{aligned}$$

It is worth mentioning that the aforementioned set cannot be obtained by using the methods of [15] and [17] and, moreover, although Corollary 4 promises an $(LN, N_r, L(N' - 1)) = (48, 4, 8)$ family, \mathbf{C} is actually a $(48, 4, 9)$ one. The larger ZCZ is due to the inherent correlation properties of MOCSS (36)

$$\sum_{k=0}^{N_r-1} \psi_{E_i, k E_j, |k \pm 1|_{N_r}}(\tau) = 0 \tag{37}$$

for $\tau = \pm(L - 1)$, $0 \leq i < N_r$, and $0 \leq j < N_r$.

VI. CONCLUSION

Three new systematic approaches—a transform domain method and two direct (time domain) synthesis methods—for generating ZCZ sequence families have been presented in this paper. The transform domain approach exploits the CC function's transform domain representation and the recursive Kronecker structure of a class of Hadamard matrices. The two other approaches begin with simple binary basic ZCZ sequences. Through progressively fine-tuning steps that include novel basic sequence-based nonuniform upsampling of unitary matrices or a collections of MOCSS's, we are able to obtain polyphase sequences that meet various ZCZ requirements.

The basic sequences are used to ensure that the required ZCZ width is satisfied during the upsampling process, while the transform domain approach uses the subperiodicity of the Hadamard product of two transform domain sequences. The orthogonality among rows of unitary matrices or MOCSS guarantees that the CC value of any two member sequences at zero lag is zero as well. We take advantage of the correlation-invariant property of the filtering-by-perfect-sequence operation to convert a nonconstant modulus sequence into a polyphase sequence. Judicious choices of the basic and perfect sequences used and the associated upsampling rate are crucial in this operation.

Our approaches are conceptually simple and require no sophisticated algebra but, in some cases, offer more flexibilities in either the choices of the sequence length, the ZCZ width, and/or the alphabet size needed. We are therefore able to produce sequence families with the same parameters as those by earlier proposals as well as some that are not achievable by related known methods. Finally, for each approach, numerical examples have been provided to further validate the proposed construction methods.

APPENDIX

PROOFS THEOREM 2 AND COROLLARY 4

Let $P = (p_0, p_1, \dots, p_{N-1})$ be a row of \mathbf{P} and $C = P \circ A = (c_0, c_1, \dots, c_{N-1})$, where $c_n = \sum_{j=0}^{N-1} p_j a_{|j-n|_N}^*$. If we can show that, for any $n \in [0, N-1]$, one and only one of the N products $\{p_j a_{|j-n|_N}^* : j = 0, \dots, N-1\}$ is nonzero, then, as both A and P consist of 0's and polyphase elements, C is a polyphase sequence as well. Because of the circular convolution nature of the filtering operation (Definition 4) and the periodic run property of P , we have only to check if this single nonzero product assertion is valid for $0 < n < N_r$.

For the first construction (23), $\gcd(N_r, N') = 1$ and both N_r and N' are positive, hence \exists unique $a, b \in \mathbb{Z}$ such that $aN' + bN_r = 1$, where one of the integer coefficients a or b must be negative [39]. Without loss of generality, we assume $b < 0$ and multiply both sides of the above Bézout's identity by s , $0 < s < N_r$, to obtain $saN' = s + sb'N_r$, $b' = -b > 0$. If $sb' \leq N' - 1$, then $saN' < N'N_r = N$ and $sa < N_r$; otherwise, subtract both sides by n_0N , where $n_0 = \lfloor \frac{sb'N_r}{N} \rfloor$ to obtain $(sa - n_0N_r)N' = s + (sb' - n_0N')N_r$. For both cases, we have, for each positive $s < N_r$, \exists unique pair of positive integers (m, n) , $0 < m \leq N_r - 1$, $0 \leq n \leq N' - 1$ such that $mN' = s + nN_r \pmod{N}$. That this property holds for $s = 0$ is obvious.

As for the second construction (24), we notice that the basic sequence admits the orthogonal tone decomposition, $B = \sum_{\ell=0}^{d-1} B_\ell$, where

$$B_\ell(i) = \begin{cases} b_i, & \ell L_0 \leq i < (\ell + 1)L_0 \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A.1})$$

When $d = \gcd(N_r, N')$, there exists positive integers a, b' such that $aN' = d + b'N_r$. Multiplying both sides by s , $0 \leq s < \frac{N}{d}$, we obtain $(sa - n_0\frac{N_r}{d})N' = sd + (sb' - n_0\frac{N_r}{d})N_r$, where $n_0 = \lfloor \frac{saN'}{L_0} \rfloor$. For all $s \in \left\{0, 1, \dots, \lfloor \frac{N_r/N'}{d} \rfloor N' + (N - N_r) \right\}$, \exists a unique integer pair (m, n) , $0 \leq m < \frac{N_r}{d}$, $0 \leq n < N'$ such that $mN' = sd + nN_r \pmod{N}$, i.e., the sequence $B_0 \circ A$ is identically zero except at indices that are multiples of d and the nonzero terms are the products of two polyphase signals whence are themselves polyphase signals.

Similarly, we can show that, for $\ell = 1, 2, \dots, d-1$, the sequence $B_\ell \circ A$ has nonzero polyphase terms at $nd - \ell$ only, where $n \in \mathbb{Z}$. Hence, the sequence $B \circ A = \sum_{\ell=0}^{d-1} B_\ell \circ A$ is a polyphase sequence.

To prove Corollary 4, we first note that the sequences generated differs from those generated by Theorem 2 in that the

perfect sequence used in Corollary 4 is the L -fold upsampled version of that used in Theorem 2 while the unfiltered ZCZ sequences for the former is an L -expanded version of those for the latter, replacing each zero entry of P by a length- L string of zeros and each nonzero entry by a complementary sequence E_{ij} of length L .

For the first construction of B , (23), we immediately have, for $0 \leq s < N_r$, \exists unique pair of positive integers (m, n) , $0 < m \leq N_r - 1$, $0 \leq n \leq N' - 1$ such that $mLN' = sL + nLN_r = k + nLN_r$. That is, in computing the filtered sequence $C = G \circ A = \{c_k\}$, where $G = \{g_k\} \stackrel{\text{def}}{=} G_i$ and $c_k = \sum_{j=0}^{LN-1} g_j a_{|j-k|_{LN}}^*$, there is only one nonzero term in the summands that add up to c_k , for $k = sL$, $s = 0, 1, \dots, N_r - 1$. That this single nonzero convolution term property holds for $sL < k < (s+1)L$ is obvious because of the special structure of G_i . The proof for the case when the second construction (24) is employed follows a similar line of argument.

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