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Free boundary problems and perpetual American strangles

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1. Introduction

An American option is an option that can be exercised at any time prior to its expiration. For an American call option (written on an underlying stock without dividends) with a finite expiration time, Merton (1973) obtained that the price coincides with the price of the corresponding European option. However, the American put option (even without dividends) presents a difficult problem. No explicit pricing formulas exist and the optimal exercise boundaries are unknown. One exception is the perpetual American put option, which is an American put option with infinite expiration time. McKean (1965) solved the perpetual American put problem by applying the Black–Scholes model. Boyarchenko and Levendorskiĭ (2002b) derived a closed formula for prices of perpetual American put and call options using Lévy-based models and the theory of pseudo-differential operators. Mordecki and Salminen (2002) utilized probabilistic techniques to obtain explicit formulas based on the assumption of mixed-exponentially distributed and arbi-

trary negative jumps for the call options, and negative mixed-exponentially distributed and arbitrary positive jumps for put options. For related work, see Boyarchenko and Levendorskiĭ (2002a), Asmussen *et al.* (2004) and references therein.

Mathematically, the problem of pricing perpetual American contracts using Lévy-based models is equivalent to the optimal stopping problem of the form

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)), \quad (1)$$

where $X = \{X_t : t \geq 0\}$ under the chosen risk-neutral probability measure, and \mathbb{P}_x is a Lévy process that starts at $X_0 = x$. Additionally, g is the non-negative continuous reward function that corresponds to a contract, $r \geq 0$ is a constant, and \mathcal{T} is a family of stopping times with respect to the natural filtration \mathcal{F} generated by X . (Here we define, on $\{\tau = \infty\}$, $e^{-r\tau} g(X_\tau) = 0$.) The object is to find the value function $V(x)$ and the optimal stopping time τ^* such that $V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$. The free boundary approach is based on the observation that, under suitable conditions, the value function $V(x)$ that solves the optimal stopping problem (1) is a solution to the free boundary (or Stephan) problem

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$$\begin{cases} (\mathcal{L}_X - r)V(x) = 0, & \text{in } \mathcal{C}, \\ V(x) = g(x), & \text{on } \mathcal{D}, \end{cases} \quad (2)$$

where $\mathcal{C} = \{x \in \mathbb{R}, V(x) > g(x)\}$ (continuation region), $\mathcal{D} = \{x \in \mathbb{R}, V(x) = g(x)\}$ (stopping region) and \mathcal{L}_X is the infinitesimal operator of X . Many authors have also observed that the boundary of the stopping region \mathcal{D} is determined by imposing the smooth pasting condition on the value function. Then, the optimal stopping problem (1) can be solved merely by solving the above free boundary problem with suitable pasting conditions and proving a verification theorem which implies that solving the free boundary problem with a smooth pasting condition (or related conditions) allows explicit solutions of the optimal stopping problem to be established. This approach yields the optimal exercise times τ^* and the rational prices $V(x)$ for perpetual American contracts. For further details and topics related to the optimal stopping and free boundary problem, refer to the monograph of Peskir and Shiryaev (2006).

This study considers the problem of pricing perpetual American strangles, which is a combination of a put and a call that are written on the same security. The log-price of the underlying asset is modeled by the hyper-exponential jump-diffusion model (HEJD), which is a jump-diffusion process with the mixture-exponential jump density function (which was introduced by Levendorskiĭ (2004)). Pricing this particular perpetual American contract requires solving the free boundary problem (2) with a two-sided reward function g . (Notably, the reward functions that are most often considered in the literature are of the American put-type or American call-type. See, for example, Kyprianou and Surya (2005); Novikov and Shiryaev (2007) and Surya (2007).) This investigation proves that, if g is the reward function for a perpetual American strangle option, then the solution of the free boundary problem that was obtained by Boyarchenko (2006) is, indeed, the solution to the optimal stopping problem (see theorem 3.2). (This finding verifies a conjecture posed by Boyarchenko (2006).) Accordingly, using the risk-neutral pricing formula, the optimal exercise boundaries and the prices of the perpetual American strangles are obtained.

The rest of the paper is organized as follows. Section 2 introduces the jump-diffusion setting and presents a verification lemma for the optimal stopping problem (1) with a general two-sided reward function. This general result is itself of interest. It can be used to price other exotic perpetual American options (such as chooser options and vertical spread options). Section 3 considers the perpetual American strangles under a geometric hyper-exponential jump-diffusion process. The optimal stopping problem that corresponds to the perpetual American strangles is solved. Section 4 presents the algorithm of Boyarchenko (2006) for finding the optimal exercise boundaries. Some numerical results obtained using this algorithm are presented. Section 5 draws conclusions.

2. Preliminaries

Throughout this paper, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a jump-diffusion process X of the form

$$X_t = ct + \sigma B_t + \sum_{n=1}^{N_t} Y_n, \quad (3)$$

where $c \in \mathbb{R}$, $\sigma > 0$, $B = (B_t, t \geq 0)$ is a standard Brownian motion, and $(N_t; t \geq 0)$ is a Poisson process with rate $\lambda > 0$. Also, $Y = (Y_n, n \geq 0)$ is a sequence of independent random variables with identical piecewise continuous density functions f . Assume further that B , N_t and Y are mutually independent. A jump-diffusion process that starts from x is simply defined as $x + X_t$ for $t \geq 0$ and the governing law is denoted by \mathbb{P}_x . For convenience, \mathbb{P} is written in place of \mathbb{P}_0 . Also, \mathbb{E}_x denotes the expectation with respect to the probability measure \mathbb{P}_x . Under these model assumptions, $\mathbb{E}(e^{zX_t}) = e^{t\psi(z)}$, $z \in i\mathbb{R}$, where ψ is called the characteristic exponent of X and is given by the formula

$$\psi(z) = \frac{\sigma^2}{2}u^2 + cz + \lambda \int e^{zy} f(y) dy - \lambda. \quad (4)$$

The generalized infinitesimal generator of X is defined by the formula

$$\begin{aligned} \mathcal{L}_X h(x) &= \frac{1}{2}\sigma^2 h''(x) + ch'(x) \\ &+ \lambda \int h(x+y)f(y) dy - \lambda h(x), \end{aligned} \quad (5)$$

for all functions h on \mathbb{R} such that h' , h'' and the integral in equation (5) exist at x .

Given a jump-diffusion process X as in (3), this section considers the optimal stopping problem (1) with the continuous reward function g given by the formula

$$g(x) = g_1(x)\mathbf{1}_{\{x \leq l_1\}} + g_2(x)\mathbf{1}_{\{x \geq l_2\}}, \quad (6)$$

for some $-\infty < l_1 \leq l_2 < \infty$. Here, $g_1(x)$ is a strictly positive C^∞ function on $(-\infty, l_1)$ and $g_2(x)$ is a strictly positive C^∞ function on (l_2, ∞) . Assume further that g_1 is continuous at l_1 with $g_1(l_1) = 0$, g_2 is continuous at l_2 with $g_2(l_2) = 0$, and $\mathbb{E}_x[\sup_{t \geq 0} e^{-rt} g(X_t)] < \infty$ for all $x \in \mathbb{R}$. For any set I in \mathbb{R} , we write $\tau_I = \inf\{t \geq 0 \mid X_t \in I\}$ and set

$$V_I(x) = \mathbb{E}_x[e^{-r\tau_I} g(X_{\tau_I})], \quad x \in \mathbb{R}. \quad (7)$$

With the special features of the reward function g , the value function of the optimal stopping problem (1) is of the form $V(x) = V_I(x)$ for some $I = (h_1, h_2)^c$ with $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$. The following proposition characterizes the function V_I in terms of solutions to a boundary value problem. (It is a special case of the Feynman–Kac theorem. See, for example, Heath and Schweizer (2001) for the diffusion case or Boyarchenko and Levendorskiĭ (2002a,b) for Lévy processes that satisfy the ACP condition.)

Proposition 2.1 *Assume that g_1 is bounded on $(-\infty, l_1)$ and the function $\int_0^\infty g_2(x+y)f(y) dy$, $x \geq l_2$, is locally bounded. Consider the interval $I = (h_1, h_2)^c$ for some $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$. If \tilde{V} is a solution of the boundary value problem*

$$\begin{cases} (\mathcal{L}_X - r)\tilde{V}(x) = 0, & x \in (h_1, h_2), \\ \tilde{V}(x) = g(x), & x \in I, \end{cases} \quad (8)$$

and \tilde{V} is in $C^2(h_1, h_2) \cap C[h_1, h_2]$, then $\tilde{V}(x) = V_I(x)$ for all $x \in \mathbb{R}$.

Remark 1 The conclusion of proposition 2.1 still holds if the functions g_1 and g_2 are C^∞ (not necessarily strictly positive) and they satisfy the conditions in proposition 2.1.

To verify the optimality of the function V_I , the following verification lemma is required. (For a proof, see Mordecki (1999) for the setting used here or Boyarchenko and Levendorskii (2002a) for a much larger class of Lévy processes.)

Theorem 2.2 (verification lemma) *Given $I = (h_1, h_2)^c$ where $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$, assume that the function $V_I(x)$ in (7) satisfies the following conditions:*

- (a) $V_I(x)$ is the difference of two convex functions;
- (b) $V_I(x)$ is a twice continuously differentiable function except possibly at h_1 and h_2 ;
- (c) the limits $V_I''(h_i \pm) = \lim_{h \rightarrow h_i \pm} V_I''(h)$, $i = 1, 2$, exist and are finite;
- (d) $(\mathcal{L}_X - r)V_I(x) \leq 0$ for all x except possibly at h_1 and h_2 ;
- (e) $V_I(x) \geq g(x)$ for all $x \in (h_1, h_2)$.

Then $V_I(x)$ is the value function of the optimal stopping problem (1) with the reward function g given in equation (6).

The most difficult application of the verification lemma is to confirm the conditions (d) and (e). Next, under some additional conditions on the candidate function, the candidate is shown to satisfy condition (e) of theorem 2.2 for a class of two-sided reward functions.

Proposition 2.3 *Assume that g_1 and g'_1 are bounded on $(-\infty, l_1)$ and the functions $\int_0^\infty g_2(x+y)f(y) dy$ and $\int_0^\infty g'_2(x+y)f(y) dy$, $x \geq l_2$, are locally bounded. Assume further that $g_1(x) - g'_1(x)$ is positive and increasing on $(-\infty, l_1)$, $g_2(x) - g'_2(x)$ is negative and decreasing on (l_2, ∞) and $\mathbb{E}_x[\sup_{t \geq 0} e^{-rt} |g'(X_t)|] < \infty$ for all x . Let $I = (h_1, h_2)^c$ for some $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$ and consider $\tilde{V}(x) = V_I(x)$ for all $x \in \mathbb{R}$. Assume further that $\tilde{V}(x)$ satisfies the following conditions:*

- (a) $(d/dx) \int \tilde{V}(x+y)f(y) dy = \int \tilde{V}'(x+y)f(y) dy$, $\forall x \in (h_1, h_2)$;
- (b) \tilde{V} is continuous at h_1 and h_2 and $\tilde{V}'(h_i)$, $i = 1, 2$, exist and are continuous there.

Then $\tilde{V}(x) \geq g(x)$ for all $x \in (h_1, h_2)$.

Proof Note that $(\mathcal{L}_X - r)\tilde{V}(x) = 0$ for $x \in (h_1, h_2)$, and by the standard theorems for ODE and the iterating technique, \tilde{V} is in $C^\infty[h_1, h_2]$ can be shown. Also, for $x \in (h_1, h_2)$,

$$0 = \frac{d}{dx}(\mathcal{L}_X - r)\tilde{V}(x) = \frac{1}{2}\sigma^2\tilde{V}'''(x) + c\tilde{V}''(x) - (\lambda + r)\tilde{V}'(x) + \lambda \int \tilde{V}'(x+y)f(y) dy,$$

which implies that $(\mathcal{L}_X - r)\tilde{V}'(x) = 0$ for $x \in (h_1, h_2)$. By condition (b), $\tilde{V}' \in C[h_1, h_2]$, and by proposition 2.1, $\tilde{V}'(x) = \mathbb{E}_x[e^{-r\tau_I} g'(X_{\tau_I})]$. This result implies that $\tilde{V}(x)$ satisfies the ODE $\tilde{V}'(x) - \tilde{V}(x) = F(x)$, where $F(x) = \mathbb{E}_x[e^{-r\tau_I} (g'(X_{\tau_I}) - g(X_{\tau_I}))]$. Note that $\tilde{V}(x) = V_I(x) \geq 0 = g(x)$ for $l_1 \leq x \leq l_2$. First, consider the case in which $h_1 \leq x \leq l_1$. By the ODE theory and application of the boundary conditions, $\tilde{V}(x) = e^x(\int_{h_1}^x e^{-t} F(t) dt + g_1(h_1)e^{-h_1})$. Set $H(x) \equiv e^{-x}(\tilde{V}(x) - g(x))$. Now, $H(x) = \int_{h_1}^x e^{-t} F(t) dt + g_1(h_1)e^{-h_1} - g_1(x)e^{-x}$ and

$$\begin{aligned} H'(x) &= e^{-x} F(x) + g_1(x)e^{-x} - g'_1(x)e^{-x} \\ &= e^{-x} \{ \mathbb{E}_x[e^{-r\tau_I} (g'(X_{\tau_I}) - g(X_{\tau_I}))] + g_1(x) \\ &\quad - g'_1(x) \} = e^{-x} \{ \mathbb{E}_x[e^{-r\tau_I^+} (g'_2(X_{\tau_I}) - g_2(X_{\tau_I}))]; \\ &\quad \{ \tau_I = \tau_I^+ \} \} + \mathbb{E}_x[e^{-r\tau_I^-} (g'_1(X_{\tau_I}) - g_1(X_{\tau_I}))]; \\ &\quad \{ \tau_I = \tau_I^- \} \} + g_1(x) - g'_1(x) \\ &\geq e^{-x} \mathbb{E}_x[e^{-r\tau_I^+} (g'_2(X_{\tau_I}) - g_2(X_{\tau_I}))]; \{ \tau_I = \tau_I^+ \} \\ &\quad + e^{-x} (g_1(x) - g'_1(x)) (1 - \mathbb{E}_x[e^{-r\tau_I^-}; \{ \tau_I = \tau_I^- \}]), \end{aligned}$$

where $\tau_I^+ = \inf\{t \geq 0 \mid X_t \geq h_2\}$ and $\tau_I^- = \inf\{t \geq 0 \mid X_t \leq h_1\}$. With respect to the last inequality, the fact that $g_1(x) - g'_1(x)$ is increasing, therefore $g_1(X_{\tau_I^-}) - g'_1(X_{\tau_I^-}) \leq g_1(h_1) - g'_1(h_1) \leq g_1(x) - g'_1(x)$ is utilized. Since $g_2(x) - g'_2(x)$ is negative and $g_1(x) - g'_1(x)$ is positive, $H'(x) \geq 0$ is obtained, implying that $H(x)$ is increasing. Therefore, $H(x) \geq H(h_1) = 0$ and so $\tilde{V}(x) \geq g(x)$. By a similar argument, we obtain $\tilde{V}(x) \geq g(x)$ for $l_2 \leq x \leq h_2$. The proof is complete. \square

To prove the optimality of the candidate \tilde{V} , the fact that the candidate function satisfies condition (d) of the verification lemma remains to be checked. In fact, this part of the optimal stopping problem is the most challenging. To do so, the reward functions for the perpetual American strangles are considered and the process X is assumed to follow the hyper-exponential jump-diffusion model introduced by Levendorskii (2004).

3. Perpetual American strangles and straddles

A strangle is a financial instrument whose reward function is a combination of a put with strike price K_1 and a call with strike price K_2 written on the same security, where $K_1 \leq K_2$. If $K_1 = K_2$, then the strangle is a straddle. The price of the underlying security under the chosen risk-neutral measure is modeled using a geometric jump-diffusion: $S_t = \exp\{X_t\}$, where X is a hyper-exponential jump-diffusion process (HEJD). Accordingly, the jump density function f is a mixture of exponential distributions

$$f(x) = \sum_{i=1}^{N^+} p_i \eta_i^+ e^{-\eta_i^+ x} 1_{\{x>0\}} + \sum_{j=1}^{N^-} q_j (-\eta_j^-) e^{-\eta_j^- x} 1_{\{x<0\}}, \quad (9)$$

where $\eta_1^- < \dots < \eta_{N^-}^- < 0 < \eta_1^+ < \dots < \eta_{N^+}^+$, and p_i and q_j are positive with $\sum_{i=1}^{N^+} p_i + \sum_{j=1}^{N^-} q_j = 1$. (A Lévy model involves infinitely many equivalent risk-neutral measures and, for pricing purposes, one of them is typically selected using the so-called Esscher transform. Notably, this transform preserves the above jump-diffusion structure. For details, see Levendorskii (2004) and the appendix of Asmussen *et al.* (2004).) The characteristic exponent of this jump-diffusion process X is given by the formula

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + cz + \lambda \left(\sum_{i=1}^{N^+} \frac{p_i \eta_i^+}{\eta_i^+ - z} + \sum_{j=1}^{N^-} \frac{q_j \eta_j^-}{\eta_j^- - z} \right) - \lambda.$$

To guarantee that $\mathbb{E}[e^{X_1}] < \infty$, assume that $\eta_i^+ > 1$ for $i = 1, 2, \dots, N^+$. Also, assume that $\psi(1) < r$ (such that

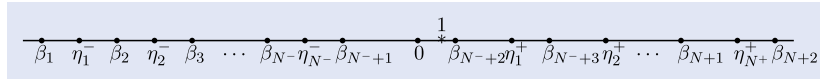


Figure 1. Relationship between $\eta_i^-, 1 \leq i \leq N^-$, $\eta_j^+, 1 \leq j \leq N^+$, and $\beta_n, 1 \leq n \leq N + 2$.

the underlying asset pays dividends continually). (Notably, if $\mathbb{E}[e^{X_1}] < e^r$ and $0 \leq g(x) \leq A + B e^x$ for some constants A and B , then $\mathbb{E}[\sup_{t \geq 0} e^{-rt} g(X_t)] < \infty$. For details, see lemma 4.1 of Mordecki and Salminen (2002).) (figure 1)

The rational price for the perpetual American strangle is the value function that solves the optimal stopping problem (1) with the reward function g , which is given by the formula

$$g(x) = (K_1 - e^x)^+ + (e^x - K_2)^+ = g_1(x)1_{\{x \leq l_1\}} + g_2(x)1_{\{x \geq l_2\}}, \tag{10}$$

where $l_1 = \ln K_1$, $l_2 = \ln K_2$, $g_1(x) = K_1 - e^x$ and $g_2(x) = e^x - K_2$. Denote by $\{\beta_n\}_{n=1}^{N+2}$ the set of all roots to the polynomial

$$\begin{aligned} \phi(x) = & \prod_{i=1}^{N^+} (\eta_i^+ - x) \prod_{j=1}^{N^-} (\eta_j^- - x) \left[\frac{1}{2} \sigma^2 x^2 + cx \right. \\ & \left. - (\lambda + r) + \lambda \left(\sum_{i=1}^{N^+} \frac{p_i \eta_i^+}{\eta_i^+ - x} + \sum_{j=1}^{N^-} \frac{q_j \eta_j^-}{\eta_j^- - x} \right) \right]. \end{aligned}$$

(Notably, $N = N^- + N^+$, $\beta_1 < \eta_1^- < \beta_2 < \eta_2^- < \dots < \beta_{N^-} < \eta_{N^-}^- < \beta_{N^-+1} < 0 < \beta_{N^-+2} < \eta_1^+ < \dots < \beta_{N+1} < \eta_{N+}^+ < \beta_{N+2}$ and $\beta_{N+2} > 1$. For details, see the investigations of Levendorskiĭ (2004) and Boyarchenko (2006).) To find a candidate for the corresponding value function, first consider the function $V(x) = V_I(x)$ in (7) for some interval $I = (h_1, h_2)$ with $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$. Boyarchenko (2006) obtained that Wiener–Hopf factorizations yield a function of the form

$$V(x) = \begin{cases} \sum_{n=1}^{N+2} C_n e^{\beta_n x}, & \text{if } x \in (h_1, h_2), \\ g(x), & \text{if } x \in (h_1, h_2)^c, \end{cases} \tag{11}$$

where C_n are constants depending on h_1 and h_2 . (For the explicit formula for C_n , see theorem 3.2 of Boyarchenko (2006). For an ODE approach, see Chang *et al.* (2013).) The function $V(x)$ will be shown to be the value function of the optimal stopping problem (1) under the smooth pasting conditions. Doing so requires the following lemma.

Lemma 3.1 (a) Assume that $V(x)$ satisfies the smooth pasting condition at $x = h_2$ and $V(x) \geq e^x - K_2$ in $(h_2 - \epsilon, h_2)$ for some $\epsilon > 0$, then $V''(h_2-) \geq e^{h_2}$.

(b) Assume that $V(x)$ satisfies the smooth pasting condition at $x = h_1$ and $V(x) \geq K_1 - e^x$ in $(h_1, h_1 + \epsilon)$ for some $\epsilon > 0$, then $V''(h_1+) \geq -e^{h_1}$.

Proof Let $F(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x} - (e^x - K_2)$. Then $F(x) \in C^\infty$. Since $V(x)$ satisfies the continuity condition and the smooth pasting condition at $x = h_2$, $F(h_2) = F'(h_2) = 0$. By Taylor’s theorem, there exists $\theta_n \in (h_2 - (1/n), h_2)$ such that

$$\begin{aligned} F\left(h_2 - \frac{1}{n}\right) - F(h_2) = & F'(h_2) \left(-\frac{1}{n}\right) \\ & + \frac{1}{2} F''(\theta_n) \left(-\frac{1}{n}\right)^2. \end{aligned} \tag{12}$$

Consider $1/n < \epsilon$. Equation (12) implies that $F''(\theta_n) \geq 0$. As n approaches ∞ , $F''(h_2-) \geq 0$, so the proof of part (a) is complete. The proof of part (b) is similar to that of part (a) and is omitted. \square

Theorem 3.2 Given an interval (h_1, h_2) with $-\infty < h_1 < \ln K_1 \leq \ln K_2 < h_2 < \infty$, set $V(x) = V_I(x)$. Assume that $V(x)$ satisfies the smooth pasting condition on the boundaries h_1 and h_2 , such that $V'(h_1) = -e^{h_1}$ and $V'(h_2) = e^{h_2}$. Then $V(x) = \mathbb{E}_x[e^{-r\tau_{(h_1, h_2)^c}} g(X_{\tau_{(h_1, h_2)^c}})]$ is the value function of the optimal stopping problem and so $\tau_{(h_1, h_2)^c}$ is an optimal stopping time for the optimal stopping problem (1).

Proof Since V_I is of the form of (11) for some constants C_n , clearly $V(x)$ satisfies conditions (b) and (c) of theorem 2.2. Notably, $V_I \in C^1$ and $V_I''(x)$ exists and is continuous except possibly at $x = h_1$ and $x = h_2$. Additionally, $\lim_{x \rightarrow h_i^\pm} V_I''(x)$ exists for $i = 1, 2$. Therefore, $V_I(x)$ is the difference of two convex functions (see problem 6.24 of Karatzas and Shreve (1991), p. 215). By direct calculation, V can easily be verified also to satisfy condition (a) of proposition 2.3. On $(-\infty, l_1)$, $g_1(x) - g_1'(x) = K_1$ is positive and increasing, and on (l_2, ∞) , $g_2(x) - g_2'(x) = -K_2$ is negative and decreasing. By proposition 2.3, $V(x) \geq g(x)$ for all $x \in (h_1, h_2)$. That $V(x)$ satisfies condition (d) of theorem 2.2 remains to be verified. Direct calculation reveals that, for $x > h_2$,

$$\begin{aligned} (\mathcal{L}_X - r)V(x) = & (\psi(1) - r)(e^x - K_2) + \lambda \\ & \sum_{j=1}^{N^-} q_j (-\eta_j^-) e^{\eta_j^- x} \int_{-\infty}^{h_1} (K_1 + K_2 - 2e^y) e^{-\eta_j^- y} dy \\ & + \lambda \sum_{j=1}^{N^-} q_j (-\eta_j^-) e^{\eta_j^- x} \\ & \times \int_{h_1}^{h_2} \left(\sum_{n=1}^{N+2} C_n e^{\beta_n y} - e^y + K_2 \right) (e^{-\eta_j^- y}) dy, \end{aligned}$$

and hence

$$\begin{aligned} \frac{d}{dx} (\mathcal{L}_X - r)V(x) = & (\psi(1) - r) e^x - \lambda \sum_{j=1}^{N^-} q_j (\eta_j^-)^2 e^{\eta_j^- x} \\ & \int_{-\infty}^{h_1} (K_1 + K_2 - 2e^y) e^{-\eta_j^- y} dy \\ & - \lambda \sum_{j=1}^{N^-} q_j (\eta_j^-)^2 e^{\eta_j^- x} \\ & \times \int_{h_1}^{h_2} \left(\sum_{n=1}^{N+2} C_n e^{\beta_n y} - e^y + K_2 \right) (e^{-\eta_j^- y}) dy. \end{aligned}$$

Notably, $\psi(1) < r$, $K_1 + K_2 - 2e^y > 0$ for $y \leq h_1$ and $\sum_{n=1}^{N+2} C_n e^{\beta_n x} = V(x) \geq g(x) \geq (e^x - K_2)^+ \geq e^x - K_2$ for $x \in (h_1, h_2)$. Therefore, $(d/dx)(\mathcal{L}_X - r)V(x) \leq 0$ and $(\mathcal{L}_X - r)V(x)$ is decreasing on (h_2, ∞) . Similarly, for $x < h_1$,

$$\begin{aligned}
 (\mathcal{L}_X - r)V(x) &= (\psi(1) - r)(K_1 - e^x) + \lambda \sum_{i=1}^{N^+} p_i \eta_i^+ e^{\eta_i^+ x} \\
 &\times \int_{h_2}^{\infty} (2e^y - K_1 - K_2) e^{-\eta_j^+ y} dy \\
 &+ \lambda \sum_{i=1}^{N^-} p_i \eta_i^+ e^{\eta_i^+ x} \int_{h_1}^{h_2} \\
 &\times \left(\sum_{n=1}^{N+2} C_n e^{\beta_n y} - K_1 + e^y \right) e^{-\eta_i^+ y} dy,
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \frac{d}{dx}(\mathcal{L}_X - r)V(x) &= -(\psi(1) - r)e^x + \lambda \sum_{i=1}^{N^+} p_i (\eta_i^+)^2 e^{\eta_i^+ x} \\
 &\times \int_{h_2}^{\infty} (2e^y - K_1 - K_2) e^{-\eta_j^+ y} dy \\
 &+ \lambda \sum_{i=1}^{N^-} p_i (\eta_i^+)^2 e^{\eta_i^+ x} \int_{h_1}^{h_2} \\
 &\times \left(\sum_{n=1}^{N+2} C_n e^{\beta_n y} - K_1 + e^y \right) e^{-\eta_i^+ y} dy.
 \end{aligned}$$

Notably, $2e^y - K_1 - K_2 > 0$ for $y \geq h_2$ and $\sum_{n=1}^{N+2} C_n e^{\beta_n x} = V(x) \geq g(x) \geq (K_1 - e^x)^+ \geq K_1 - e^x$ for $x \in (h_1, h_2)$. Therefore, $(d/dx)(\mathcal{L}_X - r)V(x) \geq 0$ and so $(\mathcal{L}_X - r)V(x)$ is increasing on $(-\infty, h_1)$.

Notably, $(\mathcal{L}_X - r)V(x) = 0$ for $x \in (h_1, h_2)$. Since $V(x)$ satisfies the continuity condition and the smooth pasting condition at $x = h_2$ and $V(x) \geq e^x - K_2$ in (h_1, h_2) , by lemma 3.1(a),

$$\begin{aligned}
 (\mathcal{L}_X - r)V(h_2+) &= (\mathcal{L}_X - r)V(h_2+) - (\mathcal{L}_X - r)V(h_2-) \\
 &= \frac{1}{2} \sigma^2 (e^{h_2} - V''(h_2-)) \leq 0.
 \end{aligned}$$

The fact that $(\mathcal{L}_X - r)V(x)$ is decreasing on (h_2, ∞) implies that $(\mathcal{L}_X - r)V(x) \leq 0$ on (h_2, ∞) . Similarly, since $V(x)$ satisfies the continuous condition and the smooth pasting condition at $x = h_1$ and $V(x) \geq K_1 - e^x$ in (h_1, h_2) , by lemma 3.1(b),

$$\begin{aligned}
 (\mathcal{L}_X - r)V(h_1-) &= (\mathcal{L}_X - r)V(h_1-) - (\mathcal{L}_X - r)V(h_1+) \\
 &= \frac{1}{2} \sigma^2 (-e^{h_1} - V''(h_1+)) \leq 0.
 \end{aligned}$$

The fact that $(\mathcal{L}_X - r)V(x)$ is increasing on $(-\infty, h_1)$ implies that $(\mathcal{L}_X - r)V(x) \leq 0$ on $(-\infty, h_1)$. Therefore, condition (d) of theorem 2.2 is verified. The proof is complete. \square

Remark 2 In the case $\sigma = 0$, the continuous pasting condition plays exactly the same role as the smooth pasting condition in the case $\sigma > 0$. Therefore, other methods are used to solve the optimal stopping problems. See, for example, Boyarchenko and Levendorskiĭ (2002a,b) and Peskir and Shiryaev (2006).

4. Numerical results

In this section, the results of Boyarchenko (2006) are first quoted to find the optimal boundaries h_1 and h_2 . Let Δh be a positive solution of the equation $\det B(h) = 0$, where $B(h)$ is an $(N + 2) \times (N + 2)$ matrix that is defined by the formula

$$B(h) = \begin{bmatrix} \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^-}} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^-}} \\ \frac{1}{\beta_1 - \eta_1^+} e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} e^{\beta_{N+2} h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^+}^+} e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^+}^+} e^{\beta_{N+2} h} \\ \frac{1}{\beta_1} (1 + \frac{K_1}{K_2} e^{\beta_1 h}) & \cdots & \frac{1}{\beta_{N+2}} (1 + \frac{K_1}{K_2} e^{\beta_{N+2} h}) \\ \frac{1}{\beta_1 - 1} (1 + e^{(\beta_1 - 1)h}) & \cdots & \frac{1}{\beta_{N+2} - 1} (1 + e^{(\beta_{N+2} - 1)h}) \end{bmatrix}. \tag{13}$$

Boyarchenko (2006) observed that the optimal boundaries h_1 and h_2 are given by the formulas $h_1 = \ln \det A_1 - \ln \det A_2$ and $h_2 = h_1 + \Delta h$. Here

$$\begin{aligned}
 A_1 &= \begin{bmatrix} \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^-}} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^-}} \\ \frac{1}{\beta_1 - \eta_1^+} e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} e^{\beta_{N+2} \Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^+}^+} e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^+}^+} e^{\beta_{N+2} \Delta h} \\ \frac{1}{\beta_1} (1 + \frac{K_1}{K_2} e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}} (1 + \frac{K_1}{K_2} e^{\beta_{N+2} \Delta h}) \\ \frac{1}{\beta_1 - 1} & \cdots & \frac{1}{\beta_{N+2} - 1} \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^-}} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^-}} \\ \frac{1}{\beta_1 - \eta_1^+} e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} e^{\beta_{N+2} \Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^+}^+} e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^+}^+} e^{\beta_{N+2} \Delta h} \\ \frac{1}{\beta_1} (1 + \frac{K_1}{K_2} e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}} (1 + \frac{K_1}{K_2} e^{\beta_{N+2} \Delta h}) \\ \frac{1}{\beta_1 K_1} & \cdots & \frac{1}{\beta_{N+2} K_1} \end{bmatrix}.
 \end{aligned}$$

For details, see Boyarchenko (2006). This optimal interval (h_1, h_2) , together with theorem 3.2 of Boyarchenko (2006), yields explicit formulas for the value function (rational price) of the strangle options.

Example 4.1 Consider the case in which $N^+ = N^- = 1$. Additionally, as in Boyarchenko (2006), set $c = -0.105$, $\sigma = 0.25$, $r = 0.06$, $\eta^+ = 1/0.4$, $\eta^- = -1/0.7$, $\lambda = 3/5$, $p = q = 0.5$ and the strike prices $K_1 = 50$ and $K_2 = 100$. Then, the value function is $V(x) = \sum_{n=1}^4 C_n e^{\beta_n x}$ in (h_1^*, h_2^*) , where $(h_1^*, h_2^*) = (2.1992, 6.1953)$, $\{\beta_1, \beta_2, \beta_3, \beta_4\} = \{-3.4812, -0.2322, 1.1995, 6.953\}$ and $\{C_1, C_2, C_3, C_4\} = \{2519.533, 61.2124, 0.2183, 1.4624 \times 10^{-18}\}$. Furthermore, if we take $N^+ = N^- = 0$ (which is the diffusion case), then $V(x) = \sum_{n=1}^2 C_n e^{\beta_n x}$ in (h_1^*, h_2^*) , where $(h_1^*, h_2^*) = (3.4151, 4.859)$, $\{\beta_1, \beta_2\} = \{-1.5607, 4.9207\}$ and $\{C_1, C_2\} = \{4037.8534, 1.1088 \times 10^{-9}\}$ (figure 2).

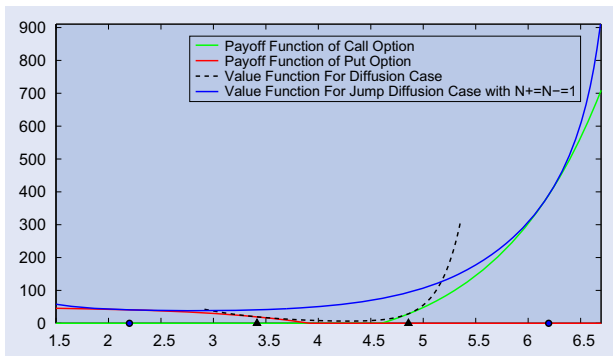


Figure 2. The solid line is the value function $V(x)$ for the jump-diffusion model with $N^+ = N^- = 1$ and the dashed line is that for the diffusion model, that is $N^+ = N^- = 0$. The optimal boundaries are marked by circles for the jump-diffusion model, and by triangles for the diffusion model.

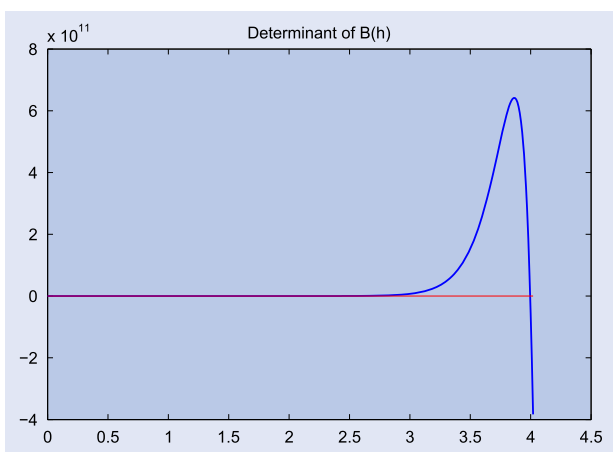


Figure 3. A graph of the determinant $B(h)$ for finding the length Δh of the optimal interval. It shows that there is only one zero for the determinant.

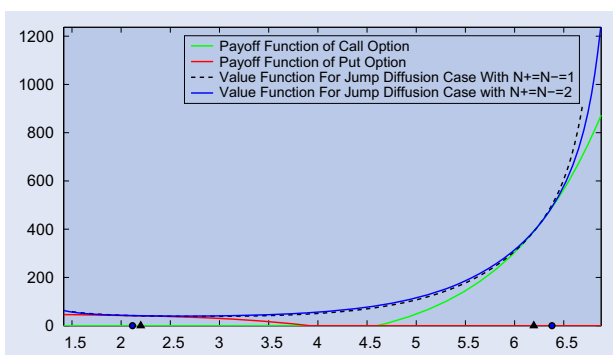


Figure 4. The solid line is the value function $V(x)$ for the jump-diffusion model with $N^+ = N^- = 2$ and the dashed line is that for the model with $N^+ = N^- = 1$. The optimal boundaries for the case $N^+ = N^- = 2$ are marked by circles and by triangles for the case $N^- = N^+ = 1$.

Interestingly, in the jump-diffusion model, the optimal interval (h_1^*, h_2^*) is much wider than that in the diffusion case. Figure 3 plots the determinant of $B(h)$ against h . The zero of the determinant (this is Δh) is observed to be unique. The graph descends sharply close to this zero of the determinant, implying

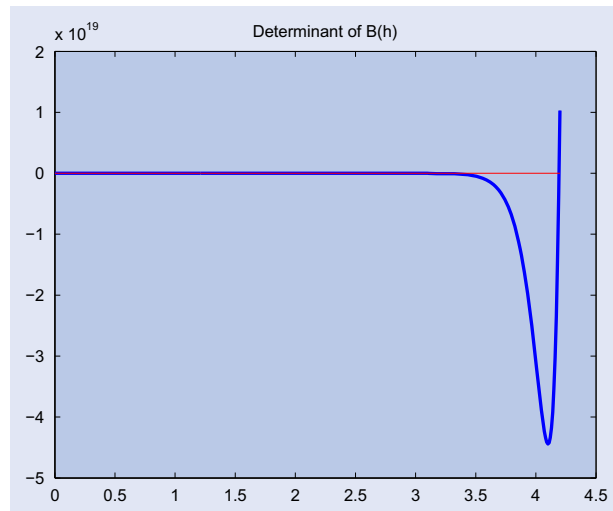


Figure 5. A graph of the determinant for finding the length of the optimal interval for the case $N^- = N^+ = 2$. The figure has similar properties as for the case $N^- = N^+ = 1$. In particular, there is only one zero for the determinant.

that the numerical result for Δh can be obtained accurately and fast.

Example 4.2 Consider the jump-diffusion model with $N^- = N^+ = 2$ and let $c = -0.105$, $\sigma = 0.25$, $r = 0.06$, $\eta_1^+ = 1/0.5$, $\eta_2^+ = 1/0.25$, $\eta_1^- = -1/2.4$, $\eta_2^- = -7.5$, $\lambda = 3/5$, $p_1 = p_2 = q_1 = q_2 = 0.25$ and the strike prices $K_1 = 50$ and $K_2 = 100$. In this model, the expected value $E[e^{X_1}]$ is the same as that with $N^- = N^+ = 1$ in example 4.1. The value function is $V(x) = \sum_{n=1}^6 C_n e^{\beta_n x}$ in (h_1^*, h_2^*) , where $(h_1^*, h_2^*) = (2.1153, 6.3801)$, $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\} = \{-7.997, -1.9409, -0.1155, 1.1642, 3.2421, 7.0931\}$ and $\{C_1, C_2, C_3, C_4, C_5, C_6\} = \{735.200.1029, 240.6048, 44.1297, 0.2679, 8.8413 \times 10^{-9}, 2.4671 \times 10^{-19}\}$ (figures 4 and 5).

5. Concluding remarks

American option contracts are more difficult to analyse than their European counterparts, because an American option can be exercised at any time prior to its expiration. Mathematically, therefore, the optimal stopping problem of the form of (1) must be solved. Unlike the corresponding PDEs for the European counterparts, such problems always lead to so-called free boundary value problems, which are not easy to solve. No explicit formulas exist for relevant value functions and the optimal exercise boundaries are unknown.

American call and put options are the simplest American contracts. The problem of pricing these options has been extensively studied and generalized since the work of McKean (1965) and Merton (1973). Recent studies on Lévy-model settings include Boyarchenko and Levendorskiĭ (2002a,b); Mordecki and Salminen (2002), Asmussen *et al.* (2004), Levendorskiĭ (2004), and the references therein.

This paper considers the perpetual American strangle and straddle options, each of which is a combination of a put and a call written on a single security. As in the studies of Asmussen *et al.* (2004) and many others, we consider the problems of pricing these options under a jump-diffusion model.

The free boundary problem approach and the work of Boyarchenko (2006) are utilized to solve the corresponding optimal stopping problems and thereby find the optimal exercise boundaries and the rational prices of the perpetual American strangle and straddle options. The present work was inspired by Boyarchenko (2006) and Boyarchenko and Boyarchenko (2011). In fact, Boyarchenko (2006) studied the same pricing problems and posed the verification of the smooth pasting principle for the value functions as an open problem. This study solves this open problem in theorem 3.2. The method presented here together with the general results in section 2 may provide an alternative method for computing the prices of other exotic options in jump-diffusion models. We leave such computations for future research.

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