Journal of Combinatorial Theory, Series B **71**, 231–243 (1997)
Article No. TB971787

Parallelogram-Free Distance-Regular Graphs

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Received November 7, 1996

Let $\Gamma = (X, R)$ denote a distance-regular graph with distance function ∂ and diameter $d \geqslant 4$. By a parallelogram of length i $(2 \leqslant i \leqslant d)$, we mean a 4-tuple $xyzu$ of vertices in X such that $\partial(x, y) = \partial(z, u) = 1$, $\partial(x, u) = i$, and $\partial(x, z) = \partial(y, z) = \partial(y, u) = i - 1$. We prove the following theorem. THEOREM. Let Γ denote a distance-regular graph with diameter $d \geqslant 4$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Suppose Γ is Q-polynomial and contains no parallelograms of length 3 and no parallelograms of length 4. Then Γ has classical parameters (d, b, α, β) with $b < -1$. By including results in [3], [9], we have the following corollary. COROLLARY. Let Γ denote a distance-regular graph with the Q-polynomial property. Suppose the diameter $d \geqslant 4$. Then the following (i)–(ii) are equivalent. (i) Γ contains no parallelograms of any length. (ii) One of the following (iia)–(iic) holds. (iia) Γ is bipartite. (iib) Γ is a generalized odd graph. (iic) Γ has classical parameters (d, b, α, β) and either $b < -1$ or Γ is a Hamming graph or a dual polar graph.

1. INTRODUCTION

It is well known that a distance-regular graph with classical parameters has the Q-polynomial property [2, Theorem 8.4.1]. For the converse, Brouwer, Cohen, Neumaier proved that a Q-polynomial regular near polygon with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$ has classical parameters [2, Theorem 8.5.1]. In [8], we proved a generalized version of above result.

In this paper, we consider a Q-polynomial distance-regular graph with $a_1 = 0$, $a_2 \neq 0$ and prove that it has classical parameters if its diameter is at least 4 and if it contains no 4-vertex configurations of a certain type (*parallelograms*). See Theorem 2.11 for details. By including some results in [3, 9], we give a classification of parallelogram-free Q-polynomial distance-regular graphs in Corollary 2.12.

For the rest of this section, we review some definitions and basic concepts. See Bannai and Ito [1] or Terwilliger [5] for more background information.

0095-8956/97 $25.00

Throughout this paper, $\Gamma = (X, R)$ will denote a finite, connected, undirected graph without loops, or multiple edges, and with vertex set X, edge set R, path length distance function ∂, and diameter $d := \max\{\partial(x, y) \mid x, y \in X\}$. Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant d)$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

is independent of x, y. The constants p^h_{ij} $(0 \leqslant h, i, j \leqslant d)$ are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1\,i-1}$ $(1 \leqslant i \leqslant d)$, $a_i := p^i_{1\,i}$ $(0 \leqslant i \leqslant d)$, $b_i := p^i_{1\,i+1} (0 \leqslant i \leqslant d-1)$, and put $b_d := 0$, $c_0 := 0$, $k = b_0$. It is immediate from the definition that $b_i \neq 0$ $(0 \leqslant i \leqslant d-1)$, $c_i \neq 0$ $(1 \leqslant i \leqslant d)$, and

$$k = b_0 = a_i + b_i + c_1 \qquad (1 \leqslant i \leqslant d). \tag{1.1}$$

We refer to k as the *valency* of Γ.

A distance-regular graph Γ is called *bipartite* whenever $a_1 = a_2 = \cdots = a_d = 0$. Γ is called a *generalized odd graph* whenever $a_1 = a_2 = \cdots = a_{d-1} = 0$, $a_d \neq 0$.

From now on, we fix a distance-regular graph with diameter $d \geqslant 3$. Let p^h_{ij} $(0 \leqslant h, i, j \leqslant d)$ denote the intersection numbers of Γ.

Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over the real number field with the rows and columns indexed by the elements of X. The *distance matrices* of Γ are the matrices $A_0, A_1, ..., A_d \in \mathrm{Mat}_X(\mathbb{R})$, defined by the rule

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X).$$

Then

$$A_0 = I, \tag{1.2}$$

$$A_0 + A_1 + \cdots + A_d = J \qquad (J = \text{all 1's matrix}), \tag{1.3}$$

$$A_i^t = A_i \qquad (0 \leqslant i \leqslant d), \tag{1.4}$$

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \qquad (0 \leqslant i, j \leqslant d), \tag{1.5}$$

$$A_i A_j = A_j A_i \qquad (0 \leqslant i, j \leqslant d). \tag{1.6}$$

Let M denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, ..., A_d$. Then M is a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{R})$, and is known as the

Bose-Mesner algebra of Γ. By [1, p. 59, p. 64], M has a second basis $E_0, E_1, ..., E_d$ such that

$$E_0 = |X|^{-1} J, \tag{1.7}$$

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leqslant i, j \leqslant d), \tag{1.8}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{1.9}$$

$$E_i^t = E_i \qquad (0 \leqslant i \leqslant d). \tag{1.10}$$

The $E_0, E_1, ..., E_d$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent. Let E denote any primitive idempotent of Γ. Then we have

$$E = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i \tag{1.11}$$

for some $\theta_0^*, \theta_1^*, ..., \theta_d^* \in \mathbb{R}$, called the *dual eigenvalues* associated with E.

Let $\circ$ denote entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij} A_i \qquad (0 \leqslant i, j \leqslant d),$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{R}$ $(0 \leqslant i, j, k \leqslant d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{ij}^k E_k \qquad (0 \leqslant i, j \leqslant d).$$

Γ is said to be *Q-polynomial* with respect to the given ordering $E_0, E_1, ..., E_d$ of the primitive idempotents, if for all integers h, i, j $(0 \leqslant h, i, j \leqslant d)$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any primitive idempotent of Γ. Then Γ is said to be Q-polynomial with respect to E whenever there exists an ordering $E_0, E_1 = E, ..., E_d$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. If Γ is Q-polynomial with respect to E, then the associated dual eigenvalues are distinct [5, p. 384].

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of V as being indexed by X. Then the Bose-Mesner algebra M acts on V by left multiplication. We call V the *standard module* of Γ. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, ..., 0, 1, 0, ..., 0)^t, \tag{1.12}$$

where the 1 is in coordinate x. Also, let $\langle\ ,\ \rangle$ denote the dot product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V). \tag{1.13}$$

Then referring to the primitive idempotent E in (1.11), we compute from (1.10)–(1.13) that

$$\langle E\hat{x}, \hat{y}\rangle = |X|^{-1}\,\theta_i^* \qquad (x, y \in X), \tag{1.14}$$

where $i = \partial(x, y)$.

2. MAIN THEOREM

Throughout this section, we will use the following notations.

DEFINITION 2.1. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$. For all $x, y \in X$, and all integers i, j, define

$$p_{ij}(x, y) := \sum_{\substack{z \in X \\ \partial(x, z) = i \\ \partial(y, z) = j}} \hat{z},$$

where the $\hat{z}$ notation is from (1.12). Further define

$$x_y^- := p_{1\,h-1}(x, y),$$
$$x_y^0 := p_{1\,h}(x, t),$$
$$x_y^+ := p_{1\,h+1}(x, y),$$

where

$$h = \partial(x, y).$$

Our work is based on the following two theorems of Leonard [4], Terwilliger [6, Theorem 3.3] and Terwilliger [7, Theorem 2.6 and Theorem 2.7].

THEOREM 2.2. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and suppose Γ is a Q-polynomial with respect to the primitive idempotent*

$$E_1 = |X|^{-1} \sum_{h=0}^{d} \theta_h^* A_h.$$

Then the following (i)–(ii) hold.

(i) ([4], [6])

$$\theta_{i-2}^* - \theta_{i-1}^* = \sigma(\theta_{i-3}^* - \theta_i^*) \qquad (3 \leqslant i \leqslant d) \tag{2.1}$$

for appropriate $\sigma \in \mathbb{R}\backslash\{0\}$.

(ii) ([6]) *For all integers* h, i, j $(1 \leqslant h \leqslant d)$, $(0 \leqslant i, j \leqslant d)$, *and all* $x, y \in X$ *such that* $\partial(x, y) = h$, *the vector*

$$p_{ij}(x, y) - p_{ji}(x, y) - r_{ij}^{h}(\hat{x} - \hat{y}) \tag{2.2}$$

is orthogonal to $E_0 V + E_1 V$, *where*

$$r_{ij}^{h} = p_{ij}^{h} \left(\frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \right). \tag{2.3}$$

THEOREM 2.3. ([7]) *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with diameter* $d \geqslant 3$, *and suppose* Γ *is Q-polynomial with respect to the primitive idempotent*

$$E_1 = |X|^{-1} \sum_{h=0}^{d} \theta_h^* A_h.$$

Then for all integers h, i, j $(0 \leqslant h, i, j \leqslant d)$, *and all* $x, y \in X$ *such that* $\partial(x, y) = h$, *the vector*

$$\begin{aligned} &p_{ij}(x, y) + p_{ji}(x, y) - s_{ij}^{h}(\hat{x} + \hat{y}) - t_{ij}^{h-1}(x_y^- + y_x^-) \\ &\quad - t_{ij}^{h}(x_y^0 + y_x^0) - t_{ij}^{h+1}(x_y^+ + y_x^+) \end{aligned} \tag{2.4}$$

is orthogonal to $E_0 V + E_1 V$, *where*

(a) $t_{ij}^{-1} = t_{ij}^{0} = t_{ij}^{d+1} = 0 \qquad (0 \leqslant i, j \leqslant d),$ (2.5)

(b) $p_{ij}^{h} = s_{ij}^{h} + c_h t_{ij}^{h-1} + a_h t_{ij}^{h} + b_h t_{ij}^{h+1} \qquad (0 \leqslant h, i, j \leqslant d),$ (2.6)

(c) $p_{ij}^{h}(\theta_i^* + \theta_j^*) = s_{ij}^{h}(\theta_0^* + \theta_h^*) + c_h t_{ij}^{h-1}(\theta_1^* + \theta_{h-1}^*) + a_h t_{ij}^{h}(\theta_1^* + \theta_h^*)$
$+ b_h t_{ij}^{h+1}(\theta_1^* + \theta_{h+1}^*) \qquad (0 \leqslant h, i, j \leqslant d),$
(θ_{-1}^*, θ_{d+1}^* *are indeterminants*). (2.7)

LEMMA 2.4. *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with diameter* $d \geqslant 4$, *and suppose* Γ *is Q-polynomial with respect to the primitive idempotent*

$$E_1 = |X|^{-1} \sum_{h=0}^{d} \theta_h^* A_h.$$

Then

$$\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^* \neq 0.$$

Proof. Suppose

$$\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^* = 0. \tag{2.8}$$

Setting $i=3$ in (2.1), we find $\sigma=1$. Evaluating (2.1) with $\sigma=1$ and $i=4$, we find

$$\theta_1^* - \theta_2^* + \theta_3^* - \theta_4^* = 0. \tag{2.9}$$

Combining (2.8), (2.9), we readily obtain $\theta_0^* = \theta_4^*$, a contradiction. This proves Lemma 2.4.

LEMMA 2.5. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 4$. Suppose Γ is Q-polynomial with respect to the primitive idempotent*

$$E_1 = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i,$$

and let the scalars s_{ij}^h, t_{ij}^h be as in (2.5)–(2.7). Then

$$s_{i\,i-2}^0 = 0, \tag{2.10}$$

$$s_{i\,i-2}^1 = 0, \tag{2.11}$$

$$s_{i\,i-2}^2 = -p_{i\,i-2}^2 \left(\frac{\theta_1^* + \theta_3^* - \theta_{i-2}^* - \theta_i^*}{\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*} \right), \tag{2.12}$$

$$t_{i\,i-2}^1 = 0, \tag{2.13}$$

$$t_{i\,i-2}^2 = 0, \tag{2.14}$$

$$t_{i\,i-2}^3 = b_2^{-1} p_{i\,i-2}^2 \left(\frac{\theta_0^* + \theta_2^* - \theta_{i-2}^* - \theta_i^*}{\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*} \right) \tag{2.15}$$

for all integers i $(2 \leqslant i \leqslant d)$.

Proof. From (2.6), (2.7) (with $h=0$ and $j=i-2$), and since $p_{i\,i-2}^0 = 0$, $c_0 = 0$, $a_0 = 0$, we find

$$0 = s_{i\,i-2}^0 + b_0 t_{i\,i-2}^1,$$

$$0 = s_{i\,i-2}^0 \theta_0^* + b_0 t_{i\,i-2}^1 \theta_1^*.$$

Lines (2.10), (2.13) follow since $\theta_0^* \neq \theta_1^*$ and $b_0 \neq 0$. From (2.6), (2.7) (with $h=1$ and $j=i-2$), and since $p^1_{i\,i-2}=0$, $t^0_{i\,i-2}=0$, $t^1_{i\,i-2}=0$ by (2.5), (2.13), we find

$$0 = s^1_{i\,i-2} + b_1 t^2_{i\,i-2},$$

$$0 = s^1_{i\,i-2}(\theta_0^* + \theta_1^*) + b_1 t^2_{i\,i-2}(\theta_1^* + \theta_2^*).$$

Lines (2.11), (2.14) follow since $\theta_0^* \neq \theta_2^*$ and $b_1 \neq 0$. From (2.6), (2.7) (with $h=2$ and $j=i-2$), and since $t^0_{i\,i-2}=0$, $t^1_{i\,i-2}=0$, $t^2_{i\,i-2}=0$ by (2.13), (2.14), we find

$$p^2_{i\,i-2} = s^2_{i\,i-2} + b_2 t^3_{i\,i-2},$$

$$p^2_{i\,i-2}(\theta_i^* + \theta_{i-2}^*) = s^2_{i\,i-2}(\theta_0^* + \theta_2^*) + b_2 t^3_{i\,i-2}(\theta_1^* + \theta_3^*).$$

Since $b_2 \neq 0$, and since $\theta_0^* + \theta_2^* \neq \theta_1^* + \theta_3^*$ by Lemma 2.4, we may solve these equations to obtain (2.12), (2.15). This proves the lemma.

LEMMA 2.6. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 4$, and suppose Γ is Q-polynomial with respect to the primitive idempotent*

$$E_1 = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i.$$

Then for any vertices $x, y \in X$ with $\partial(x, y) = 2$, and any integer i $(3 \leqslant i \leqslant d)$, the vector

$$(p^2_{i\,i-2})^{-1}\, p_{i\,i-2}(x, y) - \hat{x}\, \frac{(\theta_2^* - \theta_3^*)(\theta_0^* - \theta_{i-2}^*) - (\theta_0^* - \theta_1^*)(\theta_2^* - \theta_i^*)}{(\theta_0^* - \theta_2^*)(\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*)}$$

$$+ \hat{y}\, \frac{(\theta_1^* - \theta_2^*)(\theta_3^* - \theta_i^*) + (\theta_0^* - \theta_3^*)(\theta_1^* - \theta_{i-2}^*)}{(\theta_0^* - \theta_2^*)(\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*)}$$

$$- b_2^{-1} y_x^+\, \frac{\theta_0^* + \theta_2^* - \theta_{i-2}^* - \theta_i^*}{\theta_0^* - \theta_1^* + \theta_2^* - \theta_3^*}$$

is orthogonal to $E_0 V + E_1 V$.

Proof. From (2.4) (with $h=2$ and $j=i-2$), and since $t^1_{i\,i-2}=0$, $t^2_{i\,i-2}=0$ by (2.13), (2.14), respectively, we find the vector

$$p_{i\,i-2}(x, y) + p_{i-2\,i}(x, y) - s^2_{i\,i-2}(\hat{x} + \hat{y}) - t^3_{i\,i-2}(x_y^+ + y_x^+) \qquad (2.16)$$

is orthogonal to $E_0 V + E_1 V$, where $s^2_{i\,i-2}$, $t^3_{i\,i-2}$ are from (2.12), (2.15), respectively. From (2.2) (with $h=2$ and $j=i-2$) we find the vector

$$p_{i\,i-2}(x, y) - p_{i-2\,i}(x, y) - r^2_{i\,i-2}(\hat{x} - \hat{y}) \tag{2.17}$$

is orthogonal to $E_0 V + E_1 V$, where $r^2_{i\,i-2}$ is from (2.3). Setting $i=3$ in (2.17) we find the vector

$$y^+_x - x^+_y - r^2_{31}(\hat{x} - \hat{y}) \tag{2.18}$$

is orthogonal to $E_0 V + E_1 V$. Eliminating $p_{i-2\,i}(x, y)$, x^+_y in (2.16) using (2.17)–(2.18), we find the vector

$$2p_{i\,i-2}(x, y) - (r^2_{i\,i-2} + s^2_{i\,i-2} - t^3_{i\,i-2} r^2_{31})\,\hat{x}$$
$$+ (r^2_{i\,i-2} - s^2_{i\,i-2} - t^3_{i\,i-2} r^2_{31})\,\hat{y} - 2t^3_{i\,i-2} y^+_x$$

is orthogonal to $E_0 V + E_1 V$. The result is now obtained by evaluating the coefficients in the above line using (2.3), (2.12), (2.15), and simplifying.

THEOREM 2.7. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 4$. Suppose the intersection number $a_2 \neq 0$. Pick any 3-tuple xyz of vertices in Γ with $\partial(x, y) = \partial(x, z) = 2$, $\partial(y, z) = 1$, and set*

$$f_i(xyz) := (p^2_{i\,i-2})^{-1}\,|\{u \mid u \in X, \partial(x, u) = i,$$
$$\partial(y, u) = \partial(z, u) = i-2\}| \qquad (3 \leqslant i \leqslant d).$$

Suppose Γ is Q-polynomial with respect to the primitive idempotent

$$E_1 = |X|^{-1} \sum_{i=0}^{d} \theta^*_i A_i.$$

Then

$$f_i(xyz) = \alpha_i f_3(xyz) - \beta_i \qquad (3 \leqslant i \leqslant d), \tag{2.19}$$

where

$$\alpha_i = \frac{(\theta^*_0 + \theta^*_2 - \theta^*_{i-2} - \theta^*_i)(\theta^*_1 - \theta^*_2)}{(\theta^*_0 - \theta^*_1 + \theta^*_2 - \theta^*_3)(\theta^*_{i-2} - \theta^*_{i-1})} \qquad (3 \leqslant i \leqslant d), \tag{2.20}$$

$$\beta_i = \frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_2}\,\frac{(\theta^*_1 - \theta^*_2)(\theta^*_0 - \theta^*_i) + (\theta^*_0 - \theta^*_3)(\theta^*_2 - \theta^*_{i-2})}{(\theta^*_0 - \theta^*_1 + \theta^*_2 - \theta^*_3)(\theta^*_{i-2} - \theta^*_{i-1})}$$
$$- \frac{\theta^*_2 - \theta^*_{i-1}}{\theta^*_{i-2} - \theta^*_{i-1}} \qquad (3 \leqslant i \leqslant d). \tag{2.21}$$

Proof. To get (2.19)–(2.21), compute the inner product of $E_1\hat{z}$ and the vector in Lemma 2.6, and set the result equal 0. The computation is readily carried out once we observe by (1.14) that

$$|X|\,\langle E_1\hat{z}, p_{i\,i-2}(x,y)\rangle = p^2_{i\,i-2}(f_i(xyz)(\theta^*_{i-2}-\theta^*_{i-1})+\theta^*_{i-1}),$$

$$|X|\,\langle E_1\hat{z}, \hat{x}\rangle = \theta^*_2,$$

$$|X|\,\langle E_1\hat{z}, \hat{y}\rangle = \theta^*_1,$$

$$|X|\,\langle E_1\hat{z}, y^+_x\rangle = b_2(f_3(xyz)(\theta^*_1-\theta^*_2)+\theta^*_2).$$

DEFINITION 2.8. A distance-regular graph Γ is said to have *classical parameters* (d, b, α, β) whenever the diameter of Γ is $d \geqslant 2$, and the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1+\alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (0 \leqslant i \leqslant d), \tag{2.22}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}-\begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta-\alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \leqslant i \leqslant d), \tag{2.23}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1+b+b^2+\cdots b^{i-1}. \tag{2.24}$$

LEMMA 2.9. *Let Γ denote a distance-regular graph with diameter $d \geqslant 4$. Then the following* (i)–(ii) *are equivalent.*

(i) *Γ has classical parameters* (d, b, α, β).

(ii) *Γ is Q-polynomial with respect to a primitive idempotent*

$$E_1 = |X|^{-1}\sum_{i=0}^{d}\theta^*_i A_i$$

and $\beta_4 = 0$, *where* β_4 *is defined in* (2.21).

Proof. (i) $\Rightarrow$ (ii). Suppose Γ has classical parameters (d, b, α, β). Then Γ is Q-polynomial with respect to a primitive idempotent

$$E_1 = |X|^{-1}\sum_{i=0}^{d}\theta^*_i A_i,$$

where

$$\theta^*_i - \theta^*_0 = (\theta^*_1 - \theta^*_0) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \qquad (0 \leqslant i \leqslant d) \tag{2.25}$$

[2, p. 250]. Now $\beta_4 = 0$ is obtained by eliminating θ^*_2, θ^*_3, θ^*_4 in (2.21) for $i = 4$ and simplifying.

(ii) ⇒ (i). Suppose $\beta_4 = 0$. Then by setting $i = 4$ in (2.21),

$$(\theta^*_0 - \theta^*_1 + \theta^*_2 - \theta^*_3)(\theta^*_0 - \theta^*_2)(\theta^*_2 - \theta^*_3) - (\theta^*_1 - \theta^*_2)^2 (\theta^*_0 - \theta^*_4) = 0. \tag{2.26}$$

Set

$$b := \frac{\theta^*_1 - \theta^*_0}{\theta^*_2 - \theta^*_1}. \tag{2.27}$$

Then

$$\theta^*_2 = \theta^*_0 + \frac{(\theta^*_1 - \theta^*_0)(b+1)}{b}. \tag{2.28}$$

Eliminating θ^*_2, θ^*_3, θ^*_4 in (2.26) using (2.27) and (2.1) for $i = 3, 4$, we have

$$\frac{(\theta^*_0 - \theta^*_1)^3 (-1 + \sigma)(\sigma b^2 + \sigma b + \sigma - b)}{-b^3 \sigma^2} = 0 \tag{2.29}$$

for appropriate $\sigma \in \mathbb{R} \backslash \{0\}$. Note that $\theta^*_0 \neq \theta^*_1$, and observe that by Lemma 2.4 and by setting $i = 3$ in (2.1), $\sigma \neq -1$. Hence

$$\sigma b^2 + \sigma b + \sigma - b = 0, \tag{2.30}$$

so

$$\sigma^{-1} = \frac{b^2 + b + 1}{b}. \tag{2.31}$$

To prove Γ has classical parameters, in view or Terwilliger [6, Theorem 4.2(iii)], it suffices to prove that

$$\theta^*_i - \theta^*_0 = (\theta^*_1 - \theta^*_0) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \qquad (0 \leqslant i \leqslant d). \tag{2.32}$$

We prove (2.32) by induction on i. The cases $i = 0, 1$ are trivial and the case $i = 2$ is from (2.28). Now suppose $i \geqslant 3$. Then (2.1) implies

$$\theta^*_i = \sigma^{-1}(\theta^*_{i-1} - \theta^*_{i-2}) + \theta^*_{i-3}. \tag{2.33}$$

Evaluate (2.33) using (2.31) and the induction hypothesis, we find $\theta_i^* - \theta_0^*$ is as in (2.32). This proves Lemma 2.9.

DEFINITION 2.10. Let $\Gamma = (X, R)$ denote a distance-regular graph with distance function ∂ and diameter d. By a parallelogram of length i $(2 \leqslant i \leqslant d)$, we mean a 4-tuple $xyzu$ of vertices in X such that $\partial(x, y) = \partial(z, u) = 1$, $\partial(x, u) = i$, and $\partial(x, z) = \partial(y, z) = \partial(y, u) = i - 1$.

THEOREM 2.11. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 4$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Suppose Γ is Q-polynomial and contains no parallelograms of length* 3 *and no parallelograms of length* 4. *Then Γ has classical parameters (d, b, α, β) with $b < -1$.*

Proof. Pick any 3-tuple xyz in Γ with $\partial(x, y) = \partial(x, z) = 2$, $\partial(y, z) = 1$, and let $f_i(xyz)$ be as in Theorem 2.7. Since $a_1 = 0$, we find

$$f_3(xyz) = 0. \tag{2.34}$$

Claim 1. $f_4(xyz) = 0$.

Proof of Claim 1. Suppose $f_4(xyz) \neq 0$, and pick u such that $\partial(y, u) = \partial(z, u) = 2$ and $\partial(x, u) = 4$. Now pick $w \in X$ with $\partial(u, w) = \partial(y, w) = 1$. Observe $\partial(w, z) \neq 1$, otherwise $a_1 \neq 0$. Hence $\partial(w, z) = 2$. Now pick $v \in X$ with $\partial(x, v) = \partial(z, v) = 1$. Observe $\partial(u, v) = 3$, and $2 \leqslant \partial(v, w) \leqslant 3$. Suppose $\partial(v, w) = 2$. Then the 4-tuple $uwzv$ is a parallelogram of length 3, contradicting our assumption. Hence $\partial(v, w) = 3$. But now the 4-tuple $xvwu$ is a parallelogram of length 4, also a contradiction. Hence $f_4(xyz) = 0$.

Claim 2. Γ has classical parameters (d, b, α, β) with $b < -1$.

Proof of Claim 2. Setting $i = 4$ in (2.19) we find $\beta_4 = 0$ by (2.34) and by Claim 1. Hence Γ has classical parameters (d, b, α, β) by Lemma 2.9. Now from (1.1), (2.22), (2.23), and since $a_1 = 0$, $a_2 \neq 0$,

$$\begin{aligned} -\alpha(b+1)^2 &= a_2 - (b+1)\, a_1 \\ &= a_2 \\ &> 0. \end{aligned} \tag{2.35}$$

Hence

$$\alpha < 0. \tag{2.36}$$

By direct calculation from (2.22), we get

$$(c_2 - b)(b^2 + b + 1) = c_3 > 0. \tag{2.37}$$

Since b is an integer and $b \neq 0, -1$ [2, p. 195], we have

$$b^2 + b + 1 > 0.$$

Then from (2.37), we get

$$c_2 > b. \tag{2.38}$$

By using (2.22), (2.38), we get

$$\alpha(1 + b) = c_2 - b - 1 \geqslant 0.$$

Hence $b < -1$, by (2.36) and since $b \neq -1$.

COROLLARY 2.12. *Let Γ denote a distance-regular graph with the Q-polynomial property. Suppose the diameter $d \geqslant 4$. Then the following* (i)–(ii) *are equivalent.*

(i) *Γ contains no parallelograms of any length.*

(ii) *One of the following* (iia)–(iic) *holds.*

(iia) *Γ is bipartite.*

(iib) *Γ is a generalized odd graph.*

(iic) *Γ has classical parameters (d, b, α, β) and either $b < -1$ or Γ is a Hamming graph or a dual polar graph.*

(see [2] for the definitions and basic properties of Hamming graphs and dual polar graphs).

Proof. (ii) $\Rightarrow$ (i). It is clear that a bipartite graph and a generalized odd graph contain no parallelograms of any length. It is well known that the hamming graphs and the dual polar graphs contain no parallelograms of any length [9, Lemma 7.3]. Suppose Γ has classical parameters (d, b, α, β) with $b < -1$. Then Γ contains no parallelograms of any length by [7, Theorem 2.12] and [9, Lemma 7.3].

(i) $\Rightarrow$ (ii). If $a_2 = 0$ then (iia) or (iib) holds by [3, Lemma 2.3]. If $a_1 \neq 0$, then (iic) holds by [8, Theorem 2.6], [9, Lemma 7.3]. Suppose $a_2 \neq 0$, $a_1 = 0$. Then Γ has classical parameters (d, b, α, β) with $b < -1$ by Theorem 2.11.

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