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A Close Form Solution for the Product Acceptance Determination Based on the Popular Index C_{pk}

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Product acceptance determinations are practical tools for quality control applications involving quality contract on product orders between the vendor and the buyer. It provides the vendor and the buyer rules for product acceptance to meet the preset product quality requirement. As the rapid advancement of manufacturing technology, more than one quality characteristic must be simultaneously considered to improve the product quality because of the product design. In this article, we introduce an efficient product acceptance procedure on the basis of the generalization C_{pk}^{T} index, to deal with lot sentencing problem with very low fraction of defectives. We tabulate the required sample size n and the corresponding critical acceptance value c_0 for various α-risk, β-risk, and the levels of the lot fraction of defectives that correspond to acceptance and rejecting quality levels. Practitioners can use the proposed method to make reliable decisions in product acceptance.

Keywords: critical acceptance value; multiple characteristics; product acceptance determination; required sample size

1. Introduction

Product acceptance determination provides the vendor and the buyer a general criterion for lot sentencing while meeting their preset requirements on product quality. It basically consists of a sample size for inspection and an acceptance criterion and is usually based on the operating characteristic (OC) curve, which quantifies the risk for vendors and buyers. The OC curve plots the probability of accepting the lot against actual lot fraction defective, which displays the discriminatory power of the product acceptance $1-\alpha$ determination rule. The vendor is primarily interested in insuring that good lots would be accepted, and the buyer wants to be reasonably sure that bad product would be rejected. Therefore, two designated points, (AQL, α) and (LTPD, β), on the OC curve are focused. Acceptable quality level (AQL) presents the poorest quality for the vendor process that consumer would consider acceptable as a process average. Lot tolerance percent defective (LTPD) is the poorest quality level that the consumer is willing to accept. α and β are called the vendor's risk and the buyer's risk, respectively.

Pearn and Wu[1] developed an effective decision making method for product acceptance on the basis of the C_{pk} index. The results attended are very practical but are restricted to process with only one quality characteristic. In this article, we extend the results to cases with multiple characteristics based on the generalization C_{pk}^{T} index.

2. The generalization C_{pk}^{T} index

The generalization C_{pk}^{T} index was proposed by Pearn *et al.*,[2] designed as

$$C_{pk}^{T} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{i=1}^{m}\left(2\Phi\left(3C_{pki}\right)-1\right)+1\right]/2\right\} \tag{1}$$

where C_{pki} denotes the C_{pk} value of the ith characteristic for $i=1, 2,\ldots, m$, and m is the number of characteristics. Function $\Phi(\cdot)$ means the cumulative distribution of standard normal distribution. For a normally distributed process with a specific C_{pk}^{T} value, the lower bound on overall process yield P can be established as

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$$P \geqslant \prod_{i=1}^{m} \left[2\Phi(3C_{pki}) - 1\right] = 2\Phi\left(3C_{pk}^{T}\right) - 1 \tag{2}$$

The C_{pk}^T index provides a lower bound on the overall process yield. In practical, because the process mean and the variance for each characteristic are unknown, the C_{pk}^T index is estimated by collecting the sample data. The natural estimator of the C_{pk}^T index defined in the following is considered:

$$\hat{C}_{pk}^{T} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{i=1}^{m}\left(2\Phi(3\hat{C}_{pki}) - 1\right) + 1\right]/2\right\} \tag{3}$$

where $\hat{C}_{pki}$ denotes the natural estimator of C_{pk} value of the ith characteristic. Pearn *et al.*[2] derived the asymptotic distribution of $\hat{C}_{pk}^T$ using Taylor expansion technique for multiple variables. The asymptotic distribution of $\hat{C}_{pk}^T$ is (see[2])

$$\hat{C}_{pk}^{T} \approx N\left(C_{pk}^{T}, \frac{1}{9n} + \frac{C_{pk}^{T\,2}}{2n}\right) \tag{4}$$

For the processes with multiple characteristics, Hsu *et al.*[3] applied the bootstrap method for calculating the lower confidence bounds of the capability index C_{pu}^T and determined the sample size for the given estimation accuracy. Pearn *et al.*[4] implemented the process capability index to deal with the photolithography production control problem with multiple quality characteristics. Pearn and Cheng[5] investigated the production yield measurement for processes with multiple characteristics. Awad and Kovach[6] purposed a simple and integrated modeling methodology for robust design on the basis of multivariate process capability vector. Wu *et al.*[7] provided an overview for process capability indices practice of quality assurance. More recent studies on process capability index (PCI) include Goethals and Cho,[8] Yum and Kim,[9] and Pearn *et al.*[10]

3. Product acceptance determination

The C_{pk}^T index can be used as a quality benchmark for product acceptance. Let (AQL,$1-\alpha$) and (LTPD, β) be the two points on the OC curve of interest. Note that AQL and LTPD are levels of the product fraction of defectives that correspond to acceptable and rejectable quality levels. To determine whether a given lot is capable, we can consider the testing hypothesis as

$$H_0 : p = \text{AQL} \; ; \; H_1 : p = \text{LTPD} \tag{5}$$

where p means the process fraction of defectives. The AQL is simply a standard against which to judge the lots. It is hoped that the vendor's process will operate at a fallout level that is considerable better than the AQL. The null hypothesis with process fraction of defectives, $H_0 : p = \text{AQL}$, is equivalent to test process the capability index with $H_0 : C_{pk}^T \geqslant C_{\text{AQL}}$, where C_{AQL} is the level of acceptable quality for the C_{pk}^T index corresponding to the lot or process fraction of defectives AQL. For the production of vendors and buyers, two conditions are considered:

$$\Pr\left\{\text{reject the lot}\middle| p \geqslant \text{AQL}\right\} = \Pr\left\{\text{reject the lot}\middle| C_{pk}^{T} \geqslant C_{\text{AQL}}\right\} \leqslant \alpha \tag{6}$$

$$\Pr\left\{\text{accepting the lot}\middle| p \leqslant \text{LTPD}\right\} = \Pr\left\{\text{accepting the lot}\middle| C_{pk}^{T} \leqslant C_{\text{LTPD}}\right\} \leqslant \beta \tag{7}$$

where C_{LTPD} represents the capability requirement corresponding to the LTPD on the basis of the C_{pk}^T index.

That is, the probability of rejecting acceptable lots is no more than α. At the same time, the probability of accepting unqualified lots is no more than β. Our object is solving the two simultaneous equations mentioned earlier and then obtaining the required inspection sample size n and the critical acceptance value c_0 of C_{pk}^T. By using the approximate distribution shown in Equation (4), Equations (6) and (7) can be rewritten as

$$P\left(\hat{C}_{pk}T < c_0 \middle| C_{pk}^{T} \geqslant C_{\text{AQL}}\right) \leqslant P\left(Z < \frac{c_0 - C_{\text{AQL}}}{\sqrt{\frac{1}{9n} + \frac{C_{\text{AQL}}^2}{2n}}}\right) \leqslant \alpha \tag{8}$$

$$P\left(\hat{C}_{pk}T \geqslant c_0 \middle| C_{pk}^{T} \leqslant C_{\text{LTPD}}\right) \leqslant P\left(Z \geqslant \frac{c_0 - C_{\text{LTPD}}}{\sqrt{\frac{1}{9n} + \frac{C_{\text{LTPD}}^2}{2n}}}\right) \leqslant \beta \tag{9}$$

Equations (8) and (9) imply that

$$\frac{c_0 - C_{AQL}}{\sqrt{\frac{1}{9n} + \frac{C_{AQL}^2}{2n}}} = z_{1-\alpha} = -z_\alpha \tag{10}$$

$$\frac{c_0 - C_{LTPD}}{\sqrt{\frac{1}{9n} + \frac{C_{LTPD}^2}{2n}}} = z_\beta \tag{11}$$

From Equations (10) and (11), we have

$$c_0 - C_{AQL} = -z_\alpha\sqrt{\frac{1}{9n} + \frac{C_{AQL}^2}{2n}} \tag{12}$$

$$c_0 - C_{LTPD} = z_\beta\sqrt{\frac{1}{9n} + \frac{C_{LTPD}^2}{2n}} \tag{13}$$

Subtracting Equation (13) by Equation (12) yields

$$C_{AQL} - C_{LTPD} = \left(z_\alpha\sqrt{1/9 + C_{AQL}^2/2} + z_\alpha\sqrt{1/9 + C_{AQL}^2/2}\right)/\sqrt{n} \tag{14}$$

Consequently, from Equation (14), we establish the required inspection sample size n and the corresponding critical value c_0 as

$$n = \left\lceil \left(\frac{z_\alpha\sqrt{1/9 + C_{AQL}^2/2} + z_\beta\sqrt{1/9 + C_{LTPD}^2/2}}{C_{AQL} - C_{LTPD}} \right)^2 \right\rceil \tag{15}$$

$$c_0 = C_{AQL} - n^{-1/2} z_\alpha\sqrt{1/9 + C_{AQL}^2/2} \tag{16}$$

The symbol $\lceil n \rceil$ means the ceiling function that gains the least integer greater than or equal to n.

Remark: one-sided process

For one-sided processes with multiple characteristics, the generalization index C_{pu}^T is considered. Pearn *et al.*[11] have developed the asymptotic distribution of the natural estimator $\hat{C}_{pu}^T$ in the following[11]:

$$\hat{C}_{PU}^T \approx N\left(C_{PU}^T, \frac{1}{9n} + \frac{1}{2n}{C_{PU}^T}^2\right) \tag{17}$$

We can use the same technique to establish the close form solutions of (n, c_0) for one-sided processes with multiple characteristics similar to Equations (15) and (16) mentioned earlier.

4. Determination procedure

For practical application purpose, we calculate and tabulate the required sample size (n) and the critical acceptance values (c_0) for various α-risk, β-risk, C_{AQL}, and C_{LTPD}. Table I displays (n,c_0) values for α-risk = 0.01, 0.025, 0.05, 0.075, and 0.10 and β-risk = 0.01, 0.025, 0.05, 0.075, and 0.10, with various benchmarking quality levels, (C_{AQL},C_{LTPD}) = (1.33, 1.00), (1.50, 1.33), (1.67, 1.33), and (2.00, 1.67).

For instance, if the requirement quality level (C_{AQL},C_{LTPD}) is set to (1.50, 1.33) with α-risk = 0.01 and β-risk = 0.05, the required sample size and critical acceptance value can be obtained as (596, 1.4251). It means that the lot will be rejected if the 596 inspected product items yield measurement with $\hat{C}_{pk}^T < 1.4251$. For the proposed product acceptance determination procedure to be practical and convenient to use, a step-by-step algorithm is provided as follows

Step1: decide the process capability requirements (i.e. set the values of C_{AQL} and C_{LTPD}) and choose the α-risk and the β-risk.
Step2: check Table I to find the critical acceptance value and the required number, (n,c_0), on the basis of given α-risk, β-risk, C_{AQL}, and C_{LTPD}.
Step3: calculate the value of $\hat{C}_{pk}^T$ (sample estimator) from the n inspected samples.
Step4: make decisions to accept the entire products if $\hat{C}_{pk}^T \geqslant c_0$. Otherwise, reject the entire products.

Table I. Required sample sizes (n) and critical acceptance values (c_0) for various α- and β-risks with selected C_{AQL} and C_{LTPD}

		$C_{AQL}=1.33$, $C_{LTPD}=1.00$		$C_{AQL}=1.50$, $C_{LTPD}=1.33$		$C_{AQL}=1.67$, $C_{LTPD}=1.33$		$C_{AQL}=2.00$, $C_{LTPD}=1.67$	
α	β	n	c_0	n	c_0	n	c_0	n	c_0
0.01	0.01	158	1.1453	834	1.4104	232	1.4826	357	1.8211
	0.025	131	1.1591	701	1.4177	194	1.4973	299	1.8353
	0.05	110	1.1735	596	1.4251	163	1.5119	253	1.8497
	0.075	98	1.1849	533	1.4307	145	1.5233	225	1.8606
	0.10	89	1.1945	486	1.4354	132	1.5331	205	1.8699
0.025	0.01	137	1.1317	714	1.4031	201	1.4687	308	1.8074
	0.025	112	1.1452	592	1.4104	165	1.4828	254	1.8213
	0.05	93	1.1598	496	1.4179	137	1.4976	212	1.8359
	0.075	81	1.1704	438	1.4235	120	1.5088	186	1.8466
	0.10	73	1.1803	396	1.4284	108	1.5187	168	1.8563
0.05	0.01	120	1.1181	619	1.3960	175	1.4542	268	1.7935
	0.025	97	1.1314	506	1.4031	142	1.4682	218	1.8071
	0.05	79	1.1454	417	1.4104	116	1.4826	179	1.8214
	0.075	69	1.1571	364	1.4161	101	1.4942	156	1.8326
	0.10	61	1.1663	326	1.4211	90	1.5042	139	1.8421
0.075	0.01	110	1.1087	560	1.3907	160	1.4443	244	1.7836
	0.025	88	1.1215	453	1.3976	128	1.4574	196	1.7966
	0.05	71	1.1352	369	1.4048	104	1.4721	159	1.8105
	0.075	61	1.1461	320	1.4105	89	1.4828	137	1.8213
	0.10	54	1.1560	284	1.4155	79	1.4931	121	1.8307
0.10	0.01	102	1.1002	517	1.3862	148	1.4354	226	1.7752
	0.025	81	1.1127	414	1.3929	118	1.4486	180	1.7877
	0.05	65	1.1264	335	1.4001	95	1.4629	145	1.8015
	0.075	55	1.1363	287	1.4055	81	1.4737	124	1.8122
	0.10	48	1.1454	253	1.4104	71	1.4834	109	1.8216

5. An application example

We consider a case study to demonstrate how the product acceptance determination procedure can be used in lot sentencing problem for processes with multiple characteristics. The case we investigate involves a process manufacturing the dual-fiber tips (see[12]), which is used in making fiber optic cables. The quality characteristics and specifications are presented in Table II. The key quality characteristics include capillary diameter, length, wedge, and core diameter.

In the contract, the C_{AQL} and the C_{LTPD} are set to 1.33 and 1.00 with α-risk = 0.05 and β-risk = 0.05. First, we find the acceptance critical values and inspected sample sizes $(n, c_0) = (79, 1.1454)$ from Table I. The observations measurement and the calculated results for each characteristic are displayed in Table III. On the basis of those results, we obtain $\hat{C}_{pk}^{T} = 0.93037$. Therefore, the buyer would "reject" the entire products because the sample estimator, 0.93037, is smaller than the critical acceptance value 1.1454.

Table II. Specifications of characteristics for the dual-fiber tips

Characteristic	LSL	Target	USL
Capillary diameter (mm)	1.795	1.800	1.805
Capillary length (mm)	6.00	6.25	6.50
Wedge (°)	7.5	8	8.5
Core diameter (μm)	126	127	128

LSL: Lower Specification Limit; USL: Upper Specification Limit.

Table III. Calculated sample mean, sample standard derivation, $\hat{C}_{pki}$, and estimated nonconformity

Characteristic	$\bar{x}$	s	$\hat{C}_{pki}$	Nonconformity (ppm)
Capillary diameter	1.8008	0.00106	1.320755	74.24207
Capillary length	6.2460	0.05908	1.387949	31.29299
Wedge	8.0128	0.17414	0.932583	5146.009
Core diameter	127.02	0.13482	1.594896	1.712531

6. Conclusions

In this article, we developed an effective and clear algorithm on the basis of overall yield-measure index C^{T}_{pk} to deal with the lot sentencing problem for normally distributed processes with multiple characteristics. The explicitly close form formulae of the required sample size n and the corresponding critical acceptance value c_0 were obtained. For various given α-risk, β-risk with capability requirements C_{AQL} and C_{LTPD} values of (n,c_0) were tabulated for practitioners to make reliable decisions.

Reference

1. Pearn WL, Wu CW. An effective decision making method for product acceptance. *Omega, The International Journal of Management Science* 2007; **35**:12–21. DOI: 10.1016/j.omega.2005.01.018.
2. Pearn WL, Shiau JJH, Tai YT, Li MY. Capability assessment for processes with multiple characteristics: A generalization of the popular index C_{pk}. *Quality and Reliability Engineering International* 2011; **27**(8):1119–1129. DOI: 10.1002/qre.1200.
3. Hsu YC, Pearn WL, Chuang YF. Sample size determination for production yield estimation with multiple independent process characteristics. *European Journal of Operational Research* 2009; **196**:968–978. DOI: 10.1016/j.ejor.2008.04.029.
4. Pearn WL, Kang HY, Lee AHI, Liao MY. Photolithography control in wafer fabrication based on process capability indices with multiple characteristics. *IEEE Transactions on Semiconductor Manufacturing* 2009; **22**(3):351–356. DOI: 10.1109/TSM.2009.2024851.
5. Pearn WL, Cheng YC. Measuring production yield for processes with multiple characteristics. *International Journal of Production Research* 2010; **48**(15):4519–4536. DOI: 10.1080/00207540903036313.
6. Awad MI, Kovach JV. Multiresponse optimization using multivariate process capability index. *Quality and Reliability Engineering International* 2011; **27**:465–477. DOI: 10.1002/qre.1141.
7. Wu CW, Pearn WL, Kotz S. An overview of theory and practice on process capability indices for quality assurance. *International Journal of Production Economics* 2009; **117**:338–359. DOI: 10.1016/j.ijpe.2008.11.008.
8. Goethals PL, Cho BR. The development of a target-focused process capability index with multiple characteristics. *Quality and Reliability Engineering International* 2011; **27**:297–311. DOI: 10.1002/qre.1120.
9. Yum BJ, Kim KW. A Bibliography of the literature on process capability indices: 2000–2009. *Quality and Reliability Engineering International* 2011; **27**:251–268. DOI: 10.1002/qre.1115.
10. Pearn WL, Yen CH, Yang DY. Production yield measure for multiple characteristics processes based on S^{T}_{pk} under multiple samples. *Central European Journal of Operations Research* 2011; **20**(1):65–85. DOI: 10.1007/s10100-010-0152-9.
11. Pearn WL, Wu CH, Tsai MC. A Note on "Capability Assessment for Process with Multiple Characteristics: A Generalization of the Popular Index C_{pk}". *Quality and Reliability Engineering International* 2012; Feb 8 Online. DOI: 10.1002/qre.1295.
12. Pearn WL, Wu CH. Production quality and yield assurance for process with multiple independent characteristics. *European Journal of Operational Research* 2006; **173**:637–647. DOI: 10.1016/j.ejor.2005.02.050.

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