PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 4, December 1994

SUMS AND PRODUCTS OF CYCLIC OPERATORS

PEI YUAN WU

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is proved that every bounded linear operator on a complex separable Hilbert space is the sum of two cyclic operators. For the product, we show that an operator T is the product of finitely many cyclic operators if and only if the kernel of T^* is finite-dimensional. More precisely, if $\dim\ker T^* \leq k$ $(2 \leq k < \infty)$, then T is the product of at most $k+2$ cyclic operators. We conjecture that in this case at most k cyclic operators would suffice and verify this for some special classes of operators.

A bounded linear operator T on a complex Hilbert space H is *cyclic* if there is a vector x in H such that H is the closed linear span of the vectors $x, Tx, T^2x, \ldots$ (in this case, x is a *cyclic vector* of T). It is obvious that cyclic operators can act only on a separable space. Hence for the rest of the paper we will consider only Hilbert spaces which are separable.

For such a space H, let $\mathscr{C}(H)$ denote the set of all cyclic operators on H. Topological properties of this set have been studied quite extensively before. It was discovered that the size of $\mathscr{C}(H)$ relative to that of all operators $\mathscr{B}(H)$ depends very much on the dimension of the underlying space H. Thus if H is finite-dimensional, then it is easy to show that $\mathscr{C}(H)$ is a dense open subset of $\mathscr{B}(H)$ (cf. [6, p. 499]). It follows in particular that the set of noncyclic operators is nowhere dense, whence $\mathscr{C}(H)$ is of the second category. However, for infinite-dimensional H, exactly the opposite is true. Indeed, it was proved in [5] that the set of noncyclic operators is dense in $\mathscr{B}(H)$. A refinement of that proof together with some Fredholm index theory, moreover, yields the existence of an open dense set of noncyclic operators (cf. [1, Proposition 11.18]). This implies that $\mathscr{C}(H)$ is nowhere dense, and thus noncyclic operators form a second category set in $\mathscr{B}(H)$.

In the following, we will show the abundance of cyclic operators in another sense. Indeed, in §1, after some preliminaries, we prove that every operator is the sum of two cyclic operators. This holds on any separable space regardless of its dimension. We then consider products of cyclic operators in §2. Our main theorem (Theorem 2.12) says that an operator T is the product of finitely many cyclic operators if and only if $\ker T^*$ is finite-dimensional. To be more precise,

---

Received by the editors January 12, 1993 and, in revised form, March 2, 1993.
1991 *Mathematics Subject Classification*. Primary 47A05.
Key words and phrases. Cyclic operator, multicyclic operator, triangular operator.
This research was partially supported by the National Science Council of the Republic of China.

we show that if $\dim\ker T^* = k$ $(2 \le k < \infty)$, then T is the product of at most $k+2$ cyclic operators. We suspect that the smallest number of cyclic operators required in this case should be k, which we pose as a conjecture. We are able to verify this for several classes of operators: operators on a finite-dimensional space, multicyclic operators, operators whose spectrum does not surround zero, isometries, coisometries, and normal operators.

1. SUM

We start with three propositions, which are the main tools that we use to prove our sum and product results. The first one reduces our consideration from the (additive or multiplicative) decomposition of general operators to that of cyclic operators. It appeared in [10, p. 463] and [4, Theorem 5].

Proposition 1.1. *Every operator T can be expressed in a triangular form*

$$\begin{bmatrix} T_1 & & & * \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_n \end{bmatrix} \quad or \quad \begin{bmatrix} T_1 & & * \\ & T_2 & \\ & & \ddots \\ 0 & & \end{bmatrix},$$

where n is the multiplicity of T and the T_j's are all cyclic.

Recall that the *multiplicity* of T is the smallest cardinality of vectors $x_1, \ldots, x_n$ in H for which H is the closed linear span of the set $\{T^k x_j : k \ge 0$ and $1 \le j \le n\}$. If T is cyclic, then its multiplicity is 1.

Our second proposition is quite well known (cf. [8, Problem 167]). It presents a special matrix form for every cyclic operator.

Proposition 1.2. *An operator is cyclic if and only if it has the matrix representation*

$$\begin{bmatrix} a_1 & & & * \\ b_1 & a_2 & & \\ & b_2 & \ddots & \\ & & \ddots & \\ 0 & & & \end{bmatrix}$$

with all b_n's nonzero.

Finally, we have

Proposition 1.3. *If T has the form*

$$\begin{bmatrix} T_1 & & * \\ & T_2 & \\ & & \ddots \\ 0 & & \end{bmatrix},$$

where the T_n's are all cyclic and have mutually disjoint spectra, then T must be cyclic.

This result is an easy consequence of the following three facts: (1) if T is of the above form, then $\sum_n \oplus T_n$ is a *quasiaffine transform* of T, that is, there is an injective operator X with dense range such that $X(\sum_n \oplus T_n) = TX$ [3, Proposition 2.5]; (2) if the T_n's are cyclic and have mutually disjoint spectra, then $\sum_n \bigoplus T_n$ is cyclic [11, Corollary 1.77]; (3) if S is a quasiaffine transform of T and S is cyclic, then so is T.

The following corollary appeared in [9, Proposition 3.6].

Corollary 1.4. *If*

$$T = \begin{bmatrix} a_1 & & * \\ & a_2 & \\ & & \ddots \\ 0 & & \end{bmatrix}$$

is triangular with distinct a_n*'s, then the diagonal operator*

$$\operatorname{diag}(a_n) = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & \end{bmatrix}$$

is a quasiaffine transform of T *and* T *is cyclic.*

Now we are ready for our main result in this section.

Theorem 1.5. *Every operator is the sum of two cyclic operators.*

Proof. By Proposition 1.1, we may assume that

$$T = \begin{bmatrix} T_1 & & * \\ & T_2 & \\ & & \ddots \\ 0 & & \end{bmatrix},$$

where the T_n's are cyclic. (The proof for finitely many T_n's is the same.) Proposition 1.2 enables us to assume further that each T_n is of the form

$$\begin{bmatrix} a_{n1} & & & * \\ b_{n1} & a_{n2} & & \\ & b_{n2} & \ddots & \\ 0 & & \ddots & \end{bmatrix}$$

with nonzero b_{nj}'s. We have the decomposition

$$\begin{bmatrix} a_{n1} & & & * \\ b_{n1} & a_{n2} & & \\ & b_{n2} & \ddots & \\ 0 & & \ddots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} c_{n1} + d_n i & \overline{b}_{n1} & & 0 \\ b_{n1} & c_{n2} + d_n i & \overline{b}_{n2} & \\ & b_{n2} & \ddots & \ddots \\ 0 & & \ddots & \end{bmatrix} + \begin{bmatrix} e_{n1} & & * \\ & e_{n2} & \\ & & \ddots \\ 0 & & \end{bmatrix} \equiv S_n + R_n,$$

where the c_{nj}'s and d_n's are all real, d_n's are distinct, and e_{nj}'s are distinct. Hence

$$T = \begin{bmatrix} S_1 & & & * \\ & S_2 & & \\ & & \ddots & \\ 0 & & & \end{bmatrix} + \begin{bmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & \end{bmatrix} \equiv S + R.$$

In choosing the c_{nj}'s and d_n's, we may assume that they are such that $\{\|S_n\|\}$ is bounded by a fixed number, say, M. Then $\|R_n\| \le \|T_n\| + \|S_n\| \le \|T\| + M$ for all n, whence R is a bounded operator and therefore S is a bounded operator. We now show that both S and R are cyclic. Since S_n is the sum of the scalar $d_n i$ and a Hermitian operator, the distinctness of the d_n's implies that the spectra $\sigma(S_n)$'s are mutually disjoint. Also, by Proposition 1.2, each S_n is cyclic. Hence Proposition 1.3 implies that S is cyclic. On the other hand, by Corollary 1.4, the diagonal operator $\operatorname{diag}(e_{nj})$ is a quasiaffine transform of R_n. Hence $D = \sum_n \bigoplus \operatorname{diag}(e_{nj})$ is a quasiaffine transform of $R = \sum_n \bigoplus R_n$. Since the e_{nj}'s are all distinct, D is a cyclic operator, whence R is a cyclic operator. This completes the proof. □

2. Product

In this section, we consider products of cyclic operators. An obvious necessary condition for an operator T to be expressible as a product of k $(1 \le k < \infty)$ cyclic operators is that $\dim\ker T^* \le k$. Indeed, for $k = 1$ this is trivial. Assuming its validity for k, we prove it for $k+1$. Let $T = T_1 \cdots T_k T_{k+1}$ be a product of $k+1$ cyclic operators, and let $S = T_1 \cdots T_k$. Then $T^* = T_{k+1}^* S^*$ implies that

$$\begin{aligned} \dim\ker T^* &= \dim\ker S^* + \dim(\operatorname{ran} S^* \cap \ker T_{k+1}^*) \\ &\le \dim\ker S^* + \dim\ker T_{k+1}^* \le k+1. \end{aligned}$$

In view of the partial evidences discussed below, we suspect that, when $k \ge 2$, this necessary condition is also sufficient. This we propose as

Conjecture 2.1. *An operator T is the product of k $(2 \le k < \infty)$ cyclic operators if and only if* $\dim\ker T^* \le k$.

We remark that in the above statement we may as well replace "T is the product of k cyclic operators" by "T is the product of at most k cyclic operators" due to the following lemma.

Lemma 2.2. *Every cyclic operator is the product of two other cyclic operators.*

Proof. Assume that

$$T = \begin{bmatrix} a_1 & & & * \\ b_1 & a_2 & & \\ & b_2 & \ddots & \\ 0 & & \ddots & \end{bmatrix},$$

where the b_n's are nonzero. We decompose

$$T = \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & \end{bmatrix} \begin{bmatrix} d_1 & & & * \\ e_1 & d_2 & & \\ & e_2 & \ddots & \\ 0 & & \ddots & \end{bmatrix},$$

where c_n's are distinct and the e_n's are all nonzero. Then T is the product of two cyclic operators. □

Falling short of proving the preceding conjecture in its full strength, we are only able to verify it for several classes of special operators. We start with the finite-dimensional case.

Proposition 2.3. *On a finite-dimensional space, T is the product of k $(2 \le k < \infty)$ cyclic operators if and only if* $\dim\ker T^* \le k$.

Proof. We prove that $\dim\ker T^* = k$ implies that T is the product of k cyclic operators. Since the property of cyclicity is preserved under similarity, we may assume that T is of the form

$$\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \oplus \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \equiv T_1 \oplus \cdots \oplus T_k \oplus T_{k+1},$$

where T_j is of size n_j for $j = 1, \dots, k+1$ and the a_i's are all nonzero. Note that

$$T_1 \oplus \cdots \oplus T_k = \begin{bmatrix} 0 & 2 & & & 0 \\ & 0 & 3 & & \\ & & & \ddots & \\ & & & \ddots & N \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_N \end{bmatrix} \equiv S_1 S_2,$$

where $N = n_1 + \cdots + n_k$ and

$$b_j = \begin{cases} 1/j & \text{if } 1 \le j \le N \text{ and} \\ & j \ne n_1 + 1, n_1 + n_2 + 1, \dots, n_1 + \cdots + n_{k-1} + 1, \\ 0 & \text{otherwise}, \end{cases}$$

and that

$$T_{k+1} = \begin{bmatrix} b_{N+1} & & * \\ & \ddots & \\ 0 & & b_{N+n} \end{bmatrix} \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \equiv R_1 R_2,$$

where $b_{N+1}, \dots, b_{N+n}$ are all nonzero and distinct and the c_j's are nonzero and distinct and also distinct from the nonzero b_j's. Letting $A_j = S_j \oplus R_j$, $j = 1, 2$, we have $T = A_1 A_2$. By Proposition 1.3, A_1 is cyclic. On the other hand, since A_2 is a diagonal operator with $k-1$ zero diagonals, we can express it as a product of $k-1$ diagonal operators each with distinct diagonals. Hence A_2 is the product of $k-1$ cyclic operators, and therefore T is the product of k cyclic operators. □

Corollary 2.4. *On an* n*-dimensional space, every operator is the product of* n *cyclic operators and* n *is the smallest such number.*

Disposing of the finite-dimensional case, we move next to operators on infinite-dimensional spaces. Let the multiplicity of an operator T be denoted by $m(T)$. Since $\dim\ker T^* \leq m(T)$ for any operator T, the next proposition is weaker than what we proposed in Conjecture 2.1. Recall that a *multicyclic operator* is one that has finite multiplicity.

Proposition 2.5. *If* T *is a multicyclic operator with multiplicity* m*, then* T *is the product of* m *cyclic operators.*

Proof. We prove this by induction on m. Obviously, this is true for $m = 1$. Assuming its validity for any operator T with $m(T) = m$, we prove it for $m+1$. So let T be an operator with multiplicity $m+1$. By [10, p. 463], we have the triangulation

$$T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix},$$

where T_1 is cyclic and $m(T_2) = m$. Hence $T_2 = S_1 \cdots S_{m-1} S'_m$ is a product of m cyclic operators by the induction hypothesis. On the other hand, using Lemma 2.2 we obtain $T_1 = R_1 \cdots R_{m+1}$, where the R_j's are all cyclic and each R_j, $j = 1, \ldots, m$, is diagonal and invertible with $\sigma(R_j)$ disjoint from $\sigma(S_j)$ when $j = 1, \ldots, m-1$. Moreover, let $S'_m = S_m S_{m+1}$, where both factors are cyclic, $\sigma(S_m)$ is disjoint from $\sigma(R_m)$ and S_{m+1} is diagonal and invertible with $\sigma(S_{m+1})$ disjoint from $\sigma(R_{m+1})$. Finally, let

$$Q_j = \begin{bmatrix} R_j & 0 \\ 0 & S_j \end{bmatrix}, \qquad j = 1, \ldots, m,$$

and

$$Q_{m+1} = \begin{bmatrix} R_{m+1} & R_m^{-1} \cdots R_1^{-1} X \\ 0 & S_{m+1} \end{bmatrix}.$$

Then each Q_j is cyclic by Proposition 1.3 and $T = Q_1 \cdots Q_{m+1}$. □

To obtain other product results, we need the following lemma. It is an improvement over Lemma 2.2 for certain special cyclic operators. An operator is *triangular* if it can be represented in the matrix form

$$\begin{bmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots \end{bmatrix}.$$

Lemma 2.6. *If* T *is a cyclic operator with dense range, then* $T = T_1 T_2$*, where* T_1 *is unitary cyclic and* T_2 *is triangular cyclic.*

Proof. By Proposition 1.2, we may assume that $T = [t_{ij}]$ with $t_{i,i-1} \neq 0$ for $i \geq 2$ and $t_{ij} = 0$ for $i - j \geq 2$. In the following, we will construct a matrix $U = [u_{ij}]$ with the following properties:

(i) $UU^* = I$,
(ii) $u_{i,i+1} \neq 0$ for $i \geq 1$ and $u_{ij} = 0$ for $j - i \geq 2$, and
(iii) $UT = [a_{ij}]$ with $a_{ij} = 0$ for $i - j \geq 1$.

This is done by first letting

$$v_1 = t_{21} \tag{1}$$

and

$$v_n = -(v_1 t_{1,n-1} + \cdots + v_{n-1} t_{n-1,n-1})/t_{n,n-1} \quad \text{for } n \geq 2. \tag{2}$$

Then, for each $n \geq 1$, let

$$\alpha_n = |v_{n+1}| / \left(\sum_{j=1}^{n} |v_j|^2 \right)^{1/2} \left(\sum_{j=1}^{n+1} |v_j|^2 \right)^{1/2}, \tag{3}$$

$$w_n = \begin{cases} -(\alpha_n/\overline{v}_{n+1}) \sum_{j=1}^{n} |v_j|^2 & \text{if } v_{n+1} \neq 0, \\ 1 & \text{otherwise}, \end{cases} \tag{4}$$

and let $u_n = [u_{n1} u_{n2} \cdots]$, the nth row of U, be

$$[\alpha_n v_1 \cdots \alpha_n v_n \; w_n \; 0 \; 0 \cdots]. \tag{5}$$

To verify (i), note that

$$\begin{aligned} \|u_n\|^2 &= |\alpha_n|^2 \sum_{j=1}^{n} |v_j|^2 + |w_n|^2 \\ &= \begin{cases} |v_{n+1}|^2 / \left(\sum_{j=1}^{n+1} |v_j|^2\right) + \left(\sum_{j=1}^{n} |v_j|^2\right) / \left(\sum_{j=1}^{n+1} |v_j|^2\right) = 1 & \text{if } v_{n+1} \neq 0, \\ 0 + 1 = 1 & \text{otherwise} \end{cases} \end{aligned}$$

and, for $n > m \geq 1$, that

$$\begin{aligned} u_n \cdot u_m &= \sum_{j=1}^{m} \alpha_n v_j \overline{\alpha}_m \overline{v}_j + \alpha_n v_{m+1} \overline{w}_m \\ &= \alpha_n \left(\alpha_m \sum_{j=1}^{m} |v_j|^2 + v_{m+1} \overline{w}_m \right) = 0 \end{aligned}$$

by (3)–(5). As for (ii), we have

$$\begin{aligned} |u_{n,n+1}| &= |w_n| = (|\alpha_n|/|v_{n+1}|) \sum_{j=1}^{n} |v_j|^2 \\ &= \left(\sum_{j=1}^{n} |v_j|^2 \right)^{1/2} / \left(\sum_{j=1}^{n+1} |v_j|^2 \right)^{1/2} \geq |t_{21}| / \left(\sum_{j=1}^{n+1} |v_j|^2 \right)^{1/2} > 0 \end{aligned}$$

if $v_{n+1} \neq 0$ by (1) and (3)–(5) and $u_{ij} = 0$ for $j - i \geq 2$ by (5). Finally, for $i - j \geq 1$,

$$a_{ij} = \sum_{k=1}^{j+1} u_{ik} t_{kj} = \alpha_i \sum_{k=1}^{j+1} v_k t_{kj} = 0$$

by (2) and (5), which verifies (iii).

We next show that U is actually unitary. In view of (i) above, we need only prove that U is one-to-one. So let $x = [x_1 x_2 \cdots]^t$ be such that $Ux = 0$. We will prove that x also satisfies $T^*x = 0$. Since T has dense range, this would imply that $x = 0$ as desired. We let $y = [y_1 y_2 \cdots]^t = T^*x$. Our goal can be accomplished by showing that (a) there are infinitely many nonzero α_n's and (b) if $\alpha_n \neq 0$, then $y_1 = \cdots = y_{n-1} = 0$.

To prove (a), assume that there is some $n_0 \geq 1$ such that $\alpha_n = 0$ for all $n \geq n_0$. From (3), we have $v_{n+1} = 0$ for $n \geq n_0$; hence, $v = [\overline{v}_1 \overline{v}_2 \cdots]^t$ is a square-summable vector which satisfies $T^*v = 0$ by (2). Since T has dense range, this implies that $v = 0$, whence $t_{21} = v_1 = 0$ by (1) contradicting our assumption. To prove (b), we assume that $\alpha_n \neq 0$ for some $n \geq 2$. From $Ux = 0$, we deduce that $[x_1 \cdots x_n]$ is orthogonal to $[\overline{u}_{i1} \cdots \overline{u}_{in}]$ for $i = 1, \ldots, n-1$. These latter vectors together with $[\overline{u}_{n1} \cdots \overline{u}_{nn}]$ are mutually orthogonal by (i). Since $\alpha_n \neq 0$ and $v_1 = t_{21} \neq 0$, $[\overline{u}_{n1} \cdots \overline{u}_{nn}] = [\alpha_n \overline{v}_1 \cdots \alpha_n \overline{v}_n]$ is a nonzero vector. Hence $[x_1 \cdots x_n]$ is a multiple of $[\overline{u}_{n1} \cdots \overline{u}_{nn}]$. Since this latter vector is orthogonal to $[t_{1j} \cdots t_{nj}]$ for all $j = 1, \ldots, n-1$ by (iii), the same is true for $[x_1 \cdots x_n]$. Thus

$$y_j = \overline{t}_{1j} x_1 + \cdots + \overline{t}_{nj} x_n = 0, \qquad j = 1, \ldots, n-1.$$

Let $A = UT$. Since $T = U^*A$ and $t_{i,i-1} \neq 0$ for all i, we deduce that the diagonals of A are all nonzero. Hence it is possible to find a unitary diagonal operator D such that the diagonals of the triangular operator DA are all distinct. If $T_1 = U^*D^*$ and $T_2 = DA$, then $T = T_1 T_2$ is the asserted factorization. □

Now it is time to claim our rewards after such a laborious work. Note that if Conjecture 2.1 is indeed true, then every invertible operator should be the product of two cyclic operators. Not able to prove this, we show that the assertion is true under a more restricted condition. We say that a compact subset K of $\mathbb{C}$ *does not surround* 0 if 0 is in the unbounded component of $\mathbb{C} \backslash K$ or, equivalently, if 0 is not in the polynomially convex hull of K.

Proposition 2.7. *If the spectrum of T does not surround* 0, *then T is the product of two cyclic operators.*

Proof. By Proposition 1.1, we assume that T is of the form $[T_{ij}]$, where T_{ii} is cyclic for all i and $T_{ij} = 0$ for $i > j$. It is easily seen that $\sigma(T_{ii})$ is contained in the polynomially convex hull of $\sigma(T)$ (cf. [12, Theorem 0.8]). Hence our hypothesis implies that T_{ii} is invertible for all i. Thus Lemma 2.6 is applicable and we obtain $T_{ii} = U_i A_i$, where U_i is unitary cyclic and A_i is triangular cyclic. Let $\{r_i\}$ be a sequence of distinct real numbers between 1 and 2, and let $V_i = r_i U_i$ and $B_i = A_i / r_i$. Then $T_{ii} = V_i B_i$. Since the B_i's together with the A_i's are all invertible, we may make further adjustments, as in the end of the proof of Lemma 2.6, so that the diagonals of all the B_i's are distinct. If

$$T_1 = \begin{bmatrix} V_1 & T_{12}B_2^{-1} & T_{13}B_3^{-1} & \\ & V_2 & T_{23}B_3^{-1} & \ddots \\ & & V_3 & \ddots \\ 0 & & & \ddots \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & B_3 & \\ 0 & & & \ddots \end{bmatrix},$$

then $T = T_1T_2$. Note that the relation $B_i^{-1} = r_iA_i^{-1} = r_iT_{ii}^{-1}U_i$ implies that $\|B_i^{-1}\| \leq 2\|T^{-1}\|$ for all i, whence T_2 is invertible and therefore T_1 is indeed a bounded operator. Since the V_i's are cyclic and their spectra are mutually disjoint, Proposition 1.3 implies that T_1 is cyclic. On the other hand, if D_i is the diagonal operator whose diagonals are exactly those of B_i, then D_i is a quasiaffine transform of B_i by Corollary 1.4. Hence $\sum_i \bigoplus D_i$ is a quasiaffine transform of T_2. Since $\sum_i \bigoplus D_i$ is itself a diagonal operator with distinct diagonals, it is cyclic, whence T_2 is cyclic. This completes the proof. □

The next result is another application of Lemma 2.6.

Theorem 2.8. *Let $T = \sum_n \bigoplus T_n$, where the T_n's are cyclic. If $k \geq 2$ and T_n has dense range for all $n > k$, then T is the product of k cyclic operators.*

Proof. Using Lemma 2.2, we may express each T_n, $n = 1, \ldots, k$, as a product $T_n = T_{n1} \cdots T_{nk}$, where T_{nj}'s are all cyclic and each T_{nj}, $j \neq n$, is a diagonal operator with spectrum disjoint from the spectra of all the other T_{ij}'s. For the remaining T_n's, we use Lemma 2.6 to obtain $T_n = U_nA_n$, where U_n is unitary cyclic and A_n is triangular cyclic. We further express these T_n's as a product $T_n = T_{n1} \cdots T_{nk}$ $(n > k)$, where each T_{n1} is a distinct multiple of U_n with spectrum disjoint from the spectra of $T_{11}, \ldots, T_{k1}$, and each T_{nj}, $j = 2, \ldots, k$, is either triangular cyclic or diagonal cyclic with the closure of its distinct diagonals disjoint from the spectra of $T_{1j}, \ldots, T_{kj}$. Let $S_j = \sum_n \bigoplus T_{nj}$, $j = 1, \ldots, k$. Obviously, $T = S_1 \cdots S_k$. S_1 is cyclic by the above construction and Proposition 1.3. To prove the cyclicity of the remaining S_j's, let D_{nj} $(n > k$ and $j = 2, \ldots, k)$ be the diagonal operator with diagonals exactly those of T_{nj}. Since

$$T_{1j} \oplus \cdots \oplus T_{kj} \oplus \sum_{n=k+1}^{\infty} \bigoplus D_{nj}, \qquad j = 2, \ldots, k,$$

is cyclic and is a quasiaffine transform of S_j again by the above construction and Proposition 1.3, we conclude that S_j is cyclic as asserted. □

There are several corollaries of the preceding theorem. By the spectral theorem and the Wold decomposition, every isometry can be expressed as the direct sum of simple unilateral shifts and cyclic unitary operators. Thus Theorem 2.8 is applicable and we obtain

Corollary 2.9. *An isometry T is the product of k $(2 \leq k < \infty)$ cyclic operators if and only if* $\dim \ker T^* \leq k$.

In a similar fashion, every coisometry is the direct sum of some backward shift and cyclic unitary operators. Since the former summand is cyclic [8, Problem 160], Theorem 2.8 implies the following corollary.

Corollary 2.10. *Every coisometry is the product of two cyclic operators.*

We remark that part of Corollary 2.9 also follows from a result of Halmos [7, Theorem 2] that an isometry T with $\dim \ker T^* = k$ is the product of k simple unilateral shifts and that Corollary 2.10 follows from the result of Brown [2, Theorem 3] that every coisometry is the product of some backward shift and a simple unilateral shift.

For a normal operator T with $\dim\ker T^* = k$, we have, by the spectral theorem, the decomposition $T = \sum_{n=1}^{\infty} \oplus T_n$, where $T_1, \ldots, T_k$ are the zero operators on a one-dimensional space and every T_n $(n > k)$ is one-to-one with dense range. Hence we have

Corollary 2.11. *A normal operator* T *is the product of* k $(2 \le k < \infty)$ *cyclic operators if and only if* $\dim\ker T^* \le k$.

Concluding this paper, our final result says that products of (finitely many) cyclic operators can be characterized by the condition that the dimension of $\ker T^*$ be finite. It is obtained by combining the preceding three corollaries.

Theorem 2.12. *An operator* T *with* $\dim\ker T^* \le k$ $(2 \le k < \infty)$ *is the product of at most* $k+2$ *cyclic operators.*

Proof. If $\dim\ker T \le \dim\ker T^*$, then the polar decomposition of T yields $T = VP$, where V is an isometry with $\dim\ker V^* = \dim\ker T^* - \dim\ker T$ and $P = (T^*T)^{1/2}$ satisfies $\ker P = \ker T$. Hence, by Corollaries 2.9 and 2.11, V and P are, respectively, the products of m and n cyclic operators, where $m = \max\{\dim\ker T^* - \dim\ker T, 2\}$ and $n = \max\{\dim\ker T, 2\}$. It follows that T is the product of $k+2$ cyclic operators.

On the other hand, if $\dim\ker T > \dim\ker T^*$, then consider the decomposition $T = PV$, where $P = (TT^*)^{1/2}$ and V is a coisometry. Since $\dim\ker P = \dim\ker T^* \le k$, Corollary 2.11 implies that P is the product of k cyclic operators. Also, V is the product of two cyclic operators by Corollary 2.10. Hence in this case our assertion also follows. □

Note that in the preceding proof we actually showed that T is the product of k cyclic operators if $2 \le \dim\ker T$ and $\dim\ker T + 2 \le \dim\ker T^* \le k$ $(2 \le k < \infty)$.

Added in proof

Applying the Gram-Schmidt process to the column vectors of a matrix, we can obtain an infinite-dimensional QR decomposition: every operator with dense range is the product of a unitary operator and a triangular operator. The proof of Lemma 2.6 can then be considerably shortened by using such an argument. This observation results from a sharp remark made by Professor Tzon-Tzer Lu in a presentation of results contained herein.

References

1. C. Apostol, L. A. Fialkow, D. A. Herrero, and D. Voiculescu, *Approximation of Hilbert space operators*. II, Pitman, Boston, MA, 1984.
2. L. G. Brown, *Almost every proper isometry is a shift*, Indiana Univ. Math. J. **23** (1973), 429–431.
3. K. R. Davidson and D. A. Herrero, *The Jordan form of a bitriangular operator*, J. Funct. Anal. **94** (1990), 27–73.
4. R. G. Douglas and C. Pearcy, *A note on quasitriangular operators*, Duke Math. J. **37** (1970), 177–188.
5. P. A. Fillmore, J. G. Stampfli, and J. P. Williams, *On the essential numerical range, the essential spectrum and a problem of Halmos*, Acta Sci. Math. (Szeged) **33** (1962), 179–192.
6. I. Gohberg, P. Lancaster, and L. Rodman, *Invariant subspaces of matrices with applications*, Wiley, New York, 1986.

7. P. R. Halmos, *Products of shifts*, Duke Math. J. **39** (1972), 779–787.

8. ——, *A Hilbert space problem book*, 2nd ed., Springer-Verlag, New York, 1982.

9. D. A. Herrero, D. R. Lason, and W. R. Wogen, *Semitriangular operators*, Houston J. Math. **17** (1991), 477–499.

10. D. A. Herrero and W. R. Wogen, *On the multiplicity of* $T \oplus T \oplus \cdots \oplus T$, Rocky Mountain J. Math. **20** (1990), 445–466.

11. N. K. Nikolskii, *Multiplicity phenomenon.* I. *An introduction and maxi-formulas*, Toeplitz Operators and Spectral Function Theory (N. K. Nikolskii, ed.), OT, vol. 42, Birkhäuser-Verlag, Basel (1989), 9–57.

12. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1973.

Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, Republic of China