AIP Chaos An Interdisciplinary Journal of Nonlinear Science

Child allowances, fertility, and chaotic dynamics

Hung-Ju Chen and Ming-Chia Li

Citation: Chaos: An Interdisciplinary Journal of Nonlinear Science **23**, 023106 (2013); doi: 10.1063/1.4802034
View online: http://dx.doi.org/10.1063/1.4802034
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/23/2?ver=pdfcov
Published by the AIP Publishing

CHAOS 23, 023106 (2013)

Child allowances, fertility, and chaotic dynamics

Hung-Ju Chen[1,a)] and Ming-Chia Li[2,b)]
[1]Department of Economics, National Taiwan University, Taipei, Taiwan
[2]Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan

(Received 28 November 2012; accepted 18 March 2013; published online 24 April 2013)

This paper analyzes the dynamics in an overlapping generations model with the provision of child allowances. Fertility is an increasing function of child allowances and there exists a threshold effect of the marginal effect of child allowances on fertility. We show that if the effectiveness of child allowances is sufficiently high, an intermediate-sized tax rate will be enough to generate chaotic dynamics. Besides, a decrease in the inter-temporal elasticity of substitution will prevent the occurrence of irregular cycles. *© 2013 AIP Publishing LLC*
[http://dx.doi.org/10.1063/1.4802034]

In the past century fertility has steadily decreased in many countries over the course of economic growth. This phenomenon has led economists to worry about the future tax burden. In order to solve this problem, some countries start implementing public policies such as child allowances to raise the fertility rate to increase population growth. We follow this policy trend by assuming that government levies income tax in order to provide child allowances. With child subsidies, parents will have stronger incentives to have more children since the cost of raising children becomes lower. Fertility is an increasing function of child allowances and there exists a threshold effect of the marginal effect of child allowances on fertility. We show that if the effectiveness of child allowances is sufficiently high, an intermediate-sized tax rate will be enough to generate chaotic dynamics.

I. INTRODUCTION

The phenomenon of the decreased fertility rate in the past century has led many people to worry about the future tax burden. As the increased child rearing cost is one of the major reasons for decline in fertility, many countries started providing child subsidies (child allowances) to raise the incentive of having children. Based on an individual-level panel dataset for all married Israeli women from 1999-2005, the empirical study of Ref. 1 shows that child subsidies can cause a significant positive effect on fertility. Other empirical studies examine the relationship between child allowances and fertility can be found in Refs. 8 and 6.

In this paper, we develop an overlapping generations (OLGs) model with population growth and child allowances. Government levies income tax in order to provide child allowances. Based on the previous empirical studies, we adopt a fertility function which exhibits a positive relationship between child allowances and fertility. Theoretical studies generating a positive relationship between child allowances and fertility can be found in Refs. 10–12. In particular, there is a threshold effect of the marginal effect of child allowances on fertility. If the amount of child allowances is higher (lower) than the threshold, there will be a diminishing (increasing) marginal effect of child allowances on fertility.

We find that the effectiveness and the amount of child allowances are two important determinants to the monotonicity of the dynamics of the economy and the stability property of the non-trivial steady state. The provision of child allowances will cause two negative effects on capital accumulation. First, the tax rate reduces the after-tax income for the young and this will reduce savings and capital accumulation. Second, fertility is raised by the provision of child allowances and this will in turn raise population growth rate and reduce capital per worker in the future. The second effect is reinforced if child allowances have strong effects on fertility. We show that if the effectiveness of child allowances is large enough, an intermediate-sized tax rate will be enough to generate chaotic dynamics.

Because the formation of expectation is an important issue in an OLG model with capital accumulation, we then extend our model by considering a constant inter-temporal elasticity of substitution (CES) utility function to examine how expectation affects inter-temporal decisions and dynamics of capital accumulation. Under a CES utility function, the saving function depends on the future interest rate and we find that besides the effectiveness and amount of the child allowances, the inter-temporal elasticity of substitution in consumption is also an important factor in determining the dynamic behavior of the economy. A decrease in the inter-temporal elasticity of substitution in consumption provides for the stability of the economy. If the positive effect on stability caused by a lower inter-temporal elasticity of substitution is smaller than the negative effect on stability caused by the provision of child allowances, the dynamics of capital per worker will become non-monotonic and the complex dynamics may emerge. In both settings of the utility function, besides illustrating the dynamic property by numerical examples, we also prove the existence and uniqueness of the non-trivial equilibrium and give conditions for the (non-) monotonicity of dynamics of capital per worker. Furthermore, we also provide conditions for the occurrence of Li-Yorke chaos and the bubbling phenomenon. Therefore, this paper contributes to the existing literature of fertility by showing that the provision of child allowances may cause chaotic dynamic behavior.

a)Electronic mail: hjc@ntu.edu.tw
b)Author to whom correspondence should be addressed. Electronic mail: mcli@math.nctu.edu.tw.

The remainder of this paper is organized as follows. Section II develops an OLG model with child allowances. A logarithmic utility function is considered in this section. The stability property of the equilibrium and the monotonicity property of the dynamics are examined in Sec. III. A model with a CES utility function is developed and its dynamic behavior is analyzed in Sec. IV. Concluding remarks are given in Sec. V.

II. THE MODEL

We consider an infinite-horizon, discrete time OLG model. Agents live for three periods, corresponding to children, young agents, and old agents. Each period equals 30 years. Each agent is endowed with one unit of time in each period. Young agents use all the time for work to earn the real wage rates (w_t) for consumptions ($c_{1,t}$) and savings (s_t). When young agents become old, they spend all the time for leisure and consume ($c_{2,t+1}$) their savings from the previous period. We consider a separable utility function as follows:

$$U(c_{1t}, c_{2,t+1}) = u(c_{1,t}) + \beta u(c_{2,t+1}), \tag{1}$$

where $\beta \in (0,1)$ is the discount factor.

Government levies income tax with the rate of $\tau \in (0,1)$. The budget constraints for young and old agents are

$$c_{1,t} + s_t \le (1-\tau)w_t, \tag{2}$$

$$c_{2,t+1} \le R_{t+1}s_t, \tag{3}$$

where R_{t+1} represents the gross real return of capital in period $t+1$.

We first consider a logarithmic utility function: $u(c_i) = \log c_i$, $i = 1, 2$. Young agents maximize Eq. (1) subject to Eqs. (2) and (3), together with

$$c_{1,t} \ge 0 \quad \text{and} \quad c_{2,t+1} \ge 0.$$

Under the specification of the logarithmic utility function, the optimal saving decision given by the young agents is independent of the real interest rate; that is,

$$s_t = \frac{\beta(1-\tau)w_t}{1+\beta}. \tag{4}$$

Young agents give birth to children. Government uses the tax revenue to provide child allowances and runs a balanced budget. Since the child allowance scheme reduces the cost of raising children, it raises fertility (and therefore population growth). Thus, we assume that fertility depends on the total amount of child allowances (T_t) received by each young individual. A fertility function which is positive dependent of child allowances is also adopted by Ref. 13. With a balanced government budget constraint, that means $T_t = \tau w_t$. In particular, we assume that the relationship between fertility and child allowances is characterized by a non-decreasing and bounded function,

$$n(T_t) = \frac{a_0 + a_1\delta T_t^{\phi}}{1+\delta T_t^{\phi}}, \tag{5}$$

where $\delta, \phi > 0$, $0 \le a_0 < a_1$, $n(0) = a_0$, $\lim_{w\to\infty} n(w_t) = a_1$, $n'(T_t) = \frac{\delta\phi T_t^{\phi-1}(a_1-a_0)}{[1+\delta T_t^{\phi}]^2} > 0$. Furthermore, $n''(T_t) < 0$ if $\phi \le 1$ and for any $T_t \lesseqqgtr \bar{T} = \left[\frac{\phi-1}{\delta(1+\phi)}\right]^{1/\phi}$, $n''(T_t) \gtreqqless 0$ if $\phi > 1$. The parameter a_0 represents the fertility rate when there is no child allowance scheme and a_1 is the saturating value of fertility. Hence, $(a_1 - a_0)$ is the maximum increase in fertility with the implementation of the child allowance scheme. When $\tau = 0$ (no child allowance scheme), our model is reduced to the traditional growth OLG model with an exogenous population growth factor of a_0. Parameters δ and ϕ measure the effectiveness of child allowances. If $\phi \le 1$, there will be a diminishing marginal effect of child allowances on fertility and $n(T_t)$ is concave. However, if $\phi > 1$, there will exist a threshold of the amount of child allowances, $\bar{T}$, on the marginal effect of child allowances on fertility. When child allowances are smaller than $\bar{T}$, similar to results obtained in the studies which endogenize fertility choice, the marginal effect of child allowances is increasing with an increase in the child allowances. But when the child allowances are higher than $\bar{T}$, there is diminishing marginal effect of child allowances. That is, the function $n(T_t)$ is S-shaped when $\phi > 1$. Eq. (5) formation is used by Ref. 14 to represent the relationship between the survival probability to old age and the level of human capital to study the impact of longevity on economic growth. Eq. (5) formation is also used by Ref. 5 to represent the relationship between the efficient labor of old age and public health spending to examine the possibility of complex dynamics in an OLG model. It examines the effect of the dramatic increase in child allowances of the Israel Defense Forces for the third and higher birth-order children on fertility and finds that the fertility of Bedouin and Muslims is not affected by the increase of child allowances in Ref. 15. In the following, we will examine the impact of ϕ and focus our analysis on the case of $\phi > 1$.

Let L_t denote the number of young agents in period t. The dynamics of population of generations is therefore governed by $L_{t+1} = L_t n_t$. Output (Y_t) is produced by using capital (K_t) and labor through a Cobb-Douglas function: $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, where $A > 0$ and $\alpha \in (0,1)$, respectively, represent the total factor productivity and capital share. Thus, the output per worker (y_t) can be written as

$$y_t = \frac{Y_t}{L_t} = Ak_t^{\alpha},$$

where k_t is capital per worker in period t. It follows that the factor prices are

$$w_t = w(k_t) = A(1-\alpha)k_t^{\alpha}, \tag{6}$$

$$R_t = R(k_t) = A\alpha k_t^{\alpha-1}. \tag{7}$$

III. EQUILIBRIUM

Combining Eqs. (5) and (6) and government budget constraint, the fertility can be represented as a function of capital per worker,

$$n(k_t) = \frac{a_0 + a_1\delta[\tau A(1-\alpha)k_t^{\alpha}]^{\phi}}{1+\delta[\tau A(1-\alpha)k_t^{\alpha}]^{\phi}}. \tag{8}$$

The capital market clearing condition implies that $K_{t+1} = L_t s_t$. Using Eqs. (4), (6), and (8), the capital market clearing condition can be represented by

$$n(k_t)k_{t+1} = s(k_t).$$

Therefore, given the initial condition k_0, the equilibrium is composed by the sequence $\{k_t\}_{t\geq 0}$ that satisfies

$$k_{t+1} = \frac{\beta A(1-\tau)(1-\alpha)k_t^{\alpha}}{(1+\beta)n(k_t)}, \tag{9}$$

where $n(k_t)$ is given by Eq. (8).

Substituting Eq. (8) into Eq. (9), the dynamics of capital per worker can be represented as the following difference equation:

$$k_{t+1} = g_1(k_t) = \frac{(1-\tau)Fk_t^{\alpha}[1+\delta(\tau Bk_t^{\alpha})^{\phi}]}{D+\delta E(\tau Bk_t^{\alpha})^{\phi}}, \tag{10}$$

where $B = A(1-\alpha), F = \beta B, D = a_0(1+\beta)$ and $E = a_1(1+\beta)$ are all positive. Note that $D < E$ since $a_0 < a_1$.

A. Dynamics

Steady states of dynamics Eq. (10) are determined by $k_{t+1} = k_t$. There exists a zero (trivial) equilibrium of dynamics Eq. (10) and this zero equilibrium is unstable since $\lim_{k\to 0^+} g_1'(k) = \infty$ (see Eq. (11) below). In the following theorem, we show that there exists a unique positive (nontrivial) steady state (k^*) of dynamics Eq. (10). In addition, we also give the interval of the appearance of k^*.

Proposition 1. The dynamics $k_t \mapsto k_{t+1}$ in Eq. (10) has a unique positive steady state (k^*), which lies between $\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}}$ and $\left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}$.

Proof. For $k > 0$, define $h(k) = k^{1-\alpha}$ and

$$f(k) = \frac{(1-\tau)F}{E}\left[1 + \frac{E-D}{D+\delta E(\tau Bk)^{\phi}}\right].$$

Then the steady states are solutions of $f(k) = h(k)$. It is clear that f is strictly decreasing in k and has values from the supremum $\frac{(1-\tau)F}{D}$ to the infimum $\frac{(1-\tau)F}{E}$, and h is strictly increasing and has values from the infimum 0 to the supremum ∞. Since $0 < \tau < 1$, one gets $0 < \frac{(1-\tau)F}{E} < \frac{(1-\tau)F}{D} < \infty$, and hence there is exactly one positive solution for $f(k) = h(k)$, and the solution lies in the interval $\left(\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}}, \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}\right)$. □

In the first part of the following theorem, we provide three sufficient conditions so that $g_1(k_t)$ is monotonically increasing in k_t. In addition, although we are not able to get the closed-form solution of k^*, we show that under these three conditions, the non-trivial steady state k^* is a global attractor. In the second part of the Theorem 1, we give a sufficient condition in which $g_1(k_t)$ is not monotonically increasing in k_t. We are more interested in the second case because the non-monotonicity of $g_1(k_t)$ may generate endogenous cycles.

Theorem 1. *For the dynamics $k_t \mapsto k_{t+1}$ in Eq. (10), we have the following properties.*

1. *The unique positive steady state (k^*) is globally attracting and $g_1(k_t)$ is monotonically increasing in k_t if one of the followings holds:*
   (a) $0 < \phi \leq \bar{\phi} = \frac{E+D+2\sqrt{ED}}{E-D}$;
   (b) $\phi > \bar{\phi}$ *and* τ *is close to* 0 *or* 1;
   (c) $\left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}}$ *does not lie in the interval* $\left[\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}}, \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}\right]$, *and* ϕ *is sufficiently large.*
2. *The law of the motion of k_t is non-monotonic if $\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}} \leq \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}} \leq \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}$ and ϕ is sufficiently large.*

Proof. Let k^* denote the unique positive fixed point of $g_1(k)$. Differentiating $g_1(k)$ with respect to k gives us,

$$g_1'(k) = \frac{\alpha(1-\tau)F\{\delta^2 E(\tau Bk^{\alpha})^{2\phi} + \delta[E+D-\phi(E-D)](\tau Bk^{\alpha})^{\phi} + D\}}{k^{1-\alpha}[D+\delta E(\tau Bk^{\alpha})^{\phi}]^2}. \tag{11}$$

Since $\lim_{k\to 0} g_1'(k) = \infty$ and $\lim_{k\to\infty} g_1'(k) = 0$, we get that $g_1(k) > k$ (resp. $g_1(k) < k$) for all $0 < k < k^*$ (resp. $k > k^*$). By considering a new variable $x = \delta(\tau Bk^{\alpha})^{\phi}$, the numerator of g_1' in Eq. (11) becomes a quadratic polynomial in x. If $0 < \phi \leq \frac{E+D}{E-D}$, then $E + D - \phi(E-D) \geq 0$ and hence $g_1'(k) > 0$ for all $k > 0$. Let

$$\Delta = \phi^2(E-D)^2 - 2\phi(E-D)(E+D) + (E-D)^2.$$

Then $[\alpha(1-\tau)F]^2\Delta$ is the determinant for the existence of critical points of g_1. Solving the equation $\Delta = 0$ for ϕ gives the roots $\phi = \frac{E+D\pm 2\sqrt{ED}}{E-D}$ and hence $\Delta < 0$ for $\frac{E+D-2\sqrt{ED}}{E-D} < \phi < \frac{E+D+2\sqrt{ED}}{E-D}$. Since $\frac{E+D-2\sqrt{ED}}{E-D} < \frac{E+D}{E-D} < \frac{E+D+2\sqrt{ED}}{E-D}$, for $\frac{E+D}{E-D} < \phi \leq \frac{E+D+2\sqrt{ED}}{E-D}$, we have that $g_1'(k) \geq 0$ for all $k > 0$. Therefore, for $0 < \phi \leq \frac{E+D+2\sqrt{ED}}{E-D}$, we get that $g_1'(k) \geq 0$ for all $k > 0$. Then, $g_1'(k^*) \geq 0$. Moreover, if $0 < k < k^*$, then $\{g_1^t(k)\}_{t\geq 0}$ is an increasing sequence in the real line bounded above by k^*. By the completeness of the real numbers, $\lim_{t\to\infty} g_1^t(k)$ exists and is positive. By the continuity of $g_1(k)$, we get that $g_1(\lim_{t\to\infty} g_1^t(k)) = \lim_{t\to\infty} g_1^{t+1}(k) = \lim_{t\to\infty} g_1^t(k)$ and $\lim_{t\to\infty} g_1^t(k)$ is a positive fixed point for $g_1(k)$. By Proposition 1, $\lim_{t\to\infty} g_1^t(k) = k^*$. Similarly, if $k > k^*$, then $\lim_{t\to\infty} g_1^t(k) = k^*$. This completes the proof of (1a).

From the hypothesis of (1b), we have $\phi > \frac{E+D+2\sqrt{ED}}{E-D}$. Thus $0 < \sqrt{\Delta} < -(E+D) + \phi(E-D)$. Since the numerator of g_1' in Eq. (11) is a quadratic polynomial in $x = \delta(\tau Bk^{\alpha})^{\phi}$ with the determinant $[\alpha(1-\tau)F]^2\Delta > 0$, the critical points of g_1 are the following two positive points:

$$k_c^- = \left[\frac{-(E+D)+\phi(E-D)-\sqrt{\Delta}}{2\delta E(\tau B)^{\phi}}\right]^{\frac{1}{\alpha\phi}},$$

$$k_c^+ = \left[\frac{-(E+D)+\phi(E-D)+\sqrt{\Delta}}{2\delta E(\tau B)^{\phi}}\right]^{\frac{1}{\alpha\phi}}.$$

Moreover, g_1' is negative on the interval (k_c^-, k_c^+) and is positive on the intervals $(0, k_c^-)$ and (k_c^+, ∞). By Proposition 1, we get that $k^* < \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}$. Taking τ close enough to 0 or 1 such that

$$\left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}} < k_c^-.$$

Then $k^* < k_c^- < k_c^+$ and $g_1'(k) > 0$ for all $0 < k \le k^*$. Hence, if $0 < k < k^*$, by the same argument as above, $\{g_1^t(k)\}_{t\ge 0}$ is an increasing sequence converging to k^*. If $k > k^*$, then either $g_1^t(k) > k^*$ for all $t > 0$ or $g_1^{t_0}(k) \le k^*$ for some $t_0 > 0$. For the former case, $\{g_1^t(k)\}_{t\ge 0}$ is a decreasing sequence converging to k^*, and for the latter case, $\{g_1^t(k)\}_{t\ge t_0}$ is an increasing sequence converging to k^*. This completes the proof of (1b).

For (1c), let $\phi \ge \frac{E+D+2\sqrt{ED}}{E-D} > 1$. This implies that $\Delta \ge 0$ and hence the critical points k_c^- and k_c^+ of $g_1(k)$ are positive real numbers. As ϕ goes to ∞, we get that both k_c^- and k_c^+ converge to $\left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}}$; indeed,

$$\begin{aligned}
&\lim_{\phi\to\infty}\left[\frac{-(E+D)+\phi(E-D)\pm\sqrt{\Delta}}{2\delta E(\tau B)^{\phi}}\right]^{\frac{1}{\alpha\phi}} \\
&= \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}} \lim_{\phi\to\infty}\left(\frac{\phi}{2\delta E}\right)^{\frac{1}{\alpha\phi}}\left[\frac{-(E+D)+\phi(E-D)\pm\sqrt{\Delta}}{\phi}\right]^{\frac{1}{\alpha\phi}} \\
&= \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}} \exp\left(\lim_{\phi\to\infty}\frac{\ln\left(\frac{\phi}{2\delta E}\right)}{\alpha\phi}\lim_{\phi\to\infty}\frac{\ln\left[\frac{-(E+D)+\phi(E-D)\pm\sqrt{\Delta}}{\phi}\right]}{\alpha\phi}\right) \\
&= \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}} \exp\left(\lim_{\phi\to\infty}\left(\frac{1}{\alpha\phi}\right)\lim_{\phi\to\infty}\frac{\ln\left[\frac{-(E+D)+\phi(E-D)\pm\sqrt{\Delta}}{\phi}\right]}{\alpha\phi}\right), \quad \text{by L'Hôspital's rule} \\
&= \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}} \exp(0), \text{ since } \left|\frac{\ln\left[\frac{-(E+D)+\phi(E-D)\pm\sqrt{\Delta}}{\phi}\right]}{\alpha\phi}\right| \le \frac{\ln[2(E-D)]}{\alpha} \\
&= \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

Since $\left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}}$ is not in the interval $\left[\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}}, \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}\right]$, for all sufficiently large ϕ, neither k_c^- nor k_c^+ is in the interval $\left[\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}}, \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}\right]$. From Proposition 1, we have either $k^* < k_c^- \le k_c^+$ or $k_c^- \le k_c^+ < k^*$, and hence $g_1'(k^*) > 0$. By the same argument as above, we get that k^* is globally attracting and the law of the motion is monotonic.

To prove (2), let $\hat{k} = \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}}$ and $\phi \ge \frac{E+D+2\sqrt{ED}}{E-D} > 1$. Then, $\left(\frac{k_c^\pm}{\hat{k}}\right)^{\alpha\phi} = \frac{1}{2\delta E}[-(E+D)+\phi(E-D)\pm\sqrt{\Delta}]$. As ϕ goes to ∞, we have that

$$\lim_{\phi\to\infty} -(E+D)+\phi(E-D)+\sqrt{\Delta} = \infty$$

and

$$\begin{aligned}
&\lim_{\phi\to\infty} -(E+D)+\phi(E-D)-\sqrt{\Delta} \\
&= \lim_{\phi\to\infty}\frac{[-(E+D)+\phi(E-D)]^2-\Delta}{-(E+D)+\phi(E-D)+\sqrt{\Delta}} \\
&= \lim_{\phi\to\infty}\frac{4DE}{-(E+D)+\phi(E-D)+\sqrt{\Delta}} = 0.
\end{aligned}$$

Thus, if ϕ is sufficiently large, then $\left(k_c^- \hat{k}\right)^{\alpha\phi} < 1 < \left(\frac{k_c^+}{\hat{k}}\right)^{\alpha\phi}$ and hence $k_c^- < \hat{k} < k_c^+$. Let $\epsilon = \min(\hat{k} - k_c^-, k_c^+ - \hat{k})/2$. Then $\epsilon > 0$. By the definitions of $\hat{k}$ and $g_1(k)$, we get that $g_1(\hat{k})$ is a constant for all ϕ. Moreover, if ϕ is sufficiently large, then $g_1(k)$ will be close to $\frac{(1-\tau)Fk^\alpha}{D}$ for all $0 < k < \hat{k} - \epsilon$ and $g_1(k)$ will be close to $\frac{(1-\tau)Fk^\alpha}{E}$ for all $\hat{k} + \epsilon < k$. Since $\left[\frac{(1-\tau)F}{E}\right]^{\frac{1}{1-\alpha}} \le \hat{k} \le \left[\frac{(1-\tau)F}{D}\right]^{\frac{1}{1-\alpha}}$, we get that $\frac{(1-\tau)Fk^\alpha}{E} \le k < \frac{(1-\tau)Fk^\alpha}{D}$ for all $\hat{k} \le k \le \hat{k} + \epsilon$, $\frac{(1-\tau)Fk^\alpha}{E} < k \le \frac{(1-\tau)Fk^\alpha}{D}$ for all $\hat{k} - \epsilon \le k \le \hat{k}$, or both. By the intermediate value theorem applied to the function $g_1(k) - k$, there is a fixed point of $g_1(k)$ in $[\hat{k} - \epsilon, \hat{k} + \epsilon]$, which must be k^* due to Proposition 1. Hence, $k_c^- < k^* < k_c^+$ and $g'(k^*) < 0$. Therefore, $g_1(k)$ is not monotonically increasing in k. □

Theorem 1 indicates that the monotonicity of $g_1(k_t)$ and the stability of the non-trivial steady state (k^*) depend on the effectiveness of child allowances (ϕ) and the amount of child allowances, measured by the tax rate (τ). Note that Eq. (11) indicates that $g_1(k_t)$ will be monotonically increasing in k_t if $\phi \le 1$. Because $\bar{\phi} > 1$, the three conditions given in the first part of Theorem 1 which guarantee the monotonicity of $g_1(k_t)$ include the case of $\phi \le 1$. Based on Theorem 1, if the effectiveness of child allowances is lower than the critical value, $(\bar{\phi})$, $g_1(k_t)$ will be monotonic and k^* will be a global attractor. When the effectiveness of child allowances is higher than $\bar{\phi}$, $g_1(k_t)$ is monotonic, provided that the tax rate is sufficiently low or high. If the effectiveness of child allowances is sufficiently high, we can relax the strict requirement of tax rate for the monotonicity of $g_1(k_t)$ and only require the tax rate to not lie in a certain range.

The second part of Theorem 1 indicates that $g_1(k_t)$ will become non-monotonic if the effectiveness of child allowances is sufficiently high and the tax rate is within a certain range. We illustrate the impact of the tax rate on the shape of the dynamics Eq. (10) by figures. The following parameter settings are used: $\alpha = 0.34$, $A = 5$, and $\beta = (0.995)^{30}$. For parameters used in the fertility function, we assign $a_0 = 1$ to represent an extreme case that there is no population growth and $a_1 = 25$ to represent the maximal women's reproductive period. We set $\delta = 1$ and $\phi = 25$, a sufficiently high value to generate the non-monotonicity of the dynamics Eq. (10). We refer this parameterization as our baseline model. The graphs of $g_1(k_t)$ with $\tau = 0.1, 0.25$, and 0.35 are presented in Figure 1. It shows that an increase in the tax rate depresses the locus of $g_1(k_t)$ toward the origin and lowers the non-trivial steady state k^*. This is because the tax rate will cause two negative effects on the accumulation of capital. First, the higher tax rate reduces savings since the after-tax income for the young becomes lower. The lower saving will decrease the accumulation of capital (the *saving effect*). Second, the provision of child allowances raises fertility and this will in turn, ceteris paribus, reduces the capital per worker in the future (the *capital-dilution effect*). The effectiveness of child allowances (ϕ) strengthens the second effect. As shown in Figure 1, $g_1(k_t)$ is monotonically increasing in k_t when τ is small ($\tau = 0.1$); however, $g_1(k_t)$ becomes non-monotonic when τ becomes larger ($\tau = 0.25$ and 0.35).

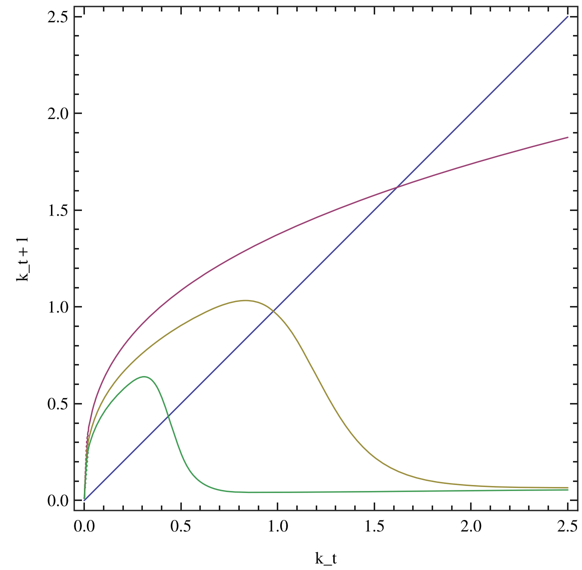

FIG. 1. The evolution of capital accumulation with $\tau = 0.1, 0.25$, and 0.35.

B. Chaotic motion

The non-monotonicity of the dynamics of k_t may behave in a way that the dynamics Eq. (10) becomes downward-sloping before the non-trivial steady state is achieved. This will cause some interesting dynamics behavior, such as regular cycles and chaos. Figure 1 also shows that as the tax rate changes from 0.25 to 0.35, the slope of the tangent line at k^* becomes steeper. Using the parameterization used in Figure 1, Figure 2 presents a bifurcation diagram of which τ lies between 0 and 1. As shown in Figure 2, when the tax rate is low, there is simple dynamics, such as a unique steady state or regular cycles. For a sufficiently low τ ($\tau < 0.26$), there is a unique limit point which is a stable steady state. As the tax rate continues rising, a period-doubling bifurcation starts emerging and the economy gets into the region of chaotic

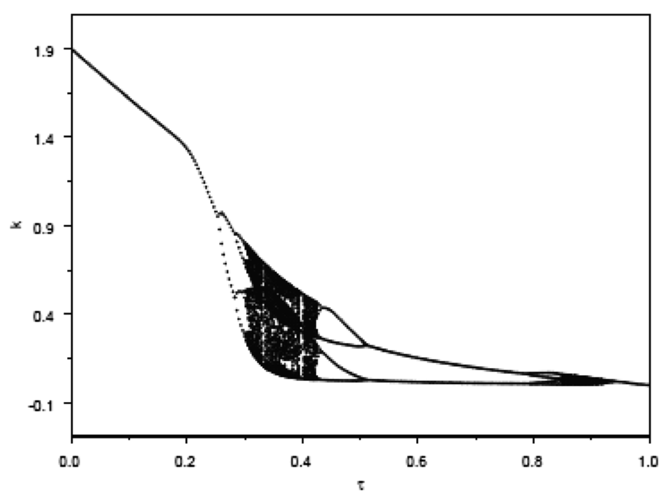

FIG. 2. The bifurcation diagram on τ.

dynamics when the tax rate is at the intermediate size ($0.31 < \tau < 0.43$). After that, a period-doubling bifurcation occurs again, followed by period-two cycles for a wide range of the tax rate ($0.515 < \tau < 0.79$). That is, the economy undergoes from simple dynamics to chaotic dynamics and then returns to simple dynamics. Note that the similar qualitative dynamic property will appear again as the tax rate continues rising. The chaotic dynamics will emerge when the tax rate is between 0.85 and 0.94 and the economy will get into the region of simple dynamics when the tax rate is higher than 0.94. Figure 2 numerically illustrates the occurrence of chaotic dynamics in our model.

We are now in the position to prove the occurrence of chaotic motion. Before doing this, we define the Li-Yorke chaos which is based on the definition given by Ref. 2. The Li-York chaos is the mostly often used definition of chaos in the one-dimensional dynamical system due to its easy verification. Also refer to Refs. 3 and 6 for more applications of the Li-Yorke chaos in economic issues.

Definition 1. Let $H: I \to I$ be a function, where I is an interval. We say that H has Li-Yorke chaos on I if

1. *H has periodic points of all periods; here by a periodic point p of period t, we mean that $H^t(p) = p$ and $H^i(p) \neq p$ for $0 < i < t$;*
2. *there exists an uncountable set $S \subset I$ such that*

   (i) *if $x, y \in S$ with $x \neq y$ then*

$$\limsup_{t\to\infty} |H^t(x) - H^t(y)| > 0$$

and

$$\liminf_{t\to\infty} |H^t(x) - H^t(y)| = 0;$$

   (ii) *if $x \in S$ and $y \in I$ is periodic then,*

$$\limsup_{t\to\infty} |H^t(x) - H^t(y)| > 0.$$

The Li-Yorke Theorem (Theorem 1, [Ref. 2, Theorem 1]) states that for a continuous function H from an interval into itself, if there is a point p such that

$$H^3(p) < p < H(p) < H^2(p),$$

then H exhibits Li-Yorke chaos. By using this condition, the following theorem shows that the law of motion in Eq. (10) will exhibit Li-Yorke chaos under certain parameterization.

Theorem 2. *If a_0 and a_1 satisfy*

$$a_0 < \frac{\beta \tau^{\frac{1-\alpha}{\alpha}}(1-\tau)[A(1-\alpha)]^{\frac{1}{\alpha}}}{1+\beta} \tag{12}$$

and

$$a_1 > \delta^{-1}\left[\frac{\beta^{\frac{\alpha+1}{\alpha}} A^{\frac{\alpha+1}{\alpha^2}} \tau^{\frac{1-\alpha^2}{\alpha^2}} (1-\tau)^{\frac{\alpha+1}{\alpha}} (1+\delta)(1-\alpha)^{\frac{\alpha+1}{\alpha^2}}}{a_0^{\frac{1}{\alpha}} (1+\beta)^{\frac{\alpha+1}{\alpha}}} - a_0\right], \tag{13}$$

then for all sufficiently large ϕ, the dynamics Eq. (10) will exhibit Li-Yorke chaos.

Notice that the right hand side of Eq. (12) (resp. (13)) dose not involve a_0 (resp. a_1).

Proof. In order to emphasize changes of the parameter ϕ, let us re-denote the function g_1 in Eq. (10) by $g_{1,\phi}$, that is,

$$g_{1,\phi}(k) = \frac{(1-\tau)Fk^\alpha}{E}\left[1 + \frac{E-D}{D+\delta E(\tau B k^\alpha)^\phi}\right].$$

Define $g_{1,\infty}$ to be a function given by the following: for $k > 0$,

$$g_{1,\infty}(k) = \frac{(1-\tau)Fk^\alpha}{D}.$$

Let

$$k_{co} = \left(\frac{1}{\tau B}\right)^{\frac{1}{\alpha}}.$$

Then, restricted to the interval $(0, k_{co})$, the limit function of $g_{1,\phi}$ is $g_{1,\infty}$ as ϕ goes to ∞. Moreover, $g_{1,\phi}(k_{co}) = \frac{(1+\delta)(1-\tau)F}{\tau B(D+\delta E)}$, $g_{1,\infty}(k_{co}) = \frac{(1-\tau)F}{\tau BD}$, and $g_{1,\infty}(g_{1,\phi}(k_{co})) = \frac{(1-\tau)F}{D}\left(\frac{(1+\delta)(1-\tau)F}{\tau B(D+\delta E)}\right)^\alpha$, which are all independent of ϕ. By plugging B, D, E, and F, the inequality Eq. (12) implies $k_{co} < g_{1,\infty}(k_{co})$, while the inequality Eq. (13) implies that $g_{1,\infty}(g_{1,\phi}(k_{co})) < k_{co}$. From the definitions of $g_{1,\phi}$ and $g_{1,\infty}$, for all $\phi > 0$, we have that $0 < g_{1,\phi}(k) < g_{1,\infty}(k)$ for all $k > 0$; in particular, we get that $0 < g_{1,\phi}(g_{1,\phi}(k_{co})) < g_{1,\infty}(g_{1,\phi}(k_{co}))$. Since $g_{1,\infty}$ is strictly increasing, $g_{1,\phi}(k_{co}) < k_{co}$; indeed, otherwise, $g_{1,\infty}(g_{1,\phi}(k_{co})) \geq g_{1,\infty}(k_{co})$, which leads a contradiction to the fact that $g_{1,\infty}(g_{1,\phi}(k_{co})) < k_{co} < g_{1,\infty}(k_{co})$. Since $g_{1,\phi}(k_{co})$ is independent of ϕ, so is the interval $I \equiv (g_{1,\infty}(g_{1,\phi}(k_{co})), k_{co})$. Since $g_{1,\phi}(k_{co}) < k_{co} < g_{1,\infty}(k_{co})$ and $\lim_{\phi\to 1,\infty} g_{1,\phi}(k) = g_{1,\infty}(k)$ for all k in I, there will exist q in I such that $g_{1,\phi}(q) = k_{co}$ if ϕ is sufficiently large. Let J denote the interval $(g_{1,\phi}(k_{co})), q)$, then J is independent of ϕ. Since $g_{1,\phi}(g_{1,\phi}(k_{co})) < g_{1,\infty}(g_{1,\phi}(k_{co}))$, we get that $g_{1,\phi}(J) \supset I$, and hence there exists p in J such that $g_{1,\phi}(p) = q$. Therefore,

$$g^3_{1,\phi}(p) = g_{1,\phi}(k_{co})) < p < g_{1,\phi}(p) = q < g^2_{1,\phi}(p) = k_{co}.$$

By applying the Li-Yorke theorem, the dynamics of Eq. (10) exhibits chaos in the sense of Li and Yorke. □

Theorem 2 provides a sufficient condition for the Li-Yorke chaos under our model setting. Figure 2 shows that the economy makes the transition from simple dynamics to chaotic dynamics, then returns to simple dynamics (*bubbling phenomenon*) as the tax rate increases. Combining the result of 1(b) in Theorem 1 and the above theorem implies the following result regarding the bubbling phenomenon.

Corollary 1. Let a_0 and a_1 satisfy Eqs. (12) and (13). *Then for each large ϕ, the dynamics Eq. (10) exhibits the bubbling phenomenon as τ increases from 0 to 1.*

Note that with the specification of the logarithmic utility function, the dynamic behavior of the economy does not depend on the agents' expectation since the saving function (4) is independent of the future interest rate. Therefore, in Sec. IV, we allow the expectation to affect the current saving decision by considering a CES utility function and examine how this change affects the dynamic behavior of the economy.

IV. A CES UTILITY FUNCTION

In this section, we consider a CES utility function as $u(c_i)=\frac{c_i^{1-\sigma}-1}{1-\sigma}$, where $\sigma>0$ and $\sigma\neq 1$ is the inverse of the inter-temporal elasticity of substitution of consumptions. When $\sigma=1$, the utility function is defined as the logarithmic utility function in Eq. (1). Under a CES utility function, the optimal saving decision becomes

$$s_t=s(w_t,R_{t+1})=\frac{(1-\tau)w_t}{1+\beta^{\frac{-1}{\sigma}}R_{t+1}^{1-\frac{1}{\sigma}}}. \quad (14)$$

Note that with $\sigma\neq 1$, the saving function in period t depends on the real interest rate in period $t+1$ (R_{t+1}).

The law of motion of capital per worker becomes

$$k_{t+1}=\frac{s(w(k_t),R(k_{t+1}))}{n(k_t)}. \quad (15)$$

Substituting Eqs. (6)–(8) into Eq. (15), the dynamics of capital per worker can be written as

$$k_{t+1}=\frac{[(1-\tau)Bk_t^{\alpha}][1+\delta(\tau Bk_t^{\alpha})^{\phi}]}{\left[1+\beta^{\frac{-1}{\sigma}}(A\alpha k_{t+1}^{\alpha-1})^{1-\frac{1}{\sigma}}\right][a_0+a_1\delta(\tau Bk_t^{\alpha})^{\phi}]}. \quad (16)$$

Although we cannot explicitly express k_{t+1} in terms of k_t from Eq. (16), we can express their relationship by an implicit function g_σ. First, for each $\sigma>0$, we define $F_\sigma:(0,\infty)\times(0,\infty)\to\mathbb{R}$ by

$$\begin{aligned}F_\sigma(k_t,k_{t+1})=&\left[k_{t+1}+\beta^{\frac{-1}{\sigma}}(A\alpha)^{1-\frac{1}{\sigma}}k_{t+1}^{1+(\alpha-1)(1-\frac{1}{\sigma})}\right]\\&\times[a_0+a_1\delta(\tau Bk_t^{\alpha})^{\phi}]-[(1-\tau)Bk_t^{\alpha}]\\&\times[1+\delta(\tau Bk_t^{\alpha})^{\phi}].\end{aligned}$$

Then for all $0<k_t,k_{t+1}<\infty$, we have F_σ that is continuously differentiable and

$$\begin{aligned}\frac{\partial F_\sigma(k_t,k_{t+1})}{\partial k_{t+1}}=&\left[1+\left(\alpha+\frac{1-\alpha}{\sigma}\right)\beta^{\frac{-1}{\sigma}}(A\alpha)^{1-\frac{1}{\sigma}}k_{t+1}^{(\alpha-1)(1-\frac{1}{\sigma})}\right]\\&\times[a_0+a_1\delta(\tau Bk_t^{\alpha})^{\phi}]\geq a_0>0.\end{aligned}$$

Thus, for each $k_t>0$, the function $k_{t+1}\mapsto F_\sigma(k_t,k_{t+1})$ is continuous on $(0,\infty)$ and attains values between $-[(1-\tau)Bk_t^{\alpha}][1+\delta(\tau Bk_t^{\alpha})^{\phi}]$ and ∞ exactly once for k_{t+1} in $(0,\infty)$. Hence, by the intermediate value theorem, there exists a unique point in $(0,\infty)$, namely $k_{t+1}=g_\sigma(k_t)$, such that $F_\sigma(k_t,g_\sigma(k_t))=0$. So far, we have obtained that g_σ forms a function from $(0,\infty)$ into itself such that

$$F_\sigma(k_t,g_\sigma(k_t))=0; \quad (17)$$

that is, $k_{t+1}=g_\sigma(k_t)$ represents the dynamics $k_t\mapsto k_{t+1}$ in Eq. (16). Note that when $\sigma=1$, the function $k_{t+1}=g_\sigma(k_t)$ coincides with the dynamics g_1 in Eq. (10) in Sec. II, therein, the utility function is logarithmic.

Furthermore, we show that g_σ is continuously differentiable. Fix $k_t>0$, the implicit function theorem implies that there exists a continuously differentiable function $\tilde{g}_\sigma(k)$ for k around k_t since $\frac{\partial F_\sigma(k_t,k_{t+1})}{\partial k_{t+1}}\neq 0$. Because $\frac{\partial F_\sigma(k_t,k_{t+1})}{\partial k_{t+1}}>0$, we get that $\tilde{g}_\sigma(k)=g_\sigma(k)$ for k around k_t and hence g_σ is continuously differentiable around k_t. Since $k_t>0$ is arbitrary, g_σ is continuously differentiable on $(0,\infty)$.

Although we are not able to get an analytical solution for the positive steady state (k_σ^*) of the dynamics Eq. (16), the following proposition demonstrates the existence and uniqueness of k_σ^*. Furthermore, it also provides conditions regarding the stability of k_σ^* and the monotonicity of the dynamics Eq. (16).

Proposition 2. For all σ sufficiently close to 1, the dynamics $k_t\mapsto k_{t+1}$ in Eq. (16) has a unique positive steady state (k_σ^*), and, as stated in the Theorem 1, has similar results of the stability property of k_σ^* and the monotonicity property of dynamics.

Proof. Let k^* denote the unique positive fixed point of g_1 in Proposition 1. First, we show that $g_1'(k^*)\neq 1$. Indeed, let $x=\delta(\tau Bk^{\alpha})^{\phi}$. Suppose that $g_1(k)=k$. Then the relation $k^{1-\alpha}=\frac{(1-\tau)F(1+x)}{D+Ex}$ holds. Plugging such a relation into $g_1'(k)$, we get that $g_1'(k)=\frac{\alpha Ex^2+\alpha[E+D-\phi(E-D)]x+\alpha D}{(1+x)(D+Ex)}$. Furthermore, suppose that $g_1'(k)=1$, we obtain that

$$(1-\alpha)Ex^2+[(1-\alpha)(E+D)+\phi\alpha(E-D)]x+(1-\alpha)D=0.$$

Since the left side of the above equation is a quadratic polynomial in x which coefficients are all positive, there are no positive roots for the equations. This shows that $g_1'(k^*)\neq 1$. Let

$$G(\sigma,k)=g_\sigma(k)-k,$$

where g_σ is given by Eq. (17). Then G is smooth jointly as a function of $\sigma>0$ and $k>0$, $G(1,k^*)=0$ and $\frac{\partial G(1,k^*)}{\partial k}\neq 0$. By the implicit function theorem, for each σ sufficiently close to 1, g_σ has a unique fixed point k_σ^* near k^*. Thus, k_σ^* is the unique positive steady state of the dynamics Eq. (16). It remains to show the stability and monotonicity properties holds. Since the function $(\sigma,k)\mapsto g_\sigma(k)$ is smooth, so is the derivative $g_\sigma'(k)$ with respect to k. Hence, replacing g_1 by g_σ and considering σ sufficiently close to 1, the same argument as in the proof of Theorem 1 proves the desired result. □

Proposition 2 indicates that the effectiveness of child allowances and the tax rate are still important factors in determining the monotonicity property of $g_\sigma(k_t)$ and the stability property of k_σ^*. When σ is close to 1, the dynamic behavior of $g_\sigma(k_t)$ is similar to that of Eq. (10). To study the impact of σ, we use the parameterization in the baseline model and present the evolution of $g_\sigma(k_t)$ with $\tau=0.1$ and $\sigma=1$, 3, and 8 in Figure 3. It shows that a rise in σ compresses the locus of $g_\sigma(k_t)$ downward and reduces k_σ^*. This is because when σ increases, households tend to save less (see Eq. (14)) and it causes another negative effect on the capital accumulation, leading to a lower k_σ^*. Figure 3 also indicates

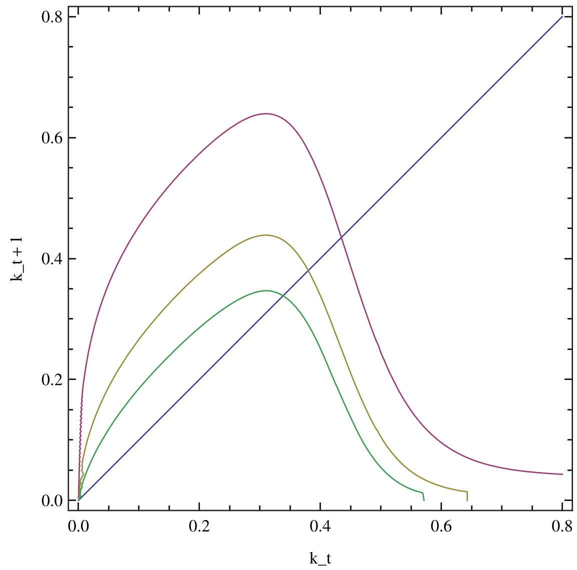

FIG. 3. The evolution of capital accumulation with $\sigma = 1, 3$, and 8.

that the slope of the tangent line at k_σ^* becomes less steeper as σ increases from 1 to 8. Therefore, besides the amount and the effectiveness of child allowances, the dynamic property of the function $g_\sigma(k_t)$ also depends on the inter-temporal elasticity of substitution. If the negative effects on the stability caused by an increase in τ outweigh the positive effect on the stability induced by an increase in σ, cycles and complex dynamics will emerge.

Figure 4 displays a bifurcation diagram with varying values of τ between 0 and 1 and $\sigma = 3$. Following Ref. 5, we use the parametric interval of the tax rate for which there exists a unique stable positive steady state to evaluate the stability of the economy. It is clear that a rise in σ increases the stability of the economy by comparing Figure 4 with Figure 2. When $\sigma = 3$, the economy is unstable (not converging to the unique stable positive steady state) for $0.3 < \tau < 0.69$ while under $\sigma = 1$, the economy is unstable for $0.26 < \tau < 0.94$. Although a higher value of σ will work for stability, period-doubling bifurcation, and chaotic motion will still occur if the tax rate is high enough, as shown by Figure 4.

In the following theorem, we provide the condition for the occurrence of Li-Yorke chaos.

Theorem 3. *Let a_0 and a_1 satisfy Eq. (12) and (13). If ϕ is sufficiently large and σ is sufficiently close to 1, the law of motion in Eq. (16) exhibits Li-Yorke chaos.*

Proof. From the proof of Theorem 3, let ϕ be sufficiently large so that there exists p such that $g_1^3(p) < p < g_1(p) < g_1^2(p)$. Since $g_\sigma(k)$ is continuous as a function of jointly σ and k, for all σ close enough to 1 and all q close enough to p, one has $g_\sigma^3(q) < q < g_\sigma(q) < g_\sigma^2(q)$. By the Li-Yorke theorem, the dynamics of Eq. (16) exhibits Li-Yorke chaos. □

Theorem 3 provides a sufficient condition for the emergence of Li-Yorke chaos in the economy with a CES utility function. In fact, it shows that chaotic motion of the dynamical system under a CES utility function is inherited from the chaotic behavior of the dynamical system under a logarithmic utility function. Similar to the case of $\sigma = 1$, Figure 4 also indicates that the dynamics of Eq. (16) exhibits the bubbling phenomenon. Together with Proposition 2, the above theorem implies the following corollary.

Corollary 2. Let a_0 and a_1 satisfy Eqs. (12) and (13). If ϕ is sufficiently large and σ is close enough to 1, the dynamics of Eq. (16) exhibits the bubbling phenomenon when τ increases from 0 to 1.

Finally, in Figure 5, we present a bifurcation diagram for varying values of σ between 1 and 5 and $\tau = 0.35$. It shows that chaotic dynamics will appear when σ is sufficiently low. Studies of Refs. 6 and 7 have shown that under perfect foresight, a standard OLG model with a CES utility function will only generate simple dynamics and the nontrivial steady state is stable. Therefore, our results indicate that a fiscal policy like the child allowances scheme can generate chaotic dynamics and bubbling phenomenon in a model where it is generally prohibited. When σ is high enough, chaotic motion will disappear and the period-doubling phenomenon will occur. An increase in σ prevents the occurrence of chaotic dynamics because of its positive effect on stability.

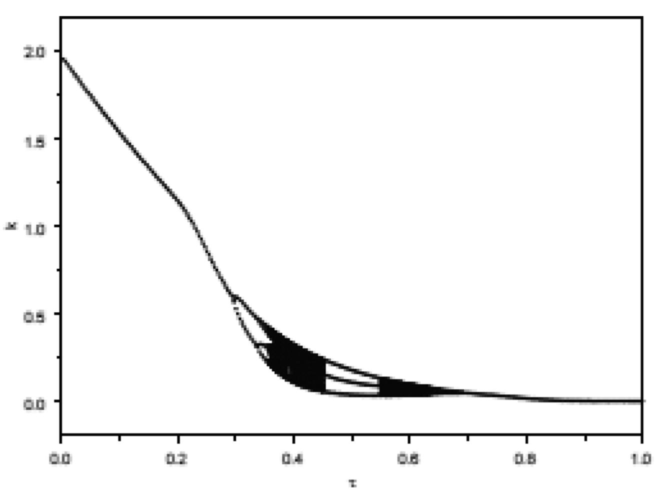

FIG. 4. The bifurcation diagram on τ when $\sigma = 3$.

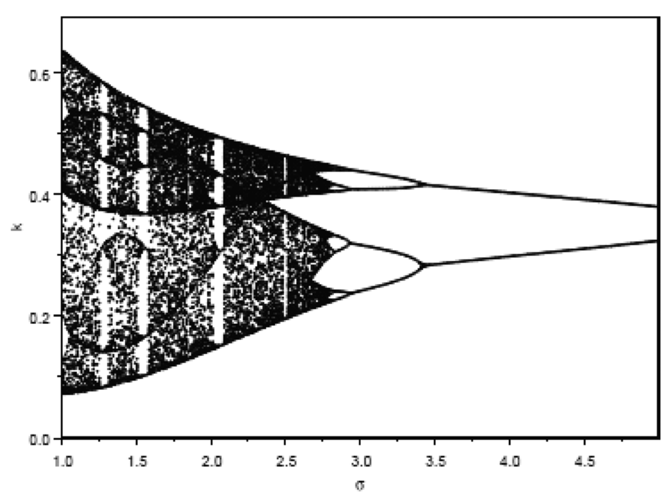

FIG. 5. The bifurcation diagram on σ when $\tau = 0.35$.

Chaotic motion will disappear if the positive effect on stability caused by an increase in σ overrides the negative effects on stability caused by the implementation of the child allowance scheme.

V. CONCLUSION

In this paper, we develop an OLG model with population growth to study the impact of child allowances on the economic dynamics. Our analysis shows that the amount and the effectiveness of child allowances as well as the inter-temporal elasticity of substitution are crucial factors in determining the dynamic property. The dynamics of capital per worker is simple if the effectiveness of child allowances on fertility is fairly small. However, cycles and even chaotic dynamics will occur if the effectiveness of child allowances is large enough and the tax rate is intermediate-sized. A decrease in the inter-temporal elasticity of substitution will prevent the economy from exposing to irregular fluctuations. Since our analysis indicates that the dynamic property depends on how child allowances affect fertility, a more precise empirical estimation of the effectiveness of child allowances is worthy of future study.

ACKNOWLEDGMENTS

The financial supports provided by the Taiwan National Science Council and the Program for Globalization Studies at the Institute for Advanced Studies in Humanities at the National Taiwan University are gratefully acknowledged.

[1]A. Cohen, R. Dehejia, and D. Romanov, "Do financial incentives affect fertility?," NBER Working Paper No. 13700 (2007).
[2]T. Li and J. Yorke, "Period three implies chaos," Am. Math. Mon. **82**, 985–992 (1975).
[3]H.-J. Chen and M.-C. Li, "Habit formation and chaotic dynamics in an n-dimensional cash-in-advance economy," Nonlinear Dyn. **58**, 49–62 (2009).
[4]H.-J. Chen and M.-C. Li, "Environmental tax policy, habit formation and nonlinear dynamics," Nonlinear Anal.: Real World Appl. **12**, 246–253 (2011).
[5]L. Fanti and L. Gori, "Public health spending, old-age productivity and economic growth: chaotic cycles under perfect foresight," J. Econ. Behav. Organ. **89**, 137–151 (2011).
[6]D. de la Croix and P. Michel, *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations* (Cambridge University Press, Cambridge, 2002).
[7]H.-J. Chen, M.-C. Li, and Y.-J. Lin, "Chaotic dynamics in an overlapping generations model with myopic and adaptive expectations," J. Econ. Behav. Organ. **67**, 48–56 (2008).
[8]C. Manski and J. Mayshar, "Private incentives and social interactions: fertility puzzles in Israel," J. Eur. Econ. Assoc. **1**, 181–211 (2003).
[9]J. Schellekens, "Family allowances and fertility: socioeconomic differences," Demography **46**, 451–468 (2009).
[10]B. van Groezen, T. Leers, and L. Meijdam, "Social security and endogenous fertility: pensions and child allowances as siamese twins," J. Public Econ. **87**, 233–251 (2003).
[11]B. van Groezen and L. Meijdam, "Growing old and staying young: population policy in an ageing closed economy," J. Popul. Econ. **21**, 573–588 (2008).
[12]L. Fanti and L. Gori, "Population and neoclassical economic growth: A new child policy perspective," Econ. Lett. **104**, 27–30 (2009).
[13]M. Momota, "The gender gap, fertility, subsidies and growth," Econ. Lett. **69**, 401–405 (2000).
[14]K. Blackburn and G. Cipriani, "A model of longevity, fertility and growth," J. Econ. Dyn. Control **26**, 187–204 (2002).
[15]R. Frish, "The effect of child allowances on fertility in Israel," Israel Econ. Rev. **6**, 1–22 (2008).