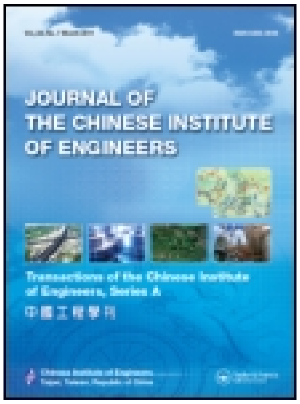

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Robust sliding control for mismatched uncertain fuzzy time-delay systems using linear matrix inequality approach

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In this article, we propose a robust sliding control design method for uncertain time-delay systems that can be represented by Takagi–Sugeno fuzzy models. The uncertain fuzzy time-delay systems under consideration have mismatched parameter uncertainties in the state matrix and external disturbances. We derive existence conditions of linear sliding surfaces guaranteeing the asymptotic stability in terms of constrained linear matrix inequalities (LMIs). We present an LMI characterization of such sliding surfaces. Also, an LMI-based algorithm is given to design the switching feedback control term so that a stable sliding motion is induced in finite time. Finally, we give two numerical design examples to show the effectiveness of the proposed method.

Keywords: uncertain time-delay systems; linear matrix inequality; sliding surface; switching feedback control

1. Introduction

Over the past two decades, fuzzy techniques have been widely and successfully exploited in nonlinear system modeling and control. The Takagi–Sugeno (T–S) model (Tagaki and Sugeno 1985) is a popular and convenient tool for handling complex nonlinear systems. Correspondingly, the fuzzy feedback control design problem for a nonlinear system has been studied extensively using T–S model where simple local linear models are combined to describe the global behavior of the nonlinear system (Tanaka *et al*. 1996, Wang *et al*. 1996, Ma *et al*. 1998, Tanaka *et al*. 1998, Korba *et al*. 2003, Nguang and Shi 2003). On the other hand, time-delay is often encountered in ical processes. Recently, the feedback stabilization problem for uncertain time-delay systems has also become a problem of interest because the existence of a delay is frequently a source of poor system performance or instability (Jafarov 2005, Peng *et al*. 2008, Zhang *et al*. 2009). Delays are sensitive to uncertainty, which directly affects the control systems.

It is known that a sliding mode control (SMC) system has various attractive features, such as fast response, good transient performance, order reduction, insensitivity to parameter variation, and invariance to external disturbances. In the SMC system, the control structure around the plant is intentionally changed using a nonlinear discontinuous controller to obtain a desired system response. Using nonlinear discontinuous control, the SMC system drives the system trajectory onto a specified and user-chosen surface, which is called the sliding or the switching surface, and maintains the trajectory on this sliding surface for all subsequent time. This motion is referred to as the sliding mode. The central feature of the SMC system is the sliding mode on the sliding surface in which the system remains insensitive to internal parameter variations and external disturbance. In sliding mode, the order of the system dynamics is reduced. This enables simplification and decoupling design procedures (Utkin 1977, DeCarlo *et al*. 1988, Walcott and Zak 1988, Choi 2003).

Considering these facts above, and utilizing the T–S model, we propose a robust sliding control design method for a mismatched uncertain T–S fuzzy delay-time model with parameter uncertainties and external disturbances. First, we derive a linear matrix inequality (LMI) condition for the existence of the fuzzy controller that guarantees a stable sliding motion on the switching surface that is insensitive to norm-bounded uncertainties. We show that the sliding surface parameter matrix can be characterized in terms of the solution of the LMI existence condition. Second, we design the nonlinear discontinuous term to drive the system trajectories so that a stable sliding motion is induced in finite time on the switching surface and the state converges to zero. Finally, two numerical design examples are given in order to show the effectiveness of the proposed method. The rest of this article is organized as follows. Section 2 describes the T–S

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fuzzy model and reviews some preliminary results. Section 3 presents an LMI existence condition of linear sliding surfaces and an explicit characterization of the sliding surface parameter matrices as well as a sliding control law. Section 4 gives two numerical design examples to demonstrate the validity and effectiveness. Finally, Section 5 offers some concluding remarks.

2. Problem formulation and preliminaries

The T–S fuzzy model is described by fuzzy IF-THEN rules, which represent local linear input–output relations of nonlinear systems. The ith rule of the T–S fuzzy time-delay model is of the following form:

Plant Rule i: IF θ_1 is μ_{i1} and... and θ_s is μ_{is}, THEN

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{A}_{\tau i}\boldsymbol{x}(t - d(t)) + \boldsymbol{B}_i\boldsymbol{u}(t), \tag{1}$$

$$\boldsymbol{x}(t) = \boldsymbol{\psi}(t), \quad t \in [-\tau, 0], \tag{2}$$

where $\boldsymbol{\psi}(t)$ is the initial condition, $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ the state, $\boldsymbol{u}(t) \in \boldsymbol{R}^m$ the control input, $\boldsymbol{A}_i \in \boldsymbol{R}^{n\times n}$ the state matrices, $\boldsymbol{A}_{\tau i} \in \boldsymbol{R}^{n\times n}$ the delayed state matrices, $\boldsymbol{B}_i \in \boldsymbol{R}^{n\times m}$ the input matrices, $\theta_j\,(j = 1,\dots,s)$ the premise variables, s the number of the premise variables, $\mu_{ij}(i = 1,\dots,r;\ j = 1,\dots,s)$ the fuzzy sets that are characterized by membership function, and r the number of the IF-THEN rules. The time-varying delay $d(t)$ is bounded as $d(t) \le \tau$. The overall fuzzy model achieved by fuzzy synthesizing of each individual plant rule is given by

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \beta_i(\theta)[\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{A}_{\tau i}\boldsymbol{x}(t - d(t)) + \boldsymbol{B}_i\boldsymbol{u}(t)], \tag{3}$$

$$\boldsymbol{x}(t) = \boldsymbol{\psi}(t), \quad t \in [-\tau, 0], \tag{4}$$

where $\theta = [\theta_1,\dots,\theta_s]$, $\beta_i(\theta) = \omega_i(\theta)/\sum_{j=1}^{r}\omega_j(\theta)$, $\omega_i : \boldsymbol{R}^s \to [0,1]$, $i = 1,\dots,r$ is the membership function of the system with respect to plant rule i. The function $\beta_i(\theta)$ can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \ge 0$, $\sum_{i=1}^{r}\beta_i(\theta) = 1$. To take into account parameter uncertainties and external disturbances, we consider the following uncertain T–S fuzzy time-delay model:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \beta_i(\theta)[&(\boldsymbol{A}_i + \boldsymbol{\Delta A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{A}_{\tau i} + \boldsymbol{\Delta A}_{\tau i}(t))\boldsymbol{x}_d(t)\\ &+ \boldsymbol{B}_i(\boldsymbol{u}(t) + \boldsymbol{h}_i(t, x, x_d, u))],\end{aligned} \tag{5}$$

$$\boldsymbol{x}(t) = \boldsymbol{\psi}(t), \quad t \in [-\tau, 0], \tag{6}$$

where $\boldsymbol{x}_d(t) = \boldsymbol{x}(t - d(t))$, $\boldsymbol{\Delta A}_i(t)$ represents the parameter uncertainties in $\boldsymbol{A}_i$, $\boldsymbol{\Delta A}_{\tau i}(t)$ the parameter uncertainties in $\boldsymbol{A}_{\tau i}$, $\boldsymbol{h}_i(t, x, x_d, u) \in \boldsymbol{R}^m$ the external disturbances. We will assume that the following assumptions are satisfied:

A1 $\boldsymbol{B}_1 = \boldsymbol{B}_2 = \cdots = \boldsymbol{B}_r := \boldsymbol{B}$ and rank $(\boldsymbol{B}) = m$.

A2 The function $\boldsymbol{h}_i(t, x, x_d, u)$ is unknown but bounded as $\|\boldsymbol{h}_i(t, x, x_d, u)\| \le \chi_i\|\boldsymbol{u}\| + \delta_i(t)$ where $\delta_i(t)$ is a known function and χ_i satisfies $\chi_i \le \chi_m < 1$ for a known constant χ_m.

A3 The time delay $d(t)$ is unknown but bounded as $d(t) \le \tau$ and $\dot{d}(t) \le d_m < 1$ where τ and d_m are known constants.

A4 $\Delta\mathrm{A}_i(t)$ and $\boldsymbol{\Delta A}_{\tau i}(t)$ are of the form $\boldsymbol{T}_i\boldsymbol{\Pi}_i(t)$ where $\boldsymbol{\Pi}_i(t)$ is a known time-varying matrix but bounded as $\|\boldsymbol{\Pi}_i(t)\| \le 1$.

Using the above assumptions, the uncertain T–S fuzzy model (5) can be written as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \beta_i(\theta)[&(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))x(t) + (\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))\boldsymbol{x}_d(t)\\ &+ \boldsymbol{B}\boldsymbol{h}_i(t, x, x_d, u)] + \boldsymbol{B}\boldsymbol{u}(t),\end{aligned} \tag{7}$$

$$\boldsymbol{x}(t) = \boldsymbol{\psi}(t), \quad t \in [-\tau, 0], \tag{8}$$

A large number of examples in the literature and various mechanical systems, such as motors and robots, fall into special cases of the above model (7), as reported in Lin *et al.* (2005), Xia and Jia (2003), Choi (2008), El-Khazali (1998), and Oucheriah (1995, 2003). The above model (7) also involves the uncertain time-delay system models considered in the previous SMC design methods (Oucheriah 1995, El-Khazali 1998, Oucheriah 2003, Xia and Jia 2003, Lin *et al.* 2005, Choi 2008). The symbol $*$ will be used in some matrix expressions to induce a symmetric structure. For given symmetric matrices, F and G of appropriate dimensions, the following holds:

$$\begin{bmatrix} \boldsymbol{F} + \boldsymbol{X} + * & * \\ \boldsymbol{Z} & \boldsymbol{G} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} + \boldsymbol{X} + \boldsymbol{X}^T & \boldsymbol{Z}^T \\ \boldsymbol{Z} & \boldsymbol{G} \end{bmatrix} \tag{9}$$

When no confusion arises, the arguments t, x, x_d, θ, etc. can be omitted for brevity.

3. Sliding control design via LMI approach

In this section, we demonstrate the problem of designing a robust sliding controller via LMI approach.

3.1. *LMI characterization of linear sliding surfaces*

The SMC design is decoupled into two independent tasks of lower dimensions. The first is concerned with the design of a sliding surface for the sliding mode such that the reduced-order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second involves choosing a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface (Utkin 1977). We first design a sliding surface that guarantees asymptotic stability of the reduced-order sliding mode dynamics using LMIs.

Theorem 1: *Consider the following LMIs*:

$$Y > 0, \quad F \geq 0,$$
$$\begin{bmatrix} N_{11} & * & * & * \\ N_{21} & N_{22} & * & * \\ \tau X_i & \tau Z_i & -\tau \Lambda^T Y \Lambda & 0 \\ N_{41} & N_{42} & 0 & -\tau \Lambda^T Y \Lambda \end{bmatrix} < 0, \quad \forall i, \tag{10}$$

where

$$N_{11} = F + \Lambda^T (A_i + T_i \Pi_i(t)) Y \Lambda + X_i + *, \tag{11}$$

$$N_{21} = \Lambda^T Y (A_{\tau i} + T_i \Pi_i(t))^T \Lambda - X_i + Z_i^T, \tag{12}$$

$$N_{22} = -(1 - d_m) F - Z_i - Z_i^T, \tag{13}$$

$$N_{41} = \tau \Lambda^T (A_i + T_i \Pi_i(t)) Y \Lambda, \tag{14}$$

$$N_{42} = \tau \Lambda^T (A_{\tau i} + T_i \Pi_i(t)) Y \Lambda. \tag{15}$$

The matrix $\Lambda \in R^{n\times(n-m)}$ is any full rank matrix such that $B^T \Lambda = 0$, $\Lambda^T \Lambda = I$. The matrices $Y \in R^{n\times n}$, $F \in R^{(n-m)\times(n-m)}$, $X_i \in R^{(n-m)\times(n-m)}$ and $Z_i \in R^{(n-m)\times(n-m)}$ are decision variables. Suppose that the LMIs equation (10) *have a solution (Y, F, X_i, Z_i) for given A_i, $A_{\tau i}$, B, d_m, τ, then there exists a linear sliding surface parameter matrix S and using a solution matrix Y to Equation* (10), *S can be parameterized as follows*:

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x \tag{16}$$

Proof: Defining a nonsingular transformation matrix M and the associated vector $v = Mx$ such that

$$M = \begin{bmatrix} (\Lambda^T Y \Lambda)^{-1} \Lambda^T \\ (B^T Y^{-1} B)^{-1} B^T Y^{-1} \end{bmatrix} = \begin{bmatrix} V \\ S \end{bmatrix}, \tag{17}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Vx \\ Sx \end{bmatrix} = Mx, \tag{18}$$

where $v_1 \in R^{n-m}$, $v_2 \in R^m$. Then we can easily see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. Let the positive definite matrix P_0 be $P_0 = \Lambda^T Y \Lambda$, where Y is a solution to the LMIs equation (10). By the above transformation we can obtain, we can transform Equation (7) into the following regular form:

$$\dot{v} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} v + \begin{bmatrix} \bar{A}_{\tau 11} & \bar{A}_{\tau 12} \\ \bar{A}_{\tau 21} & \bar{A}_{\tau 22} \end{bmatrix} v_d + \begin{bmatrix} 0 \\ I \end{bmatrix} \left(u + \sum_i \beta_i h_i \right), \tag{19}$$

where $v_d = v(t - d(t))$ and

$$\bar{A}_{11} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_i + T_i \Pi_i(t)) Y \Lambda, \tag{20}$$

$$\bar{A}_{12} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_i + T_i \Pi_i(t)) B, \tag{21}$$

$$\bar{A}_{21} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_i + T_i \Pi_i(t)) Y \Lambda, \tag{22}$$

$$\bar{A}_{22} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_i + T_i \Pi_i(t)) B, \tag{23}$$

$$\bar{A}_{\tau 11} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_{\tau i} + T_i \Pi_i(t)) Y \Lambda, \tag{24}$$

$$\bar{A}_{\tau 12} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_{\tau i} + T_i \Pi_i(t)) B, \tag{25}$$

$$\bar{A}_{\tau 21} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_{\tau i} + T_i \Pi_i(t)) Y \Lambda, \tag{26}$$

$$\bar{A}_{\tau 22} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_{\tau i} + T_i \Pi_i(t)) B. \tag{27}$$

Thus, from the above regular form, by setting $\dot{\sigma} = \sigma = 0$, we can obtain the following sliding mode dynamics:

$$\dot{\alpha} = A_o \alpha + A_d \alpha_d, \tag{28}$$

where $\alpha = v_1, \alpha_d = v_1(t - d(t))$, $A_0 = \bar{A}_{11}$, and $A_d = \bar{A}_{\tau 11}$. Let us define a Lyapunov–Krasovskii function (LKF) as

$$V(t) = \alpha^T(t) P_0 \alpha(t) + \int_{t-d}^{t} \alpha^T(s) F \alpha(s) \mathrm{d}s + \int_{-\tau}^{0} \int_{t+\eta}^{t} \dot{\alpha}^T(s) P_0 \dot{\alpha}(s) \mathrm{d}s \, \mathrm{d}\eta, \tag{29}$$

where $\boldsymbol{P}_0 = \boldsymbol{\Lambda}^T \boldsymbol{Y} \boldsymbol{\Lambda} \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{F} \in \boldsymbol{R}^{n\times n}$ are solution matrices for the LMIs equation (10). It should be noted that a large number of previous methods such as the methods given in Zhang *et al.* (2009) and Peng *et al.* (2008) have used similar LKFs to obtain less-conservative stability conditions by exploiting information on the upper bounds of delay and its time derivative. None of the previous SMC design methods (Oucheriah 1995, El-Khazali 1998, Oucheriah 2003, Xia and Jia 2003, Lin *et al.* 2005, Choi, 2008) have used the term $\int_{-\tau}^{0}\int_{t+\eta}^{t} \dot{\boldsymbol{\alpha}}^T(s)\boldsymbol{P}_0\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s\,\mathrm{d}\eta$ in stability analysis. The time derivative of the LKF is given by

$$\begin{aligned}\dot{V} &= 2\boldsymbol{\alpha}^T\boldsymbol{P}_0(\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{A}_d\boldsymbol{\alpha}_d) + \boldsymbol{\alpha}^T\boldsymbol{F}\boldsymbol{\alpha} - (1-\dot{d})\boldsymbol{\alpha}_d^T\boldsymbol{F}\boldsymbol{\alpha}_d \\ &\quad + \tau\dot{\boldsymbol{\alpha}}^T\boldsymbol{P}_0\dot{\boldsymbol{\alpha}} - \int_{t-\tau}^{t}\dot{\boldsymbol{\alpha}}^T(s)\boldsymbol{P}_0\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s. \end{aligned} \tag{30}$$

Using Equation (28) and the Newton–Leibniz formula $\boldsymbol{\alpha} - \boldsymbol{\alpha}_d - \int_{t-d}^{t}\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s = 0,$ we have

$$\begin{aligned}\dot{V} &= 2\boldsymbol{\alpha}^T\boldsymbol{P}_0(\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{A}_d\boldsymbol{\alpha}_d) + \boldsymbol{\alpha}^T\boldsymbol{F}\boldsymbol{\alpha} - (1-\dot{d})\boldsymbol{\alpha}_d^T\boldsymbol{F}\boldsymbol{\alpha}_d \\ &\quad + \tau(\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{A}_d\boldsymbol{\alpha}_d)^T\boldsymbol{P}_0(\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{A}_d\boldsymbol{\alpha}_d) \\ &\quad - \int_{t-\tau}^{t}\dot{\boldsymbol{\alpha}}^T(s)\boldsymbol{P}_0\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s + 2(\boldsymbol{\alpha}^T\boldsymbol{X}^T + \boldsymbol{\alpha}_d^T\boldsymbol{Z}^T) \\ &\quad \times\left(\boldsymbol{\alpha} - \boldsymbol{\alpha}_d - \int_{t-\tau}^{t}\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s\right), \end{aligned} \tag{31}$$

where $\boldsymbol{X} = \sum\beta_i\boldsymbol{X}_i$ and $\boldsymbol{Z} = \sum\beta_i\boldsymbol{Z}_i$. Using the inequality $2\boldsymbol{x}^T\boldsymbol{y} \le \boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{y}^T\boldsymbol{H}^{-1}\boldsymbol{Y},$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ are any vectors with appropriate dimensions and $\boldsymbol{H} > 0,$ we can obtain

$$\begin{aligned}&2\left[\boldsymbol{\alpha}^T(t)\boldsymbol{X}^T + \boldsymbol{\alpha}_d^T(t)\boldsymbol{Z}^T\right]\int_{t-\tau}^{t}\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s \\ &\quad\le \tau\left[\boldsymbol{\alpha}^T(t)\boldsymbol{X}^T + \boldsymbol{\alpha}_d^T(t)\boldsymbol{Z}^T\right]\boldsymbol{P}_0^{-1}[\boldsymbol{X}\boldsymbol{\alpha}(t) + \boldsymbol{Z}\boldsymbol{\alpha}_d(t)] \\ &\qquad + \int_{t-\tau}^{t}\dot{\boldsymbol{\alpha}}^T(s)\boldsymbol{P}_0\dot{\boldsymbol{\alpha}}(s)\mathrm{d}s, \end{aligned} \tag{32}$$

which leads to

$$\begin{aligned}\dot{V} &\le 2\boldsymbol{\alpha}^T(\boldsymbol{P}_0\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{P}_0\boldsymbol{A}_d\boldsymbol{\alpha}_d) + \boldsymbol{\alpha}^T\boldsymbol{F}\boldsymbol{\alpha} - (1-d_m)\boldsymbol{\alpha}_d^T\boldsymbol{F}\boldsymbol{\alpha}_d \\ &\quad + \tau[\boldsymbol{\alpha}^T\boldsymbol{X}^T + \boldsymbol{\alpha}_d^T\boldsymbol{Z}^T]\boldsymbol{P}_0^{-1}[\boldsymbol{X}\boldsymbol{\alpha} + \boldsymbol{Z}\boldsymbol{\alpha}_d] \\ &\quad + 2(\boldsymbol{\alpha}^T\boldsymbol{X}^T + \boldsymbol{\alpha}_d^T\boldsymbol{Z}^T)(\boldsymbol{\alpha} - \boldsymbol{\alpha}_d) + \tau(\boldsymbol{P}_0\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{P}\boldsymbol{A}_d\boldsymbol{\alpha}_d)^T \\ &\quad \times P_0^{-1}(\boldsymbol{P}_0\boldsymbol{A}_0\boldsymbol{\alpha} + \boldsymbol{P}_0\boldsymbol{A}_d\boldsymbol{\alpha}_d). \end{aligned} \tag{33}$$

By applying the Schur complement formula (Boyd *et al.* 1994) to Equation (10), we can obtain

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{N}_{11} & * \\ \boldsymbol{N}_{21} & \boldsymbol{N}_{22}\end{bmatrix} + \tau\begin{bmatrix}\boldsymbol{X}_i^T \\ \boldsymbol{Z}_i^T\end{bmatrix}\boldsymbol{P}_0^{-1}\begin{bmatrix}\boldsymbol{X}_i^T \\ \boldsymbol{Z}_i^T\end{bmatrix}^T \\ &\quad + \tau\begin{bmatrix}\boldsymbol{\Lambda}^T\boldsymbol{Y}(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))^T\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}^T\boldsymbol{Y}(\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))^T\boldsymbol{\Lambda}\end{bmatrix} \\ &\quad \times\boldsymbol{P}_o^{-1}\begin{bmatrix}\boldsymbol{\Lambda}^T\boldsymbol{Y}(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))^T\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}^T\boldsymbol{Y}(\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))^T\boldsymbol{\Lambda}\end{bmatrix}^T < 0. \end{aligned} \tag{34}$$

This implies that $\dot{V} \le -\mu(\|\boldsymbol{\alpha}\|^2 + \|\boldsymbol{\alpha}_d\|^2)$ for some $\mu > 0$. After all, we can conclude that the sliding mode dynamics equation (28) is stable.

3.2. *Sliding control law design*

After the switching surface parameter matrix $\boldsymbol{S}$ is designed so that the reduced-order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met (Utkin 1977, DeCarlo *et al.* 1988, Choi 2008). If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{x} = 0$ and maintains it there for all subsequent time. In this section, we design a sliding fuzzy control law guaranteeing that $\boldsymbol{\sigma}$ converges to zero. We will use the following nonlinear sliding switching feedback control law as the local controller:

Control Rule i: IF θ_1 is μ_{i1} and… and θ_s is μ_{is}, THEN

$$\boldsymbol{u}(t) = -\boldsymbol{S}(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))\boldsymbol{x} - \boldsymbol{S}(\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))\boldsymbol{x}_d - \rho_i(t)\frac{\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|}, \tag{35}$$

where

$$\begin{aligned}\rho_i(t) = \frac{1}{1-\boldsymbol{\chi}_m}(&\boldsymbol{\delta}_i(t) + \boldsymbol{\chi}_m\|\boldsymbol{S}(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))x \\ &+ \boldsymbol{S}(\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))\boldsymbol{x}_d\| + \kappa_i) \end{aligned} \tag{36}$$

and $\kappa_i > 0$. The final controller inferred as the weighted average of the each local controller is given by

$$\begin{aligned}\boldsymbol{u}(t) = -\sum_{i=1}^{r}\beta(\theta)\Big(&\boldsymbol{S}(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))x \\ &+ \boldsymbol{S}(\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))\boldsymbol{x}_d + \rho_i(t)\frac{\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|}\Big), \end{aligned} \tag{37}$$

and we can establish the following theorem.

Theorem 2: *Suppose that the LMIs equation* (10) *is feasible and the sliding surface is given by Equation* (16). *Then, the switching feedback control law equation* (37) *induces an ideal sliding motion on the sliding surface* $\boldsymbol{\sigma} = o$ *in finite time and the state converges to zero.*

Proof: Since Theorem 1 implies that the sliding mode dynamics restricted to $\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{x} = 0$ is stable, we only have to show that the reachability condition

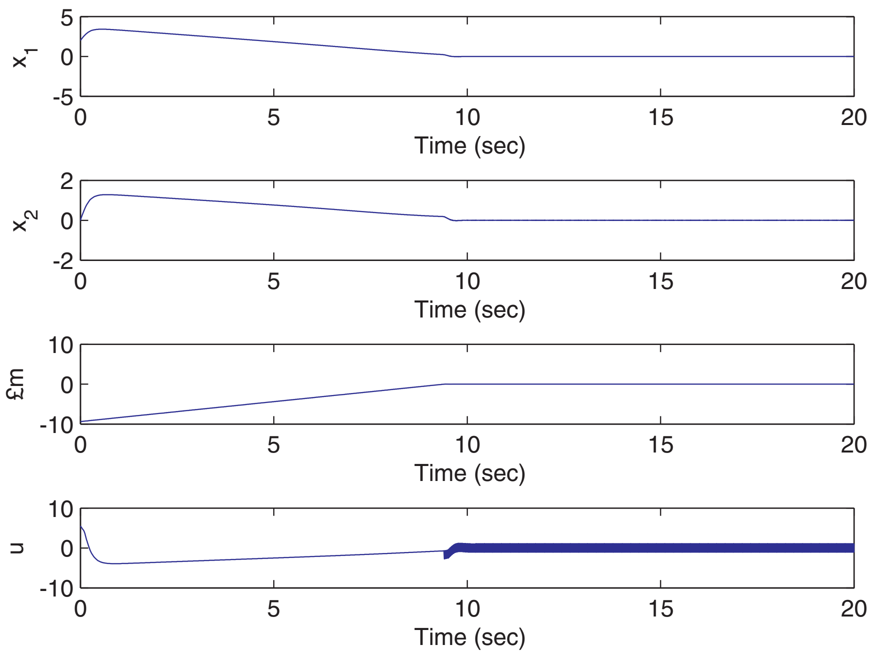

Figure 1. Control results for the system.

$\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} < -\kappa\|\boldsymbol{\sigma}\|$ is satisfied for some $\kappa > 0$. Using $\boldsymbol{SB} = \boldsymbol{I}$ and the assumption A2, we can obtain

$$\begin{aligned}\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} &= \boldsymbol{\sigma}^T\sum_{i=1}^{r}\beta_i(\boldsymbol{S}(\boldsymbol{A}_i + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))x \\ &\quad + \boldsymbol{S}(\boldsymbol{A}_{\tau i} + \boldsymbol{T}_i\boldsymbol{\Pi}_i(t))\boldsymbol{x}_d + \boldsymbol{h}_i) \\ &\quad + \boldsymbol{\sigma}^T\boldsymbol{u} \le \sum_{i=1}^{r}\beta_i(\rho_i - \chi_i\|\boldsymbol{u}\| - \delta_i)\|\boldsymbol{\sigma}\| \\ &\le -\sum_{i=1}^{r}\kappa_i\|\boldsymbol{\sigma}\|.\end{aligned}$$

After all, we can conclude that $\boldsymbol{\sigma}$ converges to zero.

4. Numerical example

In this section, two examples are used to illustrate the effectiveness of the proposed method and to compare with the existing method.

Example 1: To illustrate the performance of the proposed sliding fuzzy control design method, consider the following T–S fuzzy time-delay model (Wu and Li 2007) without mismatched parameter uncertainties and external disturbances.

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2}\beta_i(\theta)[\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{A}_{\tau i}\boldsymbol{x}_d(t)] + \boldsymbol{Bu}(t), \tag{38}$$

where $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) & x_2(t)\end{bmatrix}^T$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1\end{bmatrix}, \quad \boldsymbol{A}_{\tau 1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2\end{bmatrix}, \tag{39}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0\end{bmatrix}, \quad \boldsymbol{A}_{\tau 2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6\end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 1\end{bmatrix} \tag{40}$$

$$\beta_1 = \frac{1}{1 + e^{-2x_1(t)}}, \quad \beta_2 = 1 - \beta_1. \tag{41}$$

We assume that $d(t) = \tau = 0.4$, $\chi_i = 0$, $\delta_i = 1$, $\chi_m = 0$, $h_i = 0$ and $\kappa_i = 0.5$. Figure 1 shows the control results for system equation (38) via the proposed controller equation (37) under the initial condition $\varphi(t) = [2\ 0]^T$. In Figure 1, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of an SMC system.

Example 2: Consider a well-studied example of a continuous-time truck-trailer with time-delay proposed in Chen *et al.* (2009). The time-delay model is given by

$$\begin{aligned}\dot{x}_1(t) &= -a\frac{vT}{Lt_0}x_1(t) - (1-a)\frac{vT}{Lt_0}x_1(t-d) \\ &\quad + \frac{vT}{lt_0}[u(t) + h(t)],\end{aligned} \tag{42}$$

$$\dot{x}_2(t) = a\frac{vT}{Lt_0}x_1(t) + (1-a)\frac{vT}{Lt_0}x_1(t-d), \tag{43}$$

$$\dot{x}_3(t) = \frac{vT}{t_0}\sin\left[x_2(t) + a\frac{vT}{2L}x_1(t) + (1-a)\frac{vT}{2L}x_1(t-d)\right], \tag{44}$$

where $x_1(t)$ is the angle difference between truck and trailer (in radians), $x_2(t)$ the angle of trailer (in radians), $x_3(t)$ the vertical position of rear of trailer (in meters), $u(t)$ the steering angle (in radians), $T = 2.0$, $l = 2.8$, $L = 5.5$, $v = -1.0$ and $t_0 = 0.5$. The constant parameter a is the retarded coefficient satisfying $a \in [0, 1]$. The limits 1 and 0 correspond to a no-delay term and to a completed-delay term. We assume that the disturbance input $h(t)$ is unknown but bounded as $|h(t)| \le 1$. Using the fact that $\sin(x) \approx x$ if $x \approx 0$, we can represent the above model as the following two-rule T–S fuzzy model, including parameter uncertainties and external disturbances:

Plant rule 1: IF $\theta(t)$ is about 0, THEN

$$\dot{\boldsymbol{x}} = (\boldsymbol{A}_1 + \boldsymbol{T}_1\boldsymbol{\Pi}_1(t))x + (\boldsymbol{A}_{\tau 1} + \boldsymbol{T}_1\boldsymbol{\Pi}_1(t))\boldsymbol{x}_d + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{B}h_1 \tag{45}$$

Plant rule 2: IF $\theta(t)$ is about $\pm\pi$, THEN

$$\dot{\boldsymbol{x}} = (\boldsymbol{A}_2 + \boldsymbol{T}_2\boldsymbol{\Pi}_2(t))x + (\boldsymbol{A}_{\tau 2} + \boldsymbol{T}_2\boldsymbol{\Pi}_2(t))\boldsymbol{x}_d + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{B}h_2, \tag{46}$$

where

$$\theta(t) = x_2(t) + avTx_1(t)/2L + (1-a)vTx_1(t-d)/2L \tag{47}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} -a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{v^2T^2}{2Lt_0} & \frac{vT}{t_0} & 0 \end{bmatrix}, \tag{48}$$

$$\boldsymbol{A}_{\tau 1} = \begin{bmatrix} -(1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{v^2T^2}{2Lt_0} & 0 & 0 \end{bmatrix}, \tag{49}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{10v^2T^2}{2L\pi} & \frac{10vT}{\pi} & 0 \end{bmatrix}, \tag{50}$$

$$\boldsymbol{A}_{\tau 2} = \begin{bmatrix} -(1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{10v^2T^2}{2L\pi} & 0 & 0 \end{bmatrix}, \tag{51}$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{vT}{lt_0} \\ 0 \\ 0 \end{bmatrix}, \tag{52}$$

$$\boldsymbol{T}_1 = \boldsymbol{T}_2 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \tag{53}$$

$$\boldsymbol{\Pi}_1(t) = \boldsymbol{\Pi}_2(t) = \begin{bmatrix} \sin t & 0 & 0 \end{bmatrix}, \tag{54}$$

$$\beta_1 = \frac{1 - 1/(1 + e^{-2(\theta - 0.5\pi)})}{1 + e^{-2(\theta + 0.5\pi)}}, \tag{55}$$

$$\beta_2 = 1 - \beta_1, \tag{56}$$

$$h_1 = h_2 = h(t). \tag{57}$$

We assume that $d(t) = \tau = 0.1$. Considering LMI optimization with the data equation (45)–(57), $a = 0, \tau = 0.1$ and $d_m = 0$, we can obtain the sliding surface $\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{x}$.

Since $|h_i(t)| \le 1$, we can set $\chi_i = 0, \delta_i = 1, \chi_m = 0$ and $\kappa_i = 0.2$. We can obtain the following fuzzy controller:

Controller rule 1: IF $\theta(t)$ is about 0, THEN

$$\boldsymbol{u}(t) = -\boldsymbol{S}(\boldsymbol{A}_1 + \boldsymbol{T}_1\boldsymbol{\Pi}_1(t))x - \boldsymbol{S}(\boldsymbol{A}_{\tau 1} + \boldsymbol{T}_1\boldsymbol{\Pi}_1(t))\boldsymbol{x}_d - 1.2\mathrm{sgn}(\sigma). \tag{58}$$

Controller rule 2: IF $\theta(t)$ is about $\pm\pi$, THEN

$$u(t) = -S(A_2 + \mathrm{T}_2\Pi_2(t))x - S(A_{\tau 2} + \mathrm{T}_2\Pi_2(t))\boldsymbol{x}_d - 1.2\mathrm{sgn}(\sigma). \tag{59}$$

The final controller inferred as the weighted average of each local controller is given by

$$u(t) = -\sum_{i=1}^{2} \beta_i [S(A_i + \mathrm{T}_i \Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i \Pi_i(t))\boldsymbol{x}_d + 1.2\mathrm{sgn}(\sigma)]. \quad (60)$$

To demonstrate the controller ability, we apply the above fuzzy controller equation (60) to the system model equations (45)–(57) with $h(t) = \sin t$ and $d(t) = \tau = 0.1$. Figure 2 shows the closed-loop system responses of equations (45)–(57) and the proposed controller Equation (60) with the initial condition $\boldsymbol{\psi}(t) = [0.4\pi, 0.8\pi, -4]^T$. Moreover, the closed-loop

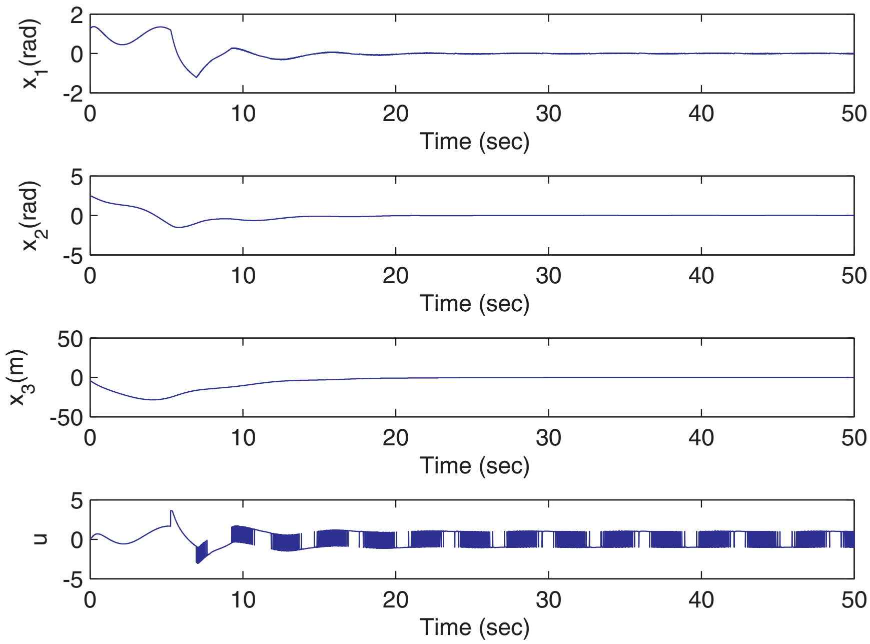

Figure 2. Simulation results with the proposed method on model equations (45)–(57).

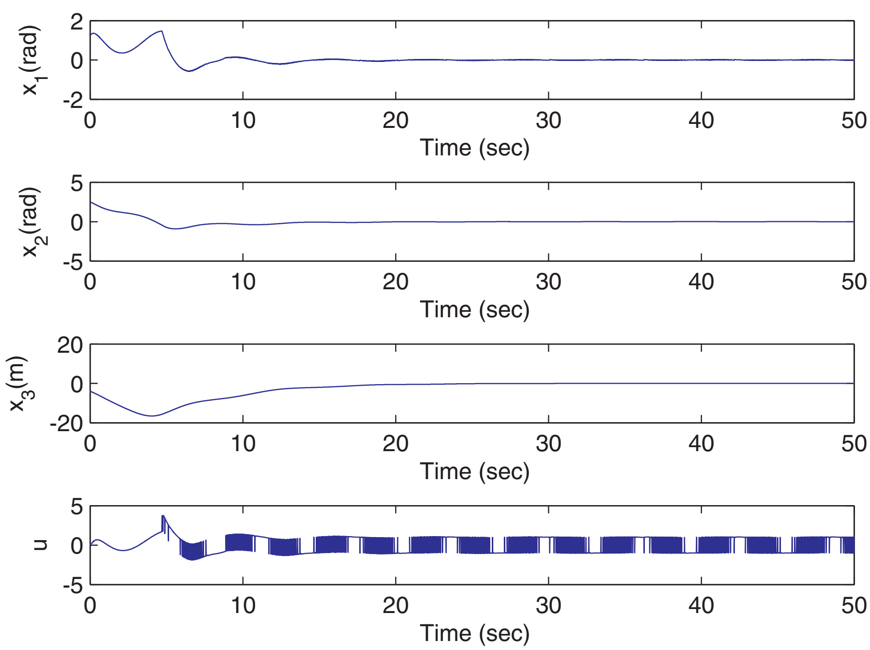

Figure 3. Simulation results with the proposed method on model equations (42)–(44).

system responses of the truth model equations (42)–(44) and the proposed controller equation (60) with the initial condition $\boldsymbol{\psi}(t) = [0.4\pi, 0.8\pi, -4]^T$ are also shown in Figure 3.

From Figures 2 and 3, the proposed controller is applicable to low-order fuzzy control synthesis for uncertain fuzzy time-delay systems with mismatched parameter uncertainties in the state matrix and external disturbances and the nonlinear truth model. The control performances of the two-rule T–S fuzzy model equations (45)–(57) and the nonlinear truth model equations (42)–(44) are satisfactory.

5. Conclusions

A robust sliding fuzzy control design method was developed for the uncertain T–S fuzzy time-delay model which includes mismatched parameter uncertainties and external disturbances. As the local controller, an SMC law with a nonlinear switching feedback control term is used. We gave an LMI condition for the existence of linear sliding surfaces guaranteeing the asymptotic stability of the reduced-order equivalent sliding mode dynamics. An explicit formula of the switching surface parameter matrix is derived in terms of the solution of the LMI existence condition and an LMI-based algorithm is developed to design the nonlinear switching feedback control term guaranteeing the reachability condition. Finally, using two numerical design examples, we have shown the effectiveness of our method.

Nomenclature

$\boldsymbol{\psi}(t), x(t), \boldsymbol{u}(t)$	initial condition, state, and control input
$\boldsymbol{A}_i, \boldsymbol{A}_{\tau i}, \boldsymbol{B}_i$	state matrices, delayed state matrices, and input matrices
$\theta_j,\ \mu_{ij},\ s,\ r$	premise variables, fuzzy sets, number of premise variables, and number of IF-THEN rules
$d(t),\ \beta_i(\theta)$	time-delay unknown function and membership function
$\tau,\ d_m$	known constants
$\boldsymbol{x}_d(t), \boldsymbol{\Delta A}_i(t),\ \boldsymbol{\Delta A}_{\tau i}(t), \boldsymbol{h}_i(t, x, x_d, u)$	delayed state, parameter uncertainties in $\boldsymbol{A}_i$, parameter uncertainties in $\boldsymbol{A}_{\tau i}$, and external disturbances
$\delta_i(t),\ \chi_i$	known function and known constant
$\boldsymbol{T}_i, \boldsymbol{\Pi}_i(t)$	constant matrix and time-varying matrix
$\boldsymbol{S}$	linear sliding surface parameter matrix
$\chi_m,\ \kappa_i$	known constants

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