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Recurrence relations of higher spin BPST vertex operators for open strings

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We calculate higher-spin Brower–Polchinski–Strassler–Tan (BPST) vertex operators for an open bosonic string and express these operators in terms of a Kummer function of the second kind. We derive an infinite number of recurrence relations among BPST vertex operators of different string states. These recurrence relations among BPST vertex operators lead to the recurrence relations among Regge string scattering amplitudes discovered recently.

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I. INTRODUCTION

Recently, there has been interest to study Regge regime (RR) string scattering amplitudes [1–6] for higher-spin string states [1,7–10]. One of the motivations was to understand their intimate link with the scattering amplitudes in the fixed-angle or Gross regime (GR) [11–15]. In the GR, a saddle-point method was used to calculate string-tree amplitudes [16–19], and the ratios of scattering amplitudes among different string states at each fixed mass level can be extracted and were found to be independent of the scattering energy and scattering angle. Alternatively, these ratios can be rederived algebraically by solving linear relations or GR stringy Ward identities from decoupling of zero-norm states (ZNS) [20–22]. More interestingly, the infinite number of these ratios for the GR can be extracted from RR string scattering amplitudes based on summation algorithms for Stirling number identities [23,24].

In contrast to the GR, an infinite number of recurrence relations among higher-spin RR string scattering amplitudes was discovered more recently [1]. Instead of RR stringy Ward identities derived from decoupling of ZNS, the calculation was based on recurrence relations of Kummer functions of the second kind [25]. These recurrence relations among RR amplitudes were considered to be dual to the linear relations among the GR amplitudes discussed above.

In this paper, we study higher-spin Regge string scattering amplitudes from a Brower–Polchinski–Strassler–Tan (BPST) vertex operator approach. Note that in the original BPST paper [2], the authors calculated the case of closed-string and thus Pomeron vertex operators. Here, for simplicity, we will calculate higher-spin BPST vertex operators at arbitrary mass levels of an open bosonic string.[1] The calculation can be easily generalized to the closedstring case. We find that all BPST vertex operators can be expressed in terms of Kummer functions of the second kind. We can then derive an infinite number of recurrence relations among BPST vertex operators of different string states. These recurrence relations among BPST vertex operators lead to the recurrence relations among Regge string scattering amplitudes discovered recently [1].

II. FOUR-TACHYON SCATTERING

We will calculate high-energy open-string scatterings in the Regge regime,

$$s \to \infty, \qquad \sqrt{-t} = \text{fixed} \quad (\text{but}\ \sqrt{-t} \neq \infty), \tag{2.1}$$

where

$$s = -(k_1 + k_2)^2 \quad \text{and} \quad t = -(k_2 + k_3)^2. \tag{2.2}$$

Note that the convention for s and t adopted here is different from the original BPST paper in Ref. [2].

We first review the calculation of tachyon BPST vertex operator [2]. The $s - t$ channel of an open-string four-tachyon amplitude can be written as

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[1]Taking advantage of Regge factorization, a Pomeron vertex operator $\mathcal{V}_P$ was introduced in Ref. [2], which allows one to calculate the coupling between the leading closed-string Regge trajectory with any n-particle external state $|\mathcal{W}\rangle$. In this paper, we only consider 4-point scattering for open strings. As such, we only need to treat the coupling of the leading open-string Reggeon to two-particle states. For brevity, we use here the term "higher-spin BPST vertex operators" collectively for the product of the vertex operator for the leading open-string Reggeon with external two-particle states, one of which has a high spin.

$$A = \int_0^1 d\omega \cdot \omega^{k_1\cdot k_2}(1-\omega)^{k_2\cdot k_3} = \int_0^1 d\omega \cdot \omega^{-2-\frac{s}{2}}(1-\omega)^{-2-\frac{t}{2}}. \tag{2.3}$$

Since $s \to \infty$, the integral is dominated around $\omega = 1$. Making the variable transformation $\omega = 1 - x$, the integral is dominated around $x = 0$, and we obtain

$$A = \int_0^1 dx \cdot (1-x)^{-2-\frac{s}{2}}x^{-2-\frac{t}{2}} \simeq \int dx \cdot x^{-2-\frac{t}{2}}e^{\frac{s}{2}x} = \Gamma\left(-1-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{1+\frac{t}{2}}. \tag{2.4}$$

Alternatively, the integral in A can be expressed as

$$A = \int d\omega \langle e^{ik_1X(0)}e^{ik_2X(\omega)}e^{ik_3X(1)}e^{ik_4X(\infty)}\rangle. \tag{2.5}$$

One can calculate the operator product expansion (OPE) in the Regge limit:

$$e^{ik_2X(w)}e^{ik_3X(z)} \sim |w-z|^{k_2\cdot k_3}e^{i(k_2+k_3)X(z)+ik_2(w-z)\partial X(z)+\cdots}.$$

This means

$$e^{ik_2X(\omega)}e^{ik_3X(1)} \sim (1-\omega)^{k_2\cdot k_3}e^{ikX(1)-ik_2(1-\omega)\partial X(1)+\text{higher power of }(1-\omega)}, \quad k = k_2 + k_3. \tag{2.6}$$

In evaluating Eq. (2.5), one can instead carry out the ω integration first in Eq. (2.6) at the operator level to obtain the BPST vertex operator [2],

$$\begin{aligned} V_{\text{BPST}} &= \int d\omega e^{ik_2X(\omega)}e^{ik_3X(1)} \\ &\sim \int d\omega(1-\omega)^{k_2\cdot k_3}e^{ikX(1)-ik_2(1-\omega)\partial X(1)} \\ &= \int dx x^{k_2\cdot k_3}e^{ikX(1)-ik_2x\partial X(1)} \\ &= \Gamma\left(-1-\frac{t}{2}\right)[ik_2\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}, \end{aligned} \tag{2.7}$$

which leads to the same amplitude as in Eq. (2.4):

$$\begin{aligned} A &= \langle e^{ik_1X(0)}V_{\text{BPST}}e^{ik_4X(\infty)}\rangle \\ &= \Gamma\left(-1-\frac{t}{2}\right)\langle e^{ik_1X(0)}[ik_2\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}e^{ik_4X(\infty)}\rangle \\ &= \Gamma\left(-1-\frac{t}{2}\right)(k_1k_2)^{1+\frac{t}{2}} \sim \Gamma\left(-1-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{1+\frac{t}{2}}. \end{aligned} \tag{2.8}$$

III. HIGHER-SPIN BPST VERTEX

A. A spin-2 state

It was shown [1,7,8] that for the 26-dimensional open bosonic string states of the leading order in energy in the Regge limit at mass level, $M_2^2 = 2(N-1)$, $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$ are of the form (we choose the second state of the four-point function to be the higher-spin string state)

$$|p_n, q_m, r_l\rangle = \prod_{n>0}(\alpha^T_{-n})^{p_n}\prod_{m>0}(\alpha^P_{-m})^{q_m}\prod_{l>0}(\alpha^L_{-l})^{r_l}|0,k\rangle, \tag{3.1}$$

where the polarizations of the second particle with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_2}(E_2, \mathrm{k}_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(\mathrm{k}_2, E_2, 0)$ as the longitudinal polarization, and $e^T = (0,0,1)$ as the transverse polarization, which lies on the scattering plane. $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1)$. The three vectors e^P, e^L, and e^T satisfy the completeness relation $\eta_{\mu\nu} = \sum_{\alpha,\beta} e^\alpha_\mu e^\beta_\nu \eta_{\alpha\beta}$, where $\mu, \nu = 0, 1, 2$ and $\alpha, \beta = P, L, T$ and $\alpha^T_{-1} = \sum_\mu e^T_\mu \alpha^\mu_{-1}$, $\alpha^T_{-1}\alpha^L_{-2} = \sum_{\mu,\nu} e^T_\mu e^L_\nu \alpha^\mu_{-1}\alpha^\nu_{-2}$ etc.

In this section, we first consider a simple case of a spin-2 state $\alpha^P_{-1}\alpha^P_{-1}|0\rangle$ corresponding to the vertex $(\partial X^P)^2 e^{ik_2X}(\omega)$. The four-point amplitude of the spin-2 state with three tachyons can be calculated by using the conventional method:

$$\begin{aligned} A^{(q_1=2)} &= \int d\omega \langle e^{ik_1X(0)}(\partial X^P)^2 e^{ik_2X}(\omega)e^{ik_3X(1)}e^{ik_4X(\infty)}\rangle \\ &= \int d\omega \omega^{k_1\cdot k_2}(1-\omega)^{k_2\cdot k_3}\left[\frac{ie^P\cdot k_1}{-\omega} + \frac{ie^P\cdot k_3}{1-\omega}\right]^2 \\ &= -(e^P\cdot k_1)^2\Gamma\left(-1-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{\frac{t}{2}-1} \\ &\quad + 2(e^P\cdot k_1)(e^P\cdot k_3)\Gamma\left(-2-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{\frac{t}{2}} \\ &\quad - (e^P\cdot k_3)^2\Gamma\left(-3-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{\frac{t}{2}+1}. \end{aligned} \tag{3.2}$$

The momenta of the four particles on the scattering plane are

$$k_1 = \left(+\sqrt{p^2+M_1^2}, -p, 0\right), \tag{3.3}$$

$$k_2 = \left(+\sqrt{p^2+M_2^2}, +p, 0\right), \tag{3.4}$$

$$k_3 = \left(-\sqrt{q^2+M_3^2}, -q\cos\phi, -q\sin\phi\right), \tag{3.5}$$

$$k_4 = \left(-\sqrt{q^2+M_4^2}, +q\cos\phi, +q\sin\phi\right), \tag{3.6}$$

where $p \equiv |\vec{p}|$, $q \equiv |\vec{q}|$, and $k_i^2 = -M_i^2$. The relevant kinematics in the Regge limit are [1,7,8]

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \tag{3.7}$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2} \tag{3.8}$$

and

$$e^T \cdot k_1 = 0, \qquad e^T \cdot k_3 \simeq -\sqrt{-t}, \tag{3.9}$$

where $\tilde{t}$ and $\tilde{t}'$ are related to t by finite mass square terms

$$\tilde{t} = t - M_2^2 - M_3^2, \qquad \tilde{t}' = t + M_2^2 - M_3^2. \tag{3.10}$$

By using Eq. (3.7), one can easily see that the three terms in Eq. (3.2) share the same order of energy in the Regge limit. We stress that this key observation on the polarizations for higher-spin states was not discussed in Refs. [2,3].

One can calculate the OPE in the Regge limit:

$$\partial X^P \partial X^P e^{ik_2 X}(w) e^{ik_3 X}(z) \sim |w - z|^{k_2 \cdot k_3} \left[\partial X(z)^P + \frac{ie^P \cdot k_3}{w - z} \right]^2 e^{ikX(z) + ik_2(w-z)\partial X(z)}.$$

This means

$$\partial X^P \partial X^P e^{ik_2 X}(\omega) e^{ik_3 X}(1) \sim (1 - \omega)^{k_2 \cdot k_3} \left[\partial X(1)^P - \frac{ie^P \cdot k_3}{1 - \omega} \right]^2 \times e^{ikX(1) - ik_2(1-\omega)\partial X(1)}, \qquad k = k_2 + k_3. \tag{3.11}$$

One can carry out the ω integration in Eq. (3.11) at the operator level to obtain the BPST vertex operator:

$$\begin{aligned}
V_{\text{BPST}}^{(q_1=2)} &= \int d\omega (\partial X^P)^2 e^{ik_2 X}(\omega) e^{ik_3 X}(1) \sim \int d\omega (1 - \omega)^{k_2 \cdot k_3} \left[\partial X(1)^P - \frac{ie^P \cdot k_3}{1 - \omega} \right]^2 e^{ikX(1) - ik_2(1-\omega)\partial X(1)} \\
&= \partial X(1)^P \partial X(1)^P \int dx x^{k_2 \cdot k_3} e^{ikX(1) - ik_2 x \partial X(1)} - 2ie^P \cdot k_3 \partial X(1)^P \\
&\quad \times \int dx x^{k_2 \cdot k_3 - 1} e^{ikX(1) - ik_2 x \partial X(1)} - (e^P \cdot k_3)^2 \int dx x^{k_2 \cdot k_3 - 2} e^{ikX(1) - ik_2 x \partial X(1)} \\
&= \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{\frac{t}{2} - 1} \partial X(1)^P \partial X(1)^P e^{ikX(1)} - 2ie^P \cdot k_3 \Gamma\left(-2 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{\frac{t}{2}} \partial X(1)^P e^{ikX(1)} \\
&\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{\frac{t}{2} + 1} e^{ikX(1)}.
\end{aligned} \tag{3.12}$$

We can use this BPST vertex operator to rederive the amplitude

$$\begin{aligned}
A^{(q_1=2)} &= \langle e^{ik_1 X(0)} V_{\text{BPST}}^{(q_1=2)} e^{ik_4 X(\infty)} \rangle \\
&= \Gamma\left(-1 - \frac{t}{2}\right) \langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{\frac{t}{2} - 1} \partial X(1)^P \partial X(1)^P e^{ikX(1)} e^{ik_4 X(\infty)} \rangle \\
&\quad - 2ie^P \cdot k_3 \Gamma\left(-2 - \frac{t}{2}\right) \langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{\frac{t}{2}} \partial X(1)^P e^{ikX(1)} e^{ik_4 X(\infty)} \rangle \\
&\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) \langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{\frac{t}{2} + 1} e^{ikX(1)} e^{ik_4 X(\infty)} \rangle \\
&\sim -\left(e^P \cdot k_1\right)^2 \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2} - 1} + 2(e^P \cdot k_1)(e^P \cdot k_3) \Gamma\left(-2 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}} \\
&\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2} + 1},
\end{aligned} \tag{3.13}$$

which is the same as the amplitude in Eq. (3.2). Note that the three terms in Eq. (3.12) lead to the three terms, respectively, in Eq. (3.13) with the same order of energy in the Regge limit.

B. Higher-spin states

We now consider the higher-spin state

$$|p_n, q_m\rangle = \prod_{n=1}(\alpha^T_{-n})^{p_n}\prod_{m=1}(\alpha^P_{-m})^{q_m}|0\rangle, \tag{3.14}$$

which corresponds to the vertex

$$V_2(\omega) = \left[\prod_{n=1}(\partial^n X^T)^{p_n}\prod_{m=1}(\partial^m X^P)^{q_m}\right]e^{ik_2X}(\omega). \tag{3.15}$$

The four-point amplitude of the above state with three tachyons was calculated to be (from now on, we set $M_2 = M$) [1,7,8]

$$\begin{aligned}A^{(p_n,q_m)} &= \int d\omega\langle e^{ik_1X(0)}V_2(\omega)e^{ik_3X(1)}e^{ik_4X(\infty)}\rangle\\ &= \left(-\frac{1}{M}\right)^{q_1}U\left(-q_1,\frac{t}{2}+2-q_1,\frac{\tilde{t}}{2}\right)B\left(-1-\frac{s}{2},-1-\frac{t}{2}\right)\cdot\prod_{n=1}[\sqrt{-t}(n-1)!]^{p_n}\\ &\quad\times\prod_{m=2}\left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m}\end{aligned} \tag{3.16}$$

$$\sim\left(-\frac{1}{M}\right)^{q_1}U\left(-q_1,\frac{t}{2}+2-q_1,\frac{\tilde{t}}{2}\right)\Gamma\left(-1-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{1+\frac{t}{2}} \tag{3.17}$$

$$\cdot\prod_{n=1}[\sqrt{-t}(n-1)!]^{p_n}\prod_{m=2}\left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m}, \tag{3.18}$$

where U is the Kummer function of the second kind and is defined to be

$$U(a,c,x) = \frac{\pi}{\sin\pi c}\left[\frac{M(a,c,x)}{(a-c)!(c-1)!} - \frac{x^{1-c}M(a+1-c,2-c,x)}{(a-1)!(1-c)!}\right], \qquad (c\neq 2,3,4\ldots). \tag{3.19}$$

In Eq. (3.19), $M(a,c,x) = \sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}\frac{x^j}{j!}$ is the Kummer function of the first kind. Here, $(a)_j = a(a+1)(a+2)\ldots(a+j-1)$ is the Pochhammer symbol. It is important to note that in Eq. (3.17), $c = c(t)$ and is not a constant as in the usual definition, so U in the Regge string scattering amplitudes is *not* a solution of the Kummer equation.

One can calculate the OPE in the Regge limit,

$$\begin{aligned}V_2(\omega)e^{ik_3X(1)} &= \left[\prod_{n=1}(\partial^n X^T)^{p_n}\prod_{m=1}(\partial^m X^P)^{q_m}\right]e^{ik_2X}(\omega)e^{ik_3X(1)}\\ &\sim\prod_{n=1}\left[\frac{(n-1)!k_3\cdot e^T}{(1-\omega)^n}\right]^{p_n}\prod_{m=2}\left[\frac{(m-1)!k_3\cdot e^P}{(1-\omega)^m}\right]^{q_m}\cdot\left[\partial X(1)\cdot e^P-\frac{ik_3\cdot e^P}{1-\omega}\right]^{q_1}(1-\omega)^{k_2\cdot k_3}e^{ikX(1)-ik_2(1-\omega)\partial X(1)}\end{aligned} \tag{3.20}$$

$$\begin{aligned}&= \left(\frac{-\tilde{t}}{2M}\right)^{q_1}\prod_{n=1}[\sqrt{-t}(n-1)!]^{p_n}\prod_{m=2}\left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m}\\ &\quad\cdot\sum_{j=0}^{q_1}\binom{q_1}{j}\left(\frac{2iM_2\partial X(1)\cdot e^P}{\tilde{t}}\right)^j(1-\omega)^{k_2\cdot k_3-N+j}e^{ikX(1)-ik_2(1-\omega)\partial X(1)},\end{aligned} \tag{3.21}$$

where $N = \sum_{n,m}(np_n + mq_m)$ is the mass level of the higher-spin vertex operator $V_2(\omega)$. As in the previous calculation, we can carry out the ω integration first in Eq. (3.21) to obtain the BPST vertex operator

$$
\begin{aligned}
V_{\rm BPST}^{(p_n;q_m)} &= \int d\omega V_2(\omega) e^{ik_3 X}(1) \\
&\sim \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m} \cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM_2 \partial X(1)\cdot e^P}{\tilde{t}}\right)^j \\
&\quad \times \int d\omega (1-\omega)^{k_2\cdot k_3 - N + j} e^{ikX(1) - ik_2(1-\omega)\partial X(1)} \\
&= \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m} \cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM \partial X(1)\cdot e^P}{\tilde{t}}\right)^j \\
&\quad \times \int dx x^{k_2\cdot k_3 - N + j} e^{ikX(1) - ik_2 x \partial X(1)} \\
&= \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m} \cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM \partial X(1)\cdot e^P}{\tilde{t}}\right)^j \\
&\quad \times \Gamma\left(-1-\frac{t}{2}+j\right) [ik_2\cdot \partial X(1)]^{1+\frac{t}{2}-j} e^{ikX(1)}. \qquad (3.22)
\end{aligned}
$$

One notes that, in Eq. (3.22), $M\partial X(1)\cdot e^P = k_2 \cdot \partial X(1)$, and the summation over j can be simplified. The BPST vertex operator can be further reduced to

$$
\begin{aligned}
V_{\rm BPST}^{(p_n;q_m)} &= \left(\frac{-\tilde{t}}{2M_2}\right)^{q_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m} \\
&\quad \cdot \sum_{j=0}^{q_1} q_1 j \left(\frac{2}{\tilde{t}}\right)^j \left(-1-\frac{t}{2}\right)_j \Gamma\left(-1-\frac{t}{2}\right) [ik_2\cdot\partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \\
&= \left(\frac{-1}{M}\right)^{q_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)!\left(-\frac{1}{2M}\right)\right]^{q_m} \\
&\quad \cdot U\left(-q_1, \frac{t}{2}+2-q_1, \frac{\tilde{t}}{2}\right) \Gamma\left(-1-\frac{t}{2}\right) [ik_2\cdot\partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)}, \qquad (3.23)
\end{aligned}
$$

where we have used

$$
\sum_{j=0}^{l} \binom{l}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(-1-\frac{t}{2}\right)_j = 2^l (\tilde{t})^{-l} U\left(-l, \frac{t}{2}+2-l, \frac{\tilde{t}}{2}\right). \qquad (3.24)
$$

One notes that the exponent of $[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}$ in Eq. (3.23) is mass level N independent. This is related to the fact that the well-known $\sim s^{\alpha(t)}$ power-law behavior of the four-tachyon string scattering amplitude in the RR can be extended to arbitrary higher-string states and is mass level independent as can be seen from Eq. (3.17). This interesting result was first pointed out in Ref. [7] and will be crucial to derive intermass level recurrence relations among BPST vertex operators to be discussed later.

The BPST vertex operator in Eq. (3.23) leads to exactly the same amplitude as in Eq. (3.18).

IV. RECURRENCE RELATIONS

For any confluent hypergeometric function $U(a, c, x)$ with parameters (a, c), the four functions with parameters $(a-1, c)$, $(a+1, c)$, $(a, c-1)$, and $(a, c+1)$ are called the contiguous functions. A recurrence relation exists between any such function and any two of its contiguous functions. There are six recurrence relations [25]:

$$
\begin{aligned}
U(a-1,c,x) - (2a-c+x)U(a,c,x) \\
+ a(1+a-c)U(a+1,c,x) = 0,
\end{aligned} \qquad (4.1)
$$

$$
\begin{aligned}
(c-a-1)U(a,c-1,x) - (x+c-1))U(a,c,x) \\
+ xU(a,c+1,x) = 0,
\end{aligned} \qquad (4.2)
$$

$$
U(a,c,x) - aU(a+1,c,x) - U(a,c-1,x) = 0, \qquad (4.3)
$$

$$
(c-a)U(a,c,x) + U(a-1,c,x) - xU(a,c+1,x) = 0, \qquad (4.4)
$$

$$
\begin{aligned}
(a+x)U(a,c,x) - xU(a,c+1,x) \\
+ a(c-a-1)U(a+1,c,x) = 0,
\end{aligned} \qquad (4.5)
$$

$$
\begin{aligned}
(a+x-1)U(a,c,x) - U(a-1,c,x) \\
+ (1+a-c)U(a,c-1,x) = 0.
\end{aligned} \qquad (4.6)
$$

From any two of these six relations, the remaining four recurrence relations can be deduced.

The confluent hypergeometric function $U(a, c, x)$ with parameters $(a \pm m, c \pm n)$ for $m, n = 0, 1, 2 \ldots$ are called associated functions. Again, it can be shown that there exist relations between any three associated functions, so that any confluent hypergeometric function can be expressed in terms of any two of its associated functions.

Recently, it was shown [1] that recurrence relations exist among higher-spin Regge string scattering amplitudes of different string states. The key to derive these relations was to use recurrence relations and the addition theorem of Kummer functions. In view of the form of higher-spin BPST vertex operators in Eq. (3.23), one can easily calculate recurrence relations among higher-spin BPST vertex operators. By using the recurrence relation of Kummer functions [1], for example,

$$U\left(-2, \frac{t}{2}, \frac{t}{2}\right) + \left(\frac{t}{2} + 1\right) U\left(-1, \frac{t}{2}, \frac{t}{2}\right) - \frac{t}{2} U\left(-1, \frac{t}{2} + 1, \frac{t}{2}\right) = 0, \tag{4.7}$$

one can obtain the following recurrence relation among BPST vertex operators at mass level $M^2 = 2$:

$$M\sqrt{-t} V_{\text{BPST}}^{(q_1=2)} - \frac{t}{2} V_{\text{BPST}}^{(p_1=1, q_1=1)} = 0. \tag{4.8}$$

Rather than constant coefficients in the RR stringy Ward identities derived in Ref. [1], the coefficients of this recurrence relation Eq. (4.8) among BPST vertex operators are kinematic variable dependent, similar to BCJ relations among field theory amplitudes [26–30]. The recurrence relation among BPST vertex operators in Eq. (4.8) leads to the recurrence relation among Regge string scattering amplitudes [1]:

$$M\sqrt{-t} A^{(q_1=2)} - \frac{t}{2} A^{(p_1=1, q_1=1)} = 0. \tag{4.9}$$

V. MORE GENERAL RECURRENCE RELATIONS

To derive more general recurrence relations, we need to calculate the BPST vertex operators corresponding to the general higher-spin states in Eq. (3.1). We first calculate the BPST vertex operator corresponding to the state

$$|p_n, r_l\rangle = \prod_{n=1} (\alpha_{-n}^T)^{p_n} \prod_{m=1} (\alpha_{-l}^L)^{r_l} |0\rangle. \tag{5.1}$$

The calculation is very similar to that of Eq. (3.14) up to some modification. One can easily get that Eq. (3.22) is now replaced by

$$\begin{aligned} V_{\text{BPST}}^{(p_n; r_l)} = & \left(\frac{-\tilde{t}'}{2M}\right)^{r_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \\ & \times \prod_{l=2} \left[\tilde{t}'(l-1)! \left(-\frac{1}{2M}\right)\right]^{r_l} \\ & \cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2iM\partial X(1) \cdot e^L}{\tilde{t}'}\right)^j \\ & \times \Gamma\left(-1 - \frac{t}{2} + j\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}-j} e^{ikX(1)}. \end{aligned} \tag{5.2}$$

One notes that, in Eq. (5.2), $M\partial X(1) \cdot e^L \neq k_2 \cdot \partial X(1)$, and, in contrast to Eq. (3.22), the two factors with exponents j and $-j$ do not cancel out. The BPST vertex operator for this case thus reduces to

$$\begin{aligned} V_{\text{BPST}}^{(p_n; r_l)} = & \left(\frac{-1}{M}\right)^{r_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \\ & \times \prod_{l=2} \left[\tilde{t}'(l-1)! \left(-\frac{1}{2M}\right)\right]^{r_l} \\ & \cdot U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}'}{2} \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)}\right) \\ & \times \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)}. \end{aligned} \tag{5.3}$$

The BPST vertex operator in Eq. (5.3) leads to the amplitude

$$\begin{aligned} A^{(p_n, r_l)} = & \left(-\frac{1}{M}\right)^{r_1} U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}'}{2}\right) \\ & \times \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{1+\frac{t}{2}} \\ & \cdot \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{l=2} \left[\tilde{t}'(l-1)! \left(-\frac{1}{2M}\right)\right]^{r_l}, \end{aligned} \tag{5.4}$$

which is consistent with the one calculated in Refs. [1,7,8]. Note that the contribution of $\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)}$ in the correlation function reduces to 1 in the Regge limit by using the first equations of Eqs. (3.7) and (3.8). One sees that Eq. (5.4) can be obtained from Eq. (3.18) by doing the replacement $\tilde{t} \to \tilde{t}'$.

We are now ready to calculate the BPST vertex operator corresponding to the most general Regge state in Eq. (3.1). Similar to the RR amplitude calculated in Ref. [1], the BPST vertex operator can be expressed in two equivalent forms:

$$V_{\mathrm{BPST}}^{(p_n;q_m;r_l)} = \prod_{n=1}[(n-1)!\sqrt{-t}]^{p_n} \cdot \prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M}\right]^{q_m} \cdot \prod_{l=2}\left[(l-1)!\frac{\tilde{t}'}{2M}\right]^{r_l} \cdot \left(\frac{1}{M}\right)^{r_1}\Gamma\left(-1-\frac{t}{2}\right)$$
$$\times[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}\cdot\sum_{i=0}^{q_1}\binom{q_1}{i}\left(\frac{2}{\tilde{t}}\right)^i\left(-\frac{t}{2}-1\right)_i U\left(-r_1,\frac{t}{2}+2-i-r_1,\frac{\tilde{t}'}{2}\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right) \tag{5.5}$$

$$= \prod_{n=1}[(n-1)!\sqrt{-t}]^{p_n} \cdot \prod_{m=2}\left[-(m-1)!\frac{\tilde{t}}{2M}\right]^{q_m} \cdot \prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M}\right]^{r_l}$$
$$\cdot\left(-\frac{1}{M}\right)^{q_1}\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}\cdot\sum_{j=0}^{r_1}\binom{r_1}{j}\left(\frac{2}{\tilde{t}'}\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right)^j$$
$$\times\left(-\frac{t}{2}-1\right)_j U\left(-q_1,\frac{t}{2}+2-j-q_1,\frac{\tilde{t}}{2}\right). \tag{5.6}$$

In the first form, Eq. (5.5), the summation $\sum_{j=0}^{r_1}$ has been carried out to produce the Kummer function while, in the second form, Eq. (5.6), the summation $\sum_{i=0}^{q_1}$ has been carried out instead to produce the Kummer function. Either form, Eq. (5.5) or (5.6), of the above BPST vertex operator leads consistently to the amplitude calculated previously [1]:

$$A^{(p_n,q_m;\,r_l)} = \prod_{n=1}[(n-1)!\sqrt{-t}]^{p_n} \cdot \prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M}\right]^{q_m} \cdot \prod_{l=2}\left[(l-1)!\frac{\tilde{t}'}{2M}\right]^{r_l} \cdot \left(\frac{1}{M}\right)^{r_1}\Gamma\left(-1-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{1+\frac{t}{2}} \tag{5.7}$$

$$\cdot\sum_{i=0}^{q_1}\binom{q_1}{i}\left(\frac{2}{\tilde{t}}\right)^i\left(-\frac{t}{2}-1\right)_i U\left(-r_1,\frac{t}{2}+2-i-r_1,\frac{\tilde{t}'}{2}\right) \tag{5.8}$$

$$= \prod_{n=1}[(n-1)!\sqrt{-t}]^{p_n} \cdot \prod_{m=2}\left[-(m-1)!\frac{\tilde{t}}{2M}\right]^{q_m} \cdot \prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M}\right]^{r_l}$$
$$\cdot\left(-\frac{1}{M}\right)^{q_1}\Gamma\left(-1-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{1+\frac{t}{2}}\cdot\sum_{j=0}^{r_1}\binom{r_1}{j}\left(\frac{2}{\tilde{t}'}\right)^j\left(-\frac{t}{2}-1\right)_j U\left(-q_1,\frac{t}{2}+2-j-q_1,\frac{\tilde{t}}{2}\right). \tag{5.9}$$

Note that, for $r_l = 0$, Eq. (5.9) reduces to Eq. (3.18) as expected. One can now derive more general recurrence relations among BPST vertex operators. As an example, the three BPST vertex operators $V_{\mathrm{BPST}}^{q_1=3}$, $V_{\mathrm{BPST}}^{p_1=1,q_1=2}$, and $V_{\mathrm{BPST}}^{q_1=2,r_1=1}$ can be calculated by using Eq. (5.6) to be

$$V_{\mathrm{BPST}}^{(q_1=3)} = \left(-\frac{1}{M}\right)^3\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}U\left(-3,\frac{t}{2}-1,\frac{t}{2}-1\right), \tag{5.10}$$

$$V_{\mathrm{BPST}}^{(p_1=1,q_1=2)} = \left(-\frac{1}{M}\right)^2\sqrt{-t}\,\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}U\left(-2,\frac{t}{2},\frac{t}{2}-1\right), \tag{5.11}$$

$$V_{\mathrm{BPST}}^{(q_1=2,r_1=1)} = \frac{t+6}{2M}\left(-\frac{1}{M}\right)^2\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}$$
$$\times\left[U\left(-2,\frac{t}{2},\frac{t}{2}-1\right)+\frac{2}{t+6}\left(-\frac{t}{2}-1\right)U\left(-2,\frac{t}{2}-1,\frac{t}{2}-1\right)\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right]. \tag{5.12}$$

The recurrence relation among Kummer functions derived from Eq. (4.4) [1],

$$U\left(-3,\frac{t}{2}-1,\frac{t}{2}-1\right)+\left(\frac{t}{2}+1\right)U\left(-2,\frac{t}{2}-1,\frac{t}{2}-1\right)-\left(\frac{t}{2}-1\right)U\left(-2,\frac{t}{2},\frac{t}{2}-1\right)=0, \tag{5.13}$$

leads to the following recurrence relation among BPST vertex operators at mass level $M^2 = 4$:

$$M\sqrt{-t}e^L\cdot\partial X(1)V_{\mathrm{BPST}}^{q_1=3}+M\sqrt{-t}e^P\cdot\partial X(1)V_{\mathrm{BPST}}^{q_1=2,r_1=1}-\left[\left(\frac{t}{2}+3\right)e^P\cdot\partial X(1)-\left(\frac{t}{2}-1\right)e^L\cdot\partial X(1)\right]V_{\mathrm{BPST}}^{p_1=1,q_1=2}=0. \tag{5.14}$$

In addition to the t dependence, the coefficients of the recurrence relation in Eq. (5.14) are operator dependent. The recurrence relation among BPST vertex operators in Eq. (5.14) leads to the recurrence relation among Regge string scattering amplitudes [1]:

$$M\sqrt{-t}A^{(q_1=3)} - 4A^{(p_1=1,q_1=2)} + M\sqrt{-t}A^{(q_1=2,r_1=1)} = 0. \tag{5.15}$$

For the next example, we construct an intermass level recurrence relation for BPST vertex operators at mass level $M^2 = 2, 4$. We begin with the addition theorem of the Kummer function [25],

$$U(a,c,x+y) = \sum_{k=0}^{\infty} \frac{1}{k!}(a)_k(-1)^k y^k U(a+k,c+k,x), \tag{5.16}$$

which terminates to a finite sum for a nonpositive integer a. By taking, for example, $a = -1$, $c = \frac{t}{2}+1$, $x = \frac{t}{2}-1$ and $y = 1$, the theorem gives [1]

$$U\left(-1,\frac{t}{2}+1,\frac{t}{2}\right) - U\left(-1,\frac{t}{2}+1,\frac{t}{2}-1\right) - U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right) = 0. \tag{5.17}$$

Equation (5.17) leads to an intermass level recurrence relation among BPST vertex operators,

$$M(2)(t+6)V_{\text{BPST}}^{(p_1=1,q_1=1)} - 2M(4)^2\sqrt{-t}V_{\text{BPST}}^{(q_1=1,r_2=1)} + 2M(4)V_{\text{BPST}}^{(p_1=1,r_2=1)} = 0, \tag{5.18}$$

where masses $M(2) = \sqrt{2}$, $M(4) = \sqrt{4} = 2$, and $V_{\text{BPST}}^{p_1=1,q_1=1}$ are BPST vertex operators at mass level $M^2 = 2$, and $V_{\text{BPST}}^{q_1=1,r_2=1}$, $V_{\text{BPST}}^{p_1=1,r_2=1}$ are BPST vertex operators at mass levels $M^2 = 4$. In deriving Eq. (5.18), it is important to use the fact that the exponent of $[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}$ in the BPST vertex operator in Eq. (5.6) is mass level N independent as mentioned in the paragraph after Eq. (3.24). The recurrence relation among BPST vertex operators in Eq. (5.18) leads to the recurrence relation among Regge string scattering amplitudes [1]:

$$M(2)(t+6)A^{(p_1=1,q_1=1)} - 2M(4)^2\sqrt{-t}A^{(q_1=1,r_2=1)} + 2M(4)A^{(p_1=1,r_2=1)} = 0. \tag{5.19}$$

In Ref. [1], it was shown that, at each fixed mass level, each Kummer function in the summation of Eq. (5.9) can be expressed in terms of Regge string scattering amplitudes $A^{(p_n,q_m;r_l)}$ at the same mass level. For general values of a, any Kummer function $U(a,c,x)$ can be expressed in terms of two of its associated functions, while for nonpositive integer values of a in the RR string amplitude case, one can further fixes $U(a,c,x)$ up to an overall factor by using Kummer function recurrence relations [1]. As a result, all Regge string scattering amplitudes can be algebraically solved by Kummer function recurrence relations up to multiplicative factors. An important application of the above properties is the construction of an infinite number of recurrence relations among Regge string scattering amplitudes. One can use the recurrence relations of Kummer functions Eqs. (4.1) to (4.6) to systematically construct recurrence relations among Regge string scattering amplitudes.

In view of the form of BPST vertex operators calculated in Eq. (5.6), one can similarly solve [1] all Kummer functions $U(a,c,x)$ in Eq. (5.6) in terms of BPST vertex operators and use the recurrence relations of Kummer functions Eqs. (4.1) to (4.6) to systematically construct an infinite number of recurrence relations among BPST vertex operators. Moreover, the forms of all BPST vertex operators can be fixed by these recurrence relations up to multiplicative factors. These recurrence relations among BPST vertex operators are dual to linear relations or symmetries among high-energy fixed-angle string scattering amplitudes discovered previously [16–19].

We illustrate the prescription here to construct other examples of recurrence relations among BPST vertex operators at mass level $M^2 = 4$. Generalization to arbitrary mass levels will be given in the next section. There are 22 BPST vertex operators for the mass level $M^2 = 4$. We first consider the group of BPST vertex operators with $q_1 = 0$, (V_{BPST}^{TTT}, V_{BPST}^{LTT}, V_{BPST}^{LLT}, V_{BPST}^{LLL}) [1]. The corresponding r_1 for each BPST vertex operator is (0, 1, 2, 3). Here, we use a new notation for the BPST vertex operator, for example, $V_{\text{BPST}}^{LLT} \equiv V_{\text{BPST}}^{(p_1=1,r_1=2)}$, $V_{\text{BPST}}^{LT} = V_{\text{BPST}}^{(p_1=1,r_2=1)}$, $V_{\text{BPST}}^{TL} = V_{\text{BPST}}^{(p_2=1,r_1=1)}$, etc. By using Eq. (5.6), one can easily calculate that

$$V_{\text{BPST}}^{TTT} = (\sqrt{-t})^3\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)} \times U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right), \tag{5.20}$$

$$V_{\text{BPST}}^{LTT} = \frac{t+6}{2M}(\sqrt{-t})^2\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)} \cdot\left[U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right) + \frac{2}{t+6}\left(-\frac{t}{2}-1\right) \times U\left(0,\frac{t}{2}+1,\frac{t}{2}-1\right)\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right], \tag{5.21}$$

$$V_{\rm BPST}^{LLT} = \left(\frac{t+6}{2M}\right)^2 (\sqrt{-t})\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}\cdot\left[U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right)\right.$$
$$\left.+\frac{4}{t+6}\left(-\frac{t}{2}-1\right)U\left(0,\frac{t}{2}+1,\frac{t}{2}-1\right)\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}+\left(\frac{2}{t+6}\right)^2\left(-\frac{t}{2}-1\right)\left(-\frac{t}{2}\right)U\left(0,\frac{t}{2},\frac{t}{2}-1\right)\left[\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right]^2\right], \quad (5.22)$$

$$V_{\rm BPST}^{LLL} = \left(\frac{t+6}{2M}\right)^3\Gamma\left(-1-\frac{t}{2}\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}\cdot\left[U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right)+\frac{6}{t+6}\left(-\frac{t}{2}-1\right)U\left(0,\frac{t}{2}+1,\frac{t}{2}-1\right)\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right.$$
$$+3\left(\frac{2}{t+6}\right)^2\left(-\frac{t}{2}-1\right)\left(-\frac{t}{2}\right)U\left(0,\frac{t}{2},\frac{t}{2}-1\right)\left[\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right]^2$$
$$\left.+\left(\frac{2}{t+6}\right)^3\left(-\frac{t}{2}-1\right)\left(-\frac{t}{2}\right)\left(-\frac{t}{2}+1\right)U\left(0,\frac{t}{2}-1,\frac{t}{2}-1\right)\left[\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right]^3\right]. \quad (5.23)$$

From the above equations, one can easily see that $U(0,\frac{t}{2}+2,\frac{t}{2}-1)$ can be expressed in terms of $V_{\rm BPST}^{TTT}$, $U(0,\frac{t}{2}+1,\frac{t}{2}-1)$ can be expressed in terms of $(V_{\rm BPST}^{TTT}, V_{\rm BPST}^{LTT})$, $U(0,\frac{t}{2},\frac{t}{2}-1)$ can be expressed in terms of $(V_{\rm BPST}^{TTT}, V_{\rm BPST}^{LTT}, V_{\rm BPST}^{LLT})$, and finally $U(0,\frac{t}{2}-1,\frac{t}{2}-1)$ can be expressed in terms of $(V_{\rm BPST}^{TTT}, V_{\rm BPST}^{LTT}, V_{\rm BPST}^{LLT}, V_{\rm BPST}^{LLL})$. So all Kummer functions can be solved and expressed in terms of BPST vertex operators. We have

$$U(0,\frac{t}{2}+2,\frac{t}{2}-1) = \Omega^{-1}(\sqrt{-t})^{-3}V_{\rm BPST}^{TTT}, \quad (5.24)$$

$$U(0,\frac{t}{2}+1,\frac{t}{2}-1) = \Omega^{-1}(\sqrt{-t})^{-3}\frac{t+6}{t+2}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]\cdot\left[V_{\rm BPST}^{TTT}-\frac{2M}{t+6}\sqrt{-t}V_{\rm BPST}^{LTT}\right],$$
$$U\left(0,\frac{t}{2},\frac{t}{2}-1\right) = \Omega^{-1}(\sqrt{-t})^{-3}\frac{(t+6)^2}{t(t+2)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^2\cdot\left[V_{\rm BPST}^{TTT}-2\frac{2M}{t+6}\sqrt{-t}V_{\rm BPST}^{LTT}+\left(\frac{2M}{t+6}\sqrt{-t}\right)^2V_{\rm BPST}^{LLT}\right], \quad (5.25)$$

$$U\left(0,\frac{t}{2}-1,\frac{t}{2}-1\right) = \Omega^{-1}(\sqrt{-t})^{-3}\frac{(t+6)^3}{t(t^2-4)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^3\cdot\left[V_{\rm BPST}^{TTT}-3\frac{2M}{t+6}\sqrt{-t}V_{\rm BPST}^{LTT}\right.$$
$$\left.+3\left(\frac{2M}{t+6}\sqrt{-t}\right)^2V_{\rm BPST}^{LLT}-\left(\frac{2M}{t+6}\sqrt{-t}\right)^3V_{\rm BPST}^{LLL}\right], \quad (5.27)$$

where $\Omega\equiv\Gamma(-1-\frac{t}{2})[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}$. To derive an example of the recurrence relation, one notes that Eq. (4.2) gives

$$\frac{t}{2}U\left(0,\frac{t}{2},\frac{t}{2}-1\right)-(t-1)U\left(0,\frac{t}{2}+1,\frac{t}{2}-1\right)+\left(\frac{t}{2}-1\right)U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right)=0, \quad (5.28)$$

which leads to the recurrence relation among BPST vertex operators:

$$\left[\left(\frac{t}{2}-1\right)-\frac{(t-1)(t+6)}{t+2}\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}+\frac{(t+6)^2}{2(t+2)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^2\right]V_{\rm BPST}^{TTT}$$
$$+\left[\frac{(t-1)}{t+2}\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}-\frac{(t+6)}{(t+2)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^2\right](2M\sqrt{-t})V_{\rm BPST}^{LTT}+\left[\frac{1}{2(t+2)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^2\right](2M\sqrt{-t})^2V_{\rm BPST}^{LLT}=0. \quad (5.29)$$

Again, one can use Eq. (5.29) to deduce the recurrence relation among Regge string scattering amplitudes:

$$(t+22)A^{(p_1=3)}-14M\sqrt{-t}A^{(p_1=2,r_1=1)}+2M^2(\sqrt{-t})^2A^{(p_1=1,r_1=2)}=0. \quad (5.30)$$

Other recurrence relations of Kummer functions can be used to derive more recurrence relations among BPST vertex operators. For example, Eq. (4.2) gives a recurrence relation of $U(0,\frac{t}{2}+1,\frac{t}{2}-1)$ and its associated functions $U(0,\frac{t}{2}-1,\frac{t}{2}-1)$ and $U(0,\frac{t}{2}+2,\frac{t}{2}-1)$

$$tU\left(0,\frac{t}{2}-1,\frac{t}{2}-1\right)-(3t-4)U\left(0,\frac{t}{2}+1,\frac{t}{2}-1\right)+2(t-2)U\left(0,\frac{t}{2}+2,\frac{t}{2}-1\right)=0, \quad (5.31)$$

which leads to the recurrence relation among BPST vertex operators:

$$\left[2(t-2)-\frac{(3t-4)(t+6)}{t+2}\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}+\frac{(t+6)^3}{(t^2-4)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^3\right]V^{TTT}_{\mathrm{BPST}}+\left[\frac{(3t-4)}{t+2}\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right.$$
$$-3\frac{(t+6)^2}{(t^2-4)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^3\left.\right](2M\sqrt{-t})V^{LTT}_{\mathrm{BPST}}+\left[\frac{3(t+6)}{(t^2-4)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^3\right](2M\sqrt{-t})^2V^{LLT}_{\mathrm{BPST}}$$
$$-\left[\frac{1}{(t^2-4)}\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\right]^3\right](2M\sqrt{-t})^3V^{LLL}_{\mathrm{BPST}}=0. \tag{5.32}$$

One can use Eq. (5.32) to deduce the recurrence relation among Regge string scattering amplitudes:

$$(3t^2+76t+92)A^{(p_1=3)}-2(23t+50)M\sqrt{-t}A^{(p_1=2,r_1=1)}+6M^2(t+6)(\sqrt{-t})^2A^{(p_1=1,r_1=2)}-4M^3(\sqrt{-t})^3A^{(r_1=3)}=0. \tag{5.33}$$

Similarly, we can consider groups of BPST vertex operators $(V^{PT}_{\mathrm{BPST}}, V^{PL}_{\mathrm{BPST}})$, $(V^{LT}_{\mathrm{BPST}}, V^{LL}_{\mathrm{BPST}})$, and $(V^{TT}_{\mathrm{BPST}}, V^{TL}_{\mathrm{BPST}})$ with $q_1=0$; a group of BPST vertex operators $(V^{PTT}_{\mathrm{BPST}}, V^{PLT}_{\mathrm{BPST}}, V^{PLL}_{\mathrm{BPST}})$ with $q_1=1$; and group of BPST vertex operators $(V^{PPT}_{\mathrm{BPST}}, V^{PPL}_{\mathrm{BPST}})$ with $q_1=2$. All the remaining 7 BPST vertex operators are with $r_1=0$, and each BPST vertex operator contains only one Kummer function. Thus, all Kummer functions involved at mass level $M^2=4$ can be algebraically solved and expressed in terms of BPST vertex operators. One can then use recurrence relations of Kummer functions to derive more recurrence relations among the BPST vertex operators.

VI. ARBITRARY MASS LEVELS

In this section, we solve the Kummer functions in terms of the highest-spin string states scattering amplitudes for arbitrary mass levels. The highest-spin string states at the mass level $M^2=2(N-1)$ are defined as

$$|N-q_1-r_1,q_1,r_1\rangle=(\alpha^T_{-1})^{N-q_1-r_1}(\alpha^P_{-1})^{q_1}\times(\alpha^L_{-1})^{r_1}|0,k\rangle, \tag{6.1}$$

where only the α_{-1} operator appears. The highest-spin string states BPST vertex operators can be easily obtained from Eq. (5.6) as

$$(V^T)^{N-q_1-r_1}(V^P)^{q_1}(V^L)^{r_1}\equiv V^{(N-q_1-r_1,q_1,r_1)}_{\mathrm{BPST}}=\Gamma\left(-\frac{t}{2}-1\right)[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}(\sqrt{-t})^{N-q_1-r_1}\left(-\frac{1}{M}\right)^{q_1}\left(\frac{\tilde{t}'}{2M}\right)^{r_1}\cdot\sum_{j=0}^{r_1}\binom{r_1}{j}\left(\frac{2}{\tilde{t}'}\frac{e^L\cdot\partial X(1)}{e^P\cdot\partial X(1)}\right)^j\left(-\frac{t}{2}-1\right)_jU\left(-q_1,\frac{t}{2}+2-j-q_1,\frac{\tilde{t}}{2}\right). \tag{6.2}$$

In view of the form of Eq. (5.27), we can solve the Kummer function from Eq. (6.2) and express it in terms of the highest-spin BPST vertex operators as

$$U\left(-q_1,\frac{t}{2}+2-q_1-r_1,\frac{\tilde{t}}{2}\right)=\frac{\Gamma(-\frac{t}{2}-1)}{(-\frac{t}{2}-1)_{r_1}}[ik_2\cdot\partial X(1)]^{1+\frac{t}{2}}e^{ikX(1)}\cdot(-MV^P)^{q_1}\left(\frac{V^T}{\sqrt{-t}}\right)^{N-q_1}\times\left[\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\left(\sqrt{-t}M\frac{V^L}{V^T}-\frac{\tilde{t}'}{2}\right)\right]^{r_1}. \tag{6.3}$$

Putting the Kummer functions (6.3) into the recurrence relations (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6), we can then obtain recurrence relations among BPST vertex operators.

Let us consider, for example, the recurrence relation

$$(c-a-1)U(a,c-1,x)-(x+c-1))U(a,c,x)+xU(a,c+1,x)=0. \tag{6.4}$$

With

$$a=-q_1,\qquad c=\frac{t}{2}+1-q_1-r_1,\qquad x=\frac{\tilde{t}}{2}=\frac{t-M^2+2}{2}, \tag{6.5}$$

the above recurrence relation becomes

$$\left(\frac{t}{2}-r_1\right)U\left(-q_1,\frac{t}{2}-q_1-r_1,\frac{\tilde{t}}{2}\right)$$
$$-\left(\frac{\tilde{t}}{2}+\frac{t}{2}-q_1-r_1\right)U\left(-q_1,\frac{t}{2}+1-q_1-r_1,\frac{\tilde{t}}{2}\right)$$
$$+\frac{\tilde{t}}{2}U\left(-q_1,\frac{t}{2}+2-q_1-r_1,\frac{\tilde{t}}{2}\right)=0. \qquad (6.6)$$

Plugging the Kummer functions (6.3) into the above recurrence relation, we obtain the recurrence relation among BPST vertex operators at general mass level N,

$$(V^P)^{q_1}(V^T)^{N-q_1}(X)^{r_1}\left[X^2+\left(\frac{\tilde{t}}{2}+\frac{t}{2}-q_1-r_1\right)X + \frac{\tilde{t}}{2}\left(\frac{t}{2}+1-r_1\right)\right]=0, \qquad (6.7)$$

where we have defined

$$X\equiv\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\left(\sqrt{-t}M\frac{V^L}{V^T}-\frac{\tilde{t}'}{2}\right) = \frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\left(\sqrt{-t}M\frac{V^L}{V^T}-\frac{t+M^2+2}{2}\right). \qquad (6.8)$$

As an example, at the mass level $M^2=4$ with $q_1=r_1=0$, we get

$$(V^T)^3\left[X^2+(t-1)X+\left(\frac{t^2}{4}-1\right)\right]=0, \qquad (6.9)$$

where

$$X=\frac{e^P\cdot\partial X(1)}{e^L\cdot\partial X(1)}\left(\sqrt{-t}M\frac{V^L}{V^T}-\frac{t+6}{2}\right). \qquad (6.10)$$

A simple calculation shows that Eq. (6.9) is exactly the same as Eq. ((5.29)), and the same recurrence relation among Regge string scattering amplitudes ((5.30)) follows.

VII. DISCUSSION

Although we focus here on the spin dependence of the four-point open-string amplitudes, it is useful to briefly recall the generality of the BPST vertex operator, which emphasizes Regge factorization and can be applied to arbitrary n-point amplitudes, $n\geq 4$. A Regge limit is defined by singling out a longitudinal direction, e.g., the z axis, along which all momenta are large while keeping transverse components, $p_\perp$, fixed. We separate particles into two groups, the right-moving and left-moving, with large p_+ and p_- large, respectively. Each can have n_R and n_L states, with $n_R+n_L=n$ and n_R, $n_L\geq 2$. Within each group, relative momenta remain finite in the Regge limit. Any n-point open-string amplitude can formally be expressed in a factorable form $A_{L,R}=\int dw\langle W_R w^{L_0-2}W_L\rangle$, where W_R and W_L are products of respective right-moving and left-moving vertex operators, with all world sheet integrations done except one, i.e., w. The last remaining integration is such that the factor w^{L_0} corresponds to overall rescaling in the world sheet coordinates in W_L. (For more details, see Ref. [2].) In the Regge limit, the amplitude $A_{L,R}$ takes on a simply factorized form, and it can be expressed in terms of the BPST vertex operator,

$$A_{L,R}=\langle W_R V^-\rangle\Pi(t)\langle V^+W_L\rangle = \langle W_{R,0}V^-\rangle\{\Pi(t)s^{\alpha(t)}\}\langle V^+W_{L,0}\rangle, \qquad (7.1)$$

where $\alpha(t)$ is the leading Regge trajectory, with $\alpha'=1/2$, and $\Pi(t)$ is a Regge propagator, given by a Gamma function. Here, $V^\pm$ are BPST vertex operators, which are "on shell" along the leading trajectory. This is the most general form of Regge factorization for any number of external particles. The factors $\langle W_{R,0}V^-\rangle$ and $\langle V^+W_{L,0}\rangle$ are generalized (n_R+1)- and (n_L+1)-point on-shell amplitudes, evaluated in the respective rest frame, with one external line being on the leading Regge trajectory. Each, due to Mobius invariance, involves n_R-2 and n_L-2 world sheet integrations.

We have studied in this paper the Regge behavior of four-point open-string scattering amplitudes, with one particle having arbitrary high spin and three others being tachyons, using the technique of the BPST vertex operator. Since we only work with four-point amplitudes in this paper, $n_R=n_L=2$, there is no integration involved for $\langle W_{R,0}V^-\rangle$ and $\langle V^+W_{L,0}\rangle$, due to Mobius invariance. In particular, W_L involves two tachyons. Since one can show that $\langle V^+W_{L,0}\rangle$ is simply a constant, therefore, what we have calculated is simply $\langle W_{R,0}V^-\rangle$, with W_R a product of two vertex operators, one for a tachyon and another for a string state with arbitrary spin. For brevity, we have collectively referred to $W_{R,0}V^-$ as BPST vertex operators. The generalization of our analysis to amplitudes for $n=5,6\ldots$ will be treated elsewhere.

We have derived in this paper an infinite number of recurrence relations among these matrix elements of the BPST vertex operator between different string states with different spins, which can be expressed in terms of a Kummer function of the second kind. These recurrence relations lead to the same recurrence relations among Regge string scattering amplitudes recently discovered in Ref. [1] by a more traditional method. We show that all Kummer functions involved at each fixed mass level can be algebraically solved and expressed in terms of BPST vertex operators. We give a prescription to construct recurrence relations among BPST vertex operators. For illustration, we calculate some examples of recurrence relations among BPST vertex operators of different string states based on recurrence relations of Kummer functions together with the addition theorem of Kummer function. We stress that, although the higher-spin BPST vertex operators were considered in Refs. [2,3], the key observation on the energy orders in the Regge limit from polarizations of higher-spin states was not discussed in Refs. [2,3]. One cannot obtain recurrence relations among higher-spin BPST vertex operators in the Regge limit without including the energy orders from these higher-spin polarizations.

The recurrence relations among BPST vertex operators lead to the recurrence relations among Regge string scattering amplitudes. They are thus both closely related to Regge stringy Ward identities [1] derived from the decoupling of Regge ZNS in the string spectrum. These recurrence relations are dual to linear relations derived from ZNS or symmetries among high-energy fixed-angle string scattering amplitudes [16–19].

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