JOURNAL OF DIFFERENTIAL EQUATIONS **114**, 57–76 (1994)

Positive Singular Solutions for Semilinear Elliptic Equations with Supercritical Growth

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Received January 13, 1992; revised July 29, 1992

1. INTRODUCTION

In this paper we are mainly interested in the existence of positive singular solutions of semilinear elliptic equations on finite balls or $\mathbb{R}^n$ when the growth of the nonlinear function is supercritical in the sense of Sobolev embedding theorems. Namely, we study the existence of positive singular and radially symmetric solutions of the equation

$$\Delta u + f(u) = 0 \qquad \text{in } \Omega - \{0\}, \tag{1.1}$$

$$\lim_{x \to 0} u(x) = \infty, \tag{1.2}$$

where $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$ is a finite ball or $\Omega = B_\infty = \mathbb{R}^n$, $n \geq 3$. The nonlinear function $f \in C^1(\mathbb{R}^1)$ (or f is in general locally Lipschitz continuous) satisfies the supercritical condition as $u \to \infty$; that is, f satisfies that following condition:

(H-1) There is a $q > (n+2)/(n-2)$ such that $(q+1)\,F(u) \leq uf(u)$ for $u \geq A$, where $F(u) = \int_0^u f(v)\,dv$ and A is a positive constant with $F(A) > 0$.

Under the assumption that f satisfies the supercritical condition, we have the following general existence result.

THEOREM 1.1. *Assume f satisfies* (H-1). *Then* (1.1) *and* (1.2) *have a positive singular solution on B_R for some $R \in (0, \infty]$.*

* Work partially supported by the National Science Council of the Republic of China.

0022-0396/94 $6.00

Furthermore, if f satisfies a certain subcritical condition at $u=0$, then the singular solution obtained in Theorem 1.1 satisfies the zero boundary condition,

$$u=0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

i.e., $u(R)=0$ if $R<\infty$ or $\lim_{r\to\infty} u(r)=0$ if $u(r)>0$ in $(0,\infty)$. More precisely, we have the following result.

THEOREM 1.2. *In addition to* (H-1), *assume f satisfies*

(H-2) $f(u)>0$ *for* $u>0$;

(H-3) (i) $f(0)>0$;

(ii) $f(u)=0$ *and* $f'(0)>0$;

(iii) $f(0)=f'(0)=0$ *and there is a* $p\in(1, n/(n-2)]$ *such that* $f(u)\geq Cu^p$ *for* $u\in(0,B)$, *where B and C are positive constants.*

Then the positive singular solution U of (1.1) *and* (1.2) *obtained in Theorem* 1.1 *satisfies* $U(R)=0$ *for some* $R<\infty$.

The existence problems of (1.1)–(1.3) have been studied by several authors in the past. When $f(0)=0$ and the growth of f is subcritical, the existence of positive singular solutions on finite balls has been shown by P. Lions [21], Ni and Sack [26], and the author [20]; see Remark 2.9 below for details. On the other hand, if f is supercritical in $(0,\infty)$, i.e., $A=0$ in (H-1), Ni and Serrin [27] have shown that there is no positive singular solution on a finite ball. Moreover, if $f(u)\leq 0$ in $(0,\varepsilon)$, for some $\varepsilon>0$, then there is no singular solution on $\mathbb{R}^n$ which satisfies (1.3). Therefore, the singular solution obtained in Theorem 1.1 is positive in $(0,\infty)$ when $A=0$ in (H-1) and does not tend to zero when $f(u)\leq 0$ in $(0,\varepsilon)$. Recently, Pan [30] studied the critical case, $f(u)=u^p+u^q$, $q=(n+2)/(n-2)>p>n/(n-2)$, and showed that there is a positive singular solution for (1.1)–(1.3) on $\mathbb{R}^n$. Furthermore, in [14], Johnson *et al.* studied the positive solutions of $\Delta u+K(|x|)\,u^p=0$, for $p>(n+2)/(n-2)$, and $n\geq 3$. Using the invariant manifold theory of dynamical systems, they proved the asymptotic behavior of ground states and the uniqueness of singular ground states provided that $K(|x|)$ satisfies some monotonicity conditions.

To prove Theorem 1.1, we consider the initial value problems:

$$u''(r)+\frac{n-1}{r}u'(r)+f(u(r))=0, \qquad r>0, \tag{1.4}$$

$$u(0,\alpha)=\alpha>0, \tag{1.5}$$

$$u'(0,\alpha)=0. \tag{1.6}$$

For $\alpha > B$, let $R(\alpha, B)$ be the first r such that $u(r, \alpha) = B$. By using Pohozaev identity and some comparison arguments, we can show that the supercritical condition (H-1) implies there are positive constants $R_*(B)$ and $R^*(B)$ such that $R_*(B) \leqslant R(\alpha, B) \leqslant R^*(B)$ for any $B \geqslant A$ and sufficiently large α. From these estimates Theorems 1.1 and 1.2 can be proved.

The estimations of $R(\alpha, B)$ in the study of the existence of singular solutions can also be applied to study the structure of sets of positive regular solutions on finite balls. Consider the solution set of equations

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega, \tag{1.7}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

where $\Omega = B_1$ is the unit ball and $\lambda > 0$ is a parameter. Note that by the well-known symmetry theorem of Gidas *et al.* [13], positive solutions on balls are necessarily radially symmetric. It is also easy to see that $(u(\cdot), \lambda)$ is a positive radial solution of (1.7) and (1.8) if and only if $u(\cdot, \alpha)$ is a positive solution of (1.4)–(1.6) with $u(r) = u(r\lambda^{1/2}, \alpha)$ and $\lambda = R^2(\alpha) < \infty$, where $R(\alpha)$ is the first zero of $u(\cdot, \alpha)$.

Equations (1.7) and (1.8) arise from many branches of mathematics and applied mathematics. They have been studied by many authors, such as Gelfand [12], Keller and Cohen [16], Amann [1], Crandall and Rabinowitz [9], P. Lions [22], Brezis and Nirenberg [3], and Ni and Serrin [27], to name just a few.

In the case where $f(u) > 0$ in $\mathbb{R}^1$, $u_0 \equiv 0$ is a subsolution of (1.7) and (1.8). A well-developed monotone iteration scheme can be applied to obtain a solution; see [1, 16, 31]. Therefore, it is known that the existence of positive solutions of (1.7) and (1.8) is equivalent to the existence of an upper bound for iterative solutions that are generated by subsolution u_0. The positive solution obtained by the iterative procedure is a minimum. Moreover, there exists a critical number $\lambda^* \leqslant \infty$ such that (1.7) and (1.8) have a positive minimum solution for each $\lambda \in (0, \lambda^*)$ and no positive solution if $\lambda > \lambda^*$. Note that $\lambda^* < \infty$ if the growth of $f(u)$ is linear or superlinear, i.e., if $\lim_{u \to \infty} \inf f(u)/u > 0$.

Apart from the study of positive minimum solutions, one of the main problems of studying (1.7) and (1.8) is the existence of positive nonminimum solutions for $\lambda \in (0, \lambda^*)$. Indeed, by using the Mountain Pass Lemma, Crandall and Rabinowitz [9] proved that when the growth of f is subcritical, there is at least one positive nonminimum solution for each $\lambda \in (0, \lambda^*)$. However, there may fail to be a second positive solution when the growth of f is supercritical. A well-known example, $f(u) = e^u$, can be used to illuminate the dependence of the solution set on the topological and geometrical properties of the domain Ω. In this case, the solution set $S = \{(u, \lambda) : u \text{ is a positive solution of (1.7) and (1.8) on } B_1\}$ is a smooth connected 1-manifold; see, e.g., [10, 12, 15]. For $n=1, 2$, there exist exactly two positive solutions for each $\lambda \in (0, 2)$, exactly one for $\lambda = 2$, and none for $\lambda > 2$. However, for $n = 3, ..., 9$, there exist $0 < \lambda_* < 2(n-2) < \lambda^*$ such that there is no positive nonminimum solution for $\lambda \in (0, \lambda_*)$. Furthermore, there are infinitely many positive solutions at $\lambda_\infty = 2(n-2)$ and a large number of positive solutions when λ is close to λ_∞. Finally, for $n \geqslant 10$, there is no positive solution for $\lambda \geqslant \lambda^* = 2(n-2)$ and exactly one positive solution for $\lambda \in (0, \lambda^*)$. Note that the growth of e^u is subcritical when $n = 1$, 2 and supercritical when $n \geqslant 3$. One of the main results in this paper is a partial generation of the above facts.

THEOREM 1.3. *Assume f satisfies* (H-1). *If $f(0) > 0$, then there exists $\lambda_* > 0$ such that there is no positive nonminimum solution for any $\lambda \in (0, \lambda_*)$. If $f(0) \leqslant 0$, then there exists $\lambda_* > 0$ such that there is no positive solution for any $\lambda \in (0, \lambda_*)$.*

It is worth remarking that the validity of Theorem 1.3 relies on the topology of the domain Ω. Indeed, when $f(u) > 0$ on $[0, \infty)$ and Ω is an annular domain, i.e., $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$, $n \geqslant 2$, there is at least one positive nonminimum solution for each $\lambda \in (0, \lambda^*)$ provided the growth of f is superlinear, i.e., $\lim_{u \to \infty} f(u)/u = \infty$; see [19].

It is known that positive regular solutions have finite energy in the supercritical case; see [11]. In this paper, we will also prove that every positive singular solution of (1.7) and (1.8) has finite energy and then demonstrate that if a sequence of positive regular solutions converges to a positive singular solution pointwise in (0, 1), then the convergence obtains in $H^1(B_1)$ and $L^{q+1}(B_1)$ as well. A combination of previous results and Theorems 1.1–1.3 enables us to give a description of the structure of the set of positive regular solutions of (1.7) and (1.8):

THEOREM 1.4. *Assume f satisfies* (H-1), (H-2), *and* (H-3). *Then the solution set $S = \{(u, \lambda) : u$ is a positive solution of (1.7) and (1.8)$\}$ is a connected C^1-smooth 1-manifold. One end of S is connected by*

(i) (0, 0) *if* (H-3)(i) *holds;*

(ii) $(0, \lambda_1 f'(0)^{-1})$ *if* (H-3(ii) *holds, where λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions on B_1;*

(iii) $(0, \infty)$ *if* (H-3)(iii) *holds.*

The other end of S is connected, in the sense of $H_0^1(B_1)$, by the singular solutions obtained in Theorem 1.2.

For related problems:

(i) For $f(u)=u^p+u^q$, $(n+2)/(n-2)\leqslant q$ and $1\leqslant p\leqslant n/(n-2)$: The uniqueness of the singular solution and the asymptotic behaviour of large regular solutions have been studied by Budd and Norbury [7], Budd [5, 6], and Merle and Peletier [23].

(ii) For $f(u)=e^u$: (1.7) and (1.8) have been studied by Bandle [2], Suzuki and Nagasaki [32], Lin [18], Mignot *et al.* [24], Veron and Veron [33], Gallouet *et al.* [11], and Bellout [4], to name just a few.

(iii) The critical case $f(u)=u^{(n+2)/(n-2)}+\lambda u$ was first studied by Brezis and Nirenberg [3] and has since been studied extensively by many authors.

The remainder of this paper is organized as follows: In Section 2, using Pohozaev identity, we estimatre $R(\alpha)$ for large α and prove Theorems 1.1–1.3. In Section 3, we establish that every positive singular solution has finite energy and prove that if positive regular solutions converge to a singular solution pointwise in (0, 1), then the convergences obtain in L^{q+1} and H^1 as well. Finally, we prove Theorem 1.4.

2. Existence of Singular Solutions

We first recall a Pohozaev identity which was obtained by Ni and Serrin [27].

Lemma 2.1. *Let $u(r)$ be a solution of (1.4) in $(r_1, r_2)\subset(0,\infty)$ and let a be an arbitrary constant. Then, for each $r\in(r_1, r_2)$ we have*

$$\frac{d}{dr}\, r^n\left\{\frac{1}{2}\,u'^2(r)+F(u(r))+\frac{a}{r}\,u(r)\,u'(r)\right\}$$

$$=r^{n-1}\left\{nF(u(r))-au(r)\,f(u(r))+\left(a+1-\frac{n}{2}\right)u'^2(r)\right\}. \tag{2.1}$$

Definition 2.2. For each $\alpha\in(0,\infty)$ and $B\geqslant 0$, let $R(\alpha, B)$ be the first r such that $u(r,\alpha)=B$. If there is no such r, we shall adopt the convention that $R(\alpha, B)=\infty$. We also stipulate that $R(\alpha)=R(\alpha, 0)$ and $R_1(\alpha)=R(\alpha, A)$.

Definition 2.3. For $q>(n+2)/(n-2)$, let $\gamma=n-2-2n/(q+1)=(1/(q+1))\{(n-2)q-(n+2)\}>0$. Define two positive functions $R_*(B)$ and $R^*(B)$ on $[A,\infty)$ by

$$R_*(B)^2=\gamma\tilde{B}M(\tilde{B})^{-1}$$

and

$$R^*(B)^2 = 2\left(\frac{n}{q+1} B\right)^2 F(B)^{-1},$$

where

$$\tilde{B} = \frac{2(n-2)}{\gamma} B \qquad \text{and} \qquad M(\tilde{B}) = \max\{f(u) : u \in [0, \tilde{B}]\}.$$

In the following theorem, we prove that for a fixed $B \geqslant A$, we can obtain an upper bound and a lower bound for $R(\alpha, B)$. This is crucial in proving Theorem 1.1 and also in later developments.

THEOREM 2.4. *Assume f satisfies* (H-1). *Then for any $B \geqslant A$ and $\alpha \in (\tilde{B}, \infty)$, we have*

$$R_*(B) \leqslant R(\alpha, B) \leqslant R^*(B), \tag{2.2}$$

and

$$\frac{q+1}{n} \frac{F(B)}{B} R_*(B) \leqslant -u'(R(\alpha, B), \alpha) \leqslant \frac{2n}{q+1} BR_*(B)^{-1}. \tag{2.3}$$

Proof. Letting $u(r) = u(r, \alpha)$ and $a = n/(q+1)$ in (2.1) and integrating (2.1) from 0 to r, by (H-1) we have

$$\frac{1}{2} u'^2(r, \alpha) + F(u(r, \alpha)) + \frac{n}{q+1} \frac{u(r, \alpha)\, u'(r, \alpha)}{r} < 0 \tag{2.4}$$

if $u(s, \alpha) > A$ for all $s \in [0, r]$. It is clear that (H-1) implies $F(u) > 0$ for all $u > A$. Hence, for any $\alpha \in (A, \infty)$, by (2.4) we have $u'(r, \alpha) < 0$ in $(0, R_1(\alpha))$. Furthermore, we have $R_1(\alpha) < \infty$ for all $\alpha \in (A, \infty)$. Indeed, by (H-1) there is a positive constant m such that

$$f(u) \geqslant m \qquad \text{for all} \quad u \geqslant A. \tag{2.5}$$

By (1.4)–(1.6) and (2.5), for $r \in (0, R_1(\alpha))$ and $\alpha \geqslant A$, we have

$$\begin{aligned} r^{n-1} u'(r, \alpha) &= -\int_0^r s^{n-1} f(s, \alpha))\, ds \\ &\leqslant -\frac{m}{n} r^n, \end{aligned} \tag{2.6}$$

which implies that

$$R_1(\alpha)^2 \leqslant \frac{2n}{m} (\alpha - A).$$

Therefore, by (H-1) and (2.4) we obtain

$$\frac{1}{2} u'^2(R(\alpha, B), \alpha) < \frac{n}{q+1} \frac{R}{R(\alpha, B)} (-u'(R(\alpha, B), \alpha)), \tag{2.7}$$

and

$$F(B) < \frac{n}{q+1} \frac{B}{R(\alpha, B)} (-u'(R(\alpha, B), \alpha)). \tag{2.8}$$

Now, (2.7) implies

$$-u'(R)(\alpha, B), \alpha) R(\alpha, B) < \frac{2n}{q+1} B, \tag{2.9}$$

or

$$-u'(R(\alpha, B), \alpha) R(\alpha, B) < (n-2-\gamma) B. \tag{2.10}$$

From (2.8) and (2.9), we obtain an upper bound for $R(\alpha, B)$,

$$R(\alpha, B)^2 \leqslant 2 \left(\frac{n}{q+1} B\right)^2 \Big/ F(B) \tag{2.11}$$

for all $\alpha \in (B, \infty)$. This proves the second inequality of (2.2).

To prove the first inequality of (2.2), there are two cases to consider: (i) $R(\alpha, \tilde{B}) \geqslant R_*(B)$ and (ii) $R(\alpha, \tilde{B}) < R_*(B)$. In case (i), since $R(\alpha, B) > R(\alpha, \tilde{B})$, we have $R(\alpha, B) > R_*(B)$. In case (ii), we need the following comparison argument.

Let $v_\alpha(r) \equiv v(r, \alpha, \tilde{B})$ be the solution of the initial value problem

$$v''(r) + \frac{n-1}{r} v'(r) + \tilde{C} = 0 \qquad \text{for} \quad r > R(\alpha, \tilde{B}), \tag{2.12}$$

$$v(R(\alpha, \tilde{B})) = \tilde{B}, \tag{2.13}$$

$$v'(R(\alpha, \tilde{B})) = u'(R(\alpha, \tilde{B}), \alpha), \tag{2.14}$$

where $\tilde{C} = M(\tilde{B})$.

Then $v_\alpha(r)$ can be solved explicitly as

$$v_\alpha(r) = \tilde{B} + \frac{1}{n-2} \left\{\tilde{R} u'(\tilde{R}, \alpha) + \frac{\tilde{C}}{n} \tilde{R}^2\right\} - \frac{1}{n-2} \left\{\tilde{R}^{n-1} u'(\tilde{R}, \alpha) + \frac{\tilde{C}}{n} \tilde{R}^n\right\} r^{2-n}$$
$$- \frac{\tilde{C}}{2n} r^2 + \frac{\tilde{C}}{2n} \tilde{R}^2, \tag{2.15}$$

where $\tilde{R}=R(\alpha, \tilde{B})$. By (2.10) and (2.15), we have

$$\begin{aligned} v_\alpha(r) &\geqslant \tilde{B}+\frac{1}{n-2}\,\tilde{R}u'(\tilde{R}, \alpha)-\frac{\tilde{C}}{n(n-2)}\,\tilde{R}^n r^{2-n}-\frac{\tilde{C}}{2n}\,r^2 \\ &\geqslant \frac{1}{n-2}\left\{\gamma\tilde{B}-\frac{1}{2}\,\tilde{C}r^2\right\} \\ &= B+\frac{\tilde{C}}{2(n-2)}\,\{\gamma\tilde{B}-\tilde{C}r^2\} \\ &\geqslant B \end{aligned} \tag{2.16}$$

for all $r\in[R(\alpha, \tilde{B}), R_*(B)]$.

Therefore, (2.2) follows if we can prove that $u(r,\alpha)\geqslant v_\alpha(r)$ on $[R(\alpha, \tilde{B}), R_*(B)]$. Let $w(r)=u(r, \alpha)-v_\alpha(r)$. Then

$$(r^{n-1}w'(r))'=r^{n-1}\{\tilde{C}-f(u(r, \alpha))\}\geqslant 0 \tag{2.17}$$

as long as $u(r, \alpha)>0$. By (2.13) and (2.14), we have

$$w(R(\alpha, \tilde{B}))=0=w'(R(\alpha, \tilde{B})). \tag{2.18}$$

Integrating (2.17) twice and using (2.18), we obtain $u(r,\alpha)\geqslant v_\alpha(r)$ on $[R(\alpha, \tilde{B}), R_*(B)]$. This proves the first inequality of (2.2).

Finaly, (2.3) follows by (2.2), (2.8), and (2.9). The proof is complete.

Remark 2.5. If the growth of f is critical, then $R(\alpha)$ may tend to 0 as $\alpha\to\infty$. Indeed, let us consider

$$f(u)=\begin{cases} n(n-2)\,u^{(n+2)/(n-2)} & \text{if } u\geqslant 1, \\ n(n-2) & \text{if } u\leqslant 1. \end{cases}$$

Then it is well known for any $\varepsilon\in(0, 1)$ that

$$U_\varepsilon(r)=\left\{\frac{\varepsilon}{\varepsilon^2+r^2}\right\}^{(n-2)/2}$$

is a solution of (1.6)–(1.8) for $U_\varepsilon(r)>1$. Note that $U_\varepsilon(0)=\varepsilon^{-(n-2)/2}\equiv\alpha$ which tends to ∞ as $\varepsilon\to 0^+$. Let $A=1$ in (H-1). Then it is easy to verify that

$$R_1(\alpha)^2=\varepsilon-\varepsilon^2,$$

and

$$-u'(R_1(\alpha), \alpha)=(n-2)(\varepsilon-\varepsilon^2)^{1/2}\,\varepsilon^{-1},$$

and so $\lim_{\alpha \to \infty} -u'(R_1(\alpha), \alpha)\, R_1(\alpha) = n-2$, which is the contrary of (2.10). Using (2.15), it is easy to see that $R(\alpha)$ behaves like $\alpha^{-1/n}$, which tends to 0 as $\alpha \to +\infty$.

An immediate consequence of the estimates of (2.2) and (2.3) is the following existence result for positive singular solutions of (1.1) and (1.2).

LEMMA 2.6. *Assume f satisfies* (H-1). *If $\{\alpha_k\}$ is a sequence with $\lim_{k \to \infty} \alpha_k = \infty$ and $\lim_{k \to \infty} R(\alpha_k) = R \leqslant \infty$, then there is a subsequence $\{\alpha'_k\}$* of *$\{\alpha_k\}$ and a nonnegative singular solution U such that $u(\cdot, \alpha'_k)$ converges to U pointwise in $(0, R)$ and also in $C^2([a, b])$ on every compact set $[a, b]$ in $(0, R)$. Moreover, if $f(0) \geqslant 0$ then U is positive in $(0, R)$, and if f satisfies* (H-2) *then U satisfies* (1.3).

Proof. We first claim that for any compact subinterval $[a, b] \subset (0, R)$, $u^{(j)}(\cdot, \alpha_k)$ are uniformly bounded on $[a, b]$, for $j = 0, 1, 2, 3$. Since f is superlinear at $u = \infty$, it is easy to see that

$$\lim_{B \to \infty} R^*(B) = 0. \tag{2.19}$$

Therefore, there is a $B \geqslant A$ such that $R^*(B) \leqslant a$. Hence, $R(\alpha_k, B) \leqslant a$ and

$$u(a, \alpha_k) \leqslant B \tag{2.20}$$

for large k. On the other hand, by the assumption that $R(\alpha_k) \to R$ as $k \to \infty$ we have $b < R(\alpha_k)$ for large k, i.e., $u(r, \alpha_k) \geqslant 0$ in $[a, b]$ for large k. We now claim that

$$u(r, \alpha_k) \leqslant B \qquad \text{on } [a, b] \tag{2.21}$$

for large k. Consider the energy function $V_\alpha(\cdot)$ along the solution $u(\cdot, \alpha)$ by

$$V_\alpha(r) = \tfrac{1}{2} u'^2(r, \alpha) + F(u(r, \alpha)). \tag{2.22}$$

Then

$$\frac{d}{dr} V_\alpha(r) = -\frac{n-1}{r} u'^2(r, \alpha) \leqslant 0, \tag{2.23}$$

i.e., $V_\alpha(r)$ is nonincreasing in $[a, b]$. If $u'(r, \alpha) > 0$ in $[a, b]$, then (2.20) implies (2.21). On the other hand, if there is $r_0 \in (a, b)$ such that $u'(r_0, \alpha) = 0$, then by (2.6) we may assume that $u(r_0, \alpha) \leqslant B$. By (2.22) and (2.23), we have

$$F(u(r, \alpha)) \leqslant F(u(r_0, \alpha)) = F(B) \tag{2.24}$$

on $[r_0, b]$. Since $F'(u)=f(u)>0$ on $[A, \infty)$, (2.24) implies that $u(r, \alpha) \leqslant B$ on $[r_0, b]$, i.e., (2.21) holds on $[a, b]$. Now for any solution $u(r)$ of (1.4) and $0<r_1<r_2<R$, $u(r)$ satisfies

$$-u'(r_2)\, r_2^{n-1} = -u'(r_1)\, r_1^{n-1} + \int_{r_1}^{r_2} t^{n-1} f(u(t))\, dt. \tag{2.25}$$

Therefore, let $r_1 = R(\alpha_k, B)$, $r_2 = r \in (R(\alpha_k, B), b]$, and $u = u(\cdot, \alpha_k)$ in (2.25), and using (2.21), we obtain

$$|u'(r, \alpha_k)| \leqslant -u'(R(\alpha_k, B), \alpha_k)\, R(\alpha_k, B) \cdot \left\{\frac{R(\alpha_k, B)}{r}\right\}^{n-2} \cdot \frac{1}{r} + \frac{M(B)}{n}\, r. \tag{2.26}$$

By (2.2), (2.3), and (2.26), there is a positive constant $C = C(a, b, B)$ such that

$$|u'(r, \alpha_k)| \leqslant C \tag{2.27}$$

on $[a, b]$ for large k. Hence, by (1.4), (2.21), and (2.27), we have shown that $u^{(j)}(\cdot, \alpha_k)$ are uniformly bounded on $[a, b]$, for $j = 0, 1, 2, 3$.

Now, using the Ascoli–Arzela theorem and the diagonal process, there is a subsequence $\{\alpha'_k\}$ of $\{\alpha_k\}$ and a C^2 nonnegative function $U(r)$ such that $u(\cdot, \alpha'_k)$ converges to U pointwise in $(0, R)$ and also in $C^2([a, b])$ for any compact subinterval $[a, b]$ of $(0, R)$.

It is clear that $U(r)$ satisfies (1.4) on $(0, R)$. Now we claim that (2.2) implies U tends to ∞ as $r \to 0^+$. For any $B > A$, if $r \leqslant R_*(B)$, then $u(r, \alpha'_k) \geqslant B$, which implies that $U(r) \geqslant B$ on $[0, R_*(B)]$. Since $R_*(B) \to 0$ as $B \to \infty$, we have $U(r) \to \infty$ as $r \to 0^+$. Therefore, $U(r)$ is a nonnegative singular solution on $(0, R)$.

Next, we claim that if $f(0) \geqslant 0$ then U is positive in $(0, R)$. If there is $\tilde{r} \in (0, R)$ such that $U(\tilde{r}) = 0$, then $U'(\tilde{r}) = 0$ since U is nonnegative in $(0, R)$. If $f(0) = 0$, then by the uniqueness of the initial value problem of the o.d.e., $U \equiv 0$ which is a contradiction. If $f(0) > 0$, then by (1.4), $U''(\tilde{r}) < 0$, which contradicts our knowledge that U is nonnegative. So we have proven that U is positive in $(0, R)$.

Finally, if $R < \infty$, then $u(R(\alpha_k), \alpha_k) = 0$ and $R(\alpha_k) \to R$ as $k \to \infty$ imply that $U(R) = 0$. If $R = \infty$, then the assumption that f is positive in $(0, \infty)$ will imply that $U(r)$ tends to 0 as $r \to \infty$. The details of this part of the proof are omitted.

The proof is complete.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{\alpha_k\}$ be a sequence with $\lim_{k\to\infty} \alpha_k = \infty$. Then either $\limsup_{k\to\infty} R(\alpha_k) = \infty$ or $R(\alpha_k)$ is bounded above. In the latter case, we may assume that $\lim_{k\to\infty} R(\alpha_k) = \tilde{R} < \infty$. Then by an argument such as that in Lemma 2.6, there is a singular solution U of (1.1) and (1.2) that is a limit of $u(\cdot, \alpha'_k)$ for some subsequence $\{\alpha'_k\}$ of $\{\alpha_k\}$, and U is positive in the neighborhood of $r = 0$. If $U(r)$ has zero in $(0, \infty)$, let R be the first zero; then U is a positive singular solution on the finite ball $B_R(0)$. Otherwise, $U(r)$ is a positive singular solution on $\mathbb{R}^n - \{0\}$.
The proof is complete.

Next, let

$$\bar{R} = \limsup_{\alpha\to\infty} R(\alpha). \tag{2.28}$$

Then Theorem 1.2 is a consequence of the following lemma.

LEMMA 2.7. *Assume f satisfies* (H-1)–(H-3). *Then $\bar{R} < \infty$.*

Proof. We first treat the cases (H-3)(i) and (ii). In these cases, there is an $\varepsilon > 0$ such that

$$f(u) \geqslant \varepsilon u \qquad \text{for all} \quad u \geqslant 0. \tag{2.29}$$

Consider the linear eigenvalue problem:

$$\Delta\phi + \lambda\phi = 0 \qquad \text{in } B_R, \tag{2.30}$$

$$\phi = 0 \qquad \text{on } \partial B_R. \tag{2.31}$$

Let $\lambda_R > 0$ be the first eigenvalue and let $\phi_R > 0$ be an associated eigenfunction of (2.30) and (2.31). It is easy to see that $R(\alpha) < \infty$ for all $\alpha > 0$. Let $R = R(\alpha)$ and $u = u(\cdot, \alpha)$. Then by using (1.3), (1.4), (2.30), and (2.31), we have

$$\int_{B_R} \phi_R\{f(u) - \lambda_R u\} = 0,$$

which implies

$$\inf\{f(u(x)) - \lambda_R u(x) : x \in B_R\} \leqslant 0.$$

Therefore, there is an $x_0 \in B_R$ that satisfies

$$f(u(x_0)) \leqslant \lambda_R u(x_0).$$

Hence, by (2.29), we obtain

$$\varepsilon \leqslant \lambda_R.$$

Since λ_R tends to 0 as $R \to \infty$ we have $\bar{R} < \infty$.

In the case of (H-3)(iii), by Theorem 2.2 of Ni and Serrin [29], (1.4) cannot have a positive solution in the exterior domain $\mathbb{R}^n - B_R$, for any $R > 0$. Hence, $R(\alpha) < \infty$ for any $\alpha > 0$. Since the same theorem also rules out the existence of a positive singular solution on $\mathbb{R}^n$, by Lemma 2.6 we have $\bar{R} < \infty$.

The proof is complete.

Proof of Theorem 1.2. The theorem follows from Lemmas 2.6 and 2.7.

Remark 2.8. In [17], Lin and Ni considered (1.4) with $f(u) = u^p + u^q$, where $(n+2)/(n-2) < q < (n+6)/(n-2)$ and $p = q/2$. In this case, $p \in (n/n-2,\ (n+2)/(n-2))$. They obtained an explicit positive entire solution in $\mathbb{R}^n$, and so there is an $\alpha \in (0, \infty)$ such that $R(\alpha) = \infty$. In this case, it is still unclear whether $\bar{R} < \infty$ (see also Merle and Peletier [23]).

Remark 2.9. Let Σ_b be the set of positive singular solutions of (1.4) on finite balls obtained by Theorem 1.2, i.e., $\Sigma_b = \{U(r) : U(r)$ is a limit of $u(\cdot, \alpha_k)$ with $R(\alpha_k) < \infty$, $\lim_{k \to \infty} \alpha_k = \infty$ and $\lim_{k \to \infty} R(\alpha_k) = R < \infty\}$. We may also study the initial value problem for backward shooting:

$$u''(r) + \frac{n-1}{r} u'(r) + f(u(r)) = 0 \qquad \text{in } (0, R),$$

$$u(R, \beta) = 0,$$

$$u'(R, \beta) = -\beta < 0.$$

Let $\Sigma_a(R) = \{u(\cdot, \beta) : u(0, \beta) = \infty$ and $u(r, \beta) > 0$ in $(0, R)\}$ and let $\Sigma_a = \bigcup \{\Sigma_a(R) : R \in (0, \infty)\}$, the set of all positive singular solutions on balls. In the case of $f(u) = u^p$ with $1 < p < (n+2)/(n-2)$, $\Sigma_b = \phi$ and $\Sigma_a(R) = \{u(\cdot, \beta) : 0 < \beta < \beta(R)\}$, for some $\beta(R) > 0$; see, e.g., Ni and Sack [26] and Lin [20]. It is clear that $\Sigma_b \subseteq \Sigma_a$. However, it is not clear whether $\Sigma_b = \Sigma_a$, i.e., whether all positive singular solutions on balls can be obtained as limits of positive regular solutions.

Proof of Theorem 1.3. As mentioned in the Introduction, $(u(\cdot), \lambda)$ is a solution of (1.7) and (1.8) if and only if $u(\cdot, \alpha)$ is a positive solution of (1.4)–(1.6) with $u(r) = u(r\lambda^{1/2}, \alpha)$ and $\lambda = R^2(\alpha) < \infty$. Therefore, it suffices to study $R(\alpha)$ for $\alpha \in (0, \infty)$.

It is clear that $R(\alpha) > 0$ for all $\alpha \in (0, \infty)$. It is also easy to see that if $\alpha_k \to \alpha_0 \in (0, \infty)$ then $R(\alpha_0) > 0$. Hence, by Theorem 2.4, the only possibility for the case where $R(\alpha)$ tends to 0 is $\alpha \to 0^+$. We shall rule out this possibility by considering the following four cases: (i) $f(0)=0$, $f'(0)>0$; (ii) $f(0)=0=f'(0)$; (iii) $f(0)=0$ and $f'(0)<0$; and (iv) $f(0)<0$. For the case where $f(0)>0$, it is well known that a positive minimum solution u_λ will tend to zero uniformly on Ω as $\lambda \to 0^+$, i.e., $R(\alpha)\to 0$ as $\alpha \to 0^+$; see, e.g., [16]. Therefore, $u(\cdot, \alpha)$ is a positive minimum solution if α is sufficiently small.

Case (i). In this case, we may assume that $f'(0)=1$, and we shall prove that $\lim_{\alpha\to\infty} R(\alpha)^2=\lambda_1$. By a bifurcation theorem of Crandall and Rabinowitz [8], there is a unique branch of bifurcation solutions $u_\lambda>0$ bifurcating from the trivial solution $u_0\equiv 0$ at $\lambda=\lambda_1$ and $\|u_\lambda\|_\infty \to 0$ as $\lambda\to\lambda_1$. Therefore, $R(\alpha)\to\lambda_1^{1/2}$ as $\alpha\to 0^+$.

Case (ii). In this case, we shall prove that $\lim_{\alpha\to 0^+} R(\alpha)=\infty$. We observe that $u(\cdot, \alpha)$ satisfies the following equation:

$$u(r, \alpha)=\alpha-\frac{1}{n-2}\int_0^r (s^{2-n}-r^{2-n})\, s^{n-1}f(u(s, \alpha))\, ds. \tag{2.32}$$

Since $f(0)=f'(0)=0$, for any $\varepsilon>0$ there exists $\delta>0$ such that $|f(u)|\leqslant \varepsilon u$ for $u\in(0, \delta)$. Therefore, if $u(r, \alpha)\in(0, 2\alpha)\subset(0, \delta)$ then $|f(u(r, \alpha))|\leqslant 2\varepsilon\alpha$. Now, it is easy to verify that

$$\left|\frac{1}{n-2}\int_0^r (s^{2-n}-r^{2-n})\, s^{n-1}f(u(s, \alpha))\, ds\right|\leqslant\frac{2\varepsilon}{n}\,\alpha r^2 \tag{2.33}$$

as far as $u(s, \alpha)\in(0, 2\alpha)$ for all $s\in(0, r)$. Hence, by (2.32) and (2.33), for any $\alpha\in(0, \delta/2)$ and $r\in(0, (n/2\varepsilon)^{1/2})$, we have $u(r, \alpha)\in(0, 2\alpha)$. This implies $\lim_{\alpha\to 0^+} R(\alpha)=\infty$.

Case (iii). In this case, there are positive constants m and δ such that $-mu\leqslant f(u)\leqslant 0$ on $[0, \delta]$. Therefore, if $u(s, \alpha)\in[0, \delta]$ for all $s\in(0, r)$, then by (2.32), we have

$$\begin{aligned} u(r, \alpha) &\leqslant \alpha+\frac{m}{n-2}\int_0^r (s^{2-n}-r^{2-n})\, s^{n-1}u(s, \alpha)\, ds \\ &\leqslant \alpha+\frac{m}{n}\, u(r, \alpha)\, r^2. \end{aligned} \tag{2.34}$$

Hence, if $u(R(\alpha, \delta), \alpha)=\delta$, then (2.34) implies that $R(\alpha, \delta)^2\geqslant (1/\delta)\{(\delta-\alpha)(n/m)\}$ and so $R(\alpha)$ has a positive lower bound for $\alpha\in(0, \delta/2)$.

Case (iv). In this case, there are $\varepsilon>0$ and $\delta>0$ such that $f(u)\leqslant -\varepsilon$ on $[0, \delta]$. Let $\tilde{C}=-\varepsilon$ in (2.12), $R(\alpha, \tilde{B})=0$, and $\tilde{B}=\alpha$ in (2.13), and

$u'(0, \alpha)=0$ in (2.14). Then (2.15) becomes $v_\alpha(r)=\alpha+(\varepsilon/2n)\,r^2$, which implies that

$$u(r, \alpha) \geqslant \alpha+\frac{\varepsilon}{2n} r^2 \tag{2.35}$$

as long as $u(r, \alpha) \in [0, \delta]$. In particular, $R(0)>0$. The continuous dependence of $u(\cdot, \alpha)$ in α and (2.35) imply there is a positive lower bound for $R(\alpha)$ for all $\alpha \in [0, \delta]$.

The proof is complete.

3. Finite Energy of Singular Solutions

In this section we first prove that the energy of positive regular solutions of (1.7) and (1.8) on the unit ball is uniformly bounded. Since the positive singular solutions in Σ_b are obtained as limits of positive regular solutions, it is reasonable to expect that Σ_b-type positive singular solutions have finite energy. Indeed, we shall prove that this is true for all positive singular solutions. Furthermore, we shall show that if u_λ is a sequence of positive regular solutions of (1.7) and (1.8) that converges to a positive singular solution U in (0, 1), then u_λ converges to U in $L^{q+1}(B_1)$ and $H_0^1(B_1)$ as well, which is a generalization of the result in Merle and Peletier [23].

We first give an energy estimate for positive regular solutions of (1.7) and (1.8) on a star-shaped domain Ω; see also [11].

LEMMA 3.1. *Assume f satisfies* (H-1). *Let Ω be a star-shaped domain. Then there is a positive constant M such that for any positive solution (u, λ) of* (1.7) *and* (1.8), *we have*

$$\int_\Omega |\nabla u|^2=\lambda \int_\Omega u f(u) \leqslant \lambda M, \tag{3.1}$$

and

$$\int_\Omega F(u) \leqslant M. \tag{3.2}$$

Proof. We may assume that Ω is a star-shaped domain with respect to the origin, i.e., $x \cdot v \geqslant 0$ for any $x \in \partial\Omega$, where $v=v(x)$ is the unit outward normal at x. If u is a solution of (1.7) and (1.8), then the following Pohozaev identity holds:

$$\lambda \int_\Omega nF(u)-\frac{n-2}{2} u f(u)=\frac{1}{2} \int_{\partial\Omega} (x \cdot v)\,|\nabla u|^2. \tag{3.3}$$

Since Ω is a star-shaped domain, the right-hand side of (3.3) is nonnegative, and so

$$\int_{\Omega} nF(u) - \frac{n-2}{2} uf(u) \geqslant 0. \tag{3.4}$$

Let $\Omega_A = \{x \in \Omega : u(x) \geqslant A\}$ and $\Omega'_A = \Omega - \Omega_A$. Let $\delta = (n-2)/2 - n/(q+1) > 0$. By (H-1), we have

$$\frac{n-2}{2} uf(u) - nF(u) \geqslant \delta uf(u) \qquad \text{for all} \quad u \geqslant A. \tag{3.5}$$

Let

$$M_1 = \max\left\{nF(u) - \frac{n-2}{2} uf(u) : u \in [0, A]\right\}, \tag{3.6}$$

and

$$M_2 = \max\{f(u) : u \in [0, A]\}. \tag{3.7}$$

Note that (3.3) implies that $M_1 > 0$ if (1.7) and (1.8) have a positive solution. Hence, by (3.4)–(3.7) we have

$$\begin{aligned}\delta \int_{\Omega_A} uf(u) &\leqslant \delta \int_{\Omega_A} uf(u) + \int_{\Omega} nF(u) - \frac{n-2}{2} uf(u)\\ &= \int_{\Omega'_A} nF(u) - \frac{n-2}{2} uf(u) - \int_{\Omega_A} \left(\frac{n-2}{2} - \delta\right) uf(u) - nF(u)\\ &\leqslant \int_{\Omega'_A} nF(u) - \frac{n-2}{2} uf(u) \leqslant M_1 \, |\Omega'_A|,\end{aligned}$$

and

$$\int_{\Omega'_A} uf(u) \leqslant AM_2 \, |\Omega'_A|,$$

where $|G|$ is the volume of domain G. The last two inequalities imply that

$$\int_{\Omega} uf(u) \leqslant (M_1 \delta^{-1} + AM_2) \, |\Omega'_A|. \tag{3.8}$$

Hence, Lemma 3.1 follows from (3.8) and (3.5). The proof is complete.

Next we need to recall two results from Ni and Serrin [27]. We first study the asymptotic behaviour of positive solutions when $r \to 0^+$.

PROPOSITION NS1. *Assume f satisfies* (H-1). *Then there is a positive constant C such that for any positive radial solution u_λ of* (1.7) *and* (1.8) *in* (0, 1) *with $\|u_\lambda\|_\infty > A$, we have*

$$u_\lambda(r) \leqslant C\lambda^{-1/(q-1)} r^{-2/(q-1)} \qquad \text{in } (0, r_0), \tag{3.9}$$

where $u_\lambda(r_0) = A$.,

Proof. The proof is a slight modification of Theorem 2.1 in [27]; here we emphasize that C is independent of u_λ. Since f satisfies (H-1), it is easy to check that there is a positive constant η such that

$$f(u) \geqslant \eta u^q \qquad \text{for all} \quad u \geqslant A. \tag{3.10}$$

Since (3.7) can be written as

$$(r^{n-1}u'(r))' = -\lambda r^{n-1} f(u(r)), \tag{3.11}$$

we have

$$u'_\lambda(r) < 0 \qquad \text{in } (0, r_0). \tag{3.12}$$

Now, for any $r \in (0, r_0)$ and $\bar{r} \in (0, r)$, if we integrate (3.11) from $\bar{r}$ to r, we obtain

$$r^{n-1}u'_\lambda(r) = \bar{r}^{n-1}u'_\lambda(\bar{r}) - \lambda \int_{\bar{r}}^{r} s^{n-1} f(u_\lambda(s))\, ds.$$

Therefore, by (3.12), we have

$$r^{n-1}u'_\lambda(r) < -\lambda \int_{\bar{r}}^{r} s^{n-1} f(u_\lambda(s))\, ds$$

for all $\bar{r} \in (0, r)$. Letting $\bar{r} \to 0$ and using (3.10) and (3.12) we obtain

$$r^{n-1}u'_\lambda(r) < -\lambda\eta \int_0^r s^{n-1} u_\lambda(s)^q\, ds < -\lambda\eta u_\lambda^q(r) \int_0^r s^{n-1}\, ds = -\lambda C_1 u_\lambda^q(r)\, r^n,$$

where $C_1 = \eta/n$. Dividing the last inequality by $u_\lambda^q(r)\, r^{n-1}$, we have

$$u_\lambda^{-q}(r)\, u'_\lambda(r) \leqslant -\lambda C_1 r \qquad \text{in } (0, r_0).$$

Integrating from 0 to r, we obtain

$$u^{1-q}(r) \geqslant \lambda C_2 r^2 \qquad \text{in } (0, r_0),$$

where $C_2 = (q-1) C_1/2$. Hence (3.9) follows, with $C = C_2^{-1/(q-1)}$. The proof is complete.

Next, we need another energy estimate, as follows.

PROPOSITION NS2. *Assume f satisfies* (H-1). *If $U(r)$ is a positive singular solution of* (1.7) *and* (1.8), *then*

$$\frac{1}{2} U'^2(r) + \lambda F(U(r)) + \frac{n}{q+1} \frac{U(r)\, U'(r)}{r} < 0, \tag{3.13}$$

as long as $U(s) > A$ in $(0, r)$.

This proposition is essentially proved by Lemmas 3.2, 3.3, and 3.4 of [27], so its proof is omitted here.

THEOREM 3.2. *Assume f satisfies* (H-1). *If U is a positive singular solution of* (1.7) *and* (1.8), *then*

$$\int_{B_1} |\nabla U|^2 = \lambda \int_{B_1} Uf(U) < \infty. \tag{3.14}$$

Proof. Let $R_1 \in (0, 1)$ such that $U(R_1) = A$. By (3.13), we have

$$-U'(r) < \frac{2n}{q+1} \frac{U(r)}{r} \tag{3.15}$$

in $(0, R_1)$. Now, substituting (3.9) into (3.15), we obtain

$$-U'(r) \leqslant C_1 r^{-2/(q+1)-1} \tag{3.16}$$

in $(0, R_1)$, where $C_1 = (2n/(q+1))C$.

For any $\varepsilon \in (0, 1)$, let $\Omega_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$ and $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$. Then

$$\int_{\Omega_\varepsilon} U \Delta U + \lambda \int_{\Omega_\varepsilon} Uf(U) = 0$$

implies

$$\int_{\partial\Omega_\varepsilon} U \frac{\partial U}{\partial \nu} - \int_{\Omega_\varepsilon} |\nabla U|^2 + \lambda \int_{\Omega_\varepsilon} Uf(U) = 0. \tag{3.17}$$

Hence, by (1.3), (3.9), and (3.16), we have

$$0 \leqslant \int_{\partial\Omega_\varepsilon} U\frac{\partial U}{\partial \nu} = -\int_{\partial B_\varepsilon} UU' \leqslant C_2\varepsilon^{n-2-4/(q-1)},$$

where $C_2 = C_1\omega_n$. Since $n-2-4/(q-1) = (1/(q-1))\{(n-2)q-(n+2)\} > 0$, we have

$$\int_{\partial\Omega_\varepsilon} U\frac{\partial U}{\partial \nu} \to 0 \qquad \text{as} \quad \varepsilon \to 0^+.$$

Similarly, (3.16) implies that for all $\varepsilon \in (0, R_1)$,

$$\int_{\Omega_\varepsilon} |\nabla U|^2 \leqslant C_3 \qquad \text{for some} \quad C_3 > 0.$$

Hence, (3.14) holds.
The proof is complete.

THEOREM 3.3. *Assume f satisfies* (H-1). *If u_λ is a sequence of positive regular solutions of* (1.7) *and* (1.8) *that converges pointwise to a positive singular solution U as $\lambda \to \lambda_0 \in (0, \infty)$, then u_λ converges to U in $L^{q+1}(B_1)$ and $H_0^1(B_1)$ as $\lambda \to \lambda_0$.*

Proof. Choose $\varepsilon > 0$ such that $u_\lambda(\varepsilon) \geqslant A$, which is guaranteed by (2.2). Then

$$\int_{B_1} |u_\lambda - U|^{q+1} = \int_{B_\varepsilon} |u_\lambda - U|^{q+1} + \int_{\Omega_\varepsilon} |u_\lambda - U|^{q+1} = I_\lambda + I'_\lambda.$$

By the argument in Lemma 2.6, u_λ converges to U uniformly in $[\varepsilon, 1]$ as $\lambda \to \lambda_0$, and so $I'_\lambda \to 0$ as $\lambda \to \lambda_0$. On the other hand, by (3.9)

$$I_\lambda \leqslant C_1 \int_0^\varepsilon r^{\nu-1}\, dr \leqslant C_2\varepsilon^\nu$$

where

$$\nu = n - 2(q+1)/q - 1 = \frac{1}{q-1}\{q(n-2)-(n+2)\} > 0,$$

and the C_i's are positive constants dependent on C, n, q, and λ_0, but not on λ. Therefore

$$\limsup_{\lambda \to \lambda_0} \int_{B_1} |u_\lambda - U|^{q+1} \leqslant C_3\varepsilon^\nu.$$

Since ε can be chosen to be arbitrarily small, we conclude that $u_\lambda \to U$ as $\lambda \to \lambda_0$ in $L^{q+1}(B_1)$. To use (2.1), (3.13), and (3.15) to prove that $u_\lambda \to U$ as $\lambda \to \lambda_0$ in $H^1(B_1)$, we proceed in the same manner; the details of the proof are omitted here.

The proof is complete.

Proof of Theorem 1.4. By Lemma 2.7, we have $R(\alpha) < \infty$ for all $\alpha > 0$. Therefore the solution set of (1.7) and (1.8) can be written as $\{(u(\cdot, \alpha), \lambda(\alpha)) : \alpha \in (0, \infty)\}$ with $\lambda(\alpha) = R^2(\alpha)$. Hence S is a C^1-smooth, connected 1-manifold. (For the case of (H-3)(i), see also Dancer [10].) The remaining parts of Theorem 1.4 follow easily from Theorems 1.1 and 3.3.

The proof is complete.

ACKNOWLEDGMENTS

This research was supported in part by the Institute for Mathematics and Its Applications (IMA) during the author's visit there in April 1991. The author is grateful for the hospitality of the IMA and for several useful conversations with Prof. Wei-Ming Ni.

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