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Brief paper

# Analysis of SDC matrices for successfully implementing the SDRE scheme ☆

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ABSTRACT

The state-dependent Riccati equation (SDRE) approach for stabilization of nonlinear affine systems was recently reported to be effective in many practical applications; however, there is no guideline on the construction of state-dependent coefficient (SDC) matrix when the SDRE solvability condition is violated, which may result in the SDRE scheme being terminated. In this study, we present several easy checking conditions so that the SDRE scheme can be successfully implemented. Additionally, when the presented checking conditions are satisfied, the sets of all feasible SDC matrices and their structures are explicitly depicted for the planar system.

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## 1. Introduction

Recently, the state-dependent Riccati equation (SDRE) approach for nonlinear system stabilization has attracted considerable attention (Bogdanov & Wan, 2007; Bracci, Innocenti, & Pollini, 2006; Çimen, 2010; Cloutier, D'Souza, & Mracek, 1996; Erdem & Alleyne, 2004; Hammett, Hall, & Ridgely, 1998; Lam, Xin, & Cloutier, 2012; Liang & Lin, 2011; Shamma & Cloutier, 2003; Sznaier, Cloutier, Hull, Jacques, & Mracek, 2000). The SDRE scheme is known to include the following benefits (Çimen, 2010): (i) the concept is intuitive and simple, and directly adopts the LQR design at every nonzero state; (ii) the design can directly affect system performance with predictable results by adjusting the state and the control weightings to specify the performance index (for instance, the engineer may modulate the weighting of the system state to speed up the response, although at the expense of increased control effort); (iii) the scheme possesses an extra design degree of freedom arising from the non-unique state-dependent coefficient (SDC) matrix representation of the nonlinear drift term, which can be utilized to enhance controller performance; and (iv) the approach preserves the essential system nonlinearities because it does not truncate any nonlinear terms. Many practical and meaningful applications successfully performed by the SDRE design have been reported (see Çimen, 2010 and the references therein). The first solid theoretical contributions on SDRE control have been provided by Cloutier et al. (1996) and Mracek and Cloutier (1998). The current study attempts to provide further theoretical support of the SDRE control strategy, as discussed in the recent survey by Çimen (2012), with rigorous mathematical proofs.

The SDRE design for nonlinear systems can be described as follows. Consider a class of nonlinear control systems and a quadratic-like performance index as (1)–(2) below:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + B(\mathbf{x})\mathbf{u} \tag{1}$$

$$\text{and} \quad J = \frac{1}{2}\int_0^\infty \left\{\mathbf{x}^T Q(\mathbf{x})\mathbf{x} + \mathbf{u}^T R(\mathbf{x})\mathbf{u}\right\} dt \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^p$ denote the system states and control inputs, respectively, $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$, $B(\mathbf{x}) \in \mathbb{R}^{n\times p}$, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $Q^T(\mathbf{x}) = Q(\mathbf{x}) \geq 0$, $R^T(\mathbf{x}) = R(\mathbf{x}) > 0$, $Q(\mathbf{x}), R(\mathbf{x}) \in C^k$, $k \geq 1$, and $(\cdot)^T$ denotes the transpose of a vector or a matrix. Note that the weighting matrices $Q(\mathbf{x})$ and $R(\mathbf{x})$ are in general state-dependent. The procedure of the SDRE scheme is summarized as the following three steps (Çimen, 2010):

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(i) Factorize $\mathbf{f}(\mathbf{x})$ into the SDC matrix representation as $\mathbf{f}(\mathbf{x}) = A(\mathbf{x})\mathbf{x}$, where $A(\mathbf{x}) \in \mathbb{R}^{n\times n}$.
(ii) Symbolically check the stabilizability of $(A(\mathbf{x}), B(\mathbf{x}))$ and the observability (resp., detectability) of $(A(\mathbf{x}), C(\mathbf{x}))$ to ensure the existence of a unique positive definite (resp., semi-definite) solution of the following SDRE:

$$A^T(\mathbf{x})P(\mathbf{x}) + P(\mathbf{x})A(\mathbf{x}) + Q(\mathbf{x}) - P(\mathbf{x})B(\mathbf{x})R^{-1}(\mathbf{x})B^T(\mathbf{x})P(\mathbf{x}) = 0 \quad (3)$$

where $C(\mathbf{x}) \in \mathbb{R}^{q\times n}$ has full rank and satisfies $Q(\mathbf{x}) = C^T(\mathbf{x})C(\mathbf{x})$.
(iii) Solve for $P(\mathbf{x})$ from (3) to produce the SDRE controller

$$\mathbf{u} = -K(\mathbf{x})\mathbf{x} \quad \text{and} \quad K(\mathbf{x}) = R^{-1}(\mathbf{x})B^T(\mathbf{x})P(\mathbf{x}). \quad (4)$$

It should be noted that the SDRE scheme is performed pointwise in $\mathbf{x}$ and the resulting closed-loop SDC matrix $A_{CL}(\mathbf{x}) := A(\mathbf{x}) - B(\mathbf{x})R^{-1}(\mathbf{x})B^T(\mathbf{x})P(\mathbf{x})$ is pointwise Hurwitz everywhere; however, it does not imply global stability of the origin (Tsiotras, Corless, & Rotea, 1996). In addition, though the SDRE approach provides satisfactory performance in many practical applications, the symbolic checking conditions stated in (ii) of the SDRE scheme are generally not easy to implement, especially when the system dynamics are complicated. Moreover, several authors have provided various guidelines on how to systematically construct SDC matrices (Çimen, 2010; Cloutier et al., 1996); however, there is no guideline on the construction of SDC matrices when the SDRE solvability condition is violated, which may result in the SDRE scheme being terminated. For instance, let $\mathbf{f}(\mathbf{x}) = [-x_2, x_1]^T$, $B(\mathbf{x}) = [0, x_2]^T$, $R(\mathbf{x}) = 1$ and $Q(\mathbf{x}) = I_2$. Suppose that an SDC matrix representation is given as $a_{11}(\mathbf{x}) = a_{22}(\mathbf{x}) = 0$, $a_{12}(\mathbf{x}) = -1$ and $a_{21}(\mathbf{x}) = 1$, where $a_{ij}(\mathbf{x})$ denotes the $(i, j)$-entry of the matrix $A(\mathbf{x})$. Then, $(A(\mathbf{x}), C(\mathbf{x}))$ is always observable, but $(A(\mathbf{x}), B(\mathbf{x}))$ is not stabilizable at the nonzero states where $x_2 = 0$. By direct calculation, the SDRE given by (3) does not have any positive semi-definite solution $P(\mathbf{x})$ when $x_2 = 0$, in which case the SDRE scheme will fail to operate. However, it will become clear later (see Theorem 1) that, at those nonzero states $\mathbf{x}$ of $x_2 = 0$, there always exists a feasible SDC matrix representation that makes the SDRE (3) solvable and the resulting $A_{CL}(\mathbf{x})$ matrix a Hurwitz matrix.

It is known that a unique positive definite (resp., semi-definite) solution $P(\mathbf{x})$ in (3) exists, rendering $A_{CL}(\mathbf{x})$ pointwise Hurwitz, if (resp., if and only if) both the conditions "$(A(\mathbf{x}), B(\mathbf{x}))$ is stabilizable" and "$(A(\mathbf{x}), C(\mathbf{x}))$ is observable (resp., has no unobservable mode on the $j\omega$-axis)" are satisfied (Zhou & Doyle, 1998). To avoid the difficulty of symbolic checking conditions, stated above, of the SDRE approach, in this article we will study the following three problems:

**Problem 1.** Let $\mathbf{x} \neq 0$ be given. Denote $\mathbf{f} = \mathbf{f}(\mathbf{x})$, $B = B(\mathbf{x})$ and $C = C(\mathbf{x})$. Explore the existence condition and, if the existence condition is satisfied, present all $A \in \mathbb{R}^{n\times n}$ that satisfy the conditions that $A\mathbf{x} = \mathbf{f}$, $(A, B)$ is stabilizable and $(A, C)$ is observable.

**Problem 2.** Same as Problem 1, except that the condition "$(A, C)$ is observable" is replaced with "$(A, C)$ is detectable".

**Problem 3.** Same as Problem 1, except that the condition "$(A, C)$ is observable" is replaced with "$(A, C)$ has no unobservable mode on the $j\omega$-axis".

From the discussions above, this study may also provide an auxiliary means to successfully continue the SDRE scheme at states in which a specific SDC matrix representation fails to operate, but where Problems 1, 2 or 3 is solvable.

To explore the existence condition of Problems 1–3 and characterize their solution matrices, we introduce the notations $W^{\perp}$ and $W_{\perp}$ as follows. Let $W \in \mathbb{R}^{p\times n}$ be given with $p < n$ and $\mathrm{rank}(W) = p$. We define $W^{\perp} = N(W)$, null space of $W$, and $W_{\perp} \in \mathbb{R}^{n\times(n-p)}$ as a selected constant matrix having orthonormal columns and satisfying $WW_{\perp} = 0$. Clearly, $W^{\perp}$ is a vector space of dimension $n - p$, and the column vectors of $W_{\perp}$ form an orthonormal basis of $W^{\perp}$. Similarly, if $W \in \mathbb{R}^{n\times q}$ and $\mathrm{rank}(W) = q < n$, we define $W^{\perp} = \{\mathbf{w}^T \mid \mathbf{w} \in N(W^T)\}$ and $W_{\perp} \in \mathbb{R}^{(n-q)\times n}$ as a selected constant matrix having orthonormal rows and satisfying $W_{\perp}W = 0$. Additionally, we denote $\mathbb{R}^{n*} = \{\mathbf{x}^T | \mathbf{x} \in \mathbb{R}^n\}$, known as the dual space of $\mathbb{R}^n$, and $\mathbb{R}^-$ as the set of negative real numbers.

The rest of this article is organized as follows: Section 2 presents the necessary and sufficient existence conditions for Problems 1–3; Section 3 includes a description of the parameterization of the solution matrices $A$ for the planar case when the existence conditions are satisfied; Section 4 presents an illustrative example; and Section 5 provides the conclusions.

## 2. Necessary and sufficient existence conditions

Necessary and sufficient existence conditions for Problems 1–3 are stated as Theorem 1 below:

**Theorem 1.**
(i) *Problem 1 is unsolvable if and only if $\{\mathbf{x}, \mathbf{f}\}$ are linearly dependent (LD) and $C\mathbf{x} = 0$.*
(ii) *Problem 2 is unsolvable if and only if $\mathbf{f} = k\mathbf{x}$ for some $k \geq 0$ and $C\mathbf{x} = \mathbf{0}$.*
(iii) *Problem 3 is unsolvable if and only if $\mathbf{f} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$.*

**Proof.** The proofs of (i) and (ii) can be found from Liang and Lin (2011), while (iii) is easily derived from the proof of (ii). Details are omitted. □

## 3. Parameterization of all solution matrices

Given that the existence condition of Problems 1, 2 or 3 is satisfied, this section explores their solution matrices. To this end, we denote $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$ and $\mathcal{A}^i$ as the sets of $A$ such that $A\mathbf{x} = \mathbf{f}$, $(A, B)$ is controllable, $(A, B)$ is stabilizable, $(A, C)$ is observable, $(A, C)$ is detectable and $(A, C)$ has no unobservable mode on the $j\omega$-axis, respectively. Additionally, we assume hereafter that, without loss of any generality, both $B$ and $C$ have full rank.

### *3.1. The solution matrices of Problems 1–3*

Define $A_p = \frac{1}{\|\mathbf{x}\|^2}\mathbf{f}\mathbf{x}^T$. It is clear that $A_p\mathbf{x} = \mathbf{f}$ and

$$\mathcal{A}_{\mathbf{xf}} = \left\{A_p + K\mathbf{x}_{\perp} \mid K \in \mathbb{R}^{n\times(n-1)}\right\} \subset \mathbb{R}^{n\times n}. \quad (5)$$

Obviously, $\mathcal{A}_{\mathbf{xf}}$ is a linear variety (i.e., a subspace through a translation) of dimension $n^2 - n$ and $K$ describes the $n^2 - n$ free parameters. Additionally, $A_p$ has the minimum Frobenius norm among the matrices in $\mathcal{A}_{\mathbf{xf}}$. To derive $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$ and $\mathcal{A}^i$, we present the following two results which can be used to reduce the dimension of checking the system's controllability, stabilizability, observability and detectability.

**Lemma 2** (Chen, 1999). *Let $\bar{A} = \begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}$ and $\bar{B} = \begin{bmatrix}0\\ \bar{B}_2\end{bmatrix}$, where $\bar{B}_2 \in \mathbb{R}^{p\times p}$ is a nonsingular matrix, $\bar{A}_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $\bar{A}_{22} \in \mathbb{R}^{p\times p}$. Then, $(\bar{A}, \bar{B})$ is controllable (resp., stabilizable) $\Leftrightarrow$ $(\bar{A}_{11}, \bar{A}_{12})$ is controllable (resp., stabilizable). In particular, when $p < n$ and $\bar{A}_{12} = 0$, then $(\bar{A}, \bar{B})$ is uncontrollable, and it is stabilizable $\Leftrightarrow \lambda(\bar{A}_{11}) \subset \mathbb{C}^-$.*

**Corollary 3.** *Let $\bar{A}$ be partitioned in the form given by Lemma 2 with $\bar{A}_{11} \in \mathbb{R}^{(n-q)\times(n-q)}$ and $\bar{A}_{22} \in \mathbb{R}^{q\times q}$. $\bar{C} = [0, \bar{C}_2]$, where $\bar{C}_2 \in \mathbb{R}^{q\times q}$ is a nonsingular matrix. Then*
(i) *$(\bar{A}, \bar{C})$ is observable (resp., detectable) $\Leftrightarrow$ $(\bar{A}_{11}, \bar{A}_{21})$ is observable (resp., detectable).*
(ii) *$(\bar{A}, \bar{C})$ has no unobservable mode on the $j\omega$-axis $\Leftrightarrow$ $(\bar{A}_{11}, \bar{A}_{21})$ has no unobservable mode on the $j\omega$-axis.*

*In particular, when $q < n$ and $\bar{A}_{21} = 0$, then $(\bar{A}, \bar{C})$ is unobservable and*

(iii) *$(\bar{A}, \bar{C})$ is detectable $\Leftrightarrow \lambda(\bar{A}_{11}) \subset \mathbb{C}^-$.*
(iv) *$(\bar{A}, \bar{C})$ has no unobservable mode on the $j\omega$-axis $\Leftrightarrow$ $\bar{A}_{11}$ has no eigenvalue on the $j\omega$-axis.*

To apply Lemma 2 and Corollary 3, we have to transform $(A, B)$ (resp., $(A, C)$) into the form of $(\bar{A}, \bar{B})$ (resp., $(\bar{A}, \bar{C})$) as stated in Lemma 2 (resp., Corollary 3). Such coordinate transformation can be chosen to be orthogonal as in the form of (6) below:

$$\mathbf{x} = M_B\bar{\mathbf{x}} \quad (\text{resp., } \mathbf{x} = M_C\bar{\mathbf{x}}) \tag{6}$$

where $M_B$ and $M_C$ are orthogonal matrices. A candidate of $M_B$ (resp., $M_C$) can be determined by the QR factorization scheme for $B$ (resp., $C^T$) and then interchanges the position of the first $p$ (resp., $q$) columns with the last $n-p$ (resp., $n-q$) columns.

Under the coordinate transformation given by Eq. (6) we have $\bar{\mathbf{x}} = M^T\mathbf{x}$ and $\bar{A} = M^TAM$, where $M = M_B$ or $M = M_C$. If we let $\bar{\mathbf{f}} = M^T\mathbf{f}$, $\bar{\mathbf{x}}_\perp = \mathbf{x}_\perp M$ and $\bar{K} = M^TK$, then $\bar{\mathbf{x}}_\perp\bar{\mathbf{x}} = \mathbf{0}$, $A\mathbf{x} = \mathbf{f} \Leftrightarrow \bar{A}\bar{\mathbf{x}} = \bar{\mathbf{f}}$, and $\frac{1}{\bar{\mathbf{x}}^T\bar{\mathbf{x}}}\bar{\mathbf{f}}\bar{\mathbf{x}}^T + \bar{K}\bar{\mathbf{x}}_\perp = M^T\left[\frac{1}{\mathbf{x}^T\mathbf{x}}\mathbf{f}\mathbf{x}^T + K\mathbf{x}_\perp\right]M$. That is, $\bar{A} \in \mathcal{A}_{\bar{\mathbf{x}}\bar{\mathbf{f}}} \Leftrightarrow A \in \mathcal{A}_{\mathbf{xf}}$. Moreover, because controllability, observability, stabilizability and detectability are invariant under equivalence transformation (Chen, 1999), we obtain the following theorem.

**Theorem 4.** *Let $M_B$ (resp., $M_C$) be an orthogonal matrix given by Eq. (6) such that $B = M_B\bar{B}$ (resp., $C^T = M_C\bar{C}^T$), $\bar{B}$ (resp., $\bar{C}$) is given by Lemma 2 (resp., Corollary 3), $\bar{\mathbf{x}}_\perp = \mathbf{x}_\perp M_B$ and $K = M_B\bar{K}$ (resp., $\bar{\mathbf{x}}_\perp = \mathbf{x}_\perp M_C$ and $K = M_C\bar{K}$). Additionally, $A = \frac{1}{\|\mathbf{x}\|^2}\mathbf{f}\mathbf{x}^T + K\mathbf{x}_\perp \in \mathcal{A}_{\mathbf{xf}}$ and $\bar{A} = \frac{1}{\|\bar{\mathbf{x}}\|^2}\bar{\mathbf{f}}\bar{\mathbf{x}}^T + \bar{K}\bar{\mathbf{x}}_\perp \in \mathcal{A}_{\bar{\mathbf{x}}\bar{\mathbf{f}}}$. Then*

(i) *$(A, B)$ is controllable (resp., $(A, C)$ is observable) $\Leftrightarrow$ $(\bar{A}, \bar{B})$ is controllable (resp., $(\bar{A}, \bar{C})$ is observable).*
(ii) *$(A, B)$ is stabilizable (resp., $(A, C)$ is detectable) $\Leftrightarrow$ $(\bar{A}, \bar{B})$ is stabilizable (resp., $(\bar{A}, \bar{C})$ is detectable).*
(iii) *$(A, C)$ has no unobservable mode on the $j\omega$-axis $\Leftrightarrow$ $(\bar{A}, \bar{C})$ has no unobservable mode on the $j\omega$-axis.*

After deriving the sets $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$ and $\mathcal{A}^i$, it is clear that the solutions of Problems 1–3 are $\mathcal{A}^{so}_{\mathbf{xf}} := \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^s \cap \mathcal{A}^o$, $\mathcal{A}^{sd}_{\mathbf{xf}} := \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^s \cap \mathcal{A}^d$ and $\mathcal{A}^{si}_{\mathbf{xf}} := \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^s \cap \mathcal{A}^i$, respectively.

### *3.2. Implementation of the case $n = 2$*

The case of $n = 1$ is trivial; therefore, we only consider the case of $n = 2$. When $\operatorname{rank}(B) = 2$ (resp., $\operatorname{rank}(C) = 2$), $(A, B)$ (resp., $(A, C)$) is controllable (resp., observable) and $\mathcal{A}^c_{\mathbf{xf}} = \mathcal{A}^s_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$ (resp., $\mathcal{A}^o_{\mathbf{xf}} = \mathcal{A}^d_{\mathbf{xf}} = \mathcal{A}^i_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$). Remaining to be considered is the case of $B = \mathbf{b} = (b_1, b_2)^T \in \mathbb{R}^2$ and $C = \mathbf{c} = (c_1, c_2) \in \mathbb{R}^{1\times 2}$. In this case, $K = \mathbf{k} \in \mathbb{R}^2$ and $\mathcal{A}_{\mathbf{xf}}$ is a 2-dimensional linear variety. To derive $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$ and $\mathcal{A}^i$, we need the following lemma.

**Lemma 5.** *Consider the two lines $L_1(\mathbf{k}) : \boldsymbol{\xi}^T\mathbf{k} = \alpha_1$ and $L_2(\mathbf{k}) : \boldsymbol{\xi}_\perp\mathbf{k} = \alpha_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Then*

(i) *$L_1(\mathbf{k})$ can be parameterized as $\mathbf{k}(\kappa) = \frac{\alpha_1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi} + \kappa\boldsymbol{\xi}_\perp^T$, where $\kappa \in \mathbb{R}$.*
(ii) *$L_1(\mathbf{k})$ and $L_2(\mathbf{k})$ are perpendicular and intersect at the point where $\mathbf{k}^* = \frac{\alpha_1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi} + \frac{\alpha_2}{\|\boldsymbol{\xi}_\perp\|^2}\boldsymbol{\xi}_\perp^T$.*
(iii) *The half line $\{\mathbf{k} \mid L_1(\mathbf{k}) = 0$ but $L_2(\mathbf{k}) \geq 0\}$ can be parameterized as $\{\mathbf{k}^* + \kappa\boldsymbol{\xi}_\perp \mid \kappa \geq 0\}$.*
(iv) *The half plane $\alpha\boldsymbol{\xi}_\perp\mathbf{k} \geq 0$, $\alpha \in \mathbb{R} \setminus \{0\}$, can be parameterized as $\mathbf{k}(\kappa_1, \kappa_2) = \kappa_1\boldsymbol{\xi}_\perp^T + \kappa_2\boldsymbol{\xi}$, where $\operatorname{sign}(\alpha)\cdot\kappa_1 \geq 0$ and $\kappa_2 \in \mathbb{R}$.*

Define $\mathbf{x}_\perp = \frac{1}{\|\mathbf{x}\|}[x_2, -x_1]$, $\mathbf{b}_\perp = \frac{1}{\|\mathbf{b}\|}[b_2, -b_1]$ and $\mathbf{c}_\perp = \frac{1}{\|\mathbf{c}\|}[c_2, -c_1]^T$. The sets $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$ and $\mathcal{A}^i$, and their structures are explicitly described in the next result.

**Theorem 6.** *Let $\mathbf{x}, \mathbf{f}, \mathbf{b}, \mathbf{c}^T \in \mathbb{R}^2$ and $\mathbf{x} \neq \mathbf{0}$. Then*

(i) $$\mathcal{A}^c_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{c}}_{\mathbf{xf}} & \text{if } \{\mathbf{x}, \mathbf{b}\} \text{ are LI;} \\ \mathcal{A}_{\mathbf{xf}} & \text{if } \{\mathbf{x}, \mathbf{b}\} \text{ are LD } \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LI;} \\ \emptyset & \text{if } \{\mathbf{x}, \mathbf{b}\} \text{ are LD } \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LD,} \end{cases}$$
*where $\mathcal{A}^{\bar{c}}_{\mathbf{xf}} := \left\{A^{\bar{c}}_p + \kappa\mathbf{b}\mathbf{x}_\perp \mid \kappa \in \mathbb{R} \ \& \ A^{\bar{c}}_p = \frac{1}{\|\mathbf{x}\|^2}\mathbf{f}\mathbf{x}^T - \frac{(\mathbf{b}_\perp\mathbf{f})(\mathbf{b}^T\mathbf{x})}{\|\mathbf{x}\|^2(\mathbf{x}_\perp\mathbf{b})}\mathbf{b}_\perp^T\mathbf{x}_\perp\right\}$ is a line in $\mathcal{A}_{\mathbf{xf}}$ in which $(A, \mathbf{b})$ is uncontrollable.*

(ii) $$\mathcal{A}^s_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} & \text{if ``}\{\mathbf{x}, \mathbf{b}\} \text{ are LI } \& \ \frac{\mathbf{b}_\perp\mathbf{f}}{\mathbf{x}_\perp\mathbf{b}} < 0\text{'' or} \\ & \text{``}\{\mathbf{x}, \mathbf{b}\} \text{ are LD } \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LI;''} \\ \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{c}}_{\mathbf{xf}} & \text{if } \{\mathbf{x}, \mathbf{b}\} \text{ are LI } \& \ \frac{\mathbf{b}_\perp\mathbf{f}}{\mathbf{x}_\perp\mathbf{b}} \geq 0; \\ \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{s}}_{\mathbf{xf}} & \text{if } \{\mathbf{x}, \mathbf{b}\} \text{ are LD } \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LD,} \end{cases}$$
*where $\mathcal{A}^{\bar{s}}_{\mathbf{xf}} := \left\{A_p + \kappa_1\mathbf{b}_\perp^T\mathbf{x}_\perp + \kappa_2\mathbf{b}\mathbf{x}_\perp \mid \kappa_2 \in \mathbb{R} \ \& \ \operatorname{sign}(\mathbf{x}_\perp\mathbf{b}_\perp^T)\cdot\kappa_1 \geq 0\right\}$ is a half plane in $\mathcal{A}_{\mathbf{xf}}$ in which $(A, \mathbf{b})$ is unstabilizable.*

(iii) $$\mathcal{A}^o_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{o}}_{\mathbf{xf}} & \text{if } \mathbf{cx} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if } \mathbf{cx} = 0 \ \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LI;} \\ \emptyset & \text{if } \mathbf{cx} = 0 \ \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LD,} \end{cases}$$
*where $\mathcal{A}^{\bar{o}}_{\mathbf{xf}} := \left\{A^{\bar{o}}_p + \kappa\mathbf{c}_\perp\mathbf{x}_\perp \mid \kappa \in \mathbb{R} \ \& \ A^{\bar{o}}_p = \frac{\mathbf{f}\mathbf{x}^T}{\|\mathbf{x}\|^2} - \frac{(\mathbf{cf})(\mathbf{x}^T\mathbf{c}_\perp)}{\|\mathbf{x}\|^2\cdot\|\mathbf{c}\|^2(\mathbf{x}_\perp\mathbf{c}_\perp)}\mathbf{c}^T\mathbf{x}_\perp\right\}$ is a line in $\mathcal{A}_{\mathbf{xf}}$ in which $(A, \mathbf{c})$ is unobservable.*

(iv) $$\mathcal{A}^d_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{d}}_{\mathbf{xf}} & \text{if } \mathbf{cx} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if ``}\mathbf{cx} = 0 \ \& \ \{\mathbf{x}, \mathbf{f}\} \text{ are LI'' or} \\ & \text{``}\mathbf{cx} = 0 \ \& \ \mathbf{f} = \mu\mathbf{x}, \ \mu < 0;\text{''} \\ \emptyset & \text{if } \mathbf{cx} = 0 \ \& \ \mathbf{f} = \mu\mathbf{x}, \ \mu \geq 0, \end{cases}$$
*where $\mathcal{A}^{\bar{d}}_{\mathbf{xf}} := \left\{A^{\bar{o}}_p + (\kappa + \mathbf{c}_\perp^T\mathbf{f})\mathbf{c}_\perp\mathbf{x}_\perp \mid \kappa \in \mathbb{R} \ \& \ \kappa\cdot\operatorname{sign}(\mathbf{x}_\perp\mathbf{c}_\perp) \geq 0\right\}$ is a half line in $\mathcal{A}^{\bar{o}}_{\mathbf{xf}}$ in which $(A, \mathbf{c})$ is undetectable.*

(v) $$\mathcal{A}^i_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{i}}_{\mathbf{xf}} & \text{if } \mathbf{cx} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if } \mathbf{cx} = 0 \ \& \ \mathbf{f} \neq \mathbf{0}; \\ \emptyset & \text{if } \mathbf{cx} = 0 \ \& \ \mathbf{f} = \mathbf{0}, \end{cases}$$
*where $\mathcal{A}^{\bar{i}}_{\mathbf{xf}} := \left\{A^{\bar{o}}_p + (\mathbf{c}_\perp^T\mathbf{f})\mathbf{c}_\perp\mathbf{x}_\perp\right\}$ is a point in $\mathcal{A}^{\bar{d}}_{\mathbf{xf}}$ in which $(A, \mathbf{c})$ has an unobservable mode on the $j\omega$-axis.*

**Proof.** Here, we only derive the sets $\mathcal{A}^c_{\mathbf{xf}}$ and $\mathcal{A}^s_{\mathbf{xf}}$. The sets $\mathcal{A}^o_{\mathbf{xf}}$, $\mathcal{A}^d_{\mathbf{xf}}$ and $\mathcal{A}^i_{\mathbf{xf}}$ can be similarly derived. Let $M_{\mathbf{b}} = [\mathbf{b}_\perp^T \vdots \frac{\mathbf{b}}{\|\mathbf{b}\|}]$. It is clear that $\bar{A} = M_{\mathbf{b}}^TAM_{\mathbf{b}}$ and $\bar{\mathbf{b}} = M_{\mathbf{b}}^T\mathbf{b}$ are in the form described in Lemma 2. By direct calculation, $\bar{A}_{12} = \frac{1}{\|\mathbf{b}\|}\left[\frac{1}{\|\mathbf{x}\|^2}(\mathbf{b}_\perp\mathbf{f})\cdot(\mathbf{b}^T\mathbf{x}) + (\mathbf{b}_\perp\mathbf{k})\cdot(\mathbf{x}_\perp\mathbf{b})\right]$ and $\bar{A}_{11} = \frac{1}{\|\mathbf{x}\|^2}(\mathbf{b}_\perp\mathbf{f})(\mathbf{b}_\perp\mathbf{x}) + (\mathbf{b}_\perp\mathbf{k})(\mathbf{x}_\perp\mathbf{b}_\perp^T)$. From Lemma 2 and Theorem 4, $(A, \mathbf{b})$ is uncontrollable $\Leftrightarrow \bar{A}_{12} = 0$, and $(A, \mathbf{b})$ is unstabilizable $\Leftrightarrow \bar{A}_{12} = 0$ and $\bar{A}_{11} \geq 0$. Now if $\mathbf{x}_\perp\mathbf{b} \neq 0$, i.e., $\{\mathbf{x}, \mathbf{b}\}$ are LI, then the set of $\mathbf{k}$ such that $\bar{A}_{12} = 0$ can be parameterized using (i) of Lemma 5 with $(\boldsymbol{\xi}, \alpha_1)$ being replaced by $\left(\mathbf{b}_\perp^T, -\frac{(\mathbf{b}_\perp\mathbf{f})(\mathbf{b}^T\mathbf{x})}{\|\mathbf{x}\|^2(\mathbf{x}_\perp\mathbf{b})}\right)$. Combining the parameterization of $\mathbf{k}$ with the expression of $\mathcal{A}_{\mathbf{xf}}$ gives the set $\mathcal{A}^{\bar{c}}_{\mathbf{xf}}$. Consequently, $\mathcal{A}^c_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{c}}_{\mathbf{xf}}$. Additionally, within $\mathcal{A}^{\bar{c}}_{\mathbf{xf}}$ (i.e., $\bar{A}_{12} = 0$), $\mathbf{k}$ satisfies the relation $\mathbf{b}_\perp\mathbf{k} = -\frac{(\mathbf{b}_\perp\mathbf{f})(\mathbf{b}^T\mathbf{x})}{(\mathbf{x}_\perp\mathbf{b})\|\mathbf{x}\|^2}$. Inserting this relation into $\bar{A}_{11}$ yields $\bar{A}_{11} = \frac{(\mathbf{b}_\perp\mathbf{f})[(\mathbf{x}^T\mathbf{b})^2 + (\mathbf{b}_\perp\mathbf{x})^2]}{(\mathbf{x}_\perp\mathbf{b})\|\mathbf{x}\|^2}$. Thus, $\bar{A}_{11} < 0 \Leftrightarrow \frac{\mathbf{b}_\perp\mathbf{f}}{\mathbf{x}_\perp\mathbf{b}} < 0$. Therefore, $\mathcal{A}^s_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$ if $\{\mathbf{x}, \mathbf{b}\}$ are LI and $\frac{\mathbf{b}_\perp\mathbf{f}}{\mathbf{x}_\perp\mathbf{b}} < 0$, and $\mathcal{A}^s_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{c}}_{\mathbf{xf}}$ if $\{\mathbf{x}, \mathbf{b}\}$ are LI and $\frac{\mathbf{b}_\perp\mathbf{f}}{\mathbf{x}_\perp\mathbf{b}} \geq 0$. We now consider the case of $\mathbf{x}_\perp\mathbf{b} = 0$, i.e., $\{\mathbf{x}, \mathbf{b}\}$ are LD. This implies that $\mathbf{b}^T\mathbf{x} \neq 0$, and $\bar{A}_{12} = 0 \Leftrightarrow \mathbf{b}_\perp\mathbf{f} = 0 \Leftrightarrow \{\mathbf{x}, \mathbf{f}\}$ are LD because $\mathbf{x}_\perp\mathbf{b} = 0$. As a result, $\mathcal{A}^s_{\mathbf{xf}} = \mathcal{A}^c_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$ if $\{\mathbf{x}, \mathbf{f}\}$ are LI. When $\{\mathbf{x}, \mathbf{f}\}$ are LD (i.e., $\bar{A}_{12} = 0$), we have $\mathcal{A}^c_{\mathbf{xf}} = \emptyset$ and $\bar{A}_{11} = (\mathbf{x}_\perp\mathbf{b}_\perp^T)\mathbf{b}_\perp\mathbf{k}$. By (iv) of Lemma 5, the set of $\mathbf{k}$ for $\bar{A}_{11} \geq 0$ is a half plane and can be parameterized as $\mathbf{k}(\kappa_1, \kappa_2) = \kappa_1\mathbf{b}_\perp^T + \kappa_2\mathbf{b}$, where $\operatorname{sign}(\mathbf{x}_\perp\mathbf{b}_\perp^T)\cdot\kappa_1 \geq 0$ and $\kappa_2 \in \mathbb{R}$. Inserting this $\mathbf{k}(\kappa_1, \kappa_2)$ into $\mathcal{A}_{\mathbf{xf}}$ yields $\mathcal{A}^{\bar{s}}_{\mathbf{xf}}$. Thus, $\mathcal{A}^s_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}^{\bar{s}}_{\mathbf{xf}}$. $\square$

It is interesting to note from Theorem 6 that the set $\mathcal{A}^s_{\mathbf{xf}}$ is always non-empty, regardless of what nonzero vector $\mathbf{b}$ is given. Moreover, it is easy to see that the results of Theorem 6 agree with those of Theorem 1. That is, $\mathcal{A}^{so}_{\mathbf{xf}} = \emptyset \Leftrightarrow \mathbf{cx} = 0$ and $\{\mathbf{x}, \mathbf{f}\}$ are LD; $\mathcal{A}^{sd}_{\mathbf{xf}} = \emptyset \Leftrightarrow \mathbf{cx} = 0$, $\mathbf{f} = \mu\mathbf{x}$ and $\mu \geq 0$; and $\mathcal{A}^{si}_{\mathbf{xf}} = \emptyset \Leftrightarrow \mathbf{cx} = 0$ and $\mathbf{f} = \mathbf{0}$.

## 4. An illustrative example

Consider the following system

$$\dot{x}_1 = x_1x_2 \quad \text{and} \quad \dot{x}_2 = -x_2 + u. \tag{7}$$

Clearly, this system is in the form of (1) with $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{f}(\mathbf{x}) = [x_1x_2, -x_2]^T$ and $B(\mathbf{x}) = [0, 1]^T$. System (7) is stabilizable and two global stabilizers, one using the Sontag formula with the control Lyapunov function $V(x_1, x_2) := (x_1^2e^{2x_2} + x_2^2)/2$ (Sontag, 1989) and the other adopting the backstepping scheme (Khalil, 1996), have the following forms:

$$u_{\mathrm{Sontag}} = \frac{x_2^2 - \sqrt{x_2^4 + (x_1^2e^{2x_2} + x_2)^4}}{x_1^2e^{2x_2} + x_2} \tag{8}$$

and

$$u_{\mathrm{BS}} = (1-\psi)x_2 - (1+\psi)x_1^2 - 2x_1^2x_2, \quad \psi > 0. \tag{9}$$

To demonstrate the SDRE design, we choose $Q(\mathbf{x}) = I_2$, $R(\mathbf{x}) = 1$ and an intuitive SDC matrix $A(\mathbf{x})$ with $a_{11}(\mathbf{x}) = a_{21}(\mathbf{x}) = 0$, $a_{12}(\mathbf{x}) = x_1$ and $a_{22}(\mathbf{x}) = -1$. Obviously, $(A(\mathbf{x}), B(\mathbf{x}))$ is stabilizable everywhere except the $X_2$-axis where the SDRE solvability condition is violated; however, by Theorem 1, $\mathcal{A}_{\mathbf{xf}}^{s\gamma} \neq \emptyset$ for $\gamma = o, d, i$ at every nonzero state because $C(\mathbf{x})\mathbf{x} = \mathbf{x} \neq \mathbf{0}$. When $\mathbf{x} = [0, x_2]^T$ and $x_2 \neq 0$, $\mathbf{f} = [0, -x_2]^T = -\mathbf{x}$ and, by (ii) of Theorem 6, $\mathcal{A}_{\mathbf{xf}}^{so} = \mathcal{A}_{\mathbf{xf}}^{sd} = \mathcal{A}_{\mathbf{xf}}^{si} = \mathcal{A}_{\mathbf{xf}}^{s} = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{s}} = \{A \mid a_{11} < 0,\ a_{12} = 0,\ a_{21} \in \mathbb{R}\ \&\ a_{22} = -1\}$. In the following, we will choose $a_{11} = -1$ and $a_{21} = 0$ for the SDC matrix of the SDRE scheme when $\mathbf{x} \in X_2$-axis.

Numerical results for initial states $\mathbf{x}(0) = [1, 1]^T$ are summarized in Fig. 1 and Table 1, where we have adopted the following three controllers: $u_{\mathrm{Sontag}}$ (labeled Sontag), $u_{\mathrm{BS}}$ with $\psi = 2$ (labeled BS) and the SDRE controller (labeled SDRE). It is observed from Fig. 1 that all of the system states of the three schemes converge to zero and, from Table 1, the SDRE scheme has better performances than the other two schemes in the performance indices that are listed in the table, where $\|u\|_\infty := \max_t \|u\|$ denotes the maximum control magnitude that required during the control period and the integration is evaluated from $t = 0$ to $t = 1000$.

It is noted that the solution trajectories of the three schemes remain on the $X_2$-axis if they start from there because $\dot{x}_1 = x_1x_2|_{x_1=0} = 0$. Thus, the trajectories of the three schemes will never reach the $X_2$-axis unless they start from there. By direct calculation, $u_{\mathrm{Sontag}} = u_{\mathrm{SDRE}} = (1-\sqrt{2})x_2$ and $u_{\mathrm{BS}} = (1-\psi)x_2$ if the system state starts from the $X_2$-axis. The resulting closed-loop dynamics for $x_2$ are $\dot{x}_2 = -\psi x_2$ for the BS design and $\dot{x}_2 = -\sqrt{2}x_2$ for both the Sontag and SDRE schemes. It is interesting to note that, when $\mathbf{x} \in X_2$-axis, $u_{\mathrm{SDRE}}$ remains unchanged regardless of the choice of $A(\mathbf{x}) \in \mathcal{A}_{\mathbf{xf}}^{s}$; however, if the weighting matrices are changed to be $Q(\mathbf{x}) = \mathrm{diag}(q_1, q_2) > 0$ and $R(\mathbf{x}) = r > 0$, then $u_{\mathrm{SDRE}} = (1 - \sqrt{1 + q_2/r})x_2$ and the resulting closed-loop dynamics for $x_2$ becomes $\dot{x}_2 = -\sqrt{1 + q_2/r} \cdot x_2$, both are independent of $q_1$. Moreover, $u_{\mathrm{SDRE}} \approx 0 = u_{\mathrm{BS}}|_{\psi=1}$ when $r \gg q_2$, which implies that the control effort should be reduced as much as possible.

## 5. Conclusions

This article has presented necessary and sufficient conditions for the existence of SDC matrices in a nonlinear system such that the SDRE scheme can be successfully implemented. These existence conditions are easy to verify, and when they are satisfied, all of the feasible SDC matrices are explicitly parameterized for the planar case. An example is also given to demonstrate the use of the main results. Nevertheless, the application of this study in SDRE design for better system performance, including optimal control recovery and basin of attraction estimation, needs further investigation.

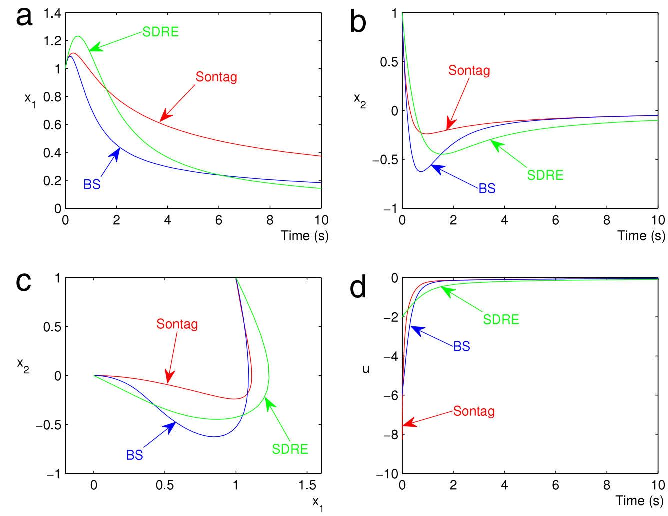


**Fig. 1.** Time history of the system states and control inputs.

**Table 1**
Performances of the three schemes.

| | Final time of $\mathbf{x}^T\mathbf{x} = 0.01$ | $\int(\mathbf{x}^T\mathbf{x} + u^2)$ | $\int u^2$ | $\|u\|_\infty$ |
|---|---|---|---|---|
| Sontag | $3.2 \times 10^3$ | 13.6 | 3.4 | 8.3 |
| BS | $8.3 \times 10^2$ | 9.7 | 5.8 | 6 |
| SDRE | 86.3 | 6.1 | 2.2 | 2 |

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