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NUMERICAL COMPUTATIONS OF INTEGRALS OVER PATHS ON RIEMANN SURFACES OF GENUS N

J.-E. Lee

This paper is a continuation of work by Forest and Lee [1,2]. In [1,2] it was proved that the function theory of periodic soliton solutions occurs on the Riemann surfaces $\Re$ of genus N, where the integrals over paths on $\Re$ play the most fundamental role. In this paper a numerical method is developed to evaluate these integrals. Precisely, the aim is to develop a computational code for integrals of the form

$$\int_\gamma f(z)\frac{dz}{R(z)}, \qquad \text{or} \qquad \int_\gamma f(z)R(z)\,dz,$$

where $f(z)$ is any single-valued analytic function on the complex plane $\mathbf{C}$, and $R(z)$ is a two-valued function on $\mathbf{C}$ of the form

$$R^2(z) = \prod_{k=1}^{2N+\delta}(z - z_0(k)), \qquad \delta = 0 \quad \text{or} \quad 1,$$

where $\{z_0(k), 1 \le k \le 2N+\delta\}$ are distinct complex numbers which play the role of the branch points of the Riemann surface $\Re = \{(z, R(z))\}$ of genus $N-1+\delta$. The integral path γ is continuous on $\Re$. The numerical code is developed in "Mathematica" [3].

1. INTRODUCTION

It is well known at present that the function theory of the periodic soliton equations occurs on Riemann surfaces of genus N (for example, [1,2]). Much numerical work has been done on the periodic soliton equations and their perturbations in order to discuss various subjects such as linearized instability analysis, bifurcation theory and chaotic motions, etc. (for example, [3, 4, 5, 6]). It is a powerful tool. In this paper, we focus on the numerical computation of integrals over paths on the Riemann surfaces of genus N since these integrals are among the most fundamental elements in the theory of Riemann surfaces, in particular, in the theory of periodic soliton equations, and are in general impossible to calculate analytically. For example, in the theory of periodic soliton equations, the followings are all in terms of integrals: the wave numbers and frequencies of the N-phase, quasi-periodic solutions, the Floquet exponents for the linearized instability analysis of the N-phase, quasi-periodic solutions, the Riemann invariants of the modulating N-phase, wavetrains for the modulational instability analysis, etc. According to the rule of the square-root function $\sqrt{z}$ in "Mathematica," which will be specified in Sec. 2, we develop the computational methods rigorously in Theorem 1 in Sec. 3 for those integrals on the Riemann surfaces with N arbitrary cut-structures. To our knowledge, such work has not appeared anywhere explicitly.

Given $2(N+\delta)(\delta = 0 \quad \text{or} \quad 1)$ distinct complex numbers $\{z_0[j], 1 \le j \le 2(N+\delta)\}$, let $\Re_N$ be the Riemann surface of the hyperelliptic curve $R(z)$, where

$$R^2(z) = \prod_{k=1}^{2N+\delta}(z - z_0[k]), \qquad \delta = 0 \quad \text{or} \quad 1. \tag{1}$$

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Each pair of the branch points $\{z_0[2k-1], z_0[2k]\}$ provides a cut in the complex plane $\mathbf{C}$ (for $\delta = 1$, $z_0[2N+2]$ is taken to be the infinite point ∞). We want to develop a numerical scheme to evaluate integrals on $\Re_n$ of the form

$$\int_\gamma f(z)\frac{dz}{R(z)}, \qquad \text{or} \qquad \int_\gamma f(z)R(z)dz \tag{2}$$

where $f(z)$ is any single-valued, analytic function in the complex plane $\mathbf{C}$, and γ is any continuous curve on $\Re_N$. For systematic argument, each pair $\{z_0[2k-1], z_0[2k]\}$ is renamed such that

$$\mathrm{Im}\big[z_0[2k]\big] < \mathrm{Im}\big[z_0[2k-1]\big] \tag{3a}$$

or

$$\mathrm{Im}\big[z_0[2k]\big] = \mathrm{Im}\big[z_0[2k-1]\big] \quad \text{and} \quad \mathrm{Re}\big[z_0[2k]\big] < \mathrm{Re}\big[z_0[2k-1]\big]. \tag{3.b}$$

For each z in $\mathbf{C}$, we denote by $(z, R^+(z))$ and $(z, R^-(z))$ (or, briefly, $R^+(z)$ and $R^-(z)$) the corresponding points in the first sheet Ω_1 and the second sheet Ω_2 of $\Re_N$ respectively, and

$$R^-(z) = -R^+(z). \tag{4}$$

For practical reasons, $R(z)$ is evaluated as

$$R(z) = \prod_{k=1}^{2N+\delta} \sqrt{z - z_0[k]}, \qquad \delta = 0 \quad \text{or} \quad 1. \tag{4a}$$

Since we shall see later that $\sqrt{\ }$ in "Mathematica" is defined as a single-valued function in $\mathbf{C}$ (which will be specified in Sec. 2), we denote

$$h(z) \;=\; \text{the value of } R(z) \text{ evaluated by "Mathematica."} \tag{4b}$$

Then, Theorem 1 in Sec. 3 gives the simple and precise rule determining $R^+(z)$ in terms of $h(z)$. It is the key theory in the entire scheme, since then we can apply the integral operator in "Mathematica" to evaluate the integrals (2) along any continuous curve γ^+ lying in Ω_1. Then, due to (4), the integrals (2) along any continuous curve γ^- lying in Ω_2 can be performed, and so does the numerical evaluation of integrals (2) along any continuous curve γ on $\Re_N$. Therefore, by the theorem in Sec. 3, it is enough to develop the numerical evaluation of the integrals (2) along any continuous curve γ lying in Ω_1, and we will do it in the following manner:

1. The path γ is replaced by its "simplest" homologous path γ^* such as a union of line segments and canonical cycles on Ω_1. Due to the homology, the integrals (2) over γ and over γ^* are identical.

2. According to criterion (11) in Theorem 1, γ^* is partitioned into two finite sets of disjoint curves $\Gamma_1 = \{\gamma_{1i}^*, 1 \le i \le m\}$ and $\Gamma_2 = \{\gamma_{2k}^*, 1 \le k \le n\}$ for some m, n such that

$$R^+(z) = \;\; h(z) \qquad \text{for} \quad z \in \gamma_{1i}^*, \forall \gamma_{1i}^* \in \Gamma_1, \tag{5a}$$

$$R^+(z) = -h(z) \qquad \text{for} \quad z \in \gamma_{2k}^*, \forall \gamma_{2k}^* \in \Gamma_2. \tag{5b}$$

3. The integral (2) over γ^* is the sum of the integrals over $\bigcup \gamma_{1i}^*$ and $\bigcup \gamma_{2k}^*$ respectively. Each integral is directly evaluated by "Mathematica" according to the proper sign in (5).

4. A numerical code for the entire scheme is completed and written in a manner which can be applied directly or easily modified for general purposes. For each γ, to make sure that the code is correct, the same integral over at least two distinct homologous paths of γ are performed. These numerical values should be almost identical.

2. THE STRUCTURE OF $\sqrt{z}$ IN "MATHEMATICA"; THE STRUCTURE OF THE RIEMANN SURFACES

By definition, the two-valued, square-root function $\sqrt{z}$ in $\mathbf{C}$ is defined as

$$\sqrt{z} = |\sqrt{r}|e^{i\theta/2} \qquad \text{whenever} \qquad z = re^{i\theta}, r \ge 0, \theta \in \mathbf{R}. \tag{6a}$$

Consider the two copies of $\mathbf{C}$, $\mathfrak{J}^+ = \{z = re^{it\pi}, -1 \le t < 1, r \ge 0\}$ and $\mathfrak{J}^- = \{z = re^{it\pi}, 1 \le t < 3, r \ge 0\}$. Define $\mathfrak{J}^+$ to be the fundamental branch of $\sqrt{z}$; then the first sheet of the Riemann surface $\mathfrak{R}_0$ of $\sqrt{z}$ is $\Omega_{01} = \{(z, \sqrt{z}), z \in \mathfrak{J}^+\}$ and the second sheet of $\mathfrak{R}_0$ is $\Omega_{02} = \{(z, \sqrt{z}), z \in \mathfrak{J}^-\}$. We denote $(z, \sqrt{z})$ in Ω_{01} as $\sqrt{z}^+$, and $(z, \sqrt{z})$ in Ω_{02} as $\sqrt{z}^-$. In "Mathematica," the value of $\sqrt{z}$ is unique and exactly identical to $\sqrt{z}^+$ except those z along the negative real line $\{z = -r, r > 0\}$ where $\sqrt{z}$ has exactly two values, i.e., for each integer n,

$$\sqrt{z} = -i|\sqrt{r}| \qquad \text{for} \qquad z = -r = re^{i(-1+4n)\pi}, \tag{6.b}$$

and

$$\sqrt{z} = \ i|\sqrt{r}| \qquad \text{for} \qquad z = -r = re^{i(1+4n)\pi}. \tag{6.c}$$

From now on, we denote the "Mathematica" value of $\sqrt{z}$ as $h_0(z)$, i.e.,

$$h_0(z) = \sqrt{z}^+ \qquad \text{whenever} \qquad z \in \mathfrak{J}^+. \tag{6.d}$$

It is clear that, in "Mathematica," $\sqrt{z} = h_0(z)$ whenever $z \in \mathbf{C}\backslash(-\infty, 0)$ or $z = re^{-i\pi}$, $r > 0$.

We now consider the Riemann surface $\mathfrak{R}_N$ of the hyperelliptic curve $R(z)$ in (1). For each branch point $z_0[k]$, $1 \le k \le 2N + \delta$, let $h_k(z)$ be the value of $\sqrt{z - z_0[k]}$ evaluated by "Mathematica," i.e.,

$$h_k(z) = h_0\big(z - z_0[k]\big), \quad \forall z \in \mathbf{C}, \quad 1 \le k \le 2N + \delta, \tag{7a}$$

$$h(z) = \prod_{k=1}^{2N+\delta} h_k(z), \quad \forall z \in \mathbf{C}. \tag{7b}$$

We define the principal branch for each cut along $\{z_0[2k-1], z_0[2k]\}$ as follows. For each pair $\{z_0[2k-1], z_0[2k]\}$, let

$$\theta_k = \operatorname{Arg}\big[z_0[2k] - z_0[2k-1]\big]. \tag{8a}$$

Due to (3), θ_k can be chosen such that

$$\theta_k \in [-\pi, 0). \tag{8b}$$

Let J_k be the straight cut from $z_0[2k]$ to $z_0[2k-1]$ in $\mathbf{C}$ parameterized as

$$J_k = \big\{z = z_0[2k-1] + te^{i\theta_k}, 0 \le t \le \big|z_0[2k] - z_0[2k-1]\big|\big\}. \tag{8c}$$

Define the initial edge of J_k lying in the first sheet Ω_1 of $\mathfrak{R}_N$ as

$$J_k^+ = \big\{(z, R(z)), z \in J_k\big\}. \tag{8d}$$

Clearly, according to (6d) and (8a), (8b), since $\big(z - z_0[2k-1]\big), \big(z - z_0[2k]\big) \in \mathfrak{J}^+$ for $z \in J_k$, so

$$\begin{aligned} \sqrt{z - z_0[2k-1]} &= h_{2k-1}(z), \\ \sqrt{z - z_0[2k]} &= h_{2k}(z) \qquad \text{for} \quad \big(z, R(z)\big) \in J_k^+. \end{aligned} \tag{8e}$$

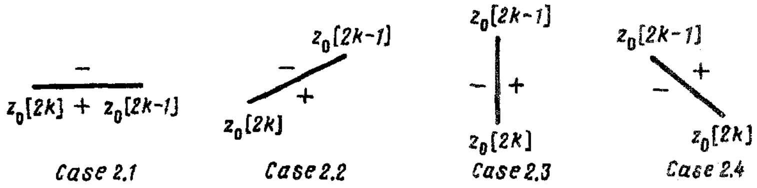

Fig. 1. The generic cut J_k.

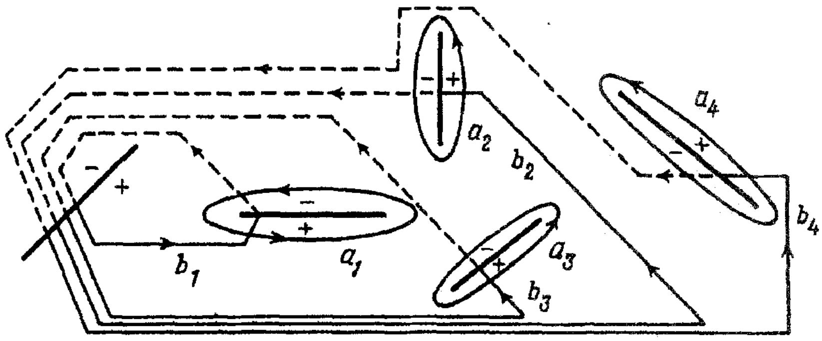

Fig. 2. The generic cut-structure and canonical a, b-cycles of $\mathfrak{R}$ for $N = 5$.

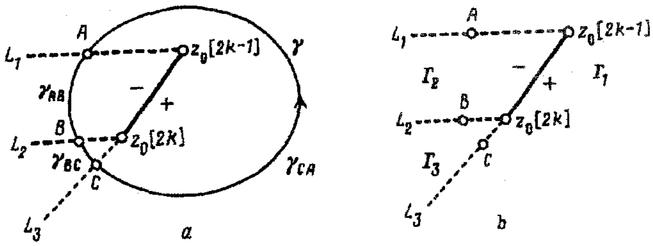

Fig. 3.

The terminal edge of J_k lying in the first sheet Ω_1 (i.e., the initial edge of J_k lying in the second sheet Ω_2) of $\Re_N$ is

$$J_k^- = \{(z, R(z)), z \in \hat{J}_k\}, \tag{8f}$$

where J_k is parameterized as

$$\hat{J}_k = \{z = z_0[2k-1] + te^{i(2\pi+\theta_k)}, 0 \le t \le |z_0[2k] - z_0[2k-1]|\}. \tag{8g}$$

It is clear that $(z - z_0[2k-1]) \in \mathfrak{J}^-$ for $z \in \hat{J}_k$. Moreover, due to the continuity of $\sqrt{\ }$ in $\Re_N$, it will become clear in Sec. 3 that

$$\begin{aligned} \sqrt{z - z_0[2k-1]} &= -h_{2k-1}(z), \\ \sqrt{z - z_0[2k]} &= h_{2k}(z) \qquad \text{for} \quad (z, R(z)) \in J_k^-. \end{aligned} \tag{8h}$$

The generic $J^\pm$ are illustrated in Fig. 1. The generic cut-structure and canonical a, b-cycles of $\Re_N$ for $N = 5$ is given in Fig. 2. Next, we determine $R^+(z)$ in terms of $\pm h(z)$.

3. DETERMINATION OF $R^+(z)$

Determination of $\sqrt{z - z_0[2k-1]}$, $\sqrt{z - z_0[2k]}$ for z in Ω_1. Now, as illustrated in Fig. 3a, let γ be a simple closed path in Ω_1 such that γ encloses the cut J_k, and the three points A, B, C in γ are such that

$$\begin{aligned} &\mathrm{Re}[A] < \mathrm{Re}\big[z_0[2k-1]\big], \qquad &&\mathrm{Im}[A] = \mathrm{Im}\big[z_0[2k-1]\big], \\ &\mathrm{Re}[B] < \mathrm{Re}\big[z_0[2k]\big], &&\mathrm{Im}[B] = \mathrm{Im}\big[z_0[2k]\big], \end{aligned}$$

and C is the intersection between γ and the line through the cut J_k such that $\mathrm{Im}[C] \le \mathrm{Im}\big[z_0[2k]\big]$. Notice that when J_k is a horizontal cut where $\mathrm{Im}\big[z_0[2k-1]\big] = \mathrm{Im}\big[z_0[2k]\big]$, then $A = B = C$. Except for this particular case, $\{A, B, C\}$ partitions the path γ into three paths, namely, γ_{CA}, γ_{AB}, and γ_{BC}. Along $\gamma_{CA}\backslash\{A\}$, since both arguments of $\big(z - z_0[2k-1]\big)$ and $\big(z - z_0[2k]\big)$ are strictly between $-\pi$ and π, according to (8e),

$$\begin{aligned} \sqrt{z - z_0[2k-1]} &= h_{2k-1}(z), \\ \sqrt{z - z_0[2k]} &= h_{2k}(z) \qquad \text{for} \quad z \in \gamma_{CA}\backslash\{A\} \quad \text{in} \quad \Omega_1. \end{aligned}$$

Notice that both $h_{2k-1}(z)$ and $h_{2k}(z)$ are continuous in $\gamma_{CA}\backslash\{A\}$. While $h_{2k}(z)$ is continuous at A, $h_{2k-1}(z) = h_0\big(z - z_0[2k-1]\big)$ has a jump at A since $\big(z - z_0[2k-1]\big)$ now has argument $-\pi$, i.e., $\big(z - z_0[2k-1]\big) \in \mathfrak{J}^-$. To assure that $\sqrt{z - z_0[2k-1]}$ is continuous through A, it is necessary that

$$\sqrt{z - z_0[2k-1]} = -h_{2k-1}(z) \qquad \text{for} \quad z \in \gamma_{AB}\backslash\{B\} \quad \text{in} \quad \Omega_1.$$

Clearly, $\sqrt{z - z_0[2k]}$ is continuous along $\gamma_{AB}\backslash\{B\}$ since the arguments of $\big(z - z_0[2k]\big)$ are strictly between $-\pi$ and π, so

$$\sqrt{z - z_0[2k]} = h_{2k}(z) \qquad \text{for} \quad z \in \gamma_{AB}\backslash\{B\} \quad \text{in} \quad \Omega_1.$$

Now, at B in γ, $(-)h_{2k-1}(z)$ is continuous while $h_{2k}(z)$ has a jump. To assure that both $\sqrt{z - z_0[2k-1]}$ and $\sqrt{z - z_0[2k]}$ are continuous through B, it is necessary that

$$\begin{aligned} \sqrt{z - z_0[2k-1]} &= -h_{2k-1}(z), \\ \sqrt{z - z_0[2k]} &= -h_{2k}(z) \qquad \text{for} \quad z \in \gamma_{BC}\backslash\{C\} \quad \text{in} \quad \Omega_1. \end{aligned}$$

For the special case where J_k is horizontal, we have $A = B = C$, and $\sqrt{z - z_0[2k-1]} = h_{2k-1}(z)$, $\sqrt{z - z_0[2k]} = h_{2k}(z)$ for $z \in \gamma$, the simplest case. In summary, we have

Proposition 1. (Determinations of $\sqrt{z - z_0[2k-1]}$, $\sqrt{z - z_0[2k]}$ along γ in Ω_1.) Let γ be a simple closed path in the first sheet Ω_1 of $\Re$ of $R(z)$ such that γ encloses a nonhorizontal cut J_k from $z_0[2k]$ to $z_0[2k-1]$. The values of $\sqrt{z - z_0[2k-1]}$ and $\sqrt{z - z_0[2k]}$ along γ are given as

$$\begin{aligned} &\sqrt{z - z_0[2k-1]} = h_{2k-1}(z), \quad &&\sqrt{z - z_0[2k]} = h_{2k}(z) \quad &&\text{for} \quad z \in \gamma_{CA}\backslash\{A\}, \qquad &&\text{(i)} \\ &\sqrt{z - z_0[2k-1]} = -h_{2k-1}(z), &&\sqrt{z - z_0[2k]} = h_{2k}(z) &&\text{for} \quad z \in \gamma_{AB}\backslash\{B\}, &&\text{(ii)} \\ &\sqrt{z - z_0[2k-1]} = -h_{2k-1}(z), &&\sqrt{z - z_0[2k]} = -h_{2k}(z) &&\text{for} \quad z \in \gamma_{BC}\backslash\{C\}. &&\text{(iii)} \end{aligned}$$

When J_k is horizontal, then $\sqrt{z - z_0[2k-1]} = h_{2k-1}(z)$, $\sqrt{z - z_0[2k]} = h_{2k}(z)$ for $z \in \gamma$.

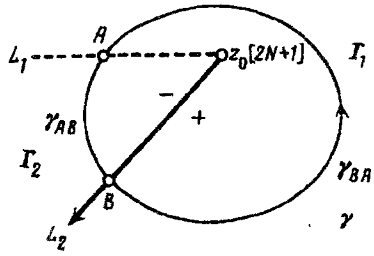

Fig. 4. The determination of $\sqrt{(z - z_0[2N+1]}$ in Ω_1.

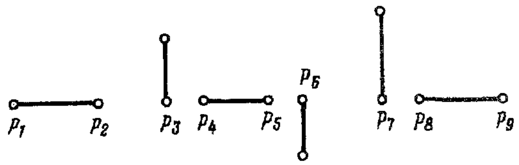

Fig. 5. The parametrization of horizontal J_k^+ in Ω_1.

From Proposition 1 for z along a curve in Ω_1, it is now easy to determine $\sqrt{z - z_0[2k-1]}$ and $\sqrt{z - z_0[2k]}$ in terms of $\pm h_{2k-1}(z)$, $\pm h_{2k}(z)$ for arbitrary number z in Ω_1. As illustrated in Fig. 3b, let L_1, L_2, L_3 be the three rays in Ω_1 where L_1 starts at $z_0[2k-1]$ and through A, L_2 starts at $z_0[2k]$ and through B, and L_3 starts at $z_0[2k-1]$ and through C. Then $\{L_1, L_2, L_3\}$ partitions Ω_1 into three regions Γ_1, Γ_2, and Γ_3 (each including its boundaries) where Γ_2 is bounded by $\{L_1, J_k^-, L_2\}$, Γ_3 is bounded by $\{L_2, L_3\}$, and $\Gamma_1 = [\Omega_1 \backslash (\Gamma_2 \bigcup \Gamma_3)] \bigcup L_1 \bigcup L_3 \bigcup J_k^+$. Proposition 1 yields

Proposition 2. (Determinations of $\sqrt{z - z_0[2k-1]}$, $\sqrt{z - z_0[2k]}$ for z in Ω_1.) Let z be a point in Ω_1 of $\Re$ of $R(z)$. Then

$$\sqrt{z - z_0[2k-1]} = h_{2k-1}(z), \quad \sqrt{z - z_0[2k]} = h_{2k}(z) \quad \text{for} \quad z \in \Gamma_1 \backslash L_1, \tag{i}$$

$$\sqrt{z - z_0[2k-1]} = -h_{2k-1}(z), \quad \sqrt{z - z_0[2k]} = h_{2k}(z) \quad \text{for} \quad z \in \Gamma_2 \backslash L_2, \tag{ii}$$

$$\sqrt{z - z_0[2k-1]} = -h_{2k-1}(z), \quad \sqrt{z - z_0[2k]} = -h_{2k}(z) \quad \text{for} \quad z \in \Gamma_3 \backslash L_3. \tag{iii}$$

When J_k is horizontal, $\sqrt{z - z_0[2k-1]} = h_{2k-1}(z)$, $\sqrt{z - z_0[2k]} = h_{2k}(z)$ for $z \in \gamma$.

Remark. In case $\delta = 1$ in $R(z)$, the determination of $\sqrt{z - z_0[2N+1]}$ in terms of $\pm h_{2N+1}(z)$ can be done similarly. As illustrated in Fig. 4 where $z_0[2N+1]$ and ∞ determine an infinite cut J_{N+1}, let γ be a simple curve in Ω_1 such that γ starts at a point $(B, R^+(B))$ in the initial edge J_{N+1}^+ of the cut, and ends at the "same point" $(B, R^-(z))$ in the terminal edge J_{N+1}^-. Let A in γ be such that $\mathrm{Re}[A] < \mathrm{Re}[z_0[2N+1]]$, $\mathrm{Im}[A] = \mathrm{Im}[z_0[2N+1]]$. Notice that when J_{N+1} is a horizontal cut, where $J_{N+1}^+ = \{z = z_0[2N+1] + se^{-i\pi}, s \geq 0\}$, we have $A = B$. Except for this particular case, $\{A, B\}$ partitions the path γ into two paths γ_{AB} and γ_{BA}. The same reason as for Proposition 1 yields

Proposition 3. (Determination of $\sqrt{z - z_0[2N+1]}$ along γ in Ω_1.) $\sqrt{z - z_0[2N+1]}$ along γ in Ω_1 is given as

$$\sqrt{z - z_0[2N+1]} = h_{2N+1}(z) \qquad \text{for} \quad z \in \gamma_{BA} \backslash \{A\}, \tag{i}$$

$$\sqrt{z - z_0[2N+1]} = -h_{2N+1}(z) \qquad \text{for} \quad z \in \gamma_{AB} \backslash \{B\}. \tag{ii}$$

In particular, when J_{N+1} is horizontal, $\sqrt{z - z_0[2N+1]} = h_{2N+1}(z)$ for $z \in \gamma$.

Again, as illustrated in Figure 4, let L_1 be the ray that starts at $z_0[2N+1]$ and passes through A. Then $\{L_1, J_{N+1}^+, J_{N+1}^-\}$ partitions Ω_1 into Γ_1, Γ_2 (each includes its boundaries), where Γ_2 is bounded by L_1 and J_{N+1}^-, and $\Gamma_1 = (\Omega_1 \backslash \Gamma_1) \bigcup L_1 \bigcup J_{N+1}^+$. Proposition 3 yields

Proposition 4. (Determination of $\sqrt{z - z_0[2N+1]}$ for z in Ω_1.) For $\delta = 1$ in $R(z)$, and z is a point in Ω_1 of $\Re$ of $R(z)$. Then

$$\sqrt{z - z_0[2N+1]} = h_{2N+1}(z) \qquad \text{for} \quad z \in \Gamma_1 \backslash L_1, \tag{i}$$

$$\sqrt{z - z_0[2N+1]} = -h_{2N+1}(z) \qquad \text{for} \quad z \in \Gamma_2 \backslash J_{N+1}^-. \tag{ii}$$

In particular, when J_{N+1} is horizontal, $\sqrt{z - z_0[2N+1]} = h_{2N+1}(z)$, $z \in \Omega_1$.

Determination of $R^+(z)$ in terms of $\pm h(z)$. Now, by observing Proposition 2 and Proposition 4, we determine $R^+(z)$, the value of $R(z)$ in Ω_1 of $\Re$. Let z be a point in Ω_1. First, according to Proposition 2 and Proposition 4, if $z_0[k]$ is a branch point of a horizontal cut, then

$$\sqrt{z - z_0[k]}^{\,+} = h_k(z) = h_0(z - z_0[k]). \tag{9}$$

For a nonhorizontal cut J_k with the two branch points $\{z_0[2k-1], z_0[2k]\}$, let L^k be the oriented line through $z_0[2k]$ in the direction of $(z_0[2k-1] - z_0[2k])$, i.e.,

$$L^k = \left\{ w : \operatorname{Im}\left[(w - z_0[2k]) / (z_0[2k-1] - z_0[2k]) \right] = 0 \right\}. \tag{10a}$$

When $\delta = 1$ in $R(z)$, if J_{N+1} is not horizontal, we take $z_0[2N+2]$ to be any finite point lying in this infinite cut J_{N+1} starting from $z_0[2N+1]$, and let L^{N+1} be the oriented line through $z_0[2N+2]$ in the direction of $(z_0[2N+1] - z_0[2N+2])$, i.e.,

$$L^{N+1} = \left\{ w : \operatorname{Im}\left[(w - z_0[2N+2]) / (z_0[2N+1] - z_0[2N+2]) \right] = 0 \right\}. \tag{10b}$$

Now, with the simplest cases (9), Proposition 2 and Proposition 4 yield

Theorem 1. (Determination of $R^+(z)$.) *Let $(z, R^+(z))$ be a point in Ω_1 of $\Re$. Then $R^+(z)$ is given by "Mathematica" as*

$$R^+(z) = (-1)^{n_z} h(z) \tag{11}$$

where n_z is the number of branch point(s) $z_0[p]$ of those nonhorizontal cuts J_k such that z and $z_0[p]$ satisfy the following two relationships:

$$\operatorname{Im}[z] \le \operatorname{Im}[z_0[p]]. \tag{12a}$$

The point z lies in the half plane to the left of L^k in the complex plane $\mathbf{C}$, i.e.,

$$z \in \left\{ w : \operatorname{Im}\left[(w - z_0[2k]) / (z_0[2k-1] - z_0[2k]) \right] > 0 \right\} \tag{12b}$$

where $p = 2k$ or $2k-1$. If none of $z_0[p]$ satisfies (12a), (12b), *then $n_z = 0$ and $R^+(z) = h(z)$.*

4. CONCLUSION

The major theory in this paper is Theorem 1 in Sec. 3. Theorem 1 states how to evaluate $R^+(z)$ by "Mathematica." Accordingly, we can write a numerical program to evaluate integrals (2). There are two delicate points for writing such a program: (i) the indications of the exact positions z where the integrand $R(z)$ change signs along the integration path γ; (ii) the parameterizations of z along the horizontal cuts J. Here, we should point out that the definition of $\sqrt{z}$ in "Mathematica" version 2.2 is incorrect. Before any correct new version appears, we should run the programs by "Mathematica" version 2.1. The subject of our next work parallels that in this paper except that the integral path γ will be a dynamical curve. For example, γ is related to some evolution equations whose function theory occurs on Riemann surfaces. Such evolution equations include periodic soliton equations and their perturbations. One further work is to perform numerical evaluations of multi-fold, multi-valued integrals.

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