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A General Approach to Synchronization of Coupled Cells*

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Abstract. This investigation presents a general framework to establish synchronization of coupled cells and coupled systems. Each individual subsystem is represented by nonlinear differential equations with or without internal or intracellular delay. A general coupling function is employed to depict the communication or interaction between subsystems or cells. Under this framework, the problem of establishing the synchronization for delayed coupled nonlinear systems is transformed to solving a corresponding linear system of algebraic equations. We start by considering a cell-to-cell system under symmetric coupling to present the main idea of the approach. The framework is then extended to the N-cell system under circulant coupling. Delay-dependent, delay-independent, and network-scale–dependent criteria for global synchronization will be established, respectively. The developed scheme can accommodate a wide range of coupled systems. We demonstrate the applications of the present approach to establish synchronization for a gene regulation model, a neuronal model, and some neural networks.

Key words. coupled system, synchronization, delay, gene regulation model, neuronal model, neural network

AMS subject classifications. 34K18, 34K20, 92B20, 92C20

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1. Introduction.

Synchronization is a crucial and common phenomenon in various biological and physical systems. In many regions of the brain, synchronization activity has been observed and implicated as a correlate of behavior and cognition [64]. It is known that synchronization encourages the strengthening of mutual connections among neurons. Synchrony and synchronous oscillations are typical activities for gene expressions in cells under interaction. For example, in somitogenesis of vertebrate embryo, the cyclic genes express synchronous oscillations in neighboring cells at the tail bud of the presomitic mesoderm [25, 42, 50]. There are many other beautiful examples, including simultaneous flashing of fireflies, crickets chirping in unison, and synchronous activity of pacemaker cells in the heart; see [45, 46, 60, 61]. There is also a large number of studies on synchronization in engineering because of its importance in applications such as synchronized chaos employed in secret communication [14].

Time delays occur in the transcription and translation processes of somitogenesis due to synthesis and trafficking of macromolecules in cells; the lags have been estimated to be around tens of minutes in cell culture [28, 42]. For connected neurons, time delay occurs in the propagation of action potentials along the axon and the transmission of signals across the synapse

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[7, 13, 16]. Modeling interacted cells or neurons with delays thus becomes an important concern in studying the collective behaviors for coupled-cell systems. On the other hand, as time lag also occurs in transmitting signals among artificial neurons, delay has been incorporated into the neural network modeling [6, 9, 37, 38, 51, 66]. Indeed, delay can modify the collective dynamics for neural networks; for example, it can induce oscillation or change the stability of the stationary solution [9]. Delay can also induce synchronization [18, 39], asynchrony [9], and oscillation death [1]. Crook et al. [13] studied a continuum model of the cortex, with excitatory coupling and distance-dependent delays, and found that for small enough delays the synchronous oscillation is stable but for larger delays this oscillation loses stability to a traveling wave. Therefore, in addition to synchronization, it is also important to investigate the synchronous phases and their transitions. Developing effective mathematical methodologies and analytic tools elucidating the synchronous activities and collective behaviors with respect to various combinations of parameters, coupling strengths, and delay magnitudes remains an important research task.

It is interesting to explore the dynamical mechanisms underlying the behavior of networks of neurons and biological oscillators. Although the real network architecture can be extremely complicated, rich dynamics arising from the interaction of simple network motifs are believed to provide sources of activities similar to those in real-life systems. Synchronization in coupled dynamical systems has attracted a lot of attention in recent decades. The coupled systems that were most studied for synchronization are various neural network systems and chaotic oscillators. Among this research, some conclude local synchronization which is concerned with the stability of synchronization manifold or solution behavior in a neighborhood of certain synchronous solution, while others obtain global synchronization by showing that all solutions converge to the synchronization manifold or some synchronous solution.

The master stability function, developed by Pecora and Carroll [43, 44], is a well-known approach to studying local synchronization of coupled chaotic systems. This method is based on computing the Lyapunov exponent of the associated variational equation to determine the stability of the synchronization manifold for the coupled systems. However, such an approach leads to a necessary (instead of sufficient) condition for local synchronization; cf. [23].

Methodologies for concluding global synchronization largely involve the notion of Lyapunov functions. For example, Belykh, Belykh, and Hasler employed the "connection graph stability method" combined with the Lyapunov function approach to studying global synchronization in small-world networks of chaotic systems [3]. From the viewpoint of feedback control, Nijmeijer and collaborators introduced the notion of passivity and semipassivity and constructed a Lyapunov–Razumikhin function to study global synchronization in coupled systems [47, 58, 59]. Other works employing the Lyapunov function/functional technique include [7, 11, 29, 33, 35, 36, 48, 49, 67, 68].

Another approach to investigating synchronous oscillation in coupled systems, especially in neural networks, is to apply bifurcation theory to obtain the existence of synchronous periodic solution and use the normal form theory and the center manifold theorem to discuss its stability [6, 57]. However, linearization of delayed equations yields another complication, as the linearized system contains exponential functions which make the analysis and computation of characteristic values difficult, especially in the case of multiple delays; see the papers by Campbell and coworkers in [9, 10].

Most of the couplings in systems, especially the delayed systems, considered in the literature are linear or linearly diffusive. Such coupling terms are expressed by a summation of connection weights multiplied to the incoming signals from other units, or by coupling strength multiplied to the difference of two corresponding components. Notice that there is a significant difference between diffusive coupling and general nonlinear coupling. For a system comprising identical subsystems under diffusive coupling, its synchronous solution is also a solution for each individual (uncoupled) subsystem, as the coupling parts are annihilated at synchronous states. This is certainly not the case for the general nonlinear coupling scheme. For coupled systems with multiple delays, the analysis is even more complicated. There are nonlinear systems with more intricate coupling and multiple delays, which are distinguished from those systems mentioned above. One such system is the segmentation clock gene model for coupled cells, which describes the gene regulation for vertebrate embryos. In the presomitic mesoderm of zebrafish embryo, neighboring cells interact through delayed, intercellular positive feedback via Delta–Notch signaling [28, 42]. A system modeling the same clock genes was proposed by Uriu, Morishita, and Iwasa in [63], where more complicated nonlinear terms accounting for the Michaelis–Menten-type degradations and general transcription and translation functions with Hill coefficients are considered. Analytic study for such kinetic models is rather difficult, as mentioned in [2]. Other examples include the neuronal models coupled through chemical synapses and neural networks with nonlinear activation functions. We shall discuss these systems in later sections.

As described above, the Lyapunov function technique, used directly or indirectly, is a commonly and largely adopted approach for tackling synchronization problems, especially global synchronization, in dynamical systems with or without delays. On the one hand, finding a Lyapunov function in highly nonlinear systems, especially with multiple delays, and/or systems with multiple components of different types, seems rather infeasible. On the other hand, for systems with delay, synchronization results concluded from the Lyapunov function approach often reduce to the situation that every solution converges asymptotically to a unique synchronous equilibrium point [9, 69]. In addition, typically only delay-independent criteria can be derived under such an approach. Therefore, a mathematical approach to tackling asymptotic behaviors and synchronization for nonlinearly coupled systems, without resorting to the Lyapunov function and computations of the characteristic values, is an appealing advance.

In this investigation, we aim at establishing a general framework based on the idea of "sequential contracting" to study synchronization for coupled systems. We start by considering the synchronization for a pair of identical subsystems under a general coupling:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}_t, t) + \mathbf{G}(\mathbf{x}_t, \mathbf{y}_t, t), \\ \dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}_t, t) + \mathbf{G}(\mathbf{y}_t, \mathbf{x}_t, t), \end{cases} \tag{1.1}$$

where $t \geq t_0$, $\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{R}^n$, and $\mathbf{x}_t, \mathbf{y}_t \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^n)$ with $\tau_M \geq 0$, are defined by $\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta), \mathbf{y}_t(\theta) = \mathbf{y}(t+\theta)$ for $\theta \in [-\tau_M, 0]$, $\mathbf{F} = (F_1, F_2, \dots, F_n)$ is a smooth function which depicts the intrinsic dynamics of each subsystem, and a smooth function $\mathbf{G} = (G_1, G_2, \dots, G_n)$ expresses the interaction between two coupled subsystems. Herein, time delays in the range $[0, \tau_M]$ are considered in the subsystems and coupling terms, $(\mathbf{x}_t, \mathbf{y}_t)$ denotes the evolution of system (1.1) at time t from $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0})$ in $\mathcal{C}([-\tau_M, 0], \mathbb{R}^n)$, and $(\mathbf{x}(t), \mathbf{y}(t))$ is the corresponding solution of system (1.1). The present framework shall cover the ODE case, i.e., when $\tau_M = 0$, and (1.1) reduces to

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, \mathbf{y}, t), \\ \dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}, t) + \mathbf{G}(\mathbf{y}, \mathbf{x}, t), \end{cases}$$

where $(\mathbf{x}, \mathbf{y})$ lies in $\mathbb{R}^{2n}$ or a positively invariant subset of $\mathbb{R}^{2n}$.

Let us denote the synchronous set by

$$\mathcal{S} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid \mathbf{x} = \mathbf{y}\}. \tag{1.2}$$

We say that a solution of (1.1) is *synchronous* if it lies in $\mathcal{S}$ completely; a solution is *asymptotically synchronous* if its ω-limit set lies in $\mathcal{S}$. The coupled system (1.1) is said to attain *global synchronization* if every solution is asymptotically synchronous, i.e.,

$$x_i(t) - y_i(t) \to 0, \text{ as } t \to \infty, \text{ for all } i = 1, \dots, n,$$

for every solution $(\mathbf{x}(t), \mathbf{y}(t)) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))$ of system (1.1).

For an arbitrary solution $(\mathbf{x}(t), \mathbf{y}(t))$ of system (1.1), where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$, $\mathbf{y}(t) = (y_1(t), \dots, y_n(t))$, by setting $z_i(t) = x_i(t) - y_i(t)$, we shall consider the following difference-differential system corresponding to (1.1):

$$\dot{z}_i(t) = F_i(\mathbf{x}_t, t) + G_i(\mathbf{x}_t, \mathbf{y}_t, t) - F_i(\mathbf{y}_t, t) - G_i(\mathbf{y}_t, \mathbf{x}_t, t), \quad i = 1, \dots, n. \tag{1.3}$$

System (1.1) attains global synchronization if and only if $z_i(t) \to 0$, $i = 1, \dots, n$, as $t \to \infty$, for every $(z_1(t), \dots, z_n(t))$ satisfying system (1.3), defined from every solution $(\mathbf{x}(t), \mathbf{y}(t))$ of (1.1).

In the literature, studying the evolution for the difference of two corresponding components, such as (1.3), has been a primary target in tackling synchronization problems. The idea of sequential contracting provides a new treatment to analyze such difference-differential systems. This approach unfolds from constructing suitable lower and upper dynamics iteratively for (1.3). Effective designs of lower and upper dynamics can then capture the asymptotic behaviors of the coupled systems (1.1). Under different formulations of lower-upper dynamics, delay-dependent criteria and delay-independent criteria for synchronization of (1.1) can be derived, respectively. This approach also leads to a network-scale–dependent criterion for synchronization in network systems. The idea of sequential contracting is quite natural in the following sense. One starts from a preliminary attracting set of $\mathcal{S}$, which usually exists from the dissipative property in coupled systems which admit synchronization. We then formulate a criterion for contraction so that the dynamics converge to $\mathcal{S}$ through iteration arguments. Such a formulation imposes mild conditions, as the nonlinear terms in the equations are not overmanipulated by linearization or other treatments.

In section 2, we analyze the asymptotic behavior for a scalar equation associated with the difference-differential equation (1.3). The analysis provides a basis for investigating the synchronization of system (1.1). We present the main theorems for system (1.1) of two coupled subsystems in section 3. In subsection 3.1, we introduce the main conditions imposed on system (1.1). In subsection 3.2, two synchronization theorems for (1.1), one under a delay-dependent criterion and the other under a delay-independent criterion, are established successively through constructing two different lower-upper dynamics for (1.3). We then implement these theorems to establish the synchronization for coupled FitzHugh–Nagumo neurons under nonlinear coupling with discrete-time delay and distribution delay, respectively, in subsection 4.1. The synchronization for a cell-to-cell kinetic model of segmentation clock genes is demonstrated in subsection 4.2. We extend the framework to treat N-cell (unit) systems under circulant coupling in subsection 5.1. In subsection 5.2, we demonstrate this extension in a K-loop neural network. We compare the present approach with the methodologies for studying synchronization in the literature in section 6.

2. Formulation and component estimate.

This section is a preparation for the main theorems in section 3. Recall the difference-differential equation (1.3),

$$\dot{z}_i(t) = F_i(\mathbf{x}_t, t) + G_i(\mathbf{x}_t, \mathbf{y}_t, t) - F_i(\mathbf{y}_t, t) - G_i(\mathbf{y}_t, \mathbf{x}_t, t),$$

where $z_i(t) = x_i(t) - y_i(t)$, and $(\mathbf{x}(t), \mathbf{y}(t))$ with $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$, $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$, is an arbitrary solution of system (1.1). The key point of the present approach is to analyze the behavior of $z_i(t)$ through a manipulation of (1.3). To this end, we first consider the following scalar delay-differential equation.

We denote by t_0 the initial time and by $\tau_M \geq 0$ the upper bound of delay magnitude. Let $w(t)$ be a bounded continuous function for $t \geq t_0$, and let $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \mathcal{C}([-\tau_M, 0]; \mathbb{R}) \times \mathcal{C}([-\tau_M, 0]; \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Let $x_t, y_t \in \mathcal{C}([-\tau_M, 0]; \mathbb{R})$ for $t \geq t_0$, and set $x(t+\theta) = x_t(\theta)$, $y(t+\theta) = y_t(\theta)$ for $\theta \in [-\tau_M, 0]$; we assume that $x(t)$ and $y(t)$ eventually enter and then remain in some closed and bounded interval $[\check{q}, \hat{q}]$; namely, $x(t)$ and $y(t)$ lie in $[\check{q}, \hat{q}]$ for all $t \geq \tilde{t}_0$, for some $\tilde{t}_0 \geq t_0$. We suppose that $z(t) = x(t) - y(t)$ satisfies the following scalar equation:

$$\dot{z}(t) = h(x(t), y(t), t) + \tilde{h}(x_t, y_t, t) + w(t), \quad t \geq t_0. \tag{2.1}$$

We shall decompose (1.3), for each i, into an equation of the form (2.1), with the spirit of collecting the instantaneous self-feedback terms in h, delayed self-feedback terms in $\tilde{h}$, and cross-coupling terms in w. How (2.1) is connected to (1.3) exactly will be seen in section 3.1. We impose the following condition on the argument structure of h and $\tilde{h}$ and the boundedness of $\tilde{h}$.

Condition (H_0). There exist $\hat{\mu}, \check{\mu}, \check{\beta}, \hat{\beta} \in \mathbb{R}$, $\rho^h > 0$, and $0 \leq \bar{\tau} \leq \tau_M$ such that for each $\phi, \psi \in \{\varphi \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}) : \varphi(\theta) \in [\check{q}, \hat{q}], \theta \in [-\bar{\tau}, 0]\}$, the following properties hold for all $t \geq t_0$:

$$(\mathrm{H}_0 - \mathrm{i}) : \begin{cases} \check{\mu} \leq h(\phi(0), \psi(0), t)/[\phi(0) - \psi(0)] \leq \hat{\mu}, & \phi(0) - \psi(0) \neq 0, \\ h(\phi(0), \psi(0), t) = 0, & \phi(0) - \psi(0) = 0, \end{cases}$$

$$(\mathrm{H}_0 - \mathrm{ii}) : |\tilde{h}(\phi, \psi, t)| \leq \rho^h, \text{ and there exists a } \tau = \tau(\phi, \psi, t) \in [0, \bar{\tau}] \text{ such that}$$
$$\begin{cases} \check{\beta} \leq \tilde{h}(\phi, \psi, t)/[\phi(-\tau) - \psi(-\tau)] \leq \hat{\beta}, & \phi(-\tau) - \psi(-\tau) \neq 0, \\ \tilde{h}(\phi, \psi, t) = 0, & \phi(-\tau) - \psi(-\tau) = 0. \end{cases}$$

Herein, τ is a function of ϕ, ψ, and t in (H_0-ii); ϕ and ψ work as x_t and y_t in (2.1), respectively; in particular, $\phi(0)$ (resp., $\psi(0)$) corresponds to $x(t)$ (resp., $y(t)$), and $\phi(\theta)$ (resp., $\psi(\theta)$) corresponds to $x(t+\theta)$ (resp., $y(t+\theta)$) for $\theta \in [-\bar{\tau}, 0]$. Thus, in (2.1), condition (H_0)

basically indicates that the dynamics of $z(t)$, or $\dot{z}(t)$, are controlled by $z(t)$ via some upper and lower factors $\hat{\mu}$ and $\check{\mu}$, and by $z(t-\tau)$ via some upper and lower factors $\hat{\beta}$ and $\check{\beta}$. Moreover, $\tilde{h}(x_t, y_t, t)$, the delay effect contributed from x_t and y_t on $\dot{z}(t)$, is bounded. Since condition (H_0) is to be imposed for more than a specific $(x(t), y(t))$, we describe it by general notation ϕ and ψ.

The main result in this section asserts that there exists a bounded and closed interval containing zero to which every solution $z(t)$ of (2.1) converges, under some delay-dependent condition. A variant of this formulation leads to the same conclusion (with a different interval) under a delay-independent condition. We note that the notation t_0, $\tilde{t}_0$, $[\check{q}, \hat{q}]$, defined before introducing system (2.1), and $\hat{\mu}$, $\check{\mu}$, $\check{\beta}$, $\hat{\beta}$, ρ^h, $\bar{\tau}$, in condition (H_0), will be used throughout this section. For all $T \geq t_0$, we set

$$|w|^{\max}(T) := \sup\{|w(t)| : t \geq T\}, \ |w|^{\max}(\infty) := \lim_{T\to\infty} |w|^{\max}(T).$$

To capture the dynamics of system (2.1), we define the following functions:

$$\hat{h}(\xi) = \begin{cases} (\hat{\mu}+\hat{\beta})\xi + 3\rho^h + |w|^{\max}(t_0) & \text{for } \xi \geq 0, \\ (\check{\mu}+\hat{\beta})\xi + 3\rho^h + |w|^{\max}(t_0) & \text{for } \xi < 0, \end{cases}$$
$$\check{h}(\xi) = -\hat{h}(-\xi).$$

Obviously, $\hat{\mu}+\hat{\beta} \geq \check{\mu}+\check{\beta}$. If $\hat{\mu}+\hat{\beta} < 0$, then $\hat{h}(\xi) \geq \check{h}(\xi)$ for all $\xi \in \mathbb{R}$; moreover, $\hat{h}$ and $\check{h}$ are piecewise linear, are decreasing, and have unique zeros at $\hat{A}^h$ and $\check{A}^h$, respectively; see Figure 1. Notably,

$$\hat{h}(\check{A}^h) = -\check{h}(\hat{A}^h) = (\hat{\mu}+\check{\mu}+\hat{\beta}+\check{\beta})(3\rho^h + |w|^{\max}(t_0))/(\hat{\mu}+\hat{\beta}) > 0. \tag{2.2}$$

The following lemma asserts that functions $\hat{h}(\cdot) - \rho^h$ and $\check{h}(\cdot) + \rho^h$ provide preliminary upper and lower bounds for the dynamics of (2.1).

Lemma 2.1. *Assume that condition* (H_0) *holds and* $\hat{\mu}+\hat{\beta} < 0$. *If* $z(t)$ *satisfies* (2.1), *then*

$$\check{h}(z(t)) + \rho^h \leq \dot{z}(t) \leq \hat{h}(z(t)) - \rho^h \text{ for all } t \geq \tilde{t}_0 + \bar{\tau}. \tag{2.3}$$

Consequently, there exists a $T_{x,y} \geq \tilde{t}_0 + 2\bar{\tau}$ *such that* $z(t) \in [\check{A}^h, \hat{A}^h]$, *and* $|\dot{z}(t)| < \hat{h}(\check{A}^h)$, *for all* $t \geq T_{x,y} - \bar{\tau}$.

Proof. Let us verify (2.3). Recall that $x(t), y(t) \in [\check{q}, \hat{q}]$ for all $t \geq \tilde{t}_0$. For all $t \geq \tilde{t}_0 + \bar{\tau}$, $z(t) = x(t) - y(t)$ satisfies

$$\dot{z}(t) = h(x(t), y(t), t) + \tilde{h}(\tilde{x}_t, \tilde{y}_t, t) + w(t) + \tilde{h}(x_t, y_t, t) - \tilde{h}(\tilde{x}_t, \tilde{y}_t, t),$$

where $\tilde{x}_t(\theta) := x(t)$, $\tilde{y}_t(\theta) := y(t)$ for all $\theta \in [-\tau_M, 0]$ are constant in θ. Note that $x_t, y_t, \tilde{x}_t, \tilde{y}_t \in \{\phi \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}) : \phi(\theta) \in [\check{q}, \hat{q}], \theta \in [-\bar{\tau}, 0]\}$. If $x(t) \geq y(t)$, then $h(x(t), y(t), t) \leq \hat{\mu} \cdot z(t)$, $\tilde{h}(\tilde{x}_t, \tilde{y}_t, t) \leq \hat{\beta} \cdot z(t)$, and $\tilde{h}(x_t, y_t, t) - \tilde{h}(\tilde{x}_t, \tilde{y}_t, t) \leq 2\rho^h$ by condition (H_0). Consequently, $\dot{z}(t) \leq (\hat{\mu}+\hat{\beta}) \cdot z(t) + 2\rho^h + |w|^{\max}(t_0) =: \hat{h}(z(t)) - \rho^h$. On the other hand, if $x(t) < y(t)$, then $h(x(t), y(t), t) \leq \check{\mu} \cdot z(t)$, and $\tilde{h}(\tilde{x}_t, \tilde{y}_t, t) \leq \check{\beta} \cdot z(t)$. Therefore, $\dot{z}(t) \leq \hat{h}(z(t)) - \rho^h$. Similar arguments lead to $\check{h}(z(t)) + \rho^h \leq \dot{z}(t)$. From (2.3), we obtain $\dot{z}(t) \leq -\rho^h$ if $z(t) \geq \hat{A}^h$, and

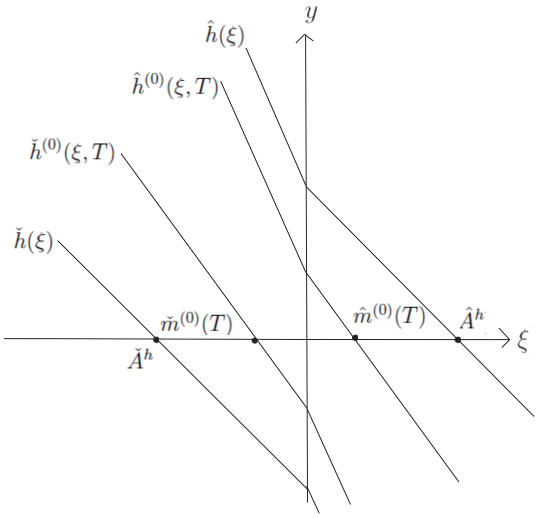

Figure 1. *Configurations of functions* $\hat{h}$, $\check{h}$, $\hat{h}^{(0)}(\cdot,T)$, *and* $\check{h}^{(0)}(\cdot,T)$.

$\dot{z}(t) \geq \rho^h$ if $z(t) \leq \hat{A}^h$, for $t \geq \tilde{t}_0 + \bar{\tau}$. Subsequently, there exists a $T_{x,y} \geq \tilde{t}_0 + 2\bar{\tau}$ such that $z(t) \in [\check{A}^h, \hat{A}^h]$, which in turn yields $|\dot{z}(t)| < \hat{h}(\check{A}^h)$ for all $t \geq T_{x,y} - \bar{\tau}$ by (2.3). ■

Herein, the notation $T_{x,y}$ indicates its dependence on $x(t)$ and $y(t)$. Now, let us consider the following condition for (2.1).

Condition (A1). $\hat{\mu} + \hat{\beta} < 0$ and $\bar{\beta}\bar{\tau} < 3\rho^h(\hat{\mu}+\hat{\beta})/[(\hat{\mu}+\check{\mu}+\hat{\beta}+\check{\beta})(3\rho^h + |w|^{\max}(t_0))]$, where $\bar{\beta} := \max\{|\check{\beta}|, |\hat{\beta}|\}$.

The latter inequality in condition (A1) requires that if the delayed effect $\tilde{h}$ in (2.1) exists, i.e., $\bar{\beta} \neq 0$, the allowable maximal magnitude of time lag $\bar{\tau}$ should be small enough. From (2.2), condition (A1) yields $\bar{\beta}\bar{\tau}\hat{h}(\check{A}^h) < 3\rho^h$, and there exists an $\varepsilon_0 > 0$ such that

$$\bar{\beta}\bar{\tau}\hat{h}(\check{A}^h) + \varepsilon_0 < 3\rho^h. \tag{2.4}$$

For each $T \geq t_0$, we further introduce the following functions:

$$\hat{h}^{(0)}(\xi,T) = \begin{cases} (\hat{\mu}+\hat{\beta})\xi + \bar{\beta}\bar{\tau}\hat{h}(\check{A}^h) + |w|^{\max}(T) + \varepsilon_0 & \text{for } \xi \geq 0, \\ (\check{\mu}+\check{\beta})\xi + \bar{\beta}\bar{\tau}\hat{h}(\check{A}^h) + |w|^{\max}(T) + \varepsilon_0 & \text{for } \xi < 0, \end{cases}$$
$$\check{h}^{(0)}(\xi,T) = -\hat{h}^{(0)}(-\xi,T).$$

Notably, condition (A1) implies (2.4), and

$$\check{h}(\xi) < \check{h}^{(0)}(\xi,T) < \hat{h}^{(0)}(\xi,T) < \hat{h}(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{2.5}$$

Let $\check{m}^{(0)}(T)$ (resp., $\hat{m}^{(0)}(T)$) be the unique solution of $\check{h}^{(0)}(\cdot,T) = 0$ (resp., $\hat{h}^{(0)}(\cdot,T) = 0$) lying in interval $[\check{A}^h, \hat{A}^h]$; see Figure 1. Notably, $\hat{m}^{(0)}(T) = -\check{m}^{(0)}(T) > 0$, and $[-\hat{m}^{(0)}(T), \hat{m}^{(0)}(T)] \subset [\check{A}^h, \hat{A}^h]$, by (2.5). Recall $T_{x,y}$ introduced in Lemma 2.1. The following lemma reveals that $\check{h}^{(0)}(\cdot,T) + \varepsilon_0$ and $\hat{h}^{(0)}(\cdot,T) - \varepsilon_0$ provide lower and upper bounds finer than $\check{h}(\cdot) + \rho^h$ and $\hat{h}(\cdot) - \rho^h$, respectively, for the dynamics of system (2.1), as time gets larger.

Lemma 2.2. *Assume that conditions* (H_0) *and* (A1) *hold. If* $z(t)$ *satisfies* (2.1), *then for each* $T \geq T_{x,y}$, *we have*

$$\check{h}^{(0)}(z(t),T)+\varepsilon_0 \leq \dot{z}(t) \leq \hat{h}^{(0)}(z(t),T)-\varepsilon_0 \quad \text{for all } t \geq T. \tag{2.6}$$

Consequently, $z(t)$ *eventually enters and stays afterward in* $[-\hat{m}^{(0)}(T), \hat{m}^{(0)}(T)]$.

Proof. By condition (H_0), there exist some $\mu(t)$ with $\check{\mu} \leq \mu(t) \leq \hat{\mu}$, $\beta(t)$ with $\check{\beta} \leq \beta(t) \leq \hat{\beta}$, and $\tau(t) := \tau(x_t, y_t, t) \leq \bar{\tau}$, such that the terms $h(x(t),y(t),t)$ and $\tilde{h}(x_t,y_t,t)$ in (2.1) become

$$h(x(t),y(t),t) = \mu(t)[x(t)-y(t)] = \mu(t)z(t),$$
$$\tilde{h}(x_t,y_t,t) = \beta(t)[x(t-\tau(t))-y(t-\tau(t))] = \beta(t)z(t-\tau(t)).$$

Thus, (2.1) can be rewritten as follows:

$$\dot{z}(t) = \mu(t)z(t)+\beta(t)z(t-\tau(t))+w(t). \tag{2.7}$$

For $t \geq T \geq T_{x,y}$, applying the mean value theorem to (2.7) yields

$$\dot{z}(t) = \mu(t)z(t)+\beta(t)[z(t)-\tau(t)\dot{z}(s)]+w(t),$$

where $s \geq t-\bar{\tau} \geq T_{x,y}-\bar{\tau}$; hence $|\dot{z}(s)| < \hat{h}(\check{A}^h)$ by Lemma 2.1. Consequently, if $z(t) \geq 0$, then $\dot{z}(t) \leq (\hat{\mu}+\hat{\beta})z(t)+\bar{\beta}\bar{\tau}\hat{h}(\check{A}^h)+|w|^{\max}(T) =: \hat{h}^{(0)}(z(t),T)-\varepsilon_0$; if $z(t)<0$, then $\dot{z}(t) \leq (\check{\mu}+\check{\beta})z(t)+\bar{\beta}\bar{\tau}\hat{h}(\check{A}^h)+|w|^{\max}(T) =: \hat{h}^{(0)}(z(t),T)-\varepsilon_0$. Hence, the right-hand inequality of (2.6) is verified. The left-hand one can be treated similarly. Since $\hat{m}^{(0)}(T)$ and $\check{m}^{(0)}(T)$ are the unique zeros of $\hat{h}^{(0)}(\cdot,T)$ and $\check{h}^{(0)}(\cdot,T)$, respectively, and $\hat{m}^{(0)}(T) = -\check{m}^{(0)}(T) > 0$, we conclude that $z(t)$ eventually enters and stays afterward in $[-\hat{m}^{(0)}(T), \hat{m}^{(0)}(T)]$, as depicted in Figure 1. ■

Lemmas 2.1 and 2.2 demonstrate the formulation of lower and upper bounds for the dynamics of (2.1) in succession. In the same spirit, we shall formulate finer lower and upper bounds iteratively to capture the asymptotic dynamics of (2.1). Now, let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a decreasing sequence with $\varepsilon_1 < \varepsilon_0$, and let $\varepsilon_k \to 0$ as $k \to \infty$. For $k \in \mathbb{N}$ and $T \geq t_0$, we define

$$\hat{h}^{(k)}(\xi,T) := \begin{cases} (\hat{\mu}+\hat{\beta})\xi+\bar{\beta}\bar{\tau}\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T)+|w|^{\max}(T)+\varepsilon_k, & \xi \geq 0, \\ (\check{\mu}+\check{\beta})\xi+\bar{\beta}\bar{\tau}\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T)+|w|^{\max}(T)+\varepsilon_k, & \xi < 0, \end{cases}$$
$$\check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T),$$

where $\check{m}^{(k)}(T) \leq 0$ is the unique zero of $\check{h}^{(k)}(\cdot,T)$, and $\hat{m}^{(k)}(T) = -\check{m}^{(k)}(T) \geq 0$ is the unique zero of $\hat{h}^{(k)}(\cdot,T)$. By arguments similar to Lemma 2.3 in [53], it can be shown that under condition (A1), for any fixed $T \geq t_0$, $\{\hat{h}^{(k)}(\cdot,T)|_{[\check{A}^h,\hat{A}^h]}\}_{k\geq 1}$ are uniformly bounded and equicontinuous; in addition, $\hat{h}^{(k)}(\cdot,T)$ is decreasing with respect to k. There exists a continuous function $\hat{h}^{(\infty)}(\cdot,T)$ defined on $[\check{A}^h,\hat{A}^h]$ such that

$$\hat{h}^{(k)}(\cdot,T) \downarrow \hat{h}^{(\infty)}(\cdot,T) \text{ uniformly on } [\check{A}^h,\hat{A}^h], \text{ as } k \to \infty, \tag{2.8}$$

by the Ascoli–Azela theorem. Since $\hat{h}^{(k)}(\cdot,T)$ is decreasing with respect to k, there exists an $m(T) \in \mathbb{R}$ such that $\hat{m}^{(k)}(T) \to m(T)$ decreasingly as $k \to \infty$, where $\hat{m}^{(k)}(T)$ is the unique zero of $\hat{h}^{(k)}(\cdot,T)$. With (2.8), $\hat{m}^{(k)}(T)\to m(T)$, and the continuity of $\hat{h}^{(k)}$ and $\hat{h}^{(\infty)}$, we can derive that $\hat{h}^{(\infty)}(\cdot,T)$ is a vertical shift of $\hat{h}^{(k)}(\cdot,T)$ and satisfies

$$\hat{h}^{(\infty)}(\xi,T)=\begin{cases}(\hat{\mu}+\hat{\beta})\xi+\bar{\beta}\bar{\tau}\hat{h}^{(\infty)}(-m(T),T)+|w|^{\max}(T), & \xi\ge 0,\\ (\check{\mu}+\check{\beta})\xi+\bar{\beta}\bar{\tau}\hat{h}^{(\infty)}(-m(T),T)+|w|^{\max}(T), & \xi<0,\end{cases}\tag{2.9}$$

where $m(T)$ is the unique zero to $\hat{h}^{(\infty)}(\cdot,T)$. Moreover, from the configuration of $\hat{h}^{(\infty)}(\cdot,T)$, it follows that

$$0\le m(T)=|w|^{\max}(T)/\{-\hat{\mu}-\hat{\beta}+\bar{\beta}\bar{\tau}(\check{\mu}+\hat{\mu}+\check{\beta}+\hat{\beta})\}.\tag{2.10}$$

The detailed computation for (2.10) is arranged in Appendix A. By (2.8) and that $|w|^{\max}(T)$ decreases with respect to T, we conclude that $\hat{m}^{(k)}(T)\to m(T)$ decreasingly, as $k\to\infty$. In addition, there exists an $\bar{m}\ge 0$, such that $m(T)\to\bar{m}$ decreasingly, as $T\to\infty$. Thus,

$$\cap_{k\ge 0,T\ge t_0}[-\hat{m}^{(k)}(T),\hat{m}^{(k)}(T)]=\cap_{T\ge t_0}[-m(T),m(T)]=[-\bar{m},\bar{m}].$$

It can be argued by induction that for arbitrarily fixed $T\ge T_{x,y}$, and $n\in\mathbb{N}$, there exists an increasing sequence $\{T_k\}_{k=0}^{n}$ with $T_{k+1}\ge T_k+\bar{\tau}$ for $k=0,1,\dots,n-1$, and $T_0\ge T+\bar{\tau}$, such that

$$\begin{cases}\check{h}^{(k)}(z(t),T)+\varepsilon_k\le\dot{z}(t)\le\hat{h}^{(k)}(z(t),T)-\varepsilon_k \text{ for } t\ge T_k+\bar{\tau},\ k=0,1,\dots,n-1,\\ z(t)\in[-\hat{m}^{(k)}(T),\hat{m}^{(k)}(T)] \text{ for } t\ge T_{k+1},\ k=0,1,\dots,n-1.\end{cases}$$

This then leads to the fact that $z(t)$ which satisfies (2.1) eventually enters and remains in $[-\hat{m}^{(k)}(T),\hat{m}^{(k)}(T)]$ for each $T\ge T_{x,y}$ and $k\in\mathbf{N}$ and hence converges to interval $[-\bar{m},\bar{m}]$ as $t\to\infty$. Based on these arguments, we conclude the following proposition.

Proposition 2.3. *Assume that conditions* (H_0) *and* (A1) *hold. If* $z(t)$ *satisfies* (2.1)*, then* $z(t)$ *converges to some interval* $[-\bar{m},\bar{m}]$ *as* $t\to\infty$*. Moreover,*

$$0\le\bar{m}\le\frac{|w|^{\max}(\infty)}{-\hat{\mu}-\hat{\beta}+\bar{\beta}\bar{\tau}(\check{\mu}+\hat{\mu}+\check{\beta}+\hat{\beta})}.$$

The conclusion in this proposition is $\bar{\tau}$-dependent. By recomposing the upper and lower functions (see Appendix B) and using arguments similar to those for Proposition 2.3, we can derive the following $\bar{\tau}$-independent conclusion.

Proposition 2.4. *If* $z(t)$ *satisfies* (2.1)*, then* $z(t)$ *converges to interval* $[-\tilde{m},\tilde{m}]$*, as* $t\to\infty$*, under condition* (H_0) *and*

$$\textit{condition (A2)}: 0\le\bar{\beta}<-\hat{\mu}/[1+|w|^{\max}(t_0)/\rho^h].$$

Moreover,

$$0\le\tilde{m}\le\frac{|w|^{\max}(\infty)}{-\hat{\mu}-\bar{\beta}}.$$

Remark 2.1. When introducing (2.1), $x(t)$ and $y(t)$ are assumed to enter and remain in $[\check{q},\hat{q}]$ eventually. Such an assumption can be weakened to that $x(t)$ and $y(t)$ converge to $[\check{q},\hat{q}]$ for the delay-independent result in Proposition 2.4.

3. Synchronization for coupled system (1.1).

We shall derive a delay-dependent criterion and a delay-independent criterion for the global synchronization of system (1.1), based on Propositions 2.3 and 2.4, respectively. Let $(\mathbf{x}_t, \mathbf{y}_t)$ be the solution evolved from an arbitrarily fixed initial condition, and let $(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))$ be the corresponding entire solution for system (1.1). Recall from (1.3) that $z_i(t) := x_i(t) - y_i(t)$ satisfies the following difference-differential system:

$$\begin{aligned} \dot{z}_i(t) &= F_i(\mathbf{x}_t, t) + G_i(\mathbf{x}_t, \mathbf{y}_t, t) - F_i(\mathbf{y}_t, t) - G_i(\mathbf{y}_t, \mathbf{x}_t, t) \\ &=: H_i(\mathbf{x}_t, \mathbf{y}_t, t), \ i = 1, \ldots, n. \end{aligned} \tag{3.1}$$

Our aim is to show that $z_i(t) \to 0$ for all $i = 1, \ldots, n$, as $t \to \infty$, to conclude the global synchronization for system (1.1). In subsection 3.1, we introduce two basic assumptions on system (1.1). In subsection 3.2, we establish the synchronization theorems for (1.1).

3.1. Dissipative and argument conditions.

We make two basic assumptions on system (1.1). The first is associated with the dissipative property of system (1.1), and the second is related to the argument structure for H_i in (3.1).

Assumption (D). All solutions of system (1.1) eventually enter and then remain in some compact set $\mathcal{Q} \times \mathcal{Q}$, where $\mathcal{Q} := [\check{q}_1, \hat{q}_1] \times \cdots \times [\check{q}_n, \hat{q}_n] \subset \mathbb{R}^n$.

Notably, under assumption (D), all solutions of system (1.1) exist on $[t_0, \infty)$. To introduce the second assumption, we decompose function H_i in (3.1) as

$$H_i(\Phi, \Psi, t) = h_i(\phi_i(0), \psi_i(0), t) + \tilde{h}_i(\phi_i, \psi_i, t) + w_i(\Phi, \Psi, t), \tag{3.2}$$

where $\Phi = (\phi_1, \ldots, \phi_n), \ \Psi = (\psi_1, \ldots, \psi_n) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^n)$. Herein, h_i (resp., $\tilde{h}_i$) refers to the instantaneous (resp., delayed) part of H_i contributed from ϕ_i and ψ_i, and w_i collects all cross-coupling terms. Such a decomposition for H_i is always achievable since a trivial case is $h_i = \tilde{h}_i \equiv 0$ and $w_i \equiv H_i$. A nontrivial decomposition for the coupled FitzHugh–Nagumo system (4.1) is illustrated in section 4.1. The following second assumption is associated with the argument structure of h_i, $\tilde{h}_i$ and the boundedness of $\tilde{h}_i$ and w_i.

Assumption (H). For $i = 1, \ldots, n$, there exist $\check{\mu}_i, \hat{\mu}_i, \hat{\beta}_i, \check{\beta}_i \in \mathbb{R}$, $\rho_i^h, \rho_i^w \geq 0$, $\bar{\mu}_{ij}$, $\bar{\beta}_{ij} \geq 0$, and $0 \leq \bar{\tau}_i, \bar{\tau}_{ij} \leq \tau_M$, $j \neq i$, such that for each $\Phi, \Psi \in \mathcal{C}_{\mathcal{Q}} := \{\tilde{\Phi} = (\phi_1, \ldots, \phi_n) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^n) : \phi_i(\theta) \in [\check{q}_i, \hat{q}_i], \text{ for all } \theta \in [-\bar{\tau}_i, 0], i = 1, \ldots, n\}$, the following three properties hold for all $t \geq t_0$:

$$(\mathrm{H-i}) : \begin{cases} \check{\mu}_i \leq h_i(\phi_i(0), \psi_i(0), t)/[\phi_i(0) - \psi_i(0)] \leq \hat{\mu}_i, & \phi_i(0) - \psi_i(0) \neq 0, \\ h_i(\phi_i(0), \psi_i(0), t) = 0, & \phi_i(0) - \psi_i(0) = 0, \end{cases}$$

$(\mathrm{H-ii})$: $|\tilde{h}_i(\phi_i, \psi_i, t)| \leq \rho_i^h$ and there exists $\tau_i = \tau_i(\phi_i, \psi_i, t) \in [0, \bar{\tau}_i]$, such that

$$\begin{cases} \check{\beta}_i \leq \tilde{h}_i(\phi_i, \psi_i, t)/[\phi_i(-\tau_i) - \psi_i(-\tau_i)] \leq \hat{\beta}_i, & \phi_i(-\tau_i) - \psi_i(-\tau_i) \neq 0, \\ \tilde{h}_i(\phi_i, \psi_i, t) = 0, & \phi_i(-\tau_i) - \psi_i(-\tau_i) = 0, \end{cases}$$

$(\mathrm{H-iii})$: $|w_i(\Phi, \Psi, t)| \leq \rho_i^w$ and there exists $\tau_{ij} = \tau_{ij}(\Phi, \Psi, t) \in [0, \bar{\tau}_{ij}], j \neq i$, such that

$$|w_i(\Phi, \Psi, t)| \leq \Sigma_{j \neq i} \{\bar{\mu}_{ij} |\phi_j(0) - \psi_j(0)| + \bar{\beta}_{ij} |\phi_j(-\tau_{ij}) - \psi_j(-\tau_{ij})|\}.$$

In practical application, assumption (H) can be realized by suitable manipulation on (3.2) through some estimates, as all solutions of system (1.1) eventually stay in compact set $\mathcal{Q} \times \mathcal{Q}$

in $\mathbb{R}^{2n}$, under assumption (D). Thus $\mathcal{C}_\mathcal{Q}$ depends on the compact set $\mathcal{Q}$ in assumption (D), and (H-i) and (H-ii) are multiple-component versions of conditions (H_0-i) and (H_0-ii) in section 2, respectively. For later use, we set

$$\bar{L}_{ij} := \bar{\mu}_{ij} + \bar{\beta}_{ij}. \tag{3.3}$$

Let us explain the connection of assumption (H) to system (3.1). In assumption (H), Φ (resp., Ψ) plays the role of $\mathbf{x}_t$ (resp., $\mathbf{y}_t$), $\phi_i(0)$ (resp., $\psi_i(0)$) of $x_i(t)$ (resp., $y_i(t)$), and $\phi_i(\theta)$ (resp., $\psi_i(\theta)$) of $x_i(t+\theta)$ (resp., $y_i(t+\theta)$) in system (3.1). Assumption (H) basically asserts that $\dot{z}_i(t)$ in system (3.1) is dominated by $z_i(t)$ via some lower and upper factors $\check{\mu}_i$ and $\hat{\mu}_i$ (see (H-i)), by $z_i(t-\tau_i)$ via some lower and upper factors $\check{\beta}_i$ and $\hat{\beta}_i$ (see (H-ii)), and by $|z_j(s)|$, $j \neq i$, when $s = t$ and $s = t - \tau_{ij}$ ($t - \tau_{ij}$ is certain uniformly bounded delayed time) via some upper factors (see (H-iii)). Actually, assumption (H) strongly relies on assumption (D) under which $\tilde{h}_i$ and w_i are bounded on set $\mathcal{C}_\mathcal{Q}$, and hence such argument conditions on h_i, $\tilde{h}_i$, and w_i can be verified by applying the mean value theorem basically. On the other hand, formulating proper forms of h_i, $\tilde{h}_i$, and w_i for a considered system is also important for assumption (H) to be met.

3.2. Synchronization for system (1.1). In this section, we shall establish the global synchronization of system (1.1) under assumptions (D) and (H). By (3.2), we can rewrite the difference-differential system (3.1) as follows:

$$\dot{z}_i(t) = h_i(x_i(t), y_i(t), t) + \tilde{h}_i((\mathbf{x}_t)_i, (\mathbf{y}_t)_i, t) + w_i(t), \tag{3.4}$$

where we regard $w_i(\mathbf{x}_t, \mathbf{y}_t, t)$ as a function of t, i.e., $w_i(t) := w_i(\mathbf{x}_t, \mathbf{y}_t, t)$, as $(\mathbf{x}_t, \mathbf{y}_t)$ is the solution evolved from a fixed initial condition, mentioned in section 3.1. Under assumptions (D) and (H), each ith component in (3.4) is in the form of (2.1) and satisfies condition (H_0) with $\check{\mu} = \check{\mu}_i$, $\hat{\mu} = \hat{\mu}_i$, $\rho^h = \rho^h_i$, $\check{\beta} = \check{\beta}_i$, $\hat{\beta} = \hat{\beta}_i$, $\bar{\tau} = \bar{\tau}_i$. Now, let us introduce the multi-component versions of conditions (A1) and (A2).

Condition (S1). $\hat{\mu}_i + \hat{\beta}_i < 0$ and $\bar{\beta}_i \bar{\tau}_i < \tau^S_i$ for all $i = 1, \dots, n$, where

$$\bar{\beta}_i := \max\{|\check{\beta}_i|, |\hat{\beta}_i|\}, \ \tau^S_i := \frac{3\rho^h_i(\hat{\mu}_i + \hat{\beta}_i)}{(\hat{\mu}_i + \check{\mu}_i + \hat{\beta}_i + \check{\beta}_i)(3\rho^h_i + \rho^w_i)}.$$

Condition (S2). $\bar{\beta}_i < -\hat{\mu}_i/(1 + \rho^w_i/\rho^h_i)$ for all $i = 1, \dots, n$.

Note that condition (S1) is delay-dependent, while condition (S2) is delay-independent. Assume that condition (S1) holds; by Proposition 2.3, for each $i = 1, \dots, n$, there exists an interval $I_i := [-\bar{m}_i, \bar{m}_i]$, to which $z_i(t)$ converges, as $t \to \infty$; moreover,

$$0 \le \bar{m}_i \le |w_i|^{\max}(\infty)/\eta_i, \tag{3.5}$$

where

$$\eta_i := -\hat{\mu}_i - \hat{\beta}_i + \bar{\beta}_i \bar{\tau}_i(\check{\mu}_i + \hat{\mu}_i + \check{\beta}_i + \hat{\beta}_i). \tag{3.6}$$

The following proposition shows that $\bar{m}_i$ can be further estimated iteratively.

Proposition 3.1. *Assume that condition* (S1) *holds. Then for each* $i = 1, \dots, n$*, there exists a sequence* $\{m_i^{(k)}\}_{k=1}^{\infty}$ *which satisfies*

$$\bar{m}_i \le m_i^{(k)} = \left[\sum_{j=1}^{i-1} \bar{L}_{ij} m_j^{(k)} + \sum_{j=i+1}^{n} \bar{L}_{ij} m_j^{(k-1)} \right] \Big/ \eta_i \tag{3.7}$$

for $k \ge 1$*, where* $m_i^{(0)} := \rho_i^w / \eta_i$ *and* $\bar{L}_{ij}$ *is defined in* (3.3).

Proof. We prove the proposition by induction and sketch the main process. Under assumption (H), $|w_i|^{\max}(\infty) \le \rho_i^w$; consequently, $\bar{m}_i \le m_i^{(0)}$ for all $i = 1, \dots, n$. Assume that $m_i^{(k)}$ in (3.7) have been defined, and hence $z_i(t)$ converges to $[-m_i^{(k)}, m_i^{(k)}]$, as $t \to \infty$, for $k = 1, \dots, k_0 - 1, i = 1, \dots, n$, and $k = k_0$, $i = 1, \dots, \ell - 1 < n$. By condition (H-iii),

$$|w_\ell(t)| = |w_\ell(\mathbf{x}_t, \mathbf{y}_t, t)| \le \Sigma_{j \ne \ell} \{ \bar{\mu}_{\ell j} |z_j(t)| + \bar{\beta}_{\ell j} |z_j(t - \tau_{\ell j}(\mathbf{x}_t, \mathbf{y}_t, t))| \};$$

then, $|w_\ell|^{\max}(\infty) \le (\Sigma_{j=1}^{\ell-1} \bar{L}_{\ell j} m_j^{(k_0)} + \Sigma_{j=\ell+1}^{n} \bar{L}_{\ell j} m_j^{(k_0-1)})$; hence

$$0 \le \bar{m}_\ell \le |w_\ell|^{\max}(\infty)/\eta_\ell \le [\Sigma_{j=1}^{\ell-1} \bar{L}_{\ell j} m_j^{(k_0)} + \Sigma_{j=\ell+1}^{n} \bar{L}_{\ell j} m_j^{(k_0-1)}]/\eta_\ell =: m_\ell^{(k_0)}.$$

This completes the proof. ∎

We observe that $\{m_i^{(k)} \mid i = 1, 2, \dots, n\}$ in Proposition 3.1 is exactly the Gauss–Seidel iteration for solving the linear system

$$\mathbf{M}\mathbf{v} = \mathbf{0}, \tag{3.8}$$

where

$$\mathbf{M} := D_{\mathbf{M}} - L_{\mathbf{M}} - U_{\mathbf{M}} = [m_{ij}]_{1 \le i,j \le n}, \; m_{ii} = \eta_i, \; m_{ij} = -\bar{L}_{ij}, \text{ for } i \ne j, \tag{3.9}$$

and $D_{\mathbf{M}}$, $-L_{\mathbf{M}}$, and $-U_{\mathbf{M}}$ represent the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\mathbf{M}$, respectively; $\bar{L}_{ij}$ and η_i are defined in (3.3) and (3.6), respectively. For each $i = 1, 2, \dots, n$, $z_i(t) = x_i(t) - y_i(t)$ converges to $[-\bar{m}_i, \bar{m}_i]$ as $t \to \infty$, and $\bar{m}_i \le m_i^{(k)}$ for all k. Thereby, the problem of synchronization for system (1.1) reduces to solving the linear problem (3.8). Restated, system (1.1) achieves global synchronization if $m_i^{(k)} \to 0$, as $k \to \infty$, for all $i = 1, 2, \dots, n$. One sufficient condition for the convergence of the Gauss–Seidel iteration for (3.8) is the strict diagonal-dominance of $\mathbf{M}$, which is straightforward to verify. However, for some systems, such as Example 4.1, such a condition is too strong a criterion for synchronization. On the other hand, it is well known that the necessary and sufficient condition for the convergence of the Gauss–Seidel iteration for (3.8) is that the absolute magnitudes of all eigenvalues of the iteration matrix $(D_{\mathbf{M}} - L_{\mathbf{M}})^{-1} U_{\mathbf{M}}$ are less than unity; see [27]. Based on such a condition, we obtain the main results in this investigation. The assertion shall be derived by computing the eigenvalues for certain corresponding matrices. Other criteria for the convergence of the Gauss–Seidel method [22, 27] may provide conditions which are easier to verify, without computing the eigenvalues of these matrices. We assume that system (1.1) satisfies assumptions (D) and (H).

Theorem 3.2. *Assume that condition* (S1) *holds. Then system* (1.1) *achieves global synchronization if the Gauss–Seidel iteration for linear system* (3.8) *converges to zero, the unique solution of* (3.8), *or, equivalently,*

$$\max_{1\leq i\leq n}\{|\lambda_i|:\ \lambda_i:\ eigenvalue\ of\ (D_{\mathbf{M}}-L_{\mathbf{M}})^{-1}U_{\mathbf{M}}\}<1.$$

By Proposition 2.4 and arguments similar to those for Proposition 3.1 and Theorem 3.2, we can derive the delay-independent criterion for the synchronization of system (1.1).

Theorem 3.3. *Assume that condition* (S2) *holds. Then system* (1.1) *achieves global synchronization if the Gauss–Seidel iteration for linear system*

$$\begin{aligned}&\tilde{\mathbf{M}}\mathbf{v}=\mathbf{0},\\&\tilde{\mathbf{M}}:=[\tilde{m}_{ij}]_{1\leq i,j\leq n},\tilde{m}_{ii}=-\hat{\mu}_i-\bar{\beta}_i,\tilde{m}_{ij}=-\bar{L}_{ij},\ for\ i\neq j,\end{aligned}\tag{3.10}$$

converges to zero, the unique solution of (3.10), *or, equivalently,*

$$\max_{1\leq i\leq n}\{|\lambda_i|:\ \lambda_i:\ eigenvalue\ of\ (D_{\tilde{\mathbf{M}}}-L_{\tilde{\mathbf{M}}})^{-1}U_{\tilde{\mathbf{M}}}\}<1,$$

where $D_{\tilde{\mathbf{M}}}$, $-L_{\tilde{\mathbf{M}}}$, and $-U_{\tilde{\mathbf{M}}}$ are the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\tilde{\mathbf{M}}$, respectively.

Remark 3.1. (i) For the delay-independent result in Theorem 3.3, assumption (D) can be relaxed to that all solutions of system (1.1) converge to $\mathcal{Q}\times\mathcal{Q}$, as $t\to\infty$; see Remark 2.1. (ii) The contents of matrices $\mathbf{M}$ and $\tilde{\mathbf{M}}$ actually reflect the structure of the coupling configuration. (iii) Let us translate the notation and theory to the ODE case. Consider $z_i(t)=x_i(t)-y_i(t)$, which satisfies

$$\begin{aligned}\dot{z}_i(t)&=F_i(\mathbf{x},t)+G_i(\mathbf{x},\mathbf{y},t)-F_i(\mathbf{y},t)-G_i(\mathbf{y},\mathbf{x},t),\\&=h_i(x_i,y_i,t)+w_i(\mathbf{x},\mathbf{y},t),\end{aligned}$$

where $\tilde{h}_i=0$ in (3.2). In assumption (H), $\mathcal{C}_{\mathcal{Q}}$ is replaced by $\mathcal{Q}$, (H-i) becomes $\check{\mu}_i\leq h_i(x_i,y_i,t)/[x_i-y_i]\leq\hat{\mu}_i$ if $x_i-y_i\neq 0$ and $h_i(x_i,y_i,t)=0$ if $x_i-y_i=0$, (H-ii) is not needed, and (H-iii) is adjusted to $|w_i(\mathbf{x},\mathbf{y},t)|\leq\rho_i^w$ and $|w_i(\mathbf{x},\mathbf{y},t)|\leq\Sigma_{j\neq i}\bar{\mu}_{ij}|x_j-y_j|$. The matrix $\mathbf{M}$ in (3.9) becomes identical to matrix $\tilde{\mathbf{M}}$ in (3.10), and Theorem 3.2 reduces to Theorem 3.3.

The idea of sequential contracting was applied to study the global synchronization and asymptotic phases in a basic neural network with nearest-neighbor coupling in [53]. Therein, each unit of the coupled system is a scalar equation. In this paper, we have extended this idea to coupled systems in the form (1.1), where each unit itself is a system of differential equations and may contain intrinsic delays. Herein, we have established a general framework to accommodate a variety of nonlinearly coupled systems for studying synchronization. Under this framework, the problem of establishing synchronization for systems under delayed and nonlinear coupling was transformed to solving a corresponding linear system of algebraic equations. In the process, we have improved the formulation and analysis so that the convergence of the corresponding Gauss–Seidel iteration is determined by the optimal condition (both sufficient and necessary). This has enhanced the applicability of the synchronization theory, as shown in the following sections.

4. Implementation of approach. We shall apply the theory developed in section 3 to establish synchronization for two models. We shall examine assumptions (D) and (H), and condition (S1) or (S2), to apply Theorem 3.2 or 3.3. We illustrate the applications with the classical FitzHugh–Nagumo neuronal model and a representative gene regulation model on the segmentation clock genes in zebrafish in subsections 4.1 and 4.2, respectively.

4.1. Coupled FitzHugh–Nagumo neurons. The FitzHugh–Nagumo model was first suggested by FitzHugh in 1961 [19], and its equivalent circuit was created by Nagumo, Arimoto, and Yoshizawa in 1962 [40] to describe a prototype of excitable systems. FitzHugh–Nagumo equations, while modified from the van der Pol equation, capture the essence of the cubic nullcline nature of the voltage-component in the simplified Hodgkin–Huxley equations; see [17].

Let us consider the excitable FitzHugh–Nagumo system coupled with time delay [6],

$$\begin{cases} \dot{x_1}(t) = -x_1^3(t) + (a+1)x_1^2(t) - ax_1(t) - x_2(t) + cf(y_1(t-\tau)), \\ \dot{x_2}(t) = bx_1(t) - \gamma x_2(t), \\ \dot{y_1}(t) = -y_1^3(t) + (a+1)y_1^2(t) - ay_1(t) - y_2(t) + cf(x_1(t-\tau)), \\ \dot{y_2}(t) = by_1(t) - \gamma y_2(t), \end{cases} \tag{4.1}$$

where a, b, $\gamma > 0$, and $c > 0$ is the coupling strength; the sigmoidal coupling function f lies in the following class:

$$\{f \in C^1 : f(0) = 0, \delta := f'(0) > f'(\xi) > 0, |f(\xi)| < \rho \text{ for } \xi \neq 0\}. \tag{4.2}$$

In system (4.1), the individual dynamics are governed by the FitzHugh–Nagumo neuron [6, 19, 40]:

$$\begin{cases} \dot{u}(t) = -u^3(t) + (a+1)u^2(t) - au(t) - v(t), \\ \dot{v}(t) = bu(t) - \gamma v(t). \end{cases} \tag{4.3}$$

In referring to the notation in (1.1), $\mathbf{F} = (F_1, F_2)$ is now

$$F_1(\Phi, t) = -\phi_1^3(0) + (a+1)\phi_1^2(0) - a\phi_1(0) - \phi_2(0), \tag{4.4}$$

$$F_2(\Phi, t) = b\phi_1(0) - \gamma\phi_2(0); \tag{4.5}$$

the two subsystems are connected via a sigmoidal coupling, a simplification of synaptic coupling, with time delay, i.e., $\mathbf{G} = (G_1, G_2)$, and

$$G_1(\Phi, \Psi, t) = cf(\psi_1(-\tau)), \tag{4.6}$$

$$G_2(\Phi, \Psi, t) = 0, \tag{4.7}$$

where $\Phi = (\phi_1, \phi_2), \Psi = (\psi_1, \psi_2) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^2)$. In (4.6), the fixed time delay τ is of discrete-time type. In reality, time delay is likely varying each time an action potential is propagated from neurons, and incorporating a distribution of delays to represent the time lags in some range of values with some associated probability distribution is an alternative formulation [7]. In this situation, the term $f(\psi_1(t-\tau))$ in (4.6) can be modified to

$$\int_0^\tau f(\psi_1(-\sigma))\mathcal{K}(\sigma)\mathrm{d}\sigma, \tag{4.8}$$

where function $\mathcal{K}$ is the kernel of the distribution representing the probability density function of time delay. Equation (4.3), a single FitzHugh–Nagumo neuron, can exhibit excitable behavior, in the sense that a small perturbation away from its quiescent state can result in a large excursion of its potential before returning to the quiescent state [24]. It was indicated in [6] that the paradigmatic example of the FitzHugh–Nagumo system in the form of (4.3) does not admit periodic solutions for any parameters a, b, γ and exhibits excitable behaviors clearly for certain parameter ranges, for instance,

$$b > \gamma^2,\ a \gg b,\ a \gg \gamma; \tag{4.9}$$

in particular, the only attractor is in the form of a stable equilibrium at the origin if

$$4b/\gamma > (a-1)^2. \tag{4.10}$$

The investigation of the behavior of the coupled FitzHugh–Nagumo system, which takes into account time delays in signal transmission, has been a subject of considerable interest. The previous works [6, 4, 5, 30] focus on the stability of the trivial equilibrium and delay-induced or coupling-induced bifurcation, which gives rise to synchronous or asynchronous oscillation. The stable synchronous periodic solution for system (4.1) was investigated in [6]. Through numerical simulation, it was shown that the system exhibits global convergence to this periodic solution. However, analytical evidence for this global dynamics has been lacking. In [70], via the method of the Lyapunov functional, synchronization conditions for the system consisting of three FitzHugh–Nagumo neurons with delayed coupling and smooth sigmoidal amplification functions were derived. However, the arguments strongly relied on additional consideration of the instantaneous self-feedback term in the coupling and hence provided a delay-independent criterion. Indeed, the existing analytical tools for studying global dynamics and synchronization for neuronal models with nonlinear and delayed coupling are rather limited. Herein, we shall derive a delay-dependent criterion and establish the global synchronization for system (4.1). Our approach can also establish delay-independent and delay-dependent global synchronization for the model considered in [70].

A nontrivial decomposition in the form of (3.2) for the coupled FitzHugh–Nagumo system (4.1) is formulated as follows: From (4.4)–(4.7),

$$\begin{aligned} H_1(\Phi,\Psi,t) = &-\phi_1^3(0) + (a+1)\phi_1^2(0) - a\phi_1(0) - [-\psi_1^3(0) + (a+1)\psi_1^2(0) - a\psi_1(0)] \\ &+ c[f(\psi_1(-\tau)) - f(\phi_1(-\tau))] - [\phi_2(0) - \psi_2(0)]; \end{aligned}$$

consequently, we set

$$h_1(\phi_1(0),\psi_1(0),t) = p(\phi_1(0)) - p(\psi_1(0)), \tag{4.11}$$

$$\tilde{h}_1(\phi_1,\psi_1,t) = c[f(\psi_1(-\tau)) - f(\phi_1(-\tau))], \tag{4.12}$$

$$w_1(\Phi,\Psi,t) = -[\phi_2(0) - \psi_2(0)], \tag{4.13}$$

where $p(\xi) := -\xi^3 + (a+1)\xi^2 - a\xi$. On the other hand, from

$$H_2(\Phi,\Psi,t) = -\gamma[\phi_2(0) - \psi_2(0)] + b[\phi_1(0) - \psi_1(0)], \tag{4.14}$$

we have

$$h_2(\phi_2(0), \psi_2(0), t) = -\gamma[\phi_2(0) - \psi_2(0)], \tag{4.15}$$
$$\tilde{h}_2(\phi_2, \psi_2, t) \equiv 0, \tag{4.16}$$
$$w_2(\Phi, \Psi, t) = b[\phi_1(0) - \psi_1(0)]. \tag{4.17}$$

Notably, by the mean value theorem, h_1 in (4.11) and $\tilde{h}_1$ in (4.12) can be written in the following form:

$$h_1(\phi_1(0), \psi_1(0), t) = [-3s_1^2 + 2(a+1)s_1 - a][\phi_1(0) - \psi_1(0)], \tag{4.18}$$
$$\tilde{h}_1(\phi_1, \psi_1, t) = -cf'(s_2)[\phi_1(-\tau) - \psi_1(-\tau)] \tag{4.19}$$

for some s_1 between $\phi_1(0)$ and $\psi_1(0)$, and s_2 between $\phi_1(-\tau)$ and $\psi_1(-\tau)$. In observing (4.11)–(4.19), we see that h_i (resp., $\tilde{h}_i$) can be transformed into a multiple of $\phi_i(0) - \psi_i(0)$ (resp., $\phi_i(-\tau) - \psi_i(-\tau)$), and the ratio can be further estimated. Notice that the terms $-3s_1^2 + 2(a+1)s_1 - a$ in (4.18) and $-cf'(s_2)$ in (4.19) are bounded when s_1, s_2 are restricted to some compact set in $\mathbb{R}$. On the other hand, roughly speaking, w_i can be transformed into a linear combination of $\phi_j(\cdot) - \psi_j(\cdot)$, $j \neq i$.

Now, we show that system (4.1) satisfies assumptions (D) and (H). We define, for $k \in \mathbb{N}$,

$$P^{(k)}(\xi) := -\xi^4 + (a+1)\xi^3 - a\xi^2 + |c\rho^{(k-1)}\xi|, \tag{4.20}$$

where $\rho^{(0)} := \rho$, and

$$\rho^{(k)} := \max\{|f(\xi)| : \xi \in [-\sqrt{\gamma^2 + b}\bar{q}^{(k)}/\gamma, \sqrt{\gamma^2 + b}\bar{q}^{(k)}/\gamma]\}, \tag{4.21}$$
$$\bar{q}^{(k)} := \max\{|\xi| : P^{(k)}(\xi) = 0\}. \tag{4.22}$$

Herein, the parameters a, b, c, γ, ρ and function f were introduced in (4.1) and (4.2).

Lemma 4.1. *All solutions of system* (4.1) *eventually enter and then remain in* $\tilde{Q}^{(k)} \times \tilde{Q}^{(k)}$, *for each* $k \in \mathbb{N}$, *where*

$$\tilde{Q}^{(k)} := [-\sqrt{\gamma^2 + b}\bar{q}^{(k)}/\gamma, \sqrt{\gamma^2 + b}\bar{q}^{(k)}/\gamma] \times [-b\bar{q}^{(k)}/\gamma, b\bar{q}^{(k)}/\gamma].$$

The proof of Lemma 4.1 is arranged in Appendix C. Actually, $\bar{q}^{(k)}$ are well defined for all $k \in \mathbb{N}$ and are strictly decreasing with respect to k. Subsequently, for larger k, $\tilde{Q}^{(k)}$ provides a smaller attracting region for the dynamics of system (4.1) and hence relaxes the conditions for our synchronization formulation. Throughout this subsection, we consider that system (4.1) satisfies assumption (D) with $\mathcal{Q} = \tilde{Q}^{(k)} =: \mathcal{Q}^*$ for some fixed k. In some cases (see Example 4.1), one does need larger k to meet the synchronization criterion. Accordingly, the evolutions for each subsystem in (4.1) will eventually enter and remain in the set:

$$\mathcal{C}^*_{\mathcal{Q}} = \{\Phi = (\phi_1, \phi_2) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^2) : \phi_i(\theta) \in [-q_i^*, q_i^*],\ i = 1, 2,\ \theta \in [-\tau, 0]\},$$

where

$$q_1^* := \sqrt{\gamma^2 + b}\bar{q}^{(k)}/\gamma, q_2^* := b\bar{q}^{(k)}/\gamma,$$

and $\bar{q}^{(k)}$ is defined in (4.22). Below, let us show that system (4.1) actually satisfies assumption (H). For all $t \geq t_0$, and $\Phi = (\phi_1, \phi_2), \Psi = (\psi_1, \psi_2) \in \mathcal{C}^*_{\mathcal{Q}}$,

$$-q_i^* \leq \phi_i(\theta), \psi_i(\theta) \leq q_i^* \quad \text{for } \theta \in [-\tau, 0],\ i = 1, 2. \tag{4.23}$$

Accordingly, by the definitions of $\tilde{h}_i$ and w_i in (4.12), (4.16) and (4.13), (4.17) for system (4.1), we obtain

$$\begin{aligned} &|\tilde{h}_1(\phi_1, \psi_1, t)| \leq 2cM_f,\ |\tilde{h}_2(\phi_2, \psi_2, t)| = 0, \\ &|w_1(\Phi, \Psi, t)| \leq 2q_2^*,\ |w_2(\Phi, \Psi, t)| \leq 2bq_1^*, \end{aligned}$$

where

$$M_f := \max\{|f(\xi)| : \xi \in [-q_1^*, q_1^*]\}. \tag{4.24}$$

This yields the boundedness of $\tilde{h}_i$ and w_i in assumption (H). The argument conditions for functions h_i, $\tilde{h}_i$, and w_i formulated in (4.13) and (4.15)–(4.19) can then be confirmed. Let us examine these conditions for h_1 and $\tilde{h}_1$, as the other cases are simpler. Note that the terms s_i, $i = 1, 2$, in h_1 and $\tilde{h}_1$ defined in (4.18) and (4.19) both satisfy $-q_1^* \leq s_i \leq q_1^*$ due to (4.23). It follows from a direct computation that

$$\begin{aligned} &\lambda \leq -3s_1^2 + 2(a+1)s_1 - a \leq (a^2 - a + 1)/3, \\ &d_f \leq f'(s_2) \leq \delta, \end{aligned}$$

where

$$\lambda := -3(q_1^*)^2 - 2(a+1)q_1^* - a, \tag{4.25}$$

$$d_f := \min\{f'(\xi) : \xi \in [-q_1^*, q_1^*]\} > 0. \tag{4.26}$$

Consequently, h_1 in (4.18) and $\tilde{h}_1$ in (4.19) satisfy, respectively,

$$\begin{aligned} &\lambda \leq h_1(\phi_1(0), \psi_1(0), t)/[\phi_1(0) - \psi_1(0)] \leq (a^2 - a + 1)/3 \text{ if } \phi_1(0) - \psi_1(0) \neq 0, \\ &-c\delta \leq \tilde{h}_1(\phi_1, \psi_1, t)/[\phi_1(-\tau) - \psi_1(-\tau)] \leq -cd_f \text{ if } \phi_1(-\tau) - \psi_1(-\tau) \neq 0. \end{aligned}$$

From these arguments, we conclude the following lemma.

Lemma 4.2. *System* (4.1) *satisfies assumption* (H) *with* $\check{\mu}_1 = \lambda$, $\hat{\mu}_1 = (a^2 - a + 1)/3$, $\check{\mu}_2 = \hat{\mu}_2 = -\gamma$, $\check{\beta}_1 = -c\delta$, $\hat{\beta}_1 = -cd_f$, $\bar{\tau}_1 = \tau$, $\rho_1^h = 2cM_f$, $\tilde{h}_2 \equiv 0$, $\rho_1^w = 2q_2^*$, $\rho_2^w = 2bq_1^*$, $\bar{\mu}_{12} = 1$, $\bar{\mu}_{21} = b$, *and* $\bar{\beta}_{12} = \bar{\beta}_{21} = 0$, *where* M_f, λ, *and* d_f *are defined in* (4.24), (4.25), *and* (4.26), *respectively.*

Assumptions (D) and (H) are thus satisfied for system (4.1) by Lemmas 4.1 and 4.2. By applying Theorem 3.2, we derive a delay-dependent criterion for synchronization of system (4.1).

Theorem 4.3. *System* (4.1), *the two FitzHugh–Nagumo neurons under delayed sigmoidal coupling, achieves global synchronization if*

$$c > [b/\gamma + (a^2 - a + 1)/3]/d_f \geq 0 \text{ and } \tau < \min\{\tau_1^F, \tau_2^F\}, \tag{4.27}$$

where

$$\tau_1^F := \frac{3M_f[(a^2-a+1)/3 - cd_f]}{\delta[(a^2-a+1)/3+\lambda - c(d_f+\delta)](3cM_f+q_2^*)},$$
$$\tau_2^F := \frac{b/\gamma - cd_f + (a^2-a+1)/3}{c\delta[(a^2-a+1)/3+\lambda-c(d_f+\delta)]},$$

and M_f, λ, and d_f are defined in (4.24), (4.25), and (4.26), respectively.

Proof. Notice that (4.27) implies $c > (a^2-a+1)/(3d_f)$ and $\tau < \tau_1^F$, which in turn lead to meeting condition (S1). Moreover, the corresponding matrices in (3.9) are

$$\mathbf{M} = \begin{pmatrix} \eta_1 & -\bar{L}_{12} \\ -\bar{L}_{21} & \eta_2 \end{pmatrix}, D_{\mathbf{M}} = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}, L_{\mathbf{M}} = \begin{pmatrix} 0 & 0 \\ \bar{L}_{21} & 0 \end{pmatrix}, U_{\mathbf{M}} = \begin{pmatrix} 0 & \bar{L}_{12} \\ 0 & 0 \end{pmatrix},$$

where $\eta_1 := -(a^2-a+1)/3 + cd_f + c\delta\tau[(a^2-a+1)/3+\lambda-c(d_f+\delta)]$, $\eta_2 := \gamma$ $\bar{L}_{12} = 1$, $\bar{L}_{21} = b$. A direct computation reveals that the corresponding matrix

$$(D_{\mathbf{M}} - L_{\mathbf{M}})^{-1}U_{\mathbf{M}} = \begin{pmatrix} 0 & \bar{L}_{12}/\eta_1 \\ 0 & \bar{L}_{12}\bar{L}_{21}/(\eta_1\eta_2) \end{pmatrix}$$

admits the eigenvalues 0 and $\bar{L}_{12}\bar{L}_{21}/(\eta_1\eta_2) = b/(\eta_1\eta_2)$. We obtain $b/(\eta_1\eta_2) < 1$ under (4.27) using $c > [b/\gamma + (a^2-a+1)/3]/d_f$ and $\tau < \tau_2^F$. The assertion thus follows from Theorem 3.2. ■

The present approach can also be applied to the case of distribution delay in (4.8). Now $\tilde{h}_1$ is modified to

$$\tilde{h}_1(\phi_1, \psi_1, t) = c\int_0^{\tau_M} [f(\psi_1(-\sigma)) - f(\phi_1(-\sigma))]\mathcal{K}(\sigma)\mathrm{d}\sigma, \tag{4.28}$$

where function $\mathcal{K}$ is the kernel of the distribution. One of the commonly used distributions is the uniform distribution: for some $\tau^{\min} > 0$, $\ell > 0$,

$$\mathcal{K}(\sigma) := \begin{cases} 0 & \text{if } 0 \le \sigma < \tau^{\min}, \\ 1/\ell & \text{if } \tau^{\min} \le \sigma \le \tau^{\min} + \ell, \\ 0 & \text{if } \tau^{\min} + \ell < \sigma \le \tau_M. \end{cases} \tag{4.29}$$

By the definition of $\mathcal{K}$, we obtain

$$\begin{aligned}\tilde{h}_1(\phi_1, \psi_1, t) &= c\left\{ \int_{\tau^{\min}}^{\tau^{\min}+\ell} [f(\psi_1(-\sigma)) - f(\phi_1(-\sigma))]\mathrm{d}\sigma \right\}\Big/ \ell \\ &= -cf'(\varsigma)[\phi_1(-s) - \psi_1(-s)]\end{aligned}$$

for some $s \in (\tau^{\min}, \tau^{\min}+\ell)$, and ς between $\phi_1(-s)$ and $\psi_1(-s)$. Thus, $-c\delta \le -cf'(\varsigma) \le -cd_f$. Basically, the arguments are valid for kernel $\mathcal{K}$ with compact support. Accordingly, we can verify that the assertions in Lemmas 4.1 and 4.2 hold, but with $\bar{\tau}_1 = \tau^{\min} + \ell$ instead. In the following, (4.1)′ denotes system (4.1) with the coupling $f(\phi_1(-\tau))$ replaced by the distribution delay (4.8) with kernel $\mathcal{K}$ in (4.29). Indeed, Lemma 4.1 also holds for system (4.1)′. By arguments similar to those in Theorem 4.3, we conclude the following theorem.

Theorem 4.4. *System* (4.1)′ *achieves global synchronization, provided*

$$c > [b/\gamma + (a^2 - a + 1)/3]/d_f \geq 0 \textit{ and } \tau^{\min} + \ell < \min\{\tau_1^F, \tau_2^F\},$$

where d_f*,* τ_1^F*, and* τ_2^F *are defined as in Theorem* 4.3*.*

It was derived in [6] that coupled system (4.1) with no time delay (i.e., $\tau = 0$) undergoes a supercritical Hopf bifurcation at $c = c_0 := a + \gamma$, which gives rise to a stable synchronous periodic solution. Numerical simulation shows that the oscillation remains a global attractor of the system for a large range of c greater than c_0. Here, we note that f in system (4.1) is merely required to satisfy $f(0) = 0$ and $f'(0) > 0$ for the local dynamics derived from the bifurcation analysis in [6], but we require f to be bounded for the consideration of global dynamics herein. The following example demonstrates that, under our synchronization framework, system (4.1) with parameters a, b, γ satisfying (4.9), (4.10), and $\tau = 0$ admits global synchronization with stable synchronous oscillation as c is larger than and near c_0. This gives an analytical support to the numerical finding in [6]. By considering τ as a bifurcation parameter, and with fixed parameters a, b, γ, c, this example illustrates that the stable oscillation sustains as τ is small enough so that no further bifurcation occurs. The system loses synchrony as τ is larger than a certain critical bifurcation value and yields to a stable antiphase periodic solution.

Example 4.1. Consider (4.1) with $a = 0.5$, $b = 0.00126$, $\gamma = 0.02$, and $f(\xi) = 5\tanh(0.2\xi)$. Choose $Q^* = \tilde{Q}^{(2)}$. This system with $\tau = 0$ undergoes a supercritical Hopf bifurcation at $c = c_0 = a + \gamma = 0.52$, which gives rise to a stable synchronous periodic solution. Let us consider the system with fixed $c = 0.52001$ which is slightly larger than c_0. In this situation, by the bifurcation analysis with respect to τ in [6] (cf. Figure 3 in [6]), the system undergoes a subcritical Hopf bifurcation at the first critical value $\tau_{1,-}^0 \approx 0.0008$, and a supercritical Hopf bifurcation at the second critical value $\tau_{2,+}^0 \approx 19.505$, where a stable antiphase periodic solution emerges. On the other hand, a direct computation gives $[b/\gamma + (a^2 - a + 1)/3]/d_f \approx 0.5049$, $\tau_1^F \approx 0.0027$, $\tau_2^F \approx 0.0004$; the system satisfies (4.27) in Theorem 4.3 and hence achieves global synchronization if $\tau < \min\{\tau_1^F, \tau_2^F\} \approx 0.0004$. Here, we note that the synchronization criterion in Theorem 4.3 does not hold if we choose $Q^* = \tilde{Q}^{(1)}$ instead. Figure 2 illustrates that the system with $\tau = 0$ and 0.0002, respectively, which are smaller than $\min\{\tau_1^F, \tau_2^F, \tau_{1,-}^0\}$ near 0.0004, admits a stable synchronous oscillation. Figure 3 demonstrates that the system with $\tau = 20$, which is slightly larger than $\tau_{2,+}^0$, admits a stable antiphase oscillation.

Remark 4.1. The present framework can also accommodate coupled FitzHugh–Nagumo systems under diffusive coupling, such as system (4.1) with the coupling terms $cf(y_1(t - \tau))$ and $cf(x_1(t - \tau))$ replaced by $c[y_1(t - \tau) - x_1(t)]$ and $c[x_1(t - \tau) - y_1(t)]$. To apply the present synchronization theories, one needs to examine assumption (D), say, via the approach in [41, 58]. Assumption (H) can then be verified subsequently.

4.2. Cell-to-cell kinetic model.

In this subsection, we consider a cell-to-cell model on the kinetics of the segmentation clock genes in zebrafish, proposed by Uriu, Morishita, and Iwasa

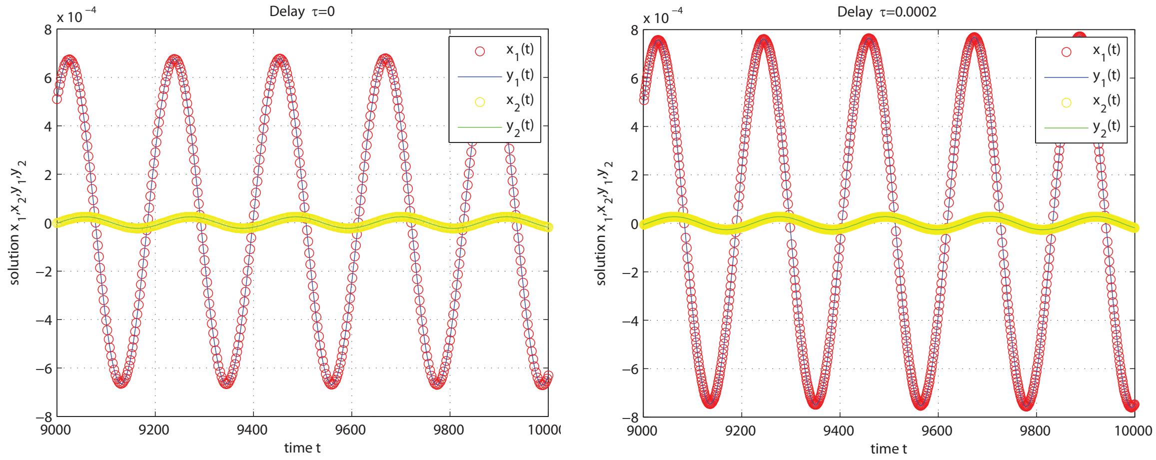

Figure 2. *Time series of the solution of system* (4.1)*, evolved from* $(0.001{+}t, -0.002{\cdot}t, 0.001{\cdot}\sin t, -0.001{\cdot}t)$ *at initial time* $t_0 = 0$*, tends to a synchronous (in-phase) oscillation. Here,* $a = 0.5$*,* $b = 0.00126$*,* $\gamma = 0.02$*,* $c = 0.52001$*,* $f(\xi) = 5\tanh(0.2\xi)$*, and* $\tau = 0$*,* $\tau = 0.0002$*, respectively.*

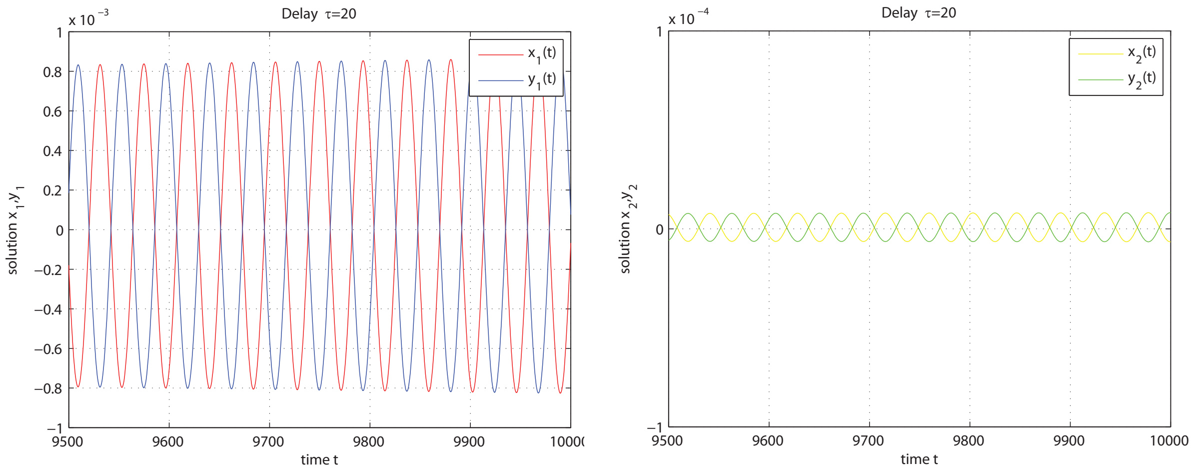

Figure 3. *Time series of the solution of system* (4.1)*, evolved from* $(-0.0001, -0.0002, 0.0001, 0.0002)$ *at initial time* $t_0 = 0$*, tends to an antiphase oscillation, with* $a = 0.5$*,* $b = 0.00126$*,* $\gamma = 0.02$*,* $c = 0.52001$*,* $f(\xi) = 5\tanh(0.2\xi)$*, and* $\tau = 20$*.*

in [62, 63]:

$$
(4.30)\qquad \begin{cases}
\dot{x}_1(t) &= g_H(x_3(t), y_4(t)) - f_1(x_1(t)),\\
\dot{x}_2(t) &= \nu_3 x_1(t) - f_2(x_2(t)),\\
\dot{x}_3(t) &= \nu_5 x_2(t) - f_3(x_3(t)),\\
\dot{x}_4(t) &= g_D(x_3(t)) - f_4(x_4(t)),\\
\dot{y}_1(t) &= g_H(y_3(t), x_4(t)) - f_1(y_1(t)),\\
\dot{y}_2(t) &= \nu_3 y_1(t) - f_2(y_2(t)),\\
\dot{y}_3(t) &= \nu_5 y_2(t) - f_3(y_3(t)),\\
\dot{y}_4(t) &= g_D(y_3(t)) - f_4(y_4(t)).
\end{cases}
$$

In this system, x_1, x_2, x_3, and x_4 (resp., y_1, y_2, y_3, and y_4) represent the concentrations of *her* mRNA, Her protein in cytoplasm, Her protein in nucleus, and Delta protein of the first cell (resp., the second cell), respectively; $\nu_3 > 0$ is the synthesis rate of Her protein in cytoplasm and $\nu_5 > 0$ is the transportation rate of Her protein from cytoplasm to nucleus. The transcription initiation rates are described by g_H and g_D as

$$g_H(u,v) = \frac{(k_1)^n}{(k_1)^n + u^n} \cdot (\nu_1 + \nu_c v),\ g_D(u) = \frac{\nu_7 (k_7)^h}{(k_7)^h + u^h}, \tag{4.31}$$

where nonnegative integers h and n are the Hill coefficients, $\nu_1 > 0$ is the Basal transcription rate of *her* mRNA, $\nu_c > 0$ is the activation rate of *her* mRNA transcription by Delta–Notch signal, k_1 is the threshold constant for the suppression of *her* mRNA transcription by Her protein in nucleus, $\nu_7 > 0$ is the synthesis rate of Delta protein, and $k_7 > 0$ is the threshold constant for the suppression of Delta protein synthesis by Her protein. The degradations are depicted by f_i as

$$f_1(u) = \frac{\nu_2 u}{k_2 + u},\ f_2(u) = \frac{\nu_4 u}{k_4 + u} + \nu_5 u, \tag{4.32}$$

$$f_3(u) = \frac{\nu_6 u}{k_6 + u},\ f_4(u) = \frac{\nu_8 u}{k_8 + u}, \tag{4.33}$$

where $i = 1, 2, 3, 4$, and ν_2, ν_4, ν_6, $\nu_8 > 0$ (resp., k_2, k_4, k_6, $k_8 > 0$) are the maximum degradation rates (resp., Michaelis constants for degradation) of *her* mRNA, Her protein in cytoplasm, Her protein in nucleus, and Delta protein, respectively. This model (4.30) introduces an intermediate process, namely, the transportation of Her protein from cytoplasm to nucleus, to avoid taking into account time delay in transcription and translation. Although time delay is not modeled in (4.30), the nonlinear transcription functions g_H, g_D and degradations f_i are much more complicated than those adopted in the delayed model [28, 32]. We remark that system (4.30) is representative, as the equations modeling other gene regulations admit similar forms.

For coupled system (4.30), we consider the evolution $X(t, X_0)$ from initial condition $X_0 \in \mathbb{R}^8_+$ at initial time t_0, where $\mathbb{R}^8_+ := \{(x_1, \ldots, x_4, y_1, \ldots, y_4) : x_i \geq 0, y_i \geq 0, i = 1, 2, 3, 4\}$. To ensure that (4.30) is a proper model for modeling gene regulations, we note that if the Hill coefficients h and n are nonnegative even integers, then $\mathbb{R}^8_+$ is positively invariant under the flow generated by system (4.30).

Below, we shall establish an attracting region for solutions of system (4.30) evolved from

$\mathbb{R}^8_+$. To this end, we first introduce the following quantities:

$$\hat{\varrho}_1 = \frac{(\nu_1 + \nu_c\hat{\varrho}_4)k_2}{\nu_2 - (\nu_1 + \nu_c\hat{\varrho}_4)}, \quad \check{\varrho}_1 = \frac{(k_1)^n k_2 \nu_1}{\nu_2(\hat{\varrho}_3)^n + (k_1)^n(\nu_2 - \nu_1)},$$
$$\hat{\varrho}_2 = \frac{\nu_3\hat{\varrho}_1 - \nu_4 - k_4\nu_5 + \sqrt{4k_4\nu_3\nu_5\hat{\varrho}_1 + (\nu_4 + k_4\nu_5 - \nu_3\hat{\varrho}_1)^2}}{2\nu_5},$$
$$\check{\varrho}_2 = \frac{\nu_3\check{\varrho}_1 - \nu_4 - k_4\nu_5 + \sqrt{4k_4\nu_3\nu_5\check{\varrho}_1 + (\nu_4 + k_4\nu_5 - \nu_3\check{\varrho}_1)^2}}{2\nu_5},$$
$$\hat{\varrho}_3 = \frac{k_6\nu_5\hat{\varrho}_2}{\nu_6 - \nu_5\hat{\varrho}_2}, \quad \check{\varrho}_3 = \frac{k_6\nu_5\check{\varrho}_2}{\nu_6 - \nu_5\check{\varrho}_2},$$
$$\hat{\varrho}_4 = \frac{k_8\nu_7}{\nu_8 - \nu_7}, \quad \check{\varrho}_4 = \frac{(k_7)^h k_8 \nu_7}{\nu_8(\hat{\varrho}_3)^h + (k_7)^h(\nu_8 - \nu_7)}.$$

Note that $\hat{\varrho}_4$ is determined by k_8, ν_7, and ν_8, and then $\hat{\varrho}_1$, $\hat{\varrho}_2$, $\hat{\varrho}_3$, $\check{\varrho}_1$, $\check{\varrho}_2$, $\check{\varrho}_3$, $\check{\varrho}_4$ can be computed successively. In addition, $0 < \check{\varrho}_i < \hat{\varrho}_i$ for $i = 1, 2, 3, 4$ under the condition

$$\nu_8 > \nu_7, \ \nu_2 > \nu_1 + \nu_c\hat{\varrho}_4, \ \text{and} \ \nu_6 - \nu_5\hat{\varrho}_2. \tag{4.34}$$

We thus define a subset $\mathcal{Q}_\varrho \times \mathcal{Q}_\varrho$ of $\mathbb{R}^8_+$, where $\mathcal{Q}_\varrho := [\check{\varrho}_1, \hat{\varrho}_1] \times \cdots \times [\check{\varrho}_4, \hat{\varrho}_4]$. We further define the following quantities for later use:

$$\check{d}_i := \min\{f_i'(\xi) : \xi \in [\check{\varrho}_i, \hat{\varrho}_i]\} \quad \text{for } i = 1, 2, 3, 4, \tag{4.35}$$
$$\hat{d}_i := \max\{f_i'(\xi) : \xi \in [\check{\varrho}_i, \hat{\varrho}_i]\} \quad \text{for } i = 1, 2, 3, 4, \tag{4.36}$$
$$\rho_1 := \max\left\{\left|\frac{\partial g_H}{\partial u}(u, v)\right| : u \in [\check{\varrho}_3, \hat{\varrho}_3], v \in [\check{\varrho}_4, \hat{\varrho}_4]\right\}, \tag{4.37}$$
$$\rho_2 := \max\left\{\left|\frac{\partial g_H}{\partial v}(u, v)\right| : u \in [\check{\varrho}_3, \hat{\varrho}_3], v \in [\check{\varrho}_4, \hat{\varrho}_4]\right\}, \tag{4.38}$$
$$\rho_3 := \max\{|g_D'(\xi)| : \xi \in [\check{\varrho}_3, \hat{\varrho}_3]\}. \tag{4.39}$$

Proposition 4.5. *Assume that* (4.34) *holds and the Hill coefficients h and n are nonnegative even integers. Then the solution of system* (4.30) *evolved from any point in $\mathbb{R}^8_+$ exists for all time larger than t_0 and converges to $\mathcal{Q}_\varrho \times \mathcal{Q}_\varrho$.*

Proof. Suppose that $X(t) = (x_1(t), \dots, x_4(t), y_1(t), \dots, y_4(t))$ is an arbitrary solution. Then $x_i(t) \geq 0$ and $y_i(t) \geq 0$ for $i = 1, 2, 3, 4$, as long as the solution exists, as noted above. The proof will proceed via successive component estimates. As seen from the equation for $x_4(t)$ in (4.30),

$$\dot{x}_4(t) \leq \hat{f}_4(x_4(t)), \tag{4.40}$$

where $\hat{f}_4(\xi) := \nu_7 - f_4(\xi)$. It is not difficult to verify that $\hat{f}_4(\xi)$ is strictly decreasing and has a unique zero at $\hat{\varrho}_4 > 0$ if $\nu_8 > \nu_7$, where $\hat{\varrho}_4$ is defined above. Accordingly, $x_4(t)$ exists for all $t \geq t_0$ and converges to $[0, \hat{\varrho}_4]$ as $t \to \infty$ due to (4.40). Similarly, we can prove that $y_4(t)$ converges to $[0, \hat{\varrho}_4]$ as $t \to \infty$. Thus, for any $\varepsilon > 0$ there exists a $t_1^\varepsilon \geq t_0$ such that

$$\dot{x}_1(t) \leq \hat{f}_1(x_1(t)) + \varepsilon =: \hat{f}_1^\varepsilon(x_1(t)) \quad \text{for all } t \geq t_1^\varepsilon, \tag{4.41}$$

where $\hat{f}_1(\xi) := \nu_1 + \nu_c\hat{\varrho}_4 - f_1(\xi)$ is strictly decreasing with a unique zero at $\hat{\varrho}_1 > 0$ and $\hat{f}_1^\varepsilon(\xi) = \hat{f}_1(\xi) + \varepsilon$ is also strictly decreasing with a unique zero at $\hat{\varrho}_1^\varepsilon$ if $\nu_2 > \nu_1 + \nu_c\hat{\varrho}_4$; in addition, $\hat{\varrho}_1^\varepsilon \downarrow \hat{\varrho}_1$ as $\varepsilon \downarrow 0$. Based on (4.41), we verify that $x_1(t)$ exists for all $t \geq t_0$ and converges to $[0, \hat{\varrho}_1^\varepsilon]$ for any $\varepsilon > 0$, and hence to $[0, \hat{\varrho}_1]$ as $t \to \infty$. Similarly, we can prove that $y_1(t)$ converges to $[0, \hat{\varrho}_1]$ as $t \to \infty$. Applying arguments similar to those for $x_1(t)$, we can show that $x_i(t)$ and $y_i(t)$, $i = 2, 3$, exist for all $t \geq t_0$ and converge to $[0, \hat{\varrho}_i]$ as $t \to \infty$. Indeed, since $x_1(t)$ converges to $[0, \hat{\varrho}_1]$ as $t \to \infty$, for any $\varepsilon > 0$, there exists a $t_2^\varepsilon \geq t_1^\varepsilon$ such that

$$\dot{x}_2(t) \leq \hat{f}_2(x_2(t)) + \varepsilon \quad \text{for all } t \geq t_2^\varepsilon,$$

where $\hat{f}_2(\xi) := \nu_3\hat{\varrho}_1 - f_2(\xi)$ is strictly decreasing and has a unique zero at $\hat{\varrho}_2 > 0$. Accordingly, $x_2(t)$ exists for all $t \geq t_0$ and converges to $[0, \hat{\varrho}_2]$ as $t \to \infty$. Subsequently, for any $\varepsilon > 0$ there exists a $t_3^\varepsilon \geq t_2^\varepsilon$ such that

$$\dot{x}_3(t) \leq \hat{f}_3(x_3(t)) + \varepsilon \quad \text{for all } t \geq t_3^\varepsilon,$$

where $\hat{f}_3(\xi) := \nu_5\hat{\varrho}_2 - f_3(\xi)$ is strictly decreasing with a unique zero at $\hat{\varrho}_3 > 0$ if $\nu_6 - \nu_5\hat{\varrho}_2$. Accordingly, $x_3(t)$ exists for all $t \geq t_0$ and converges to $[0, \hat{\varrho}_3]$ as $t \to \infty$. Similarly, we can prove that y_i exist for all $t \geq t_0$ and converge to $[0, \hat{\varrho}_i]$ as $t \to \infty$, for $i = 2, 3$.

Next, we further verify that $x_i(t)$ and $y_i(t)$ converge to $[\check{\varrho}_i, \hat{\varrho}_i]$ as $t \to \infty$, for $i = 1, 2, 3, 4$. Since $x_3(t)$ converges to $[0, \hat{\varrho}_3]$ and $y_4(t)$ converges to $[0, \hat{\varrho}_4]$ as $t \to \infty$, for any $\varepsilon > 0$, there exists a $t_4^\varepsilon \geq t_3^\varepsilon$ such that

$$\dot{x}_1(t) > \check{f}_1(x_1(t)) - \varepsilon \quad \text{for all } t \geq t_4^\varepsilon, \tag{4.42}$$

where $\check{f}_1(\xi) := \nu_1(k_1)^n/[(k_1)^n + (\hat{\varrho}_3)^n] - f_1(\xi)$ is strictly decreasing with a unique zero at $\check{\varrho}_1 > 0$. From (4.42), it follows that $x_1(t)$ converges to $[\check{\varrho}_1, \hat{\varrho}_1]$ as $t \to \infty$. Consequently, for any $\varepsilon > 0$ there exists a $t_5^\varepsilon \geq t_4^\varepsilon$ such that

$$\dot{x}_2(t) > \check{f}_2(x_1(t)) - \varepsilon \quad \text{for all } t \geq t_5^\varepsilon,$$

where $\check{f}_2(\xi) := \nu_3\check{\varrho}_1 - f_2(\xi)$ is strictly decreasing and has a unique zero at $\check{\varrho}_2 > 0$. It follows that $x_2(t)$ converges to $[\check{\varrho}_2, \hat{\varrho}_2]$ as $t \to \infty$. Subsequently, for any $\varepsilon > 0$ there exists a $t_6^\varepsilon \geq t_5^\varepsilon$ such that

$$\dot{x}_3(t) > \check{f}_3(x_1(t)) - \varepsilon \quad \text{for all } t \geq t_6^\varepsilon,$$

where $\check{f}_3(\xi) := \nu_5\check{\varrho}_2 - f_3(\xi)$ is strictly decreasing with a unique zero at $\check{\varrho}_3 > 0$, which yields that $x_3(t)$ converges to $[\check{\varrho}_3, \hat{\varrho}_3]$ as $t \to \infty$. Then, for any $\varepsilon > 0$ there exists a $t_7^\varepsilon \geq t_6^\varepsilon$ such that

$$\dot{x}_4(t) > \check{f}_4(x_4(t)) - \varepsilon \quad \text{for all } t \geq t_7^\varepsilon,$$

where $\check{f}_4(\xi) := \nu_7(k_7)^h/[(k_7)^h + (\hat{\varrho}_3)^h] - f_4(\xi)$ is strictly decreasing and has a unique zero at $\check{\varrho}_4 > 0$. Accordingly, $x_4(t)$ converges to $[\check{\varrho}_4, \hat{\varrho}_4]$ as $t \to \infty$. Similarly, we can prove that $y_i(t)$ converges to $[\check{\varrho}_i, \hat{\varrho}_i]$ as $t \to \infty$, for $i = 1, 2, 3, 4$. ■

Below, we shall establish the global synchronization for system (4.30). Since this is an ODE system, we shall take $\mathcal{Q} = \mathcal{Q}_\varrho$ in assumption (D), replace $\mathcal{C}_\mathcal{Q}$ by $\mathcal{Q}$ in assumption (H), and apply Theorem 3.3; see Remark 3.1(iii).

Theorem 4.6. *Assume that* (4.34) *holds and the Hill coefficients* h *and* n *are nonnegative even integers. Then system* (4.30) *achieves global synchronization if*

$$\check{d}_1\check{d}_2\check{d}_3\check{d}_4 > \nu_3\nu_5(\rho_1\check{d}_4 + \rho_3\rho_2), \tag{4.43}$$

where $\check{d}_i$, $i = 1, 2, 3, 4$, *and* ρ_i, $i = 1, 2, 3$, *are defined in* (4.35) *and* (4.37)–(4.39), *respectively.*

Proof. By Proposition 4.5, system (4.30) satisfies assumption (D) with $\mathcal{Q} = \mathcal{Q}_\varrho$. Next, let us examine assumption (H). Setting $z_i = x_i - y_i$, we consider the difference-differential equation corresponding to (4.30):

$$\begin{cases} \dot{z}_1(t) &= g_H(x_3(t), y_4(t)) - g_H(y_3(t), x_4(t)) - [f_1(x_1(t)) - f_1(y_1(t))], \\ \dot{z}_2(t) &= \nu_3[x_1(t) - y_1(t)] - [f_2(x_2(t)) - f_2(y_2(t))], \\ \dot{z}_3(t) &= \nu_5[x_2(t) - y_2(t)] - [f_3(x_3(t)) - f_3(y_3(t))], \\ \dot{z}_4(t) &= g_D(x_3(t)) - g_D(y_3(t)) - [f_4(x_4(t)) - f_4(y_4(t))]. \end{cases} \tag{4.44}$$

Then following the notation in (3.1),

$$H_i(\Phi, \Psi, t) = h_i(\phi_i(0), \psi_i(0), t) + \tilde{h}_i(\phi_i, \psi_i, t) + w_1(\Phi, \Psi, t), \tag{4.45}$$

where $i = 1, 2, 3, 4$,

$$\begin{aligned} h_i(\phi_i(0), \psi_i(0), t) &= -[f_i(\phi_i(0)) - f_i(\psi_i(0))] = -f_i'(\xi_i)[\phi_i(0) - \psi_i(0)], \\ \tilde{h}_i(\phi_i, \psi_i, t) &\equiv 0, \\ w_1(\Phi, \Psi, t) &= g_H(\phi_3(0), \psi_4(0)) - g_H(\psi_3(0), \phi_4(0)) \\ &= \frac{\partial g_H}{\partial u}(\xi_5, \xi_6)[\phi_3(0) - \psi_3(0)] - \frac{\partial g_H}{\partial v}(\xi_5, \xi_6)[\phi_4(0) - \psi_4(0)], \\ w_2(\Phi, \Psi, t) &= \nu_3[\phi_1(0) - \psi_1(0)], \\ w_3(\Phi, \Psi, t) &= \nu_5[\phi_2(0) - \psi_2(0)], \\ w_4(\Phi, \Psi, t) &= g_D(\phi_3(0)) - g_D(\psi_3(0)) = g_D'(\xi_7)[\phi_3(0) - \psi_3(0)], \end{aligned}$$

and ξ_i is between $\phi_i(0)$ and $\psi_i(0)$ for $i = 1, 2, 3, 4$, ξ_5 and ξ_7 are between $\phi_3(0)$ and $\psi_3(0)$, and ξ_6 is between $\phi_4(0)$ and $\psi_4(0)$. Notably,

$$-\hat{d}_i \le -f_i'(\xi_i) \le -\check{d}_i \quad \text{for } i = 1, 2, 3, 4, \tag{4.46}$$

$$\left|\frac{\partial g_H}{\partial u}(\xi_5, \xi_6)\right| \le \rho_1, \ \left|\frac{\partial g_H}{\partial v}(\xi_5, \xi_6)\right| \le \rho_2, \ |g_D'(\xi_7)| \le \rho_3, \tag{4.47}$$

where $\check{d}_i$, $\hat{d}_i$, and ρ_i are defined in (4.35), (4.36), and (4.37)–(4.39), respectively. From (4.45)–(4.47), we can verify that system (4.30) satisfies assumption (H) with $\check{\mu}_i = -\hat{d}_i$, $\hat{\mu}_i = -\check{d}_i$, and $\tilde{h}_i \equiv 0$, $i = 1, 2, 3, 4$, $\bar{\beta}_{ij} = \bar{\tau}_{ij} = 0$ if $i \ne j$ and $i, j = 1, 2, 3, 4$, $\bar{\mu}_{12} = \bar{\mu}_{23} = \bar{\mu}_{24} = \bar{\mu}_{31} = \bar{\mu}_{34} = \bar{\mu}_{41} = \bar{\mu}_{42} = 0$, $\bar{\mu}_{13} = \rho_1$, $\bar{\mu}_{14} = \rho_2$, $\bar{\mu}_{21} = \nu_3$, $\bar{\mu}_{32} = \nu_5$, and $\bar{\mu}_{43} = \rho_3$. Applying Theorem 3.3 yields that system (4.30) attains global synchronization if the Gauss–Seidel iteration for linear system

$$\begin{pmatrix} \check{d}_1 & 0 & -\rho_1 & -\rho_2 \\ -\nu_3 & \check{d}_2 & 0 & 0 \\ 0 & -\nu_5 & \check{d}_3 & 0 \\ 0 & 0 & -\rho_3 & \check{d}_4 \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{4.48}$$

converges to zero. Moreover, the Gauss–Seidel iteration for (4.48) converges to zero if and only if (4.43) holds. This completes the proof. ■

By studying the stationary equations associated with system (4.30), we can establish the existence of synchronous equilibrium $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4)$. By considering the difference-differential equations for $z_i = x_i - \overline{x}_i, z_{i+4} = y_i - \overline{x}_i,\ i = 1, 2, 3, 4$, the global convergence to $\overline{\mathbf{x}}$ can be obtained by the present technique (sequential contracting), as in [32] for Lewis's model. There are three major dynamical phases in this gene regulation model: synchronous oscillation, oscillation-arrested, and traveling wave. Our Theorem 4.6 on the global synchronization and global convergence to the steady state corresponds to the oscillation-arrested phase. This is the first analytical study on the collective behavior of Iwasa's model, (4.30).

5. Further extensions.

Basically, the present approach is applicable to dissipative coupled systems whose difference-differential equations admit a structure which captures the difference of two corresponding components (depicted in assumption (H)). In subsection 5.1, we extend the synchronization framework to coupled systems comprising N subsystems. A neural network system comprising a ring of K loops demonstrates this extension in subsection 5.2. In this application, our approach leads to a network-scale–dependent criterion for synchronization, where a smaller network is more favored for synchronization.

5.1. N-cell system under circulant coupling.

Let us consider the following N-cell system of general form:

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i^t, t) + \mathbf{G}_i(\mathbf{x}_1^t, \dots, \mathbf{x}_N^t, t), \quad i = 1, \dots, N, \tag{5.1}$$

where $t \geq t_0$, $\mathbf{x}_i(t) = (x_{i,1}(t), \dots, x_{i,n}(t)) \in \mathbb{R}^n$, $\mathbf{x}_i^t \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^n)$ are defined by $\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t + \theta)$, $\mathbf{F} = (F_1, \dots, F_n)$, and $\mathbf{G}_i$ are continuous functions. Basically, our approach can be extended to N-cell system (5.1) under circulant coupling; namely, $\mathbf{G}_i$ satisfies

$$\mathbf{G}_i(\Phi_1, \dots, \Phi_N, t) = \mathbf{G}(\Phi_i, \dots, \Phi_{N+(i-1)}, t), \quad i = 1, \dots, N, \tag{5.2}$$

for some function $\mathbf{G}$, where $\Phi_\ell = \Phi_{\ell (\mathrm{mod}\ N)}$.

Systems of neural network and neuronal network in the literature largely admit the following form:

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t), t) + c\Sigma_{j=1}^N w_{ij} \tilde{\mathbf{G}}(\mathbf{x}_j(t)), \quad i = 1, \dots, N, \tag{5.3}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\tilde{\mathbf{G}} : \mathbb{R}^n \to \mathbb{R}^n$, c, $w_{ij} \in \mathbb{R}$, or

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t), t) + c\Sigma_{j=1}^N w_{ij} \tilde{\mathbf{G}}(\mathbf{x}_j(t - \tau)), \quad i = 1, \dots, N, \tag{5.4}$$

or $\Sigma_{j=1, j\neq i}^N w_{ij} \tilde{\mathbf{G}}(\mathbf{x}_i(t), \mathbf{x}_j(t - \tau))$ in the coupling terms, if transmission delay τ is taken into account. If the coupling matrix $[w_{ij}]$ satisfies the diffusive condition

$$w_{ii} = -\Sigma_{j=1, j\neq i}^N w_{ij}, \quad i = 1, \dots, N, \tag{5.5}$$

then the coupling terms in (5.4) can be put into

$$c\Sigma_{j=1, j\neq i}^N w_{ij} \cdot [\tilde{\mathbf{G}}(\mathbf{x}_j(t - \tau)) - \tilde{\mathbf{G}}(\mathbf{x}_i(t - \tau))]; \tag{5.6}$$

see [33, 68]. Notice that under diffusive condition (5.5), the coupling terms in (5.3) and (5.4) (i.e., (5.6)) will vanish at synchronous solutions. In addition, the coupling is linear in (5.3), (5.4), and (5.6) if $\tilde{\mathbf{G}}$ is a linear function, i.e., an $n \times n$ real matrix (see [26, 35]). The gap-junctional (linear diffusive) coupling for N neurons is described by

$$\mathbf{G}_i(\Phi_1, \dots, \Phi_N, t) = c\Sigma_{j=1, j\neq i}^{N} w_{ij} \cdot [\Phi_j(0) - \Phi_i(0)], \tag{5.7}$$

or $c\Sigma_{j=1}^{N} w_{ij} \cdot \Phi_j(0)$, if w_{ij} satisfy (5.5). Systems (5.3) and (5.4) and those with couplings (5.6) or (5.7) are obviously in the form of (5.1). The couplings in these systems are circulant if $[w_{ij}]$ is a circulant constant matrix, i.e., $[w_{ij}] = \mathrm{circ}(w_1, \dots, w_N)$; see [15, 65]. Moreover, in the case of gap-junctional coupling, the function $\mathbf{G}$ in (5.2) satisfies $\mathbf{G}(\Phi_1, \dots, \Phi_N, t) = c\sum_{i=1}^{N} w_i \Phi_i(0)$, or $c\sum_{i=1}^{N} w_i \Phi_i(-\tau)$ if time delay is considered. Such a connection includes all-excitatory, all-inhibitory, symmetrically connected excitatory rings and symmetrically connected excitatory rings of neurons [65]. The nearest-neighbor coupling between subsystems, i.e., $[w_{ij}] = \mathrm{circ}(a, \beta, 0, \dots, 0, \alpha)$, $\alpha, a, \beta \in \mathbb{R}$, is a basic example of a circulant matrix; see [9, 12, 51, 53]. Note that a circulant matrix is not necessarily symmetric.

In establishing the synchronization of coupled systems such as system (5.3) or (5.4), the diffusive condition (5.5) is commonly imposed on the coupling matrix $[w_{ij}]$ in the literature; see [26, 33, 35, 68]. Such a condition is unnecessary in our approach. In some previous papers including [11, 29], the components of the coupling function $\tilde{\mathbf{G}}$ are required to have large enough slopes. Our approach is free from this requirement.

Now let us extend the formulation for the synchronization of two-cell system (1.1) to that of N-cell system (5.1) satisfying (5.2). It will be shown that the arguments for the synchronization of (1.1) are parallel to those for (5.1). Therefore, the settings for these two that work in parallel will share the same notation.

By setting $\mathbf{z}_i(t) = (z_{i,1}(t), \dots, z_{i,n}(t)) := \mathbf{x}_i(t) - \mathbf{x}_{i+1}(t)$, $i = 1, \dots, N$, we consider the difference-differential system corresponding to (5.1): for $i = 1, \dots, N$ and $j = 1, \dots, n$,

$$\dot{z}_{i,j}(t) = H_j(\mathbf{x}_i^t, \dots, \mathbf{x}_{N+(i-1)}^t, t), \quad \mathbf{x}_\ell^t = \mathbf{x}_{\ell(\mathrm{mod}\ N)}^t, \tag{5.8}$$

where

$$\begin{aligned} &H_j(\Phi_1, \dots, \Phi_N, t) \\ &:= F_j(\Phi_1, t) + G_j(\Phi_1, \dots, \Phi_N, t) - F_j(\Phi_2, t) - G_j(\Phi_2, \dots, \Phi_N, \Phi_1, t). \end{aligned}$$

We can decompose function H_j as follows for $j = 1, \dots, n$:

$$H_j(\Phi_1, \dots, \Phi_N, t) = h_j(\phi_{1,j}(0), \phi_{2,j}(0), t) + \tilde{h}_j(\phi_{1,j}, \phi_{2,j}, t) + w_j(\Phi_1, \dots, \Phi_N, t), \tag{5.9}$$

where $\Phi_i = (\phi_{i,1}, \dots, \phi_{i,n}) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^n)$ for $i = 1, \dots, N$. Then each $z_{i,j}(t)$ in (5.8), $1 \le i \le N$, $1 \le j \le n$, satisfies

$$\dot{z}_{i,j}(t) = h_j(x_{i,j}(t), x_{i+1,j}(t), t) + \tilde{h}_j((\mathbf{x}_i^t)_j, (\mathbf{x}_{i+1}^t)_j, t) + w_{i,j}(t), \tag{5.10}$$

in the form of (2.1), where $w_{i,j}(t) := w_j(\mathbf{x}_i^t, \mathbf{x}_{i+1}^t, \dots, \mathbf{x}_{N+(i-1)}^t, t)$. Therefore, transferring our formulation of synchronization from two-cell system (1.1) to N-cell system (5.1) amounts to relabeling the two-dimensional indices in (5.10) to one-dimensional indices just as those in (3.4); more precisely, by setting $z_{i,j} = \tilde{z}_{(i-1)n+j} =: \tilde{z}_\ell$, $x_{i,j} = \tilde{x}_{(i-1)n+j} =: \tilde{x}_\ell$ and $w_{i,j} = \tilde{w}_{(i-1)n+j} =: \tilde{w}_\ell$, system (5.10) can be rewritten as

$$\dot{\tilde{z}}_\ell(t) = h_j(\tilde{x}_\ell(t), \tilde{x}_{\ell+n}(t), t) + \tilde{h}_j(\tilde{x}^t_\ell, \tilde{x}^t_{\ell+n}, t) + \tilde{w}_\ell(t), \tag{5.11}$$

where $\tilde{z}_\ell(t) = \tilde{x}_\ell(t) - \tilde{x}_{\ell+n}(t)$. We thus proceed to establish the synchronization of (5.1) as we consider (1.1) in section 3. The difference-differential equations (5.11) induced from (5.1) play the same role as (3.4) induced from (1.1). We introduce the following assumptions for (5.1), which resemble assumptions (D) and (H) in section 3; they are equivalent, respectively, after relabeling indices.

Assumption (D)*. All solutions of system (5.1) eventually enter and then remain in some compact set $\mathcal{Q}^N := \mathcal{Q} \times \cdots \times \mathcal{Q}$, where $\mathcal{Q} := [\check{q}_1, \hat{q}_1] \times \cdots \times [\check{q}_n, \hat{q}_n] \subset \mathbb{R}^n$.

Assumption (H)*. For each $j = 1, \ldots, n$, there exist $\check{\mu}_j, \hat{\mu}_j, \hat{\beta}_j, \check{\beta}_j \in \mathbb{R}$, $\rho^h_j, \rho^w_j > 0$, $\bar{\mu}^{(i)}_{jk}$, $\bar{\beta}^{(i)}_{jk} \geq 0$, and $0 \leq \bar{\tau}_j, \bar{\tau}^{(i)}_{jk} \leq \tau_M$ for $(i,k) \in \mathcal{A}_j := \{1, \ldots, N\} \times \{1, \ldots, n\} - \{1\} \times \{j\}$, such that for each $(\Phi_1, \ldots, \Phi_N) \in \mathcal{C}_\mathcal{Q}$, where $\Phi_i = (\phi_{i,1}, \ldots, \phi_{i,n})$, $i = 1, \ldots, N$, and

$$\begin{aligned} \mathcal{C}_\mathcal{Q} := \{&(\Psi_1, \ldots, \Psi_N) : \Psi_i = (\psi_{i,1}, \ldots, \psi_{i,n}) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^n), \\ &\psi_{i,j}(\theta) \in [\check{q}_j, \hat{q}_j],\ \theta \in [-\bar{\tau}_j, 0], i = 1, \ldots, N, j = 1, \ldots, n\}, \end{aligned}$$

the following three properties hold for all $t \geq t_0$:

$$(\mathrm{H-i})^* : \begin{cases} \check{\mu}_j \leq h_j(\phi_{1,j}(0), \phi_{2,j}(0), t)/[\phi_{1,j}(0) - \phi_{2,j}(0)] \leq \hat{\mu}_j, & \phi_{1,j}(0) - \phi_{2,j}(0) \neq 0, \\ h_j(\phi_{1,j}(0), \phi_{2,j}(0), t) = 0, & \phi_{1,j}(0) - \phi_{2,j}(0) = 0, \end{cases}$$

$(\mathrm{H-ii})^* : |\tilde{h}_j(\phi_{1,j}, \phi_{2,j}, t)| \leq \rho^h_j$, and there exists $\tau_j = \tau_j(\phi_{1,j}, \phi_{2,j}, t) \in [0, \bar{\tau}_j]$, such that

$$\begin{cases} \check{\beta}_j \leq \tilde{h}_j(\phi_{1,j}, \phi_{2,j}, t)/[\phi_{1,j}(-\tau_j) - \phi_{2,j}(-\tau_j)] \leq \hat{\beta}_j, & \phi_{1,j}(-\tau_j) - \phi_{2,j}(-\tau_j) \neq 0, \\ \tilde{h}_j(\phi_{1,j}, \phi_{2,j}, t) = 0, & \phi_{1,j}(-\tau_j) - \phi_{2,j}(-\tau_j) = 0, \end{cases}$$

$(\mathrm{H-iii})^* : |w_j(\Phi_1, \ldots, \Phi_N, t)| \leq \rho^w_j$, and there exists $\tau^{(i)}_{jk} = \tau^{(i)}_{jk}(\Phi_1, \ldots, \Phi_N, t) \in [0, \bar{\tau}^{(i)}_{jk}]$, $(i,k) \in \mathcal{A}_j$, such that

$$|w_j(\Phi_1, \ldots, \Phi_N, t)| \leq \sum_{(i,k) \in \mathcal{A}_j} \{\bar{\mu}^{(i)}_{jk} |\phi_{i,k}(0) - \phi_{i+1,k}(0)| + \bar{\beta}^{(i)}_{jk} |\phi_{i,k}(-\tau^{(i)}_{jk}) - \phi_{i+1,k}(-\tau^{(i)}_{jk})|\}.$$

Set

$$\bar{L}^{(i)}_{jk} := \bar{\mu}^{(i)}_{jk} + \bar{\beta}^{(i)}_{jk}. \tag{5.12}$$

Let us introduce the condition imposed for the synchronization of system (5.1), which is exactly parallel to condition (S1) for the synchronization of system (1.1).

Condition (S^*_1). $\hat{\mu}_j + \hat{\beta}_j < 0$ and $\bar{\beta}_j \bar{\tau}_j < \tau^*_j$ for all $j = 1, \ldots, n$, where

$$\bar{\beta}_j := \max\{|\check{\beta}_j|, |\hat{\beta}_j|\},\ \tau^*_j := \frac{3\rho^h_j(\hat{\mu}_j + \hat{\beta}_j)}{(\hat{\mu}_j + \check{\mu}_j + \hat{\beta}_j + \check{\beta}_j)(3\rho^h_j + \rho^w_j)}.$$

Under condition (S_1^*), we can capture the asymptotic behavior for each $\tilde{z}_\ell$ in (5.11) by Proposition 2.3. More precisely, there exist nN intervals $[-a_\ell, a_\ell]$ to which $z_\ell(t)$ converges, where $1 \le \ell = (i-1)n + j \le nN$; moreover,

$$a_\ell = a_{(i-1)n+j} \le |\tilde{w}_\ell|^{\max}(\infty)/\eta_j,$$

where

$$\eta_j := -\hat{\mu}_j - \hat{\beta}_j + \bar{\beta}_j \bar{\tau}_j(\check{\mu}_j + \hat{\mu}_j + \check{\beta}_j + \hat{\beta}_j). \tag{5.13}$$

Applying arguments similar to those in Proposition 3.1, we can show that for each $\ell = (i-1)n + j$, there exists a sequence $\{\tilde{a}_\ell^{(k)}\}_{k=1}^\infty$ with $\tilde{a}_\ell^{(k)} \ge a_\ell$ satisfying

$$\begin{aligned} \tilde{a}_\ell^{(k)} &= \tilde{a}_{(i-1)n+j}^{(k)} \\ &= \Bigg\{ \sum_{1\le\sigma<i,\ 1\le l\le n} \bar{L}_{jl}^{(\sigma)} \tilde{a}_{(\sigma-1)n+l}^{(k)} + \sum_{1\le l<j} \bar{L}_{jl}^{(i)} \tilde{a}_{(i-1)n+l}^{(k)} \\ &\quad + \sum_{n\ge l>j} \bar{L}_{jl}^{(i)} \tilde{a}_{(i-1)n+l}^{(k-1)} + \sum_{N\ge\sigma>i,\ 1\le l\le n} L_{jl}^{(\sigma)} \tilde{a}_{(\sigma-1)n+l}^{(k-1)} \Bigg\} \Bigg/ \eta_j. \end{aligned}$$

Actually $\{\tilde{a}_\ell^{(k)}\}_{k=1}^\infty$ is the Gauss–Seidel iteration for solving the linear system

$$\operatorname{circ}(M^{(1)}, M^{(2)}, \ldots, M^{(N)})\mathbf{v} = \mathbf{0}, \tag{5.14}$$

with $M^{(i)} = [m_{jl}^{(i)}]_{1\le j,l\le n}$ for $1 \le i \le N$, satisfying

$$m_{jl}^{(1)} = \begin{cases} \eta_j, & 1 \le j = l \le n, \\ -\bar{L}_{jl}^{(1)}, & 1 \le j \ne l \le n, \end{cases} \qquad m_{jl}^{(i)} = -\bar{L}_{jl}^{(i)} \quad \text{for } i = 2, \ldots, N,$$

where $\bar{L}_{jl}^{(i)}$ and η_j are defined in (5.12) and (5.13), respectively. Similar to Theorem 3.2, we can establish the synchronization for coupled N-cell system.

Theorem 5.1. *Consider system (5.1) which satisfies (5.2) and assumptions* (D)* *and* (H)*. *Then the system globally synchronizes if condition* (S_1^*) *holds and the Gauss–Seidel iterations for linear system (5.14) converge to zero, the unique solution.*

Although we formulated the synchronization theory for systems under circulant coupling, the idea of sequential contracting is not restricted to such a coupling. In fact, the operation relies on suitable manipulation of the difference-differential equations which are sure to be formulated according to the coupling configuration. For instance, consider system (5.4) under the diffusion condition (5.5) with $\tilde{\mathbf{G}}(\mathbf{x}_i) = (\tilde{G}_1(x_{i,1}), \ldots, \tilde{G}_n(x_{i,n}))$ where $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n})$, a setting largely adopted in the literature. We can consider the difference-differential equations $\dot{z}_{i,j}^{(k)}(t) = \dot{x}_{i,k}(t) - \dot{x}_{j,k}(t)$ instead of the previous $\dot{z}_{i,j}(t) = \dot{x}_{i,j}(t) - \dot{x}_{i+1,j}(t)$, corresponding to system (5.4). The coupling terms of the ith component, i.e., (5.6), can be rewritten as follows:

$$
\begin{aligned}
&c\Sigma_{j=1,j\neq i}^{N} w_{ij}\cdot[\tilde{\mathbf{G}}(\mathbf{x}_j(t-\tau))-\tilde{\mathbf{G}}(\mathbf{x}_i(t-\tau))]\\
&= c\Sigma_{j=1,j\neq i}^{N} w_{ij}\cdot\begin{pmatrix}\tilde{G}_1(x_{j,1}(t-\tau))-\tilde{G}_1(x_{i,1}(t-\tau))\\ \vdots\\ \tilde{G}_n(x_{j,n}(t-\tau))-\tilde{G}_n(x_{i,n}(t-\tau))\end{pmatrix}\\
&= -c\Sigma_{j=1,j\neq i}^{N} w_{ij}\cdot\begin{pmatrix}\tilde{G}_1'(\xi_{i,j}^{(1)})\cdot z_{i,j}^{(1)}(t-\tau)\\ \vdots\\ \tilde{G}_n'(\xi_{i,j}^{(n)})\cdot z_{i,j}^{(n)}(t-\tau)\end{pmatrix},
\end{aligned}
$$

where $\xi_{i,j}^{(k)}$ are between $x_{i,k}(t-\tau)$ and $x_{j,k}(t-\tau)$. Subsequently, the corresponding difference-differential equations can be represented by a linear combination of $z_{i,j}^{(k)}(\cdot)$, $i,j=1,\dots,N$, $k=1,\dots,n$, and showing $z_{i,j}^{(k)}(t)\to 0$, as $t\to\infty$ reduces to solving a homogeneous linear system, as in Theorem 5.1. The popular coupling configurations considered in the literature, global, nearest-neighbor, star, small-world network, and scale-free network, if formulated to satisfy the diffusive condition, as in [71], can therefore be treated by our approach. The application of our approach is determined by the setting of the difference-differential equation. Thus it is also possible to consider systems with couplings other than diffusive and circulant types.

5.2. A ring of K loops. Let us apply Theorem 5.1 to a coupled neural network that consists of a ring of K loops. Suppose there are K groups of neurons, and in each group there are n neurons which connect themselves into a loop,

$$\dot{x}_j(t)=-\mu_j x_j(t)+g(b_{j-1}x_{j-1}(t-\tau_I)),\quad j=1,\dots,n, \tag{5.15}$$

where $g(\xi)=\tanh(\xi)$, $\tau_I\geq 0$, $x_\ell=x_{\ell(\text{mod } n)}$, and $b_\ell=b_{\ell(\text{mod } n)}$. These K groups structure themselves into a network in the form of a ring which is coupled as

$$
\begin{cases}
\dot{x}_{i,j}(t)=-\mu_j x_{i,j}(t)+g(b_{j-1}x_{i,j-1}(t-\tau_I)),\ j=1,\dots,n-1 \ (\text{mod } n),\\
\dot{x}_{i,n}(t)=-\mu_n x_{i,n}(t)+g(b_{n-1}x_{i,n-1}(t-\tau_I))+c[g(b_n x_{i-1,n}^{\tau}(t))+g(b_n x_{i+1,n}^{\tau}(t))],
\end{cases} \tag{5.16}
$$

where $i=1,2,\dots,K$, $K\geq 3$, $c>0$, $\mu_j>0$, $b_j>0$, $j=1,\dots,n$; $x_{i,j}$ stands for the jth component in the ith loop and $x_{i,j}^{\tau}(t):=x_{i,j}(t-\tau_T)$, $\tau_T\geq 0$. Obviously, the coupling matrix $\text{circ}(0,c,0,\dots,0,c)$ in system (5.16) admits the circulant structure (5.2) but does not satisfy the diffusive condition (5.5). The case for system (5.16) with $K=2$, and $\mu_j=1$ and $b_j=b$ for $j=1,\dots,n$, has been studied in [8]. Therein, the stability of the trivial equilibrium was obtained via linear stability analysis, and the existence of in-phase oscillation for the symmetric coupling case was predicted. We note that as the slope condition is not met, the approach in [11, 29] does not apply to the coupling function $g(\xi)=\tanh(\xi)$ herein.

Based on Theorem 5.1, we shall establish the following network-scale–dependent and delay-dependent synchronization for system (5.16).

Theorem 5.2. *System* (5.16) *attains global synchronization if*

$$\begin{aligned}
&\mu_j > b_{j-1} \quad \textit{for } j = 1, \dots, n-1 \pmod{n},\\
&\mu_n + b_n c\hat{L} - b_{n-1} - (K-3)b_n c > 0,\\
&\tau_T < \min\left\{\frac{3(\mu_n + b_n c\hat{L})}{b_n[2\mu_n + b_n c(1+\hat{L})](1+cK)}, \frac{\mu_n + b_n c\hat{L} - b_{n-1} - (K-3)b_n c}{b_n c[2\mu_n + c b_n(1+\hat{L})]}\right\},
\end{aligned}$$

where $\hat{L} := \min\{(\tanh)'(\xi) : \xi \in [-(1+2c)b_n/\mu_n, (1+2c)b_n/\mu_n]\}$.

Proof. By setting $z_{i,j}(t) = x_{i,j}(t) - x_{i+1,j}(t)$, $1 \le i \le K$, $1 \le j \le n$, we consider the difference-differential system induced from (5.16): for $i = 1, \dots, K$,

$$\text{(5.17)} \quad \begin{cases} \dot{z}_{i,j}(t) = -\mu_j[x_{i,j}(t) - x_{i+1,j}(t)] + w_{i,j}(t), \ j = 1, \dots, n-1,\\ \dot{z}_{i,n}(t) = -\mu_n[x_{i,n}(t) - x_{i+1,n}(t)] - c[g(b_n x^\tau_{i,n}(t)) - g(b_n x^\tau_{i+1,n}(t))] + w_{i,n}(t), \end{cases}$$

where

$$\begin{aligned}
w_{i,j}(t) &= g(b_{j-1}x_{i,j-1}(t-\tau_I)) - g(b_{j-1}x_{i+1,j-1}(t-\tau_I)), j = 1, \dots, n-1 \pmod{n},\\
w_{i,n}(t) &= g(b_{n-1}x_{i,n-1}(t-\tau_I)) - g(b_{n-1}x_{i+1,n-1}(t-\tau_I))\\
&\quad - c\Sigma_{\ell \in J_i}[g(b_n x^\tau_{\ell,n}(t)) - g(b_n x^\tau_{\ell+1,n}(t))],
\end{aligned}$$

and $J_i := \{1, \dots, K\} \backslash \{i, i-1, i+1 \pmod{K}\}$. Obviously, for each $1 \le i \le K$, $x_{i,j}(t)$ eventually enters and then remains in $[-1/\mu_j, 1/\mu_j]$ for $j = 1, \dots, n-1$, and $x_{i,n}(t)$ eventually enters and then remains in $[-(1+2c)/\mu_n, (1+2c)/\mu_n]$. Consequently, system (5.16) satisfies assumption (D)* with $[\check{q}_j, \hat{q}_j] = [-1/\mu_j, 1/\mu_j]$, $j = 1, \dots, n-1$, and $[\check{q}_n, \hat{q}_n] = [-(1+2c)/\mu_n, (1+2c)/\mu_n]$. On the other hand, functions h_j, $\tilde{h}_j$, and w_j in (5.9) are now

$$\begin{aligned}
h_j(\phi_{1,j}(0), \phi_{2,j}(0), t) &= -\mu_j[\phi_{1,j}(0) - \phi_{2,j}(0)], \quad j = 1, \dots, n,\\
\tilde{h}_j(\phi_{1,j}, \phi_{2,j}, t) &= \begin{cases} 0, \quad j = 1, \dots, n-1,\\ -c[g(b_n\phi_{1,n}(-\tau_T)) - g(b_n\phi_{2,n}(-\tau_T))], \quad j = n, \end{cases}\\
w_j(\Phi_1, \dots, \Phi_K, t) &= \begin{cases} g(b_{j-1}\phi_{1,j-1}(-\tau_I)) - g(b_{j-1}\phi_{2,j-1}(-\tau_I)), j = 1, \dots, n-1 \pmod{n},\\ g(b_{n-1}\phi_{1,n-1}(-\tau_I)) - g(b_{n-1}\phi_{2,n-1}(-\tau_I))\\ -c\Sigma_{l \ne 1,2,K}[g(b_n\phi_{l,n}(-\tau_T)) - g(b_n\phi_{l+1,n}(-\tau_T))], \ j = n. \end{cases}
\end{aligned}$$

Accordingly, it can be verified that system (5.16) satisfies assumption (H)* with $\check{\mu}_j = \hat{\mu}_j = -\mu_j$ for $j = 1, \dots, n$, $\tilde{h}_j \equiv 0$, $\rho^w_j = 2$ for $j = 1, \dots, n-1$, $\check{\beta}_n = -cb_n$, $\hat{\beta}_n = -cb_n\hat{L}$, $\rho^h_n = 2c$, $\bar{\tau}_n = \tau_T$,

$\rho_n^w = 2 + 2c(K-3)$, and

$$\bar{\mu}_{jk}^{(i)} = 0 \ \text{ for all } i = 1, 2, \ldots, K, \ \text{ and } 1 \le j, k \le n,$$

$$\bar{\beta}_{jk}^{(1)} = \begin{cases} b_n, & (j,k) = (1,n), \\ b_{l-1}, & (j,k) = (l, l-1), \ l = 2, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\bar{\beta}_{jk}^{(i)} = 0 \ \text{ for all } i = 2, K, \ \text{ and } 1 \le j, k \le n,$$

$$\bar{\beta}_{jk}^{(i)} = \begin{cases} 0 & \text{otherwise}, \\ cb_n, & (j,k) = (n,n), \end{cases} \quad i \ne 1, 2, K,$$

$$\bar{\tau}_{jk}^{(i)} = \begin{cases} \tau_T, & i \ne 1, 2, K, \text{ and } (j,k) = (n,n), \\ \tau_I, & i = 1, \text{ and } (j,k) = (1,n) \text{ or } (j,k) = (l, l-1) \text{ for } l = 2, \ldots, n, \\ 0 & \text{otherwise}. \end{cases}$$

System (5.16) satisfies condition (S_1^*) due to the inequality $\tau_T < 3(\mu_n + b_n c\hat{L})/\{b_n[2\mu_n + b_n c(1+\hat{L})](1+cK)\}$. According to Theorem 5.1, the synchronization of system (5.16) follows from the convergence of the Gauss–Seidel iteration for solving the linear system

$$\text{circ}(M^{(1)}, \mathbf{0}^n, M^{(2)}, \ldots, M^{(2)}, \mathbf{0}^n)\mathbf{v} = \mathbf{0}, \tag{5.18}$$

where $M^{(1)} = [M_{jk}^{(1)}]_{1\le j,k\le n}$, $\mathbf{0}^n = [\mathbf{0}_{jk}^n]_{1\le j,k\le n}$, $M^{(2)} = [M_{jk}^{(2)}]_{1\le j,k\le n}$ are defined by

$$M_{jk}^{(1)} = \begin{cases} \mu_l, & (j,k) = (l,l), \ l = 1, \ldots, n-1, \\ \mu_n + b_n c\hat{L} - \tau_T b_n c[2\mu_n + cb_n(1+\hat{L})], & (j,k) = (n,n), \\ -b_n, & (j,k) = (1,n), \\ -b_{l-1}, & (j,k) = (l, l-1), \ l = 2, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathbf{0}_{jk}^n = 0, \quad 1 \le j, k \le n,$$

$$M_{jk}^{(2)} = \begin{cases} -b_n c, & (j,k) = (n,n), \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\text{circ}(M^{(1)}, \mathbf{0}^n, M^{(2)}, \ldots, M^{(2)}, \mathbf{0}^n)$ is strictly diagonally dominant, due to inequalities $\mu_j > b_{j-1}$ for $j = 1, \ldots, n-1 \pmod n$, $\mu_n + b_n c\hat{L} - b_{n-1} - (K-3)b_n c > 0$, and $\tau_T < [\mu_n + b_n c\hat{L} - b_{n-1} - (K-3)b_n c]/\{b_n c[2\mu_n + cb_n(1+\hat{L})]\}$. Consequently, the Gauss–Seidel iteration of linear system (5.18) converges to zero. ∎

The conditions in Theorem 5.2 depend on the scale of the network (K), coupling strength (c), coupling function (g), coupling delay (τ_T), and μ_i, b_j which determine the intrinsic dynamics in the loop. Moreover, the inequalities in the condition favor smaller network scale K. Actually, the present approach can also establish network-scale–independent and delay-independent synchronization criterion for system (5.16). The modified arguments based on setting $z_{i,j}(t) = x_{i,j}(t) - x_{i+1,j}(t)$, $1 \le i \le K-1$, $1 \le j \le n$, and the components $z_{i,n}(t)$ in (5.17) can also be regarded as follows:

$$\dot{z}_{i,n}(t) = -\mu_n z_{i,n}(t) + w_{i,n}(t),$$

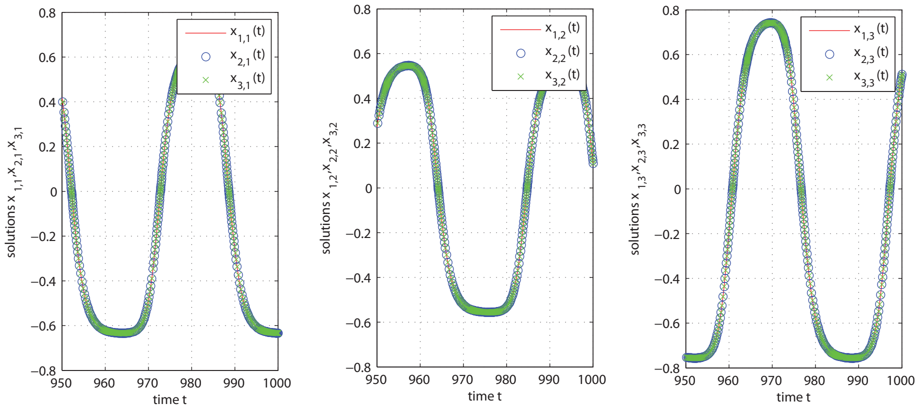

Figure 4. *Evolutions of components* $(x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$, $i = 1, 2, 3$, *for the solution of* (5.16) *with* $K = 3$, $n = 3$, $b_i = 0.99$, $i = 1, 2, 3$, $\tau_I = 11$, $\tau_T = 0.0004$, *and* $c = 20/99$, *starting from* $(-\sin t, t, 0, -t, 0, \sin t, -t \cdot \sin t, t, \sin t)$. *This solution converges to a synchronous periodic solution.*

where $w_{i,n}(t) = g(b_{n-1}x_{i,n-1}(t-\tau_I)) - g(b_{n-1}x_{i+1,n-1}(t-\tau_I)) + c\Sigma_{\ell=i-1,i+1}[g(b_n x^{\tau}_{\ell,n}(t)) - g(b_n x^{\tau}_{\ell+1,n}(t)]$. Then, by applying Theorem 3.3 and arguments similar to those for Theorem 5.2, we can derive the following result.

Theorem 5.3. *System* (5.16) *attains global synchronization if* $\mu_1 > b_n$, $\mu_j > b_{j-1}$, $j = 2, \ldots, n-1$, *and* $\mu_n > b_{n-1} + 2b_n c$.

For a synchronized coupled system, it is appealing to see its asymptotic states. Our numerical computations show that the possible global asymptotic states of system (5.16) include the origin, multiple equilibria, and a nontrivial periodic solution.

Example 5.1. (i) Consider system (5.16) with $K = 3$, $n = 3$, $\tau_I = 11$, $\tau_T = 0.0004$, $c = 20/99$, and $\mu_i = 1$ and $b_i = 0.99$, $i = 1, 2, 3$. The conditions of Theorem 5.2 are met for such parameters and delays, and hence the system achieves global synchronization. Figure 4 illustrates that the evolution from an asynchronous initial state converges to a nontrivial synchronous periodic solution.

(ii) Consider system (5.16) with the same parameters and delays except that c is changed to $c = 400/99$. The system still satisfies the conditions of Theorem 5.2. Figure 5 demonstrates that solutions originating from two different initial points converge to two distinct nontrivial synchronous steady states.

Example 5.2. Let us illustrate that synchronization depends on the network scale as well as the delay magnitude. In [8], the authors considered a ring of two loops ($K = 2$) comprising (5.15) with $n = 3, \tau_I = 0$:

$$(5.19)\qquad \begin{cases} \dot{x}_j(t) = -x_j(t) + g(bx_{j-1}(t)),\ j = 1, 2 \pmod 3, \\ \dot{x}_3(t) = -x_3(t) + g(bx_2(t)) + c_1 g(by_3(t-\tau_T)), \\ \dot{y}_j(t) = -y_j(t) + g(by_{j-1}(t)),\ j = 1, 2 \pmod 3, \\ \dot{y}_3(t) = -y_3(t) + g(by_2(t)) + c_2 g(bx_3(t-\tau_T)). \end{cases}$$

It was concluded that the origin of the system is globally asymptotically stable for all $\tau_T \geq 0$ if $b \cdot \max_{i=1,2}\{1 + |c_i|\} < 1$. Accordingly, system (5.19) with $b = 0.3$ and $c = 2$ achieves global

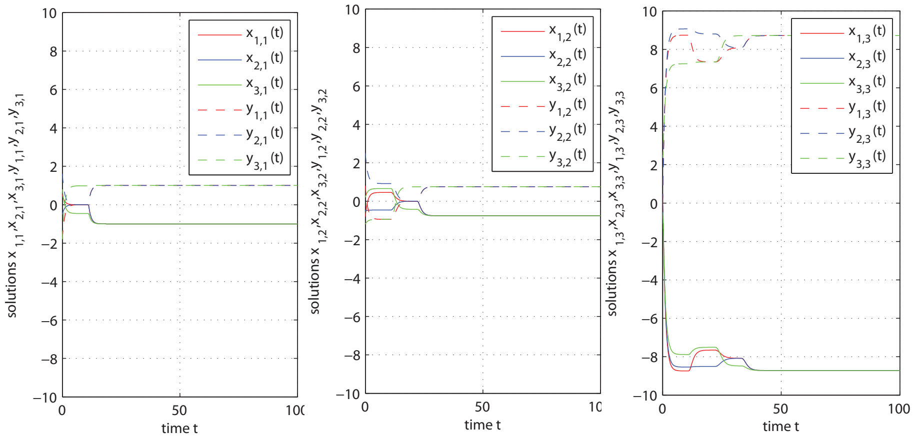

Figure 5. *Evolution of components* $(x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$ *(resp.,* $(y_{i,1}(t), y_{i,2}(t), y_{i,3}(t))$*),* $i = 1, 2, 3$*, for the solution of* (5.16) *with* $K = 3$*,* $n = 3$*,* $b_i = 0.99$ *for* $i = 1, 2, 3$*,* $\tau_I = 11$*,* $\tau_T = 0.0004$*, and* $c = 400/99$ *starting from* $(0.5, -0.8, 0, -0.5, -0.5, 0, 0.8, 0.2, -0.5)$ *(resp.,* $(-1.8, 0.8, 0, 1.6, 2.5, 0, -1.8, -1.2, 2.5)$*). These two solutions converge to different nontrivial synchronous equilibria.*

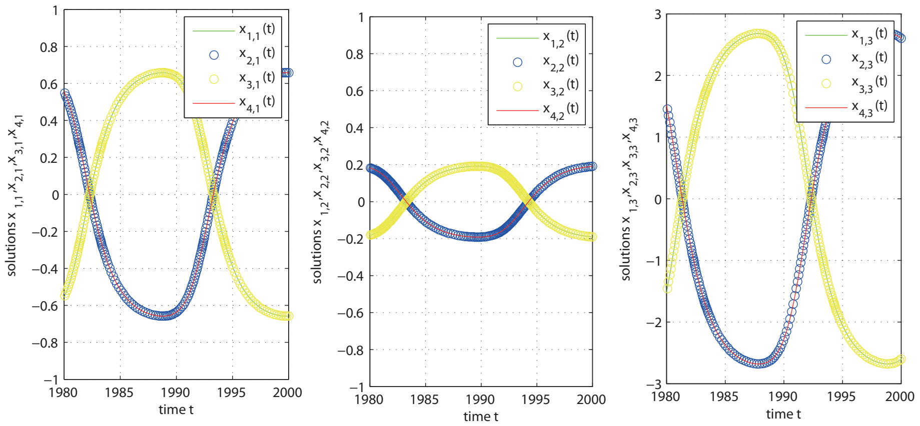

Figure 6. *Evolution of components* $(x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$*,* $i = 1, 2, 3, 4$*, for the solution of* (5.16) *with* $K = 4$*,* $n = 3$*,* $\mu_i = 1$*,* $b_i = 0.3$ *for* $i = 1, 2, 3$*,* $c = 2$*,* $\tau_I = 0$*, and* $\tau_T = 10$ *starting from* $(t \cdot \sin t, t, \sin t, -t \cdot \sin t, -t, -\sin t, t \cdot \sin t, t, \sin t, -t \cdot \sin t, -t, -\sin t)$*. This evolution tends to an asynchronous oscillation.*

convergence to the origin, and hence global synchronization, for all $\tau_T \geq 0$. If we consider the coupled loops with larger scale K, then the synchrony may be lost. More precisely, consider system (5.16) with $n = 3$, $K = 4$, $\mu_i = 1$, and $b_i = 0.3$ for $i = 1, 2, 3$, $c = 2$, and $\tau_I = 0$; then the system satisfies the criteria of Theorem 5.2, and hence attains global synchronization, if τ_T is smaller than some critical value near 0.1501. The numerical simulation in Figure 6 shows that the synchrony is lost if we increase τ_T to $\tau_T = 10$, while the other parameters remain. This is in contrast to the case $K = 2$ with the same parameters in (5.19).

On the other hand, it appears that system (5.16) satisfying the condition of Theorem 5.3

tends to achieve global convergence to the origin. For a systematic study of the asymptotic synchronous states, one can analyze the synchronous equations associated with system (5.16):

$$\begin{cases} \dot{x}_j(t) = -\mu_j x_j(t) + g(b_{j-1}x_{j-1}(t-\tau_I)), \ j = 1, \ldots, n-1 \ (\mathrm{mod}\ n), \\ \dot{x}_n(t) = -\mu_n x_n(t) + g(b_{n-1}x_{n-1}(t-\tau_I)) + 2cg(b_n x_n^{\tau}(t)). \end{cases}$$

However, the analysis may become challenging if n is large.

6. Discussion and conclusion.

Previous studies on synchronization have been primarily focused on dynamical systems with linear or linearly diffusive coupling. Those works include employing the master stability function or analyzing the stability of the synchronous set to study local synchronization [23, 35, 36, 43, 44] and using the Lyapunov function technique to study global synchronization [3, 29, 35, 36, 47, 48, 59, 58]. There are some papers, including [11, 29], which considered nonlinear coupling but are subject to the diffusive condition and the slope condition, which requires the coupling functions to have positive lower bounds on their slopes.

The collective behaviors between systems with diffusive coupling and nondiffusive coupling bear completely different senses. For a coupled system comprising identical subsystems under diffusive coupling, its synchronous solution is also a solution for each individual subsystem in isolation, because the coupling parts are annihilated at synchronous states. This is certainly not the case for the nondiffusive coupling scheme. For example, in the excitable FitzHugh–Nagumo neurons under nonlinear and nondiffusive coupling, discussed in section 4.1, there exists a synchronous oscillation, while each isolated subsystem does not have any periodic orbit. Laying aside the modeling issue, that the diffusive condition has been largely imposed in concluding synchronization is due to its need as a mathematical technicality in the derivation. Our approach requires neither this diffusive condition nor the slope condition on the coupling functions and has thus established new collective behaviors for coupled systems.

Indeed, the current challenge of the mathematical approach in concluding synchronization is to treat systems under nonlinear and delayed coupling. This investigation presented a new approach, named *sequential contracting*, to study global synchronization of coupled systems. The analysis finds its innovative capacity especially in systems under nonlinear and delayed coupling. The first key step of this approach is to seek a formulation of the difference-differential equations corresponding to the coupled systems, which can be manipulated to construct effective upper and lower dynamics. Through studying these upper-lower dynamics iteratively, the problem of synchronization is transformed, via sequential contracting, into solving a homogeneous linear system of algebraic equations. The present approach can be implemented to establish delay-dependent, delay-independent, network-scale–dependent, and network-scale–independent criteria for synchronization of coupled systems, through suitable designs of sequential upper and lower dynamics. We note that delay-dependent and network-scale–dependent criteria for synchronization are rare and even lacking in systems under delayed and nonlinear coupling in the literature.

The present approach can treat synchronization for systems comprising multiple subsystems coupled in a symmetric or asymmetric manner. The subsystems can be of arbitrary dimension, and thus this framework is suitable for models with multiple components such as signaling pathways in cell biology. We have applied the present approach to a variety of coupled systems, including classical neuronal models under synaptic coupling and neural networks under nonlinear and delayed coupling. In addition to new findings, applications to these various systems illustrate the assorted technicalities associated with applying this approach. Our methodology can also treat chaotic synchronization [54].

By applying Theorems 3.2 and 3.3 and arguments similar to those for Theorem 4.3, we can derive delay-independent and delay-dependent criteria for the synchronization in networks of oscillators under nonlinear and delayed couplings [56, 57, 67]. Moreover, the derived global synchronization criteria are compatible with the existence of stable in-phase periodic solution established by bifurcation theory or the existence of multiple synchronous equilibria and hence provide a theoretical result to the global dynamics for the system considered in [56, 57].

As applied to a network of oscillators, the present framework can accommodate a variety of coupling configurations, although we demonstrated only the circulant coupling. Basically, the difference-differential equations are composed according to the coupling configuration or network topology, as mentioned in section 5.1. The problem that solutions evolve toward the synchronous set can then be solved by analyzing the difference-differential equations.

If a positively invariant set for a coupled system can be located, then the idea of sequential contracting can also be applied to study local dynamics and local synchronization in that set. The analysis can also be adapted to investigate antiphase behaviors for coupled systems, via considering $x_i(t) + y_i(t) \to 0$, as $t \to \infty$, for each i.

A so-called *contraction analysis* was proposed by Slotine and collaborators [34, 55] to study convergence and synchronization for coupled oscillators. The formulation is based on a linearization setting and the criterion for synchronization is in terms of eigenvalue for certain corresponding matrices, including the Jacobian of the vector field for each unit. The present approach employs upper-lower dynamics iteratively to avoid overmanipulating the nonlinearity by linearization. In studying synchronization and asymptotic behaviors in dynamical systems, a dissipative condition, such as assumption (D) in section 3.1, is usually a basic requirement, although concluding such a property in a nonlinear system is already a nontrivial task. Through studying the upper-lower dynamics iteratively, our analysis actually incorporates the notion of attracting set into the framework. Such a consideration is lacking in [34, 55].

The synchronization considered in our manuscript may be called "perfect synchronization"; i.e., the corresponding components of all subsystems tend to be identical as time evolves. Underlying such a scenario is the existence of some synchronous solution $(\mathbf{x}(t), \mathbf{y}(t)) = (\mathbf{z}(t), \mathbf{z}(t))$ for system (1.1). For example, if the coupled system globally synchronizes to a periodic solution, then there must exist such a stable periodic orbit on the synchronous set. For coupled systems comprising nonidentical subsystems, it is natural to relax the notion of perfect synchronization to "approximate synchronization" (see [3]). On the other hand, it is possible to extend our approach to identical subsystems coupled under asymmetric coupling. In fact, in the expression in (5.1) (without assuming (5.2)), $\mathbf{G}_i$ is allowed to be disparate for each i. The key point of such an extension is whether the induced difference-differential system (5.8) can be analyzed under the present framework. An example for such an extension is the chaotic oscillators, such as the Lorenz oscillator, coupled in a driven-and-response manner [54].

There are synchronization problems for some neural networks under delayed and nonlinear coupling which cannot be solved by previous methodologies, including the Lyapunov function technique [53]. On the other hand, while looking for a Lyapunov function seems difficult in some systems with complicated nonlinear terms and delays, our approach provides a new alternative to tackle asymptotic behaviors including synchronization. The idea of sequential contracting has also been applied to study multistability in delayed neural networks [52], the asymptotic phases in an integro-differential equation modeling T-cell differentiation [20, 21], and the preimages of a snapback repeller in multidimensional maps [31].

Appendices A–C contain detailed verifications which are supplementary to sections 2 and 4.

Appendix A. Proof of Proposition 2.3. From

$$\hat{h}^{(\infty)}(-m(T),T) = -(\check{\mu}+\check{\beta})m(T) + \bar{\beta}\bar{\tau}\hat{h}^{(\infty)}(-m(T),T) + |w|^{\max}(T),$$

it follows that

$$0 \le \hat{h}^{(\infty)}(-m(T),T) = \frac{-(\check{\mu}+\check{\beta})m(T) + |w|^{\max}(T)}{1-\bar{\beta}\bar{\tau}}.$$

Consequently, for $\xi \ge 0$,

$$\hat{h}^{(\infty)}(\xi,T) = (\hat{\mu}+\hat{\beta})\xi + \frac{\bar{\beta}\bar{\tau}[-(\check{\mu}+\check{\beta})m(T) + |w|^{\max}(T)]}{1-\bar{\beta}\bar{\tau}} + |w|^{\max}(T). \tag{A.1}$$

Using that $m(T)$ is a zero to $\hat{h}^{(\infty)}(\cdot,T)$ in (A.1) yields

$$m(T) = |w|^{\max}(T)/\{-\hat{\mu} - \hat{\beta} + \bar{\beta}\bar{\tau}(\check{\mu}+\hat{\mu}+\check{\beta}+\hat{\beta})\}.$$

Appendix B. Upper-lower dynamics for Proposition 2.4.

$$\begin{aligned}
\hat{h}(\xi) &:= \begin{cases} \hat{\mu}\xi + \rho^h + |w|^{\max}(t_0) & \text{for } \xi \ge 0, \\ \check{\mu}\xi + \rho^h + |w|^{\max}(t_0) & \text{for } \xi < 0, \end{cases}\\
\hat{h}^{(0)}(\xi,T) &:= \begin{cases} \hat{\mu}\xi + \bar{\beta}\hat{A}^h + |w|^{\max}(T) + \varepsilon_0 & \text{for } \xi \ge 0, \\ \check{\mu}\xi + \bar{\beta}\hat{A}^h + |w|^{\max}(T) + \varepsilon_0 & \text{for } \xi < 0, \end{cases}\\
\hat{h}^{(k)}(\xi,T) &:= \begin{cases} \hat{\mu}\xi + \bar{\beta}\hat{m}^{(k-1)}(T) + |w|^{\max}(T) + \varepsilon_k, & \xi \ge 0, \\ \check{\mu}\xi + \bar{\beta}\hat{m}^{(k-1)}(T) + |w|^{\max}(T) + \varepsilon_k, & \xi < 0, \end{cases}\\
\check{h}(\xi) &= -\hat{h}(-\xi,T), \check{h}^{(0)}(\xi,T) = -\hat{h}^{(0)}(-\xi,T), \check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T).
\end{aligned}$$

Appendix C. Proof of Lemma 4.1. Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be an arbitrary solution of (4.1). We first show that $(x_1(t), x_2(t))$ converge to $\tilde{Q}^{(1)}$. By setting $V(t) := [x_1^2(t) + x_2^2(t)/b]/2$, we obtain

$$\dot{V}(t) = -x_1^4 + (a+1)x_1^3 - ax_1^2 + cf(y_1^\tau)x_1 - (\gamma/b)x_2^2,$$

where $x_i = x_i(t)$, $i = 1, 2$, and $y_1^\tau = y_1(t-\tau)$. Notably, $|f(y_1^\tau)| < \rho$; thus

$$\dot{V}(t) \le -x_1^4 + (a+1)x_1^3 - ax_1^2 + \rho|cx_1| - (\gamma/b)x_2^2 = P^{(1)}(x_1) - (\gamma/b)x_2^2, \tag{C.1}$$

where $P^{(1)}$ is defined in (4.20). Obviously, $P^{(1)}(\xi) < 0$ for $|\xi| > \bar{q}^{(1)}$. Thus $V(t)$ is decreasing with respect to t should $(x_1(t), x_2(t))$ stay in $\{(\xi,\zeta) \in \mathbb{R}^2, |\xi| > \bar{q}^{(1)}\}$. On the other hand, $-\gamma x_2(t) - b\bar{q}^{(1)} \le \dot{x}_2(t) \le -\gamma x_2(t) + b\bar{q}^{(1)}$; hence $x_2(t)$ approaches $[-(b/\gamma)\bar{q}^{(1)}, (b/\gamma)\bar{q}^{(1)}]$ should $(x_1(t), x_2(t))$ stay in $\{(\xi,\zeta) \in \mathbb{R}^2, |\xi| \le \bar{q}^{(1)}, |\zeta| > (b/\gamma)\bar{q}^{(1)}\}$. Now, let us consider the following set:

$$\Omega^{(1)} := \left\{ (\xi,\zeta) \in \mathbb{R}^2 : \begin{cases} \xi^2 + \zeta^2/b \le (1 + b/\gamma^2)(\bar{q}^{(1)})^2, & |\xi| > \bar{q}^{(1)}, \\ |\zeta| \le (b/\gamma)\bar{q}^{(1)}, & |\xi| \le \bar{q}^{(1)} \end{cases} \right\}.$$

Notably, $\max\{V(\xi,\zeta) : (\xi,\zeta) \in \Omega^{(1)}\} = V(\tilde{\xi},\tilde{\zeta}) = (1 + b/\gamma^2)(\bar{q}^{(1)})^2$, where $(\tilde{\xi},\tilde{\zeta})$ lies on the boundary of $\Omega^{(1)}$ with $|\tilde{\xi}| \ge \bar{q}^{(1)}$. Based on these arguments, $\Omega^{(1)}$ is positively invariant under the flow generated by system (4.1). Moreover, $(x_1(t), x_2(t))$ enters $\Omega^{(1)}$, and hence $\tilde{Q}^{(1)}$, as $t \to \infty$. The assertion exactly holds for $(y_1(t), y_2(t))$. Note that (C.1) refers to the first estimation on $\dot{V}(t)$ via function $P^{(1)}$. As restricted to region $\tilde{Q}^{(1)}$, we can construct function $P^{(2)}$ defined in (4.20), which provides finer estimation on $\dot{V}(t)$. Accordingly, we can conclude that $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ both enter $\tilde{Q}^{(2)}$, which is defined in Lemma 4.1. Iteratively, for all $k \ge 3$, we can construct $\bar{q}^{(k)}$ as defined in (4.22), which is strictly decreasing, and then conclude that both $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ eventually enter, and then remain in, $\tilde{Q}^{(k)}$, defined in Lemma 4.1.

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