PHYSICAL REVIEW A **88**, 051601(R) (2013)

Roton instabilities and correlated Wigner crystals of rotating dipolar fermions in the fractional quantum Hall regime

Shih-Da Jheng,[1] T. F. Jiang,[1] and Szu-Cheng Cheng[2,*]

[1]*Institute of Physics, National Chiao Tung University, Hsinchu 30010, Taiwan, Republic of China*

[2]*Department of Physics, Chinese Culture University, Taipei 11114, Taiwan, Republic of China*

(Received 17 May 2013; published 4 November 2013)

We point out that the Wigner crystal (WC) state of rotating dipolar fermions in the lowest Landau level is strongly correlated. From intra-Landau-level excitations of the fractional quantum Hall effect (FQHE) liquid, we find that the roton minimum of the excitation spectrum at $\nu = 1/7$ becomes negative, indicating an absence of the FQHE state, which is supposed to be the ground state of the system. Therefore, we show that the Hartree-Fock theory of the WC does not predict the correct ground-state energy in the lowest Landau level. There exists a strong correlation effect on the Wigner crystallization of rotating dipolar gases.

DOI: 10.1103/PhysRevA.88.051601 PACS number(s): 67.85.−d, 03.75.Ss, 73.43.Lp, 73.43.Nq

The cold quantum gas with the dipole-dipole interaction (DDI) was first realized in Cr atoms [1]. The quantum gases with the DDI are qualitatively different from nondipolar ones [2]. The novel anisotropic and long-range nature of the DDI offers a broad range of strong correlated many-body physics [3]. New quantum phases were predicted for the dipolar Bose-Einstein condensate (BEC) [4]. The influence of the trapping geometry on the stability of the BEC and the effect of the DDI on the excitation spectrum were studied [5]. The vortex lattice of the rotating dipolar BEC exhibits novel bubble, stripe, and square structures [6]. For the dipolar Fermi gases, the s-wave scattering is prohibited due to the Pauli exclusion principle, and the Fermi surface is distorted by dipolar effects [7]. Bond pairs of fermions with resonant interactions are formed, and the system of a Fermi gas behaves as a bosonic gas of molecules [8]. The observed pairing of fermions provides the crossover between the weakly paired strongly overlapping Bardeen-Cooper-Schrieffer regime and the tightly bound weakly interacting diatomic molecular BEC regime [9]. The strong correlations of fermions induced by the DDI can then be explored, such as the dipolar-induced superfluidity [10] and fractional quantum Hall effect (FQHE) states in rotating dipolar Fermi gases [11,12].

Rotating gases feel the Coriolis force in the rotating frame. The Coriolis force on rotating gases is identical to the Lorentz force of a charged particle in a magnetic field. Quantum mechanically, energy levels of a charged particle in a uniform magnetic field show discrete Landau levels. In the lowest Landau level (LLL), the kinetic energy of rotating dipolar gases is frozen, and the DDI creates strong correlations on particles. Therefore, in the lowest Landau level, the potential energy from the DDI dominates the kinetic energy, and rotating dipolar fermions will crystallize into a Wigner crystal (WC) or will become the FQHE liquid. Baranov *et al.* [12] have shown that the FQHE states have lower energies than the WCs as filling factors $\nu \geqslant 1/7$ and in the zero-extension limit along the rotating axial direction where $\nu = 2\pi\rho a^2$. Here ρ and a are the average density and the magnetic length, respectively. They also investigated the stability of the WC and found that the WC states were stable in the regime of $\nu < 1/7$, but there was no test on the stability of the FQHE liquid. It is still a question whether the FQHE liquid for rotating dipolar fermions is stable.

The phase transition point of the WC and FQHE states is usually determined by comparing their energies. But this is not enough for a consistent theoretical description of the FQHE. A consistent theory should tell us when the FQHE will not occur and becomes unstable [13]. One possible signature for instability can be revealed from the softening of the roton gap [14], which is an energy minimum at a finite wave vector of the collective excitations of the FQHE state. Whereas, the roton gap decreases with ν, the collective excitations of the FQHE states for a two-dimensional (2D) electron gas do not show any instability at the small ν regime where the actual ground state is the WC [13–15]. Such a shortcoming of the theory of the FQHE is repaired by applying the composite-fermion scheme to observe the collapse of the roton gap of the FQHE liquid at small ν's [13]. For the present theoretical description of rotating dipolar fermions, it is not clear whether the excitation gaps do cross zero at small ν's. It would be interesting to look for a similar instability, which may provide a clue to the true ground states, in rotating dipolar fermions.

In this Rapid Communication, we apply the single-mode approximation (SMA) [14] to calculate intra-Landau-level excitation energies of FQHE states in rotating 2D polarized dipolar gases whose mass is M. We focus on low-energy excitations of the density oscillations in the LLL. To avoid the inhomogeneous effect in the 2D direction, we studied the system rotating in the limit of critical rotation where the magnitude of the rotating frequency Ω of the system is close to but still smaller than the trapping frequency. Under the limit of critical rotation, the density of the trapped gas becomes uniform except at the boundary given by a trapping potential [16]. The excitation spectra of FQHE states exhibit a finite excitation-energy gap at $k = 0$ and a roton minimum Δ (roton) at wave vector $k_{\min}$, which is the wave vector corresponding to the roton minimum. Δ decreases with decreasing ν and surprisingly becomes negative when $\nu \leqslant 1/7$, indicating an absence of the FQHE state. It was argued by Baranov *et al.* [12] that the ground states of rotating dipolar fermions for $\nu > 1/7$ and $\nu < 1/7$ are supposed to be the FQHE state and the

*sccheng@faculty.pccu.edu.tw

WC, respectively. They do not discuss what really happens around $\nu = 1/7$. In this Rapid Communication, we show that the Hartree-Fock (HF) WC state does not provide the correct estimate of the ground-state energy at $\nu = 1/7$. There exists a strong correlation effect on the Wigner crystallization of rotating dipolar gases at small ν.

The formal development of mathematics within the subspace of the LLL has been presented elsewhere [14]. We will apply previous formalisms and will take the magnetic length $a = 1$. The projected density operator with N particles within the LLL is given by $\bar{\rho}_{\mathbf{k}} = \sum_{j=1}^{N} \exp(-ik\partial/\partial z_j)\exp(-ik^* z_j/2)$, where $z_j = x_j + iy_j$ and $k = k_x + ik_y$ are the complex representations of the jth particle position and the wave vector of the density oscillations. The projected density operators satisfy a commutation relation defined by [14] $[\bar{\rho}_{\mathbf{q}}, \bar{\rho}_{\mathbf{k}}] = \Lambda(k,q)\bar{\rho}_{\mathbf{k}+\mathbf{q}}$, where $\Lambda(k,q) = (e^{q^* k/2} - e^{qk^*/2})$. The kinetic energy is constant and neglected due to the LLL approximation. The projected Hamiltonian is written as

$$H = \frac{1}{2(2\pi)^2} \int d^2\mathbf{q}\, V(\mathbf{q})\left(\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} - \rho e^{-q^2/2}\right), \qquad (1)$$

where $V(\mathbf{q}) = -2\pi q \exp(-\xi^2)\mathrm{erfc}(\xi) D/a^3$ is the Fourier transform of the 2D dipolar-interaction potential with the coupling constant D. $\xi = \sqrt{q\ell}/2a$, where ℓ is the extension of the system along the rotating axial direction and $\mathrm{erfc}(\xi)$ is the complementary error function. Because of the Pauli exclusion of fermions, we have ignored the contact interaction term to obtain $V(\mathbf{q})$, whose negative sign was due to the contact term being ignored. The interaction would still be repulsive if the contact term was included.

From Ref. [14], the excitation state $|\mathbf{k}\rangle$ with momentum $\mathbf{k}$ is given by $|\mathbf{k}\rangle = N^{-1/2}\bar{\rho}_{\mathbf{k}}|0\rangle$, where $|0\rangle$ is the ground state with energy E_0. From $H|\mathbf{k}\rangle = E_k|\mathbf{k}\rangle$, we can evaluate the excitation energy $\Delta(k) = E_k - E_0$ by the formula $\Delta(k) = \bar{f}(k)/\bar{s}(k)$, where $\bar{f}(k)$ and $\bar{s}(k)$ are the projected oscillator strength and static structure factor, respectively. Using Eq. (1), $\bar{f}(k) = \langle 0|[\bar{\rho}_{-\mathbf{k}}, [H, \bar{\rho}_{\mathbf{k}}]]|0\rangle/2N$ is readily evaluated, and we have

$$\bar{f}(k) = \frac{1}{2}\sum_{\mathbf{q}} V(\mathbf{q})\Lambda(k,q)\left[\bar{s}(q)e^{-k^2/2}\Lambda^*(-k,q) - \bar{s}(k+q)\Lambda(k,q)\right]. \qquad (2)$$

Note that $\bar{s}(k) = \rho \int d^2\mathbf{r}\, \exp(-i\mathbf{k}\cdot\mathbf{r})g(\mathbf{r}) + \exp(-k^2/2)$, where

$$g(\mathbf{r}) = 1 - e^{-2R} + 2\sum_{m=0}^{\infty} \frac{C_{2m+1}}{(2m+1)!}(R)^{2m+1}e^{-R}, \qquad (3)$$

and $R = r^2/4$. $g(\mathbf{r})$ is the radial distribution function which measures the correlation between pairs of particles and becomes unity if there is no correlation between particles. $g(\mathbf{r})$ can be different from unity due to the ground-state wave function containing the correlation from the Pauli exclusion principle. The coefficients C_{2m+1} in Eq. (3) were derived by the sum rules from the physical properties of Laughlin's wave functions [15] and are shown in Table I for different fractional filling factors. $C_{2m+1} = 0$ for $m \geqslant 7$. Having obtained an analytic form of the radial function, ground-state energies and the projected static structure factors of the FQHE states are readily evaluated.

TABLE I. Coefficients of the radial distributions of FQHE states.

	C_1	C_3	C_5	C_7	C_9	C_{11}	C_{13}
$\nu = 1/3$	-1	$17/32$	$1/16$	$-3/32$	0	0	0
$\nu = 1/5$	-1	-1	$7/16$	$11/8$	$-13/16$	0	0
$\nu = 1/7$	-1	-1	-1	$-25/32$	$79/16$	$-85/32$	0
$\nu = 1/9$	-1	-1	-1	-1	$-29/8$	$47/4$	$-49/8$

For $\ell = 0$, the ground-state energies are $E_0(\nu)/(D/a^3)$'s, for $\nu = 1/3$, $1/5$, $1/7$, and $1/9$, the ground-state energies are $0.3690(1/3)^{3/2}$, $0.3361(1/5)^{3/2}$, $0.3228(1/7)^{3/2}$, and $0.3105(1/9)^{3/2}$, respectively. These energies are in excellent agreement with the energies from the Monte Carlo method [12] where the $E_0(\nu)/(D/a^3)$'s are equal to $0.3665(1/3)^{3/2}$, $0.3348(1/5)^{3/2}$, $0.3216(1/7)^{3/2}$, and $0.3145(1/9)^{3/2}$ for $\nu = 1/3$, $1/5$, $1/7$, and $1/9$, respectively.

From the Fourier transform of Eq. (3), we compute $\bar{s}(k)$, then use this to evaluate $\bar{f}(k)$ and to finally calculate the collective-excitation energy $\Delta(k)$. $\bar{s}(k) \sim k^4$ for small k is a phenomenon of the lack of density fluctuations or the incompressibility of FQHE states at long wavelengths. This is the source of the finite-energy gap observed in the FQHE of a 2D electron gas in a strong magnetic field. Therefore, we compute $\Delta(k)$ at small k using the exact leading term in $\bar{s}(k)$, which is $\bar{s}(k) = (1-\nu)k^4/8\nu$ in the long-wavelength limit. We studied the collective-mode dispersion for $\ell = 0$. The evaluated $\Delta(k)$ versus wave vector k for the FQHE states is shown in Fig. 1. The essential features of excitation dispersions exhibit a finite excitation-energy gap at $k = 0$ and a roton. An energy gap at $k = 0$ implies that the density fluctuations cost energy, and a rotating-dipolar-Fermi gas is an incompressible fluid. The excitation-energy gap at $k = 0$ is decreasing with decreasing ν and becomes very small at $\nu = 1/9$, indicating that the FQHE is easily destroyed by the quantum fluctuations at very low ν.

The roton structure is caused by a peak in $\bar{s}(k)$ associated with the short-range order in Laughlin's liquid [14]. The susceptibility of Laughlin's liquid is sharply peaked at $k_{\min}$, which is close to the primitive reciprocal-lattice wave vector G of a WC. The location of G is indicated by the arrow for the individual filling factor in Fig. 1. The magnitude of the susceptibility also rises very rapidly as the filling factor approaches $\nu = 1/7$, which is usually interpreted as the starting signal for crystallization. The closeness of $k_{\min}$ and G indicates that the Laughlin liquid is very sensitive to fluctuations whose wave vectors are comparable with the wave vector of the WC [14]. The WC state is shown to be the ground state at very low ν. This fact is revealed by the roton energies decreasing with decreasing ν in Fig. 1. The roton gap is finite and reveals a significant reduction in going from $\nu = 1/3$ and $1/5$, respectively. This behavior of the roton gap indicates that it costs energy at higher ν to make the FQHE state unstable and there is a signature of incipient crystallization near $\nu = 1/7$.

If we lower ν further, the roton minimum is continuously decreasing and becomes negative as $\nu \leqslant 1/7$ [see Figs. 1(c) and 1(d)]. This collapse of the gap shows that the FQHE state with excitations has lower energy than the FQHE state itself and no FQHE in the regime of $\nu \leqslant 1/7$. We know that no FQHE but the Wigner crystallization exists in the

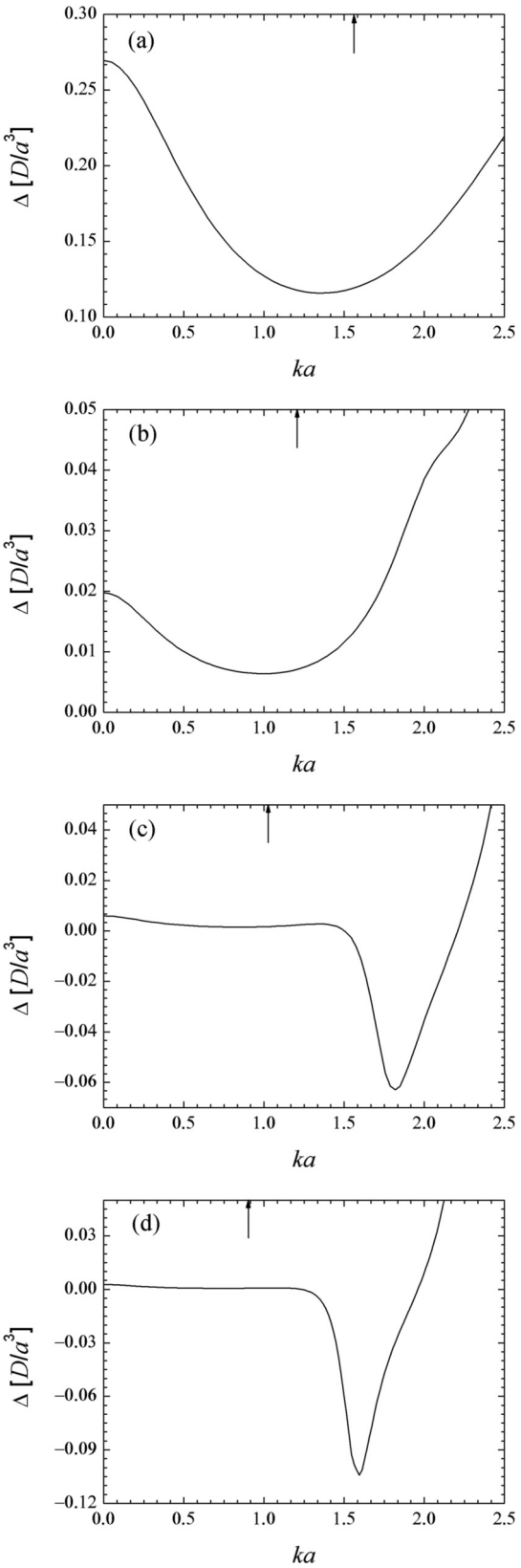

FIG. 1. Collective-excitation dispersions for FQHE states with zero thicknesses at (a) $\nu = 1/3$, (b) $\nu = 1/5$, (c) $\nu = 1/7$, and (d) $\nu = 1/9$, respectively. The location of the primitive reciprocal-lattice wave vector of the corresponding Wigner crystal is marked by an arrow inside the figure.

regime of $\nu \leqslant 1/9$ [12], which is contradictory with our study. We conclude, from the roton instability, that the phase boundary between the FQHE and the WC states is at $\nu = 1/7$. Therefore, our study shows that the FQHE states below $\nu = 1/7$ are unstable for a spontaneous creation of excitations. This unexpected instability suggests that there exists a strong correlation effect on the WC state which is ignored in the previous HF theory of the WC [12]. Note that the G's of the WC states at $\nu = 1/7$ and $1/9$ are given by $G_{1/7}a = 1.018$ and $G_{1/9}a = 0.898$, respectively. Instead of near the G of a WC, we find that the roton minima for $\nu = 1/7$ and $1/9$ are far from this wave vector and are located at $ka = 1.833$ and 1.583, respectively. It seems that the collapse of the roton gap only indicates the instability of the FQHE state and does not show any signature of the Wigner crystallization at very low ν. In fact, we believe that one must introduce Laughlin-Jastrow correlations into the WC state to interpret the Wigner crystallization in terms of the softening of the roton minimum [17]. As shown in Ref. [17], density fluctuations caused by displacements of particles from the lattice sites effectively are suppressed by the Laughlin-Jastrow correlation. This delocalization of particles from the lattice sites makes the average lattice constant, and the G of the WC becomes shorter and larger, respectively. The fact that the wave vector, at which the roton minimum becomes negative, is far from the G of the WC indicates the importance of the Laughlin-Jastrow correlation on the Wigner crystallization of dipolar fermions.

We analyze the Laughlin-Jastrow correlation of the WC using the following trial wave function [17]:

$$\Psi_C(\{\mathbf{r}_i\}) = A\prod_{i<j}(z_i - z_j)^m \prod_i \phi_{R_i}(r_i), \tag{4}$$

where A stands for complete antisymmetry and m is a variational parameter to be optimized. To antisymmetrize the wave function, we choose to use either a Slater determinant (even m) or a symmetric sum (odd m) for all possible permutations of the single-particle wave functions. Because the particles of a finite system will be pushed radially away from their lattice sites due to a nonuniform repulsive force from other particles, some ghost particles will be introduced to balance the nonuniform repulsive force exerted by other particles inside the physical disk [17]. Therefore, the single particle, which is located at the triangular lattice site $\mathbf{R}_i$, is given as [17] $\phi_{R_i}(\mathbf{r}_i) = \exp[-\frac{1}{4}|\mathbf{r}_i - (1-\nu m)\mathbf{R}_i|^2 + \frac{i}{2}\mathbf{r}_i \times (1-\nu m)\mathbf{R}_i \cdot \mathbf{e}_z - \nu m(1-\nu m)|R_i|^2/4]$. For $\ell = 0$, ground-state energies $E_{C\mathrm{WC}}(\nu)/(D/a^3)$ of the correlated WCs are evaluated by the Monte Carlo method and are equal to 0.3056, 0.3006, and 0.2973 for $\nu = 1/9$, $1/11$, and $1/13$, respectively. These energies are lower than the ground-state energies $E_{\mathrm{HF\,WC}}(\nu)/(D/a^3)$ of the HF WC states [12], where the $E_{\mathrm{HF\,WC}}(\nu)/(D/a^3)$'s are 0.3117, 0.3053, and 0.3012 for $\nu = 1/9$, $1/11$, and $1/13$, respectively. That the correlated WC has a lower energy than the HF WC is consistent with the excitation spectra, indicating the non-negligible effect of the Laughlin-Jastrow correlation on the Wigner crystallization of the dipolar fermions.

The SMA theory provides an energy upper bound to the lowest excited state. For large k, the quasiparticle and the

quasihole, which are a pair of quasiparticles created by the density-wave excitation, are far apart. The upper bound of the excitation energy of the pair is given by the first excitation moment $\Gamma = 2[\nu E_0(1) - E_0(\nu)]/(1-\nu)$ [14], where $E_0(1) = 0.6267D/a^3$ is the ground-state energy at $\nu = 1$. Suppose the binding energy of a far apart quasiparticle-quasihole pair is very small due to the $1/r^3$ of the dipole-dipole interaction. The upper bound of the excitation energies of a quasiparticle plus a quasihole is also provided by the moment Γ. The moments for $\nu = 1/3$ and $1/5$ are $\Gamma_{1/3} = 0.4136D/a^3$ and $\Gamma_{1/5} = 0.2382D/a^3$, respectively. These values are above the roton gap but lie considerably below the result of the quasiparticle excitation spectrum of Baranov *et al.* [11]: The excitation energy of a quasihole is $\Delta\varepsilon_{qh} = 0.9271D/a^3$, and the same order of magnitude is expected in the excitation energy of quasiparticles. Although we do not have an explanation for this large discrepancy, we can certainly say that there are no low-lying excitations of quasiparticles and quasiholes below the roton mode. The lack of low-lying single-particle excitations means that the dynamics of the system is completely described in terms of collective modes. The validity of the SMA is proved.

In conclusion, we investigated intra-Landau-level excitations of a rotating quasi-2D dipolar system in the FQHE regime. We show that the excitation gap at $k = 0$ and the roton minimum are decreasing with decreasing ν. We find that the roton minimum becomes negative and the FQHE states are unstable as $\nu \leqslant 1/7$. The wave vector at which the roton gap is collapsed is far from the primitive reciprocal-lattice wave vector of the WC. This discrepancy indicates that one must introduce Laughlin-Jastrow correlations into the ground state of the WC to find the phase transition from the WC to the FQHE liquid. We believe that this transition will occur between $\nu = 1/5$ and $1/7$.

This work has been supported by the NSC of the Republic of China (Grant No. NSC102-2112-M-034-001-MY3) and the NCTS of Taiwan.

[1] J. Stuhler, A. Griesmaier, T. Koch, M. Fattori, T. Pfau, S. Giovanazzi, P. Pedri, and L. Santos, Phys. Rev. Lett. **95**, 150406 (2005).

[2] M. Baranov *et al.*, Phys. Scr., T **102**, 74 (2002).

[3] M. A. Baranov, Phys. Rep. **464**, 71 (2008); T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. **72**, 126401 (2009)

[4] H. P. Büchler *et al.*, Phys. Rev. Lett. **98**, 060404 (2007); G. E. Astrakharchik, J. Boronat, I. L. Kurbakov, and Y. E. Lozovik, *ibid.* **98**, 060405 (2007).

[5] U. R. Fischer, Phys. Rev. A **73**, 031602(R) (2006).

[6] N. R. Cooper, E. H. Rezayi, and S. H. Simon, Phys. Rev. Lett. **95**, 200402 (2005); J. Zhang and H. Zhai, *ibid.* **95**, 200403 (2005).

[7] T. Miyakawa, T. Sogo, and H. Pu, Phys. Rev. A **77**, 061603(R) (2008).

[8] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004); M. W. Zwierlein *et al.*, *ibid.* **92**, 120403 (2004).

[9] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).

[10] M. A. Baranov, L. Dobrek, and M. Lewenstein, Phys. Rev. Lett. **92**, 250403 (2004); G. M. Bruun and E. Taylor, *ibid.* **101**, 245301 (2008).

[11] M. A. Baranov, K. Osterloh, and M. Lewenstein, Phys. Rev. Lett. **94**, 070404 (2005).

[12] M. A. Baranov, H. Fehrmann, and M. Lewenstein, Phys. Rev. Lett. **100**, 200402 (2008).

[13] V. W. Scarola, K. Park, and J. K. Jain, Phys. Rev. B **62**, R16259 (2000).

[14] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. Lett. **54**, 581 (1985); Phys. Rev. B **33**, 2481 (1986).

[15] M. O. Goerbig and C. M. Smith, Phys. Rev. B **66**, 241101(R) (2002).

[16] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science **292**, 476 (2001).

[17] H. Yi and H. A. Fertig, Phys. Rev. B **58**, 4019 (1998).