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The construction of dual-trace factor in Yang-Mills theory

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Abstract: Recently, a BCJ dual of the color-ordered formula for Yang-Mills amplitude was proposed, where the dual-trace factor satisfies cyclic symmetry and KK-relation. In this paper, we present a systematic construction of the dual-trace factor based on its proposed relations to kinematic numerators in dual-DDM form. We show that the construction presented respects relabeling symmetry. In addition, we show that using relabeling symmetry as conditions, the same construction can be solved independently.

Keywords: Scattering Amplitudes, Gauge Symmetry

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1 Introduction

In recent years, a significant progress in the study of scattering amplitudes is the discovery of color-kinematic duality [1]. At tree-level, the duality states that the complete Yang-Mills tree amplitude $\mathcal{A}_{\text{tot}}$ can always be written into the following formula

$$\mathcal{A}_{\text{tot}} = \sum_i \frac{c_i n_i}{D_i}, \tag{1.1}$$

where the sum runs over all distinct cubic tree diagrams. In the formulation of Bern, Carrasco and Johansson (BCJ), the kinematic factors n_i, which we will call "BCJ numerator", satisfy the same algebraic relations as those of the color factors c_i, i.e.,

$$\begin{aligned} \text{antisymmetry}: &\quad c_i \to -c_i \Rightarrow n_i \to -n_i \\ \text{Jacobi}-\text{like identity}: &\quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0. \end{aligned} \tag{1.2}$$

The duality between color and kinematic factors provides strong constraints on color-ordered Yang-Mills tree amplitudes. Specifically, the antisymmetry of kinematic factors implies Kleiss-Kuijf [2, 3] relations, while the Jacobi-like identity implies BCJ relations [1]. The newly discovered BCJ relations have been understood both from string [4–6] and field theory [7–10] perspectives. BCJ relations also serve as the key to the understanding of KLT relations [11], which express gravity tree amplitude in terms of products of two color-ordered Yang-Mills tree amplitudes (See [12–15]). Although a proof at loop-levels is currently absent, explicit calculations show that the duality (1.2) is also satisfied at the first few loops [16–25].

Because of the applications in the study of Yang-Mills and gravity amplitudes, the construction of BCJ numerators has become an important problem, and there are many discussions in the literature. In [26], BCJ numerators were constructed by string pure-spinor method. In [27, 28], a light-cone gauge approach for the kinematic algebra was suggested, which provides a natural algebraic explanation to BCJ duality. Using this approach BCJ numerators of MHV Yang-Mills amplitude and all amplitudes in self-dual Yang-Mills theory can indeed be expressed as structure constants of a diffeomorphism algebra [27, 28]. Based on this idea, we have proposed a more general kinematic algebra in [29], from which one can construct BCJ numerators at tree-level in arbitrary D-dimensions and for arbitrary helicity configurations.

The fact that the color factors c_i and the kinematic numerators n_i share the same algebraic structure, also suggests that the existing decompositions of Yang-Mills tree amplitudes may have color-kinematic counterparts. Traditionally, we have two different decompositions of Yang-Mills amplitudes [3]

$$\text{Trace form}: \quad \mathcal{A}_{\text{tot}} = g^{n-2} \sum_{\sigma \in S_{n-1}} \text{Tr}(T^1 \dots T^{\sigma_n}) A(1, \sigma_2, \dots, \sigma_n), \tag{1.3}$$

$$\text{DDM form}: \quad \mathcal{A}_{\text{tot}} = g^{n-2} \sum_{\sigma \in S_{n-2}} c_{1|\sigma(2,\dots,n-1)|n} A(1, \sigma, n), \tag{1.4}$$

where g is the coupling constant, and A's are the color-ordered amplitudes. In Trace form the generator T^a is given by fundamental representation of U(N) group, while in DDM form $c_{1|\sigma(2,\dots,n-1)|n}$ is constructed using structure constants f^{abc} as

$$c_{1|\sigma(2,\dots,n-1)|n} = f^{1\sigma_2 x_1} f^{x_1 \sigma_3 x_2} \dots f^{x_{n-3} \sigma_{n-1} n} \ . \tag{1.5}$$

The equivalence between BCJ form (1.1) and DDM form was shown in [30], where both forms were proven to be equivalent to the KLT relation of color ordered scalar theory. To show the equivalence between DDM form and Trace form [3], the following two properties of U(N) Lie algebra are essential

$$\text{Property One}: \quad (f^a)_{ij} = f^{aij} = \text{Tr}(T^a[T^i, T^j]), \tag{1.6}$$

$$\begin{aligned} \text{Property Two}: \quad & \sum_a \text{Tr}(XT^a)\text{Tr}(T^a Y) = \text{Tr}(XY) \\ & \sum_a \text{Tr}(XT^a Y T^b) = \text{Tr}(X)\text{Tr}(Y) \ . \end{aligned} \tag{1.7}$$

Based on BCJ duality (1.1), it is natural to exchange the role between c_i and n_i and consider the following two dual forms

$$\text{Dual Trace form :} \quad \mathcal{A}_{\text{tot}} = g^{n-2} \sum_{\sigma \in S_{n-1}} \tau_{1\sigma_2 \dots \sigma_n} \widetilde{A}(1, \sigma_2, \dots, \sigma_n), \tag{1.8}$$

$$\text{Dual DDM form :} \quad \mathcal{A}_{\text{tot}} = g^{n-2} \sum_{\sigma \in S_{n-2}} n_{1|\sigma(2,\dots,n-1)|n} \widetilde{A}(1, \sigma, n), \tag{1.9}$$

where $\widetilde{A}$'s are color ordered tree amplitudes of scalar theory with f^{abc} as its cubic coupling constants (see references [30, 31]) and τ (which we will call "Dual-trace factor" or simply τ-function) is required to be cyclic invariant. Indeed, the Dual-DDM form was given in [32] while the Dual-Trace form was conjectured in [33] with explicit constructions for the first few lower-point amplitudes and a general construction was suggested in [28].

Although the existence of the above two dual forms were established, a systematic Feynman rule-like prescription to τ-functions and BCJ numerators n_σ is not yet known at this moment. The dual DDM-form was studied in [29], where the construction of BCJ numerators $n_{1|\sigma(2,\dots,n-1)|n}$ was given (see (2.5)). Although the result in [29] for BCJ numerators is just a small step towards the systematic local diagram construction, it does give us some useful applications.

In this paper, we use BCJ numerators to systematically construct the dual-trace factor τ and realize the proposed Dual-trace form (1.8). Unlike the trace factor $\text{Tr}(T^a \dots)$ which satisfies cyclic symmetry by construction, there is no relation presumed among these τ-functions, thus in principle there are $n!$ τ-functions we need to determine. Any solution to these $n!$ τ-functions will be a rightful choice as long as it gives the right total amplitude (1.8). However, from the way dual-DDM forms are labeled, we see that there are only $(n-2)!$ BCJ numerators $n_{1\sigma n}$ needed to completely fix the total amplitude, thus it is very natural to impose some relations to reduce the number of independent τ's. These imposed relations are [33]:

- (A) *Cyclic symmetry:*

$$\tau_{12\dots n} = \tau_{n1\dots(n-1)}. \tag{1.10}$$

  Using the cyclic symmetry, we can fix the first index to be any particular number, for example, 1, thus the number of independent τ-functions is reduced to $(n-1)!$.

- (B) *KK-relation:*

$$\tau_{1,\alpha,n,\beta^T} = (-1)^{n_\beta} \sum_{\{\sigma\} \in OP(\{\alpha\} \bigcup \{\beta\})} \tau_{1,\sigma,n}, \tag{1.11}$$

  where n_β is the number of elements in the set β, and β^T denotes the inverse of the ordering of set β. The sum in (1.11) is over all permutations of the set $\{\alpha\} \bigcup \{\beta\}$ where relative ordering inside both subsets α and β are kept. Using this relation, we can fix two particular numbers, for example, 1 and n, at the first and last positions, thus the number of independent τ-functions is $(n-2)!$.

Having imposed the above two relations, we need to find these $(n-2)!$ independent τ-functions, such that relation (1.8) is satisfied. This problem is solved by imposing the following relations between $(n-2)!$ BCJ numerators $n_{1\sigma n}$[1] and $(n-2)!$ $\tau_{1\sigma n}$-functions (where $\sigma \in S_{n-2}(23\ldots(n-1))$)

$$n_{1\sigma_2\ldots\sigma_{(n-1)}n} = \tau_{1[\sigma_2,[\ldots,[\sigma_{n-1},n]\ldots]]}, \tag{1.12}$$

here $[\,,\,]$ denotes the antisymmetric combination, for example $n_{123} = \tau_{1[2,3]} = \tau_{123} - \tau_{132}$.

After solving $\tau_{1\sigma n}$ as linear combinations of $n_{1\sigma n}$ using (1.12), we can use (1.11) and (1.10) to obtain all other τ-functions. The claim is that if we put these τ's back to the right hand side of (1.8), we do get the left hand side. In fact the proof of equivalence of Trace form (1.3) and DDM form (1.4) only relies on two facts: (1) Partial amplitudes $A(1,\sigma,n)$ are cyclic symmetric and satisfy KK-relations. (2) We have two different factors (in this case the trace and the color factor) satisfying the relation

$$c_{1\sigma_2\ldots\sigma_{(n-1)}n} = \mathrm{Tr}(T^1[T^{\sigma_2},[\ldots,[T^{\sigma_{n-1}},T^n]\ldots]]). \tag{1.13}$$

Noting that the imposed relation (1.12) is exactly the same as (1.13) and that partial amplitudes $\widetilde{A}(1,\sigma,n)$ in (1.8) and (1.9) are cyclic symmetric and satisfy KK-relations, we see that the claim is true.

The above logic is perfectly right, but there are two unclear points. The first is that each solution is based on (1.12) where a pair of numbers ($(1,n)$ in the example above) are fixed. Choosing a different pair will result in a different solution in principle. For example, the same $\tau_{123\ldots n}$-function can have two different expressions corresponding to two different choices of the fixed pair (i_1,j_1) and (i_2,j_2). Secondly, for a given fixed pair, $(1,n)$ for example, are the expressions for any two τ-functions related to each other by a corresponding relabeling? Specifically, let us assume two τ-functions are $\tau_{\sigma_1} = \sum_{i=1}^{(n-2)!} c_i n_{1\alpha_i n}$ and $\tau_{\sigma_2} = \sum_{j=1}^{(n-2)!} d_j n_{1\alpha_j n}$, where $n_{1\alpha n}$'s are the $(n-2)!$ independent numerators in KK-basis with $(1,n)$ fixed at two ends. If two orderings are related to each other by a permutation P of n-elements, i.e., $\sigma_2 = P(\sigma_1)$, we can get another expression $\sum_{i=1}^{(n-2)!} c_i n_{P(1\alpha_i n)}$ by relabeling from the expression of τ_{σ_1}. Then the question is that whether we have

$$\sum_{j=1}^{(n-2)!} d_j n_{1\alpha_j n} \stackrel{?}{=} \sum_{i=1}^{(n-2)!} c_i n_{P(1\alpha_i n)}\,. \tag{1.14}$$

These two points are related to each other. In fact, if (1.14) is satisfied, it should be expected that τ-functions are unique no matter which pair is taken fixed in (1.12) to get them, since different choices can be related to each other by a permutation. In this paper, we will show that the solution obtained by our algorithm based on (1.12) will have this *natural relabeling property*. In other words, conditions (1.12) and natural relabeling property are in fact consistent with each other. With this understanding we present another algorithm that uses relabeling property to solve τ-functions.

[1]Since in our whole paper, we will only use BCJ numerators of the DDM-chain form, to simplify the notation we will use $n_{123\ldots n} \equiv n_{1|23\ldots|n}$.

The structure of this paper is the following. In section 2, we provide a short review of the kinematic algebra proposed in [29]. Then we provide an algorithm to construct dual-trace factors in section 3. To demonstrate the idea outlined in section 3, we present several examples in section 4. We discuss the natural relabeling property and prove that the solution obtained from our algorithm in section 3 does satisfy the relabeling symmetry. A short summary of this work is given in section 6. Finally, details of the proof of the relabeling property are given in the appendix.

2 Useful properties of BCJ numerators

Before presenting the construction of dual-trace factors, let us review some useful properties of BCJ numerators discussed in [29]. Especially we will use the Jacobi identity to establish some relations among BCJ numerators n_α with different orderings. To show these relations we will follow the method given in [3], but there is a small difference. The proof done in [3] is for color factors c_σ constructed using structure constant f^{abc} of U(N) Lie algebra. The construction is local in the sense that it is given by a chain-shaped Feynman diagram with a set of Feynman rules prescribing the contribution of each vertex. For BCJ numerator n_σ, there is still no local construction based on Feynman diagram with Feynman-like rules prescribing its vertices. In fact, finding a such local construction is one motivation of our work [29], where some progress has been made towards this goal, which we review in this section.

To construct local expressions of BCJ numerators n_α, we need to use structure constants of kinematic Lie algebra with the generator given by

$$T^{k,a} \equiv e^{ik\cdot x}\partial_a. \tag{2.1}$$

Using these generators, we can calculate commutation relations and find the kinematic structure constant $f^{ab}{}_c$

$$\begin{aligned}[T^{k_1,a}, T^{k_2,b}] &= (-i)(\delta_a{}^c k_{1b} - \delta_b{}^c k_{2a})\, e^{i(k_1+k_2)\cdot x}\partial_c \\ &= f^{(k_1,a),(k_2,b)}{}_{(k_1+k_2,c)}\, T^{(k_1+k_2,c)}.\end{aligned} \tag{2.2}$$

These kinematic structure constants satisfy antisymmetry property

$$f^{12}{}_3 = -f^{21}{}_3, \tag{2.3}$$

and Jacobi identity

$$f^{1_a,2_b}{}_{(1+2)^e} f^{(1+2)_e,3_c}{}_{(1+2+3)^d} + f^{2_b,3_c}{}_{(2+3)^e} f^{(2+3)_e,1_a}{}_{(1+2+3)^d} + f^{3_c,1_a}{}_{(1+3)^e} f^{(1+3)_e,2_b}{}_{(1+2+3)^d} = 0. \tag{2.4}$$

It is worth noticing that unlike structure constants of group Lie algebra, for the $f^{12}{}_3$ given in (2.2) there is no natural way to lift index 3 up thus indices $1,2,3$ cannot be put on the same footing. To distinguish these three indices, we use arrows and the Jacobi identity (2.4) can be represented as the cyclic sum over three incoming arrow legs illustrated in figure 1.

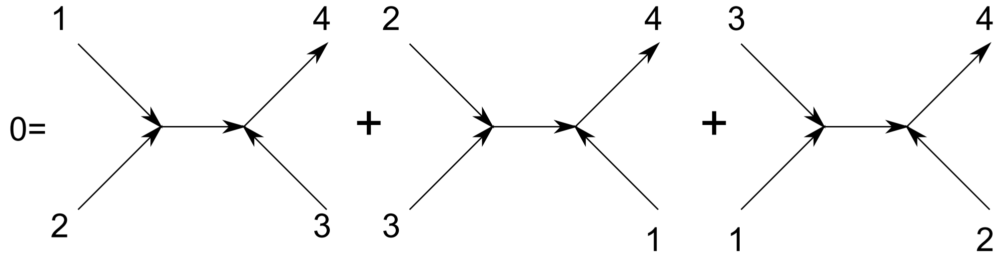
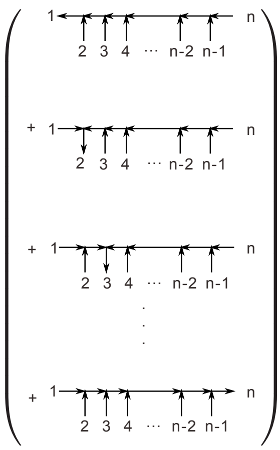


Figure 1. The Jacobi identity (2.4) of kinematic structure constants can be represented by the sum over cyclic orderings of three incoming arrows $1, 2, 3$.

Using the above kinematic algebra, we have shown in previous work [29] that the total tree-level amplitude of YM-theory can be written as dual-DDM form (1.9) where the *BCJ numerator* is given by

$$n_{12\ldots(n-1)n} = \sum_{j=1}^{N} c_j \epsilon(q_j) \cdot \tag{2.5}$$



where each term in the bracket is constructed using kinematic structure constants as coupling for each cubic vertex. The $\epsilon(q_j)$ is defined as $\prod_{t=1}^{n} \epsilon_t^{\mu_t}(q_{tj})$ where $\epsilon_t^{\mu_t}(q_{tj})$ is the polarization vector of the t-th external particle with gauge choice q_{tj}. The c_j's are coefficients solved by our averaging procedure given in [29]. The explicit expressions of c_j's are not important for our purpose here and the only useful fact we need is that c_j's are *independent of color orderings*. In other words, for all color orderings of n_α, the c_j's are same. Because of this structure, when we discuss BCJ numerators, the $\sum_j c_j \epsilon(q_j)$ part can be neglected and we will focus only on the part inside the bracket.

Now the part inside the bracket has a Feynman diagram-like structure, much like the color structure discussed in [3]. Using the same method given in [3] (i.e., using the Jacobi identity (2.4) and antisymmetry property (2.3)), we can find some nice relations among BCJ numerators n_α with different orderings. For example, we have the following two identities

$$n_{n\alpha_1\ldots\alpha_i 1\rho_1\ldots\rho_j(n-1)} = \sum_{\{\rho\}\in OP(\{\beta\}\bigcup\{\gamma\})} (-1)^{r+1} n_{n\alpha_1\ldots\alpha_i\gamma_1\ldots\gamma_s(n-1)\beta_r\ldots\beta_1 1}. \tag{2.6}$$

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and

$$n_{n\alpha_1\ldots\alpha_i 1\rho_1\ldots\rho_j(n-1)} = \sum_{\{\rho\}\in OP(\{\beta\}\bigcup\{\gamma\})} (-1)^{i+s} n_{1\beta_1\ldots\beta_r(n-1)\gamma_s\ldots\gamma_1\alpha_i\ldots\alpha_1 n}. \tag{2.7}$$

From now, without explicit explanations, all sets are ordered in all manipulations, thus $OP(\{\beta\}\bigcup\{\gamma\})$ denotes all possible unions of two sets with arbitrary relative ordering between them, but relative ordering inside each set has been kept (see the explanation after equation (1.11)). The sum in (2.6) and (2.7) can alternatively be regarded as over all possible splittings of the ordered set $\{\rho\}$ into two ordered subsets $\{\beta\}$ and $\{\gamma\}$ (both sets $\{\beta\}$ and $\{\gamma\}$ can be empty set). In each ordered subset, the relative ordering must be the same as the relative ordering in the mother set $\{\rho\}$. These two identities are relabeling invariant, i.e., if we act a permutation $P\in S_n$ on n-indices on both sides, there two identities still hold.

Although identities (2.6) and (2.7) are well known for experts in the field, it nevertheless did not appear in the literature explicitly. Since these two identities are very important to the discussion of the relabeling property of τ-functions constructed by the algorithm outlined in previous sections, to be self-contained, we would like to review their proofs using the method given in [3]. As we have mentioned, the $c_j\epsilon(q_j)$-part is the same for all orderings, so we just need to prove that the part in the bracket of (2.5) satisfies above two identities. For this purpose we draw the diagram representing a typical term in part (a) of the figure 2 and apply Jacobi identity to the part framed by a box. The result is give in part (b) of figure 2 for a particular arrow configuration. It can be shown that the same result is true for all other possible arrow configurations. The result in part (b) tells us that we can move the ρ_1 attached to the $(n-1)$-th block to two places. At the first place ρ_1 will be attached to n-th block with $+$ sign and at the second place ρ_1 will be attached to leg 1 with $-$ sign. Repeating above manipulations, we can move down ρ_2, ρ_3 and finally ρ_j. A typical final configuration will be the ordering $\{n,\alpha_1,\ldots,\alpha_i,\gamma_1,\ldots,\gamma_s,n-1,\beta_r,\ldots,\beta_1,1\}$ with sign $(-)^{r+1}$ (The extra $-$ sign comes from pulling $n-1$ from up to down by antisymmetry), where sets $\{\gamma_1,\ldots,\gamma_s\}$ and $\{\beta_1,\ldots,\beta_r\}$ come from the splitting of original set $\{\rho_1,\ldots,\rho_j\}$ with relative ordering kept in each subset. By now, we have proved the identity (2.6). Finally we reverse the ordering to get $\{1,\beta,n-1,\gamma_s^T,\alpha^T,n\}$ with sign $(-)^{r+1+(n-2)}$. Using $n-3=i+j$ and $j=r+s$ we get the sign to be $(-)^{i+s}$ (where T means the reversing of ordering). Thus the identity (2.7) is proved.

Identities (2.6) and (2.7) can also be replaced by the following two forms (where for later applications we have switched the $1\leftrightarrow n$)

$$n_{1\alpha_1\ldots\alpha_i n\rho_1\ldots\rho_j(n-1)} = -n_{1\alpha_1\ldots\alpha_i[\rho_1,[\rho_2,\ldots[\rho_j,(n-1)]\ldots]]n}. \tag{2.8}$$

and

$$n_{\rho_1\ldots\rho_j 1,\alpha_1,\ldots,\alpha_i,n} = (-)\, n_{1[[[\rho_1,\rho_2],\ldots],\rho_j],\alpha_1,\ldots,\alpha_i,n}\,, \tag{2.9}$$

and $[\,,\,]$ is the anti-commutative bracket, i.e., $n_{\ldots[a,b]\ldots}=n_{\ldots ab\ldots}-n_{\ldots ba\ldots}$. Furthermore, if the set ρ is empty, $[\rho_1,[\rho_2,\ldots[\rho_j,(n-1)]\ldots]]$ is just $(n-1)$. Using formula (2.8) and (2.9) we can move one index to the last or to the first position.

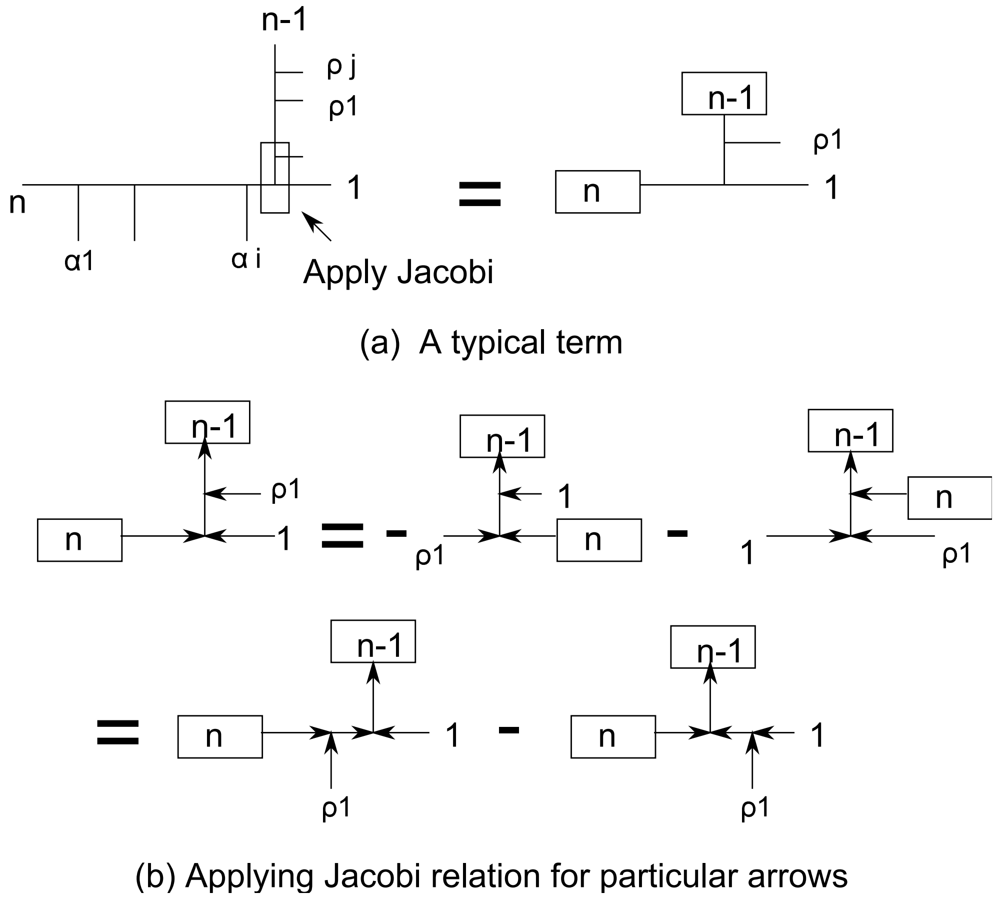

Figure 2. (a) Diagrammatic representation of a typical term $n_{n\alpha_1\ldots\alpha_i 1\rho_1\ldots\rho_j(n-1)}$ and its simplified schematic. On the left hand side we use a square to highlight the place where Jacobi identity is subsequently applied in the graphs below. (b) The manipulation using Jacobi identity on a particular arrow assignment. For simplicity, we use an "n-box" to schematically represent the chain $\{n, a_1, \ldots, a_i\}$ drawn in (a) and similarly an "$(n-1)$-box" to represent the chain $\{n-1, p_j, \ldots, p_2\}$.

There is one remark before we conclude this section. It will be clear soon that all discussions in this paper are based on the above two identities (2.6) and (2.7). Although the derivation has explicitly used a local diagram construction following the approach in [3], it seems that the local diagram construction is not essential since all we need are Jacobi identities of these BCJ numerators n_α plus antisymmetry.

3 Construction of τ by BCJ numerators

In this section, we present the construction of τ-functions using (1.12), (1.10) and (1.11). With $(1, n)$ fixed and a given ordering σ_2,. . . ,σ_{n-2}, the τ and n are related by (1.12)

$$n_{1\sigma_2\ldots\sigma_{n-1}n} = \tau_{1[\sigma_2,[\ldots,[\sigma_{n-1},n]\ldots]]}, \tag{3.1}$$

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where the bracket $[A,B]\equiv AB-BA$. For example, we have

$$\tau_{1[2,[3,4]]}=\tau_{1234}-\tau_{1243}-\tau_{1342}+\tau_{1432}.$$

The expansion at the right handed side of (3.1) includes cases where n is not at the last index. Thus we should use KK-relation (1.11) to put n to the last position. After such manipulations we get

$$n_{1\sigma_2\ldots\sigma_{n-1}n}=\sum_{\sigma'\in S_{n-2}}G_{1,n}(\sigma|\sigma')\tau_{1\sigma' n}. \tag{3.2}$$

To solve τ by n, we need to understand the matrix $G_{1,n}(\sigma|\sigma')$. The $(n-2)!\times(n-2)!$ matrix can be obtained by following steps. First it is easy to see that

$$\tau_{1[\sigma_2,[\ldots,[\sigma_{n-1},n]\ldots]]}=\sum_{\{\sigma\}\in OP(\{\alpha\}\bigcup\{\beta\})}(-1)^{n_\beta}\tau_{1\alpha n\beta^T} \tag{3.3}$$

where n_β is the number of elements of the set β and the sum has the same meaning as the sum in (2.6) and (2.7), i.e., it is over all possible splittings of the ordered set $\{\sigma_2,\ldots,\sigma_{n-1}\}$ in two subsets α and β (empty sets are allowed) such that inside each subset the relative ordering defined by the set σ is kept. For example, the set $\sigma=\{234\}$ has the following eight splittings

$(\alpha,\beta)=$
$(\{234\},\emptyset)/(\{23\},\{4\})/(\{24\},\{3\})/(\{34\},\{2\})/(\emptyset,\{234\})/(\{4\},\{23\})/(\{3\},\{24\})/(\{2\},\{34\})$

Secondly using the imposed KK relation on τ (1.11), any $(-1)^{n_\beta}\tau_{1\alpha n\beta^T}$ can be expressed as $\sum\limits_{\rho\in OP(\{\alpha\}\bigcup\{\beta\})}\tau_{1\rho n}$. Combining these two steps we arrive at

$$\tau_{1[\sigma_2,[\ldots,[\sigma_{n-1},n]\ldots]]}=\sum_{\{\sigma\}\in OP(\{\alpha\}\bigcup\{\beta\})}\;\sum_{\{\rho\}\in OP(\{\alpha\}\bigcup\{\beta\})}\tau_{1\rho n}. \tag{3.4}$$

From this formula we can see that the matrix element $G_{1,n}(\sigma|\sigma')$ is the number of splittings of σ into two subsets α,β, such that the ordering σ' can be obtained by recombining two subsets α,β arbitrarily with relative ordering kept inside each subset.

Let us give a few examples of $G_{1,n}(\sigma|\sigma')$ with $\sigma=\{234\}$. For $\sigma'=\{234\}$, there are eight splittings of σ, which can be used to recombine to get σ', so $G_{1,5}(\{234\}|\{234\})=8$. For $\sigma'=\{243\}$ there are four splittings $(\{23\},\{4\})$, $(\{24\},\{3\})$, $(\{4\},\{23\})$, $(\{3\},\{24\})$ available, so $G_{1,5}(\{234\}|\{243\})=4$. For $\sigma'=\{423\}$ there are only two splittings $(\{23\},\{4\})$, $(\{4\},\{23\})$ available, so $G_{1,5}(\{234\}|\{423\})=2$. For $\sigma'=\{432\}$, there is no any splitting, so $G_{1,5}(\{234\}|\{432\})=0$. In fact, one can show that for three numbers i,j,k, if their ordering inside the set σ is $i>j>k$ and their ordering inside the set σ' is $i<j<k$, the $G_{1,n}(\sigma|\sigma')=0$. The reason is the following. To reproduce the right ordering inside σ', when we split σ into two subsets, elements i,j must belong to different subsets. Similar distributions must hold for pairs i,k and j,k, but these three conditions can not be satisfied simultaneously. From this general argument, we can see that majority elements of matrix $G_{1,n}(\sigma|\sigma')$ are zero.

Now let us discuss some general properties of matrix $G_{1,n}(\sigma|\sigma')$:

- (1) First the value can be written as

$$G_{1,n}(\sigma|\sigma') = \sum_{\{\sigma'\}\in OP(\{\alpha'\}\{\beta'\})} \sum_{\{\sigma\}\in OP(\{\alpha\}\bigcup\{\beta\})} \delta(\{\alpha\},\{\beta\}|\{\alpha'\},\{\beta'\}), \tag{3.5}$$

  where these two sums are over all the possible ordered sets α, β such that $\{\sigma\} \in OP(\{\alpha\}\bigcup\{\beta\})$ and all the possible ordered sets $\{\alpha'\}$, $\{\beta'\}$ such that $\{\sigma'\} \in OP(\{\alpha'\}\{\beta'\}$. In other word, we sum over all the possible splittings of σ and σ' into two ordered subsets respectively. This means elements of matrix $G_{1,n}(\sigma|\sigma')$ can be calculated by another way. First we find all 2^{n_σ} splittings of the set σ to two subsets (α,β) with relative ordering kept and all splittings of the set σ' into two subsets (α',β') with relative ordering kept. Then we find all subsets such that $(\alpha,\beta)=(\alpha',\beta')$. The total number is $G_{1,n}(\sigma|\sigma')$.

- (2) From the above property (3.5), it is easy to see that G is symmetric

$$G_{1,n}(\sigma|\sigma') = G_{1,n}(\sigma'|\sigma). \tag{3.6}$$

- (3) Some special elements can be obtained. When $\sigma=\sigma'$, all possible splittings of $\sigma \in S_{n-2}$ should be counted, i.e., $G_{1,n}(\sigma|\sigma) = 2^{n-2}$. If σ,σ' are different only by one permutation of two adjacent numbers, then $G_{1,n}(\sigma|\sigma') = 2^{n-3}$. If σ,σ' have the following orderings: $\sigma = (\ldots, i_1, i_2, \ldots, j_1, j_2, \ldots)$ and $\sigma' = (\ldots, i_2, i_1, \ldots, j_2, j_1, \ldots)$, we have $G_{1,n}(\sigma|\sigma') = 2^{n-4}$. Similar pattern holds for more interchanges of adjacent pairs.

- (4) The fourth observation is that

$$G_{1,n}(\sigma|\sigma') = G_{1,n}(P(\sigma)|P(\sigma')) \tag{3.7}$$

  where P is any permutation of $(n-2)$ elements. The reason is that (3.5) cares only relative orderings between sets σ,σ', so it does not matter if a element is called x or y.

Having discussed the matrix $G_{1,n}(\sigma|\sigma')$, we can solve $\tau_{1,\sigma,n}$ by BCJ numerators using (3.2) and obtain

$$\tau_{1\sigma' n} = \frac{\det \mathbb{G}'}{\det \mathbb{G}}, \tag{3.8}$$

where $\mathbb{G} = G_{1,n}(\sigma|\sigma')$ is the $(n-2)! \times (n-2)!$ matrix, and $\mathbb{G}'$ is obtained by replacing the column σ' of $\mathbb{G}$ by the column of BCJ numerators $n_{1\sigma n}$. Now we have provided a general construction of $(n-2)!$ τ's with 1, n fixed at the two ends (3.8). Other $(n! - (n-2)!)$ τ's can be obtained by KK relations and cyclic symmetry (see (1.10) and (1.11)).

The above construction for τ's is complete and there is no any ambiguity. The only subtle point is that in the construction, two special elements have been fixed, for example

1 and n. Then it is natural to ask what is the relation between two solutions obtained by fixing two different pairs (i_1, j_1) and (i_2, j_2)? There are two possibilities. The first possibility is that these two solutions do not have any relation. Another possibility is that these two solutions will relate to each other by some manipulations. As we will show in later sections, there is a *natural relabeling property* for solutions obtained by above construction. This property tells us that if we have an expression for just one τ-function, expressions for all other τ's can be obtained by relabeling. Furthermore, all solutions coming from different fixed pairs will give same answer.

4 Examples

In this section, we will use several examples to explicitly demonstrate the algorithm for τ-functions and check that the solution satisfies the natural relabeling property.

4.1 Three-point case

In this case, with $1, 3$ fixed we have $G_{1,3}(\{2\}|\{2\}) = 2$,

$$n_{123} = 2\tau_{123} \Longrightarrow \tau_{123} = \frac{1}{2}n_{123}. \tag{4.1}$$

Other five τ's can be obtained from τ_{123} by cyclic symmetry and KK-relations

$$\begin{aligned} \text{Cyclic}: &\quad \tau_{312} = \tau_{231} = \tau_{123} \\ \text{KK}-\text{rel}: &\quad \tau_{213} = \tau_{321} = \tau_{132} = -\tau_{123} \end{aligned} \tag{4.2}$$

To check the relabeling, noticing that if we exchange $3 \leftrightarrow 2$ in (4.1), we get

$$\widetilde{\tau}_{132} = \frac{1}{2}n_{132} \tag{4.3}$$

Because for three-point case, BCJ numerator n is cyclic symmetric and anti-symmetric under exchanging of pair, i.e., $n_{123} = -n_{132}$, we see that $\widetilde{\tau}_{132}$ is equal to τ_{132} in (4.2), i.e., τ_{132} can be obtained from τ_{123} by relabeling indices. It is easy to check the relabeling property for other τ's.

4.2 Four-point case

For four-point case, we have $4! = 24$ τ's. Now let us use our algorithm to determine all of them and check the relabeling property:

Solving the τ's in KK-basis: we have $(4-2)! = 2$ equations and $(4-2)! = 2$ independent τ's. With ordering $(\{23\}, \{32\})$ the matrix is

$$G_{1,4} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \tag{4.4}$$

From this we can solve

$$\tau_{1234} = \frac{\begin{vmatrix} n_{1234} & 2 \\ n_{1324} & 4 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{1}{3} n_{1234} - \frac{1}{6} n_{1324} \tag{4.5}$$

and

$$\tau_{1324} = \frac{\begin{vmatrix} 4 & n_{1234} \\ 2 & n_{1324} \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}} = -\frac{1}{6} n_{1234} + \frac{1}{3} n_{1324} \tag{4.6}$$

Find remaining τ's: having τ_{1234} and τ_{1324}, we can construct other 22 τ's. Using KK-relation, we can obtain the following four τ with 1 fixed:

$$\begin{aligned}
\tau_{1243} &\equiv -\tau_{1234} - \tau_{1324} = -\frac{1}{6} n_{1234} - \frac{1}{6} n_{1324} \\
\tau_{1342} &\equiv -\tau_{1234} - \tau_{1324} = -\frac{1}{6} n_{1234} - \frac{1}{6} n_{1324} \\
\tau_{1423} &\equiv +\tau_{1324} = -\frac{1}{6} n_{1234} + \frac{1}{3} n_{1324} \\
\tau_{1432} &\equiv +\tau_{1234} = \frac{1}{3} n_{1234} - \frac{1}{6} n_{1324}
\end{aligned} \tag{4.7}$$

Having obtained all $\tau_{1\sigma}$, the remaining 18 τ's are obtained by imposed cyclic symmetry. For example, we have

$$\tau_{1234} = \tau_{4123} = \tau_{3412} = \tau_{2341} = \frac{1}{3} n_{1234} - \frac{1}{6} n_{1324} \tag{4.8}$$

Relabeling property: using these explicit result, we can check the relabeling property. First it is easy to see that expression (4.6) can be obtained from (4.5) by replacing $2 \to 3, 3 \to 2$. In principle, this fact does not need to hold. However, by symmetric property and property (3.7) of matrix G, it is natural to get $\tau_{1P(\sigma)n} = P(\tau_{1\sigma n})$.

Now we check the relabeling property for $\tau_{1\sigma}$. From solution τ_{1234} by index relabeling $(3,4) \to (4,3)$ we can write down

$$\widetilde{\tau}_{1243} = \frac{1}{3} n_{1243} - \frac{1}{6} n_{1423} \tag{4.9}$$

To check if $\widetilde{\tau}_{1243}$ is equal to τ_{1243} obtained in (4.7), we use relation (2.6) to get

$$n_{1243} = -n_{1234}, \quad n_{1423} = -n_{1234} + n_{1324} \tag{4.10}$$

Putting it back, it is easy to check $\widetilde{\tau}_{1243} = \tau_{1243}$. In other words, two expressions, i.e., the one obtained by our imposed KK-relation and the one obtained by index relabeling from known solution $\tau_{1\sigma 4}$, are same!

Next, we check the relabeling property for τ's where 1 is not the first index. By relabeling $(1, 2, 3, 4) \to (4, 1, 2, 3)$ from τ_{1234} we have

$$\widetilde{\tau}_{4123} = \frac{1}{3}n_{4123} - \frac{1}{6}n_{4213}. \tag{4.11}$$

Using relation (2.7), this expands into

$$n_{4123} = n_{1234} - n_{13(24)}, \quad n_{4213} = -n_{1324}, \tag{4.12}$$

and we do find $\widetilde{\tau}_{4123}$ is equal to the τ_{4123} written down in (4.8).

Although we have presented only three examples, using the Mathematica we have checked that indeed the result obtained from our algorithm (including using KK and cyclic relations) does agree with the one obtained from relabeling of single τ_{1234} expression.

4.3 Five-point case

Having seen the above four-point example, we will not present too much detail for cases with five, six and seven points. The rest can be easily recovered via the same algorithm.

Solutions from our algorithm: first we can solve τ's in KK-basis using the following equation

$$\begin{pmatrix} n_{12345} \\ n_{12435} \\ n_{13245} \\ n_{14235} \\ n_{13425} \\ n_{14325} \end{pmatrix} = \begin{pmatrix} 2^3 & 2^2 & 2^2 & 2^1 & 2^1 & 0 \\ 2^2 & 2^3 & 2^1 & 2^2 & 0 & 2^1 \\ 2^2 & 2^1 & 2^3 & 0 & 2^2 & 2^1 \\ 2^1 & 2^2 & 0 & 2^3 & 2^1 & 2^2 \\ 2^1 & 0 & 2^2 & 2^1 & 2^3 & 2^2 \\ 0 & 2^1 & 2^1 & 2^2 & 2^2 & 2^3 \end{pmatrix} \begin{pmatrix} \tau_{12345} \\ \tau_{12435} \\ \tau_{13245} \\ \tau_{14235} \\ \tau_{13425} \\ \tau_{14325} \end{pmatrix}. \tag{4.13}$$

and obtain an expression of τ_{12345} as

$$\tau_{12345} = \frac{1}{4}n_{12345} - \frac{1}{10}n_{12435} - \frac{1}{20}n_{14235} - \frac{1}{10}n_{13245} - \frac{1}{20}n_{13425} + \frac{1}{10}n_{14325}. \tag{4.14}$$

Other five τ's can be obtained by relabeling $2, 3, 4$ from (4.14). Having these six τ solutions for KK-basis, we can use the KK-relations and cyclic relations to find other 114 τ's.

Relabeling property: now we need to check if the above result has the relabeling property. Although we have used Mathematica to check that all 120 τ's are related to each other by relabeling property, here we will give only a few examples. The first example is[2]

$$\tau_{12534} = -\frac{1}{20}n_{12345} + \frac{1}{10}n_{12435} - \frac{1}{20}n_{13425} + \frac{1}{20}n_{14235} + \frac{1}{10}n_{14325} \tag{4.15}$$

which is obtained from KK-relation. Let us relabel the solution (4.14) with $3 \to 5, 4 \to 3, 5 \to 4$. After plugging

$$n_{12534} = -n_{12345} + n_{12435}$$

[2]It is worth noticing that the expression of τ_{12534} is simpler than (4.14) because it has only 5 terms and simpler coefficients. It will be interesting to investigate if it is general.

$$
\begin{aligned}
n_{12354} &= -n_{12345} \\
n_{13254} &= -n_{13245} \\
n_{15234} &= -n_{12345} + n_{12435} + n_{13425} - n_{14325} \\
n_{15324} &= n_{12435} - n_{13245} + n_{13425} - n_{14235} \\
n_{13524} &= -n_{13245} + n_{13425}
\end{aligned}
$$

it can be checked that the relabeling (4.14) does reproduce (4.15).

The second example is $\tau_{51234} \equiv \tau_{12345}$ by our definition. The relabeling of (4.14) gives

$$
\widetilde{\tau}_{51234} = \frac{1}{4} n_{51234} - \frac{1}{10} n_{51324} - \frac{1}{20} n_{53124} - \frac{1}{10} n_{52134} - \frac{1}{20} n_{52314} + \frac{1}{10} n_{53214}. \tag{4.16}
$$

Using the result

$$
\begin{aligned}
n_{51234} &= n_{12345} - n_{12435} - n_{13425} + n_{14325}, \\
n_{51324} &= n_{13245} - n_{13425} - n_{12435} + n_{14235}, \\
n_{53124} &= -n_{12435} + n_{14235}, \\
n_{52134} &= -n_{13425} + n_{14325}, \\
n_{52314} &= n_{14325}, \\
n_{53214} &= n_{14235}.
\end{aligned} \tag{4.17}
$$

it is easy to see that $\tau_{51234} = \widetilde{\tau}_{51234}$, i.e., the relabeling property is satisfied.

4.4 Six-point case

For six points, using the algorithm we can solve 24 τ's in KK-basis. Since they are related to each other by relabeling, we simply write down one solution τ_{123456},

$$
\begin{aligned}
\tau_{123456} = \frac{1}{630}\Big[& 126 n_{123456} - 44 n_{123546} - 44 n_{124356} - 19 n_{124536} - 19 n_{125346} + 36 n_{125436} \\
& -44 n_{132456} + 16 n_{132546} - 19 n_{134256} - 19 n_{134526} + n_{135246} + 11 n_{135426} \\
& -19 n_{142356} + n_{142536} + 36 n_{143256} + 11 n_{143526} - 9 n_{145236} + 16 n_{145326} \\
& -19 n_{152346} + 11 n_{152436} + 11 n_{153246} + 16 n_{153426} + 16 n_{154236} - 44 n_{154326} \Big]
\end{aligned} \tag{4.18}
$$

Using KK-basis of τ we can get the remaining 696 τ's. Now let us check the relabeling property. We have used the Mathematica to check that all 720 τ's are related to each other by the relabeling property. Here we give only one example $\tau_{612345} \equiv \tau_{123456}$ by cyclic symmetry. The relabeling from τ_{123456} gives

$$
\begin{aligned}
\widetilde{\tau}_{612345} = \frac{1}{630}\Big[& 126 n_{612345} - 44 n_{612435} - 44 n_{613245} - 19 n_{613425} - 19 n_{614235} + 36 n_{614325} \\
& -44 n_{621345} + 16 n_{621435} - 19 n_{623145} - 19 n_{623415} + n_{624135} + 11 n_{624315} \\
& -19 n_{631245} + n_{631425} + 36 n_{632145} + 11 n_{632415} - 9 n_{634125} + 16 n_{634215} \\
& -19 n_{641235} + 11 n_{641325} + 11 n_{642135} + 16 n_{642315} + 16 n_{643125} - 44 n_{643215} \Big]
\end{aligned} \tag{4.19}
$$

Using the result

$$
n_{61ijk5} = n_{1ijk56} - n_{1jk5i6} - n_{1ik5j6} - n_{1ij5k6} + n_{1k5ji6} + n_{1i5kj6} + n_{1j5ki6} - n_{15kji6}
$$

$$
\begin{aligned}
n_{6i1jk5} &= -n_{1jk5i6} + n_{1j5ki6} + n_{1k5ji6} - n_{15kji6} \\
n_{6ij1k5} &= n_{1k5ji6} - n_{15kji6} \\
n_{6ijk15} &= -n_{15kji6}.
\end{aligned} \tag{4.20}
$$

Indeed we see that $\widetilde{\tau}_{612345} = \tau_{612345}$.

4.5 Seven-point case

At seven-points, the expression for $\tau_{1234567}$ with $1, 7$ fixed is

$$
\begin{aligned}
\tau_{1234567} &= \\
\frac{1}{24192} & \Big(4032n_{1234567} - 1284n_{1234657} - 1284n_{1235467} - 513n_{1235647} - 513n_{1236457} + 940n_{1236547} \\
& -1284n_{1243567} + 393n_{1243657} - 513n_{1245367} - 438n_{1245637} + 27n_{1246357} + 259n_{1246537} \\
& -513n_{1253467} + 27n_{1253647} + 940n_{1254367} + 259n_{1254637} - 213n_{1256347} + 337n_{1256437} \\
& -438n_{1263457} + 259n_{1263547} + 259n_{1264357} + 344n_{1264537} + 337n_{1265347} - 940n_{1265437} \\
& -1284n_{1324567} + 421n_{1324657} + 393n_{1325467} + 205n_{1325647} + 205n_{1326457} - 337n_{1326547} \\
& -513n_{1342567} + 205n_{1342657} - 438n_{1345267} - 513n_{1345627} + 32n_{1346257} + 205n_{1346527} \\
& +27n_{1352467} - 56n_{1352647} + 259n_{1354267} + 184n_{1354627} + 23n_{1356247} + 157n_{1356427} \\
& +32n_{1362457} - 6n_{1362547} - 51n_{1364257} + 143n_{1364527} + 6n_{1365247} - 259n_{1365427} \\
& -513n_{1423567} + 205n_{1423657} + 27n_{1425367} + 32n_{1425637} - 56n_{1426357} - 6n_{1426537} \\
& +940n_{1432567} - 337n_{1432657} + 259n_{1435267} + 205n_{1435627} - 6n_{1436257} - 148n_{1436527} \\
& -213n_{1452367} + 23n_{1452637} + 337n_{1453267} + 157n_{1453627} - 213n_{1456237} + 337n_{1456327} \\
& +23n_{1462357} - 53n_{1462537} + 6n_{1463257} + 11n_{1463527} + 148n_{1465237} - 205n_{1465327} \\
& -438n_{1523467} + 32n_{1523647} + 259n_{1524367} - 51n_{1524637} + 23n_{1526347} + 6n_{1526437} \\
& +259n_{1532467} - 6n_{1532647} + 344n_{1534267} + 143n_{1534627} - 53n_{1536247} + 11n_{1536427} \\
& +337n_{1542367} + 6n_{1542637} - 940n_{1543267} - 259n_{1543627} + 148n_{1546237} - 205n_{1546327} \\
& -213n_{1562347} + 148n_{1562437} + 148n_{1563247} + 107n_{1563427} + 107n_{1564237} - 393n_{1564327} \\
& -513n_{1623457} + 205n_{1623547} + 184n_{1624357} + 143n_{1624537} + 157n_{1625347} - 259n_{1625437} \\
& +205n_{1632457} - 148n_{1632547} + 143n_{1634257} + 344n_{1634527} + 11n_{1635247} - 184n_{1635427} \\
& +157n_{1642357} + 11n_{1642537} - 259n_{1643257} - 184n_{1643527} + 107n_{1645237} - 421n_{1645327} \\
& +337n_{1652347} - 205n_{1652437} - 205n_{1653247} - 421n_{1653427} - 393n_{1654237} + 1284n_{1654327} \Big).
\end{aligned} \tag{4.21}
$$

By relabeling the above expression $(1, 2, 3, 4, 5, 6, 7) \to (7, 1, 2, 3, 4, 5, 6)$, we get $\widetilde{\tau}_{7123456}$. To show that it is equal to $\tau_{7123456} = \tau_{1234567}$ from cyclic symmetry, we just need to use the following expressions

$$
\begin{aligned}
n_{71ijkl6} &= n_{1ijkl67} - n_{1ijk6l7} - n_{1ijl6k7} + n_{1ij6lk7} - n_{1ikl6j7} + n_{1ik6lj7} + n_{1il6kj7} - n_{1i6lkj7} \\
&\quad -n_{1jkl6i7} + n_{1jk6li7} + n_{1jl6ki7} - n_{1j6lki7} + n_{1kl6ji7} - n_{1k6lji7} - n_{1l6kji7} + n_{16lkji7} \\
n_{7i1jkl6} &= -n_{1jkl6i7} + n_{1jk6li7} + n_{1jl6ki7} - n_{1j6lki7} + n_{1kl6ji7} - n_{1k6lji7} - n_{1l6kji7} + n_{16lkji7}
\end{aligned}
$$

$$
\begin{aligned}
n_{7ij1kl6} &= n_{1kl6ji7} - n_{1k6lji7} - n_{1l6kji7} + n_{16lkji7} \\
n_{7ijk1l6} &= -n_{1l6kji7} + n_{16lkji7} \\
n_{7ijkl16} &= n_{16lkji7}
\end{aligned}
\tag{4.22}
$$

5 Relabeling property

In the previous section, we have demonstrated the consistency between our algorithm and the relabeling property. In this section, we will give a general understanding of this property. Before doing so, we need to address a technical issue concerning the definition of matrix G. Note that generically relabeling is a permutation of n elements while the definition of G involves only permutation of $(n-2)$ elements since two of them have been fixed. To relate matrix G with different fixed pairs, we enlarge the definition of matrix $G_{1,n}(\sigma|\sigma')$ to $G_{1,n}(\sigma|\sigma') \equiv G(1,\sigma,n|1,\sigma',n)$ and require that when we split the set $\{1,\sigma,n\}$ to two subsets, the first element 1 must be at the first subset and the last element n must be at the second subset. It is easy to see that the new definition is equivalent to the old one. More importantly, the relabeling property (3.7) of $(n-2)$ elements can be enlarged to incorporate the relabeling of n elements, i.e, we will have that

$$
G(\sigma|\sigma') = G(P(\sigma)|P(\sigma')), \qquad P \in S_n\,, \tag{5.1}
$$

where now σ, σ' are lists of n elements.

Having enlarged the definition of matrix G, we can start our discussions. The structure of this section is the following. First we will set up a general framework for discussions. Then we will discuss how the relabeling property can be used to solve τ-function.

5.1 The proof of relabeling property

We note that the relabeling property can be proven without knowing the solution explicitly. Since by our algorithm three equations (1.12), (1.10) and (1.11) fix the solution uniquely, we can prove the property by showing that it is a property possessed by these three equations. The imposed cyclic symmetry (1.10) and KK-relations (1.11) have this property obviously, thus the key is to show that it is a property of equation (1.12) as well.

Consider the relabeling $x_s \to z_s$ with $s = 1, \ldots, n$. Under the relabeling, an ordering of n elements becomes another ordering of n elements, $X_i \to Z_i$ with $i = 1, \ldots, (n-2)!$ running through the whole KK-basis. Thus equation (3.2) becomes

$$
\sum_{j=1}^{(n-2)!} G_{X_iX_j}\tau_{X_j} = n_{X_i} \Longrightarrow \sum_{j=1}^{(n-2)!} G_{Z_iZ_j}\tau_{Z_j} = n_{Z_i} \tag{5.2}
$$

where the sum is over KK-basis X_j. However, by cyclic and KK-relations, we have

$$
\tau_{Z_i} = \sum_{j=1}^{(n-2)!} Q_{Z_iX_j}\tau_{X_j} \tag{5.3}
$$

and by (2.7) and (2.6) we have

$$n_{Z_i} = \sum_{j=1}^{(n-2)!} P_{Z_i X_j} n_{X_j} \tag{5.4}$$

thus

$$G_{Z_i Z_j}\tau_{Z_j} = n_{Z_i} \Longrightarrow G_{Z_i Z_j} Q_{Z_j X_k}\tau_{X_k} = P_{Z_i X_l} n_{X_l} \tag{5.5}$$

where to make the notation simpler, we have used Einstein summation convention.

To see that solution from our algorithm have the relabeling property, we just need to show in addition,

$$[P_{Z_i X_l}]^{-1} G_{Z_i Z_j} Q_{Z_j X_k} = G_{X_l X_k} \tag{5.6}$$

Now let us demonstrate the use of (5.6) by several examples. First let us consider the four-point case where the matrix G is given by (4.4). For relabeling given by permutation $P(3,4)$, we have $X_1 = (1,2,3,4) \to Z_1 = (1,2,4,3)$ and $X_2 = (1,3,2,4) \to Z_2 = (1,4,2,3)$. Under this permutation we find

$$\begin{pmatrix} \tau_{Z_1} \\ \tau_{Z_2} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_{X_1} \\ \tau_{X_2} \end{pmatrix} \tag{5.7}$$

and

$$\begin{pmatrix} n_{Z_1} \\ n_{Z_2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} n_{X_1} \\ n_{X_2} \end{pmatrix} \tag{5.8}$$

Thus it is easy to check that

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \tag{5.9}$$

As a second example, we consider the permutation $P(1,2)$ at five-point, where the G matrix is given by (4.13). For this permutation, we find the matrix representation of P through the following manipulation

$$\begin{pmatrix} n_{12345} \\ n_{12435} \\ n_{13245} \\ n_{13425} \\ n_{14235} \\ n_{14325} \end{pmatrix} \to \begin{pmatrix} n_{21345} \\ n_{21435} \\ n_{23145} \\ n_{23415} \\ n_{24135} \\ n_{24315} \end{pmatrix} = (-) \begin{pmatrix} n_{12345} \\ n_{12435} \\ n_{1[2,3]45} \\ n_{1[[2,3],4]5} \\ n_{1[2,4]35} \\ n_{1[[2,4],3]5} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} n_{12345} \\ n_{12435} \\ n_{13245} \\ n_{13425} \\ n_{14235} \\ n_{14325} \end{pmatrix}. \tag{5.10}$$

Similar manipulation gives the matrix of Q as

$$Q_{Z_i X_j} = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.11}$$

It is straightforward to see that $[P_{Z_i X_l}]^{-1} G_{Z_i Z_j} Q_{Z_j X_k} = G_{X_l X_k}$ is indeed satisfied.

Having shown two examples, now we discuss how we could give a general proof. To do so, we need the following observation. Assuming the relabeling from $X_i \to Z_i$ can be broken into two steps, $X_i \to Y_i$ and $Y_i \to Z_i$, it is easy to see that

$$\tau_{Z_i} = Q_{Z_iY_t}\tau_{Y_t} = Q_{Z_iY_t}Q_{Y_tX_j}\tau_{X_j}, \quad n_{Z_i} = P_{Z_iY_t}n_{Y_t} = P_{Z_iY_t}P_{Y_tX_j}n_{X_j}, \tag{5.12}$$

and the condition (5.6) becomes

$$[P_{Z_iY_{t'}}P_{Y_{t'}X_l}]^{-1}G_{Z_iZ_j}Q_{Z_jY_t}Q_{Y_tX_k} = G_{X_lX_k} \quad . \tag{5.13}$$

If the relabeling property holds for both steps $X_i \to Y_i$ and $Y_i \to Z_i$, i.e.

$$[P_{Y_{t'}X_l}]^{-1}G_{Y_{t'}Y_t}Q_{Y_tX_k} = G_{X_lX_k} \tag{5.14}$$

and

$$[P_{Z_iY_{t'}}]^{-1}G_{Z_iZ_j}Q_{Z_jY_t} = G_{Y_{t'}Y_t} \tag{5.15}$$

then (5.13) also holds. This observation reflects the structure of group, so that if we can show (5.6) is true for all generators of permutation group S_n, it will be true for the whole group.

For permutation group S_n, there are $(n-1)$ generators. For our convenience, we choose the following $(n-2)$ generators[3]

$$\mathcal{P}_n = (1, n), \quad \mathcal{P}_i = (i, i+1), \quad i = 2, 3, \ldots, n-2; \tag{5.16}$$

plus any one permutation of the form $\mathcal{P}_{1i} = (1, i)$ or $\mathcal{P}_{ni} = (n, i)$ with $i = 2, 3, \ldots, n-1$. For permutations $\mathcal{P}_i$ with $i = 2, \ldots, n-2$, they are permutations among KK-basis with $(1, n)$ fixed and it is easy to see that corresponding matrixes P, Q satisfy $P = Q$ and $P^{-1} = P^T$. By the general property (3.7), $P^{-1}GP = G(\mathcal{P}(\sigma)|\mathcal{P}(\sigma')) = G(\sigma|\sigma')$, i.e., the condition (5.6) is satisfied.

For permutation $\mathcal{P}_n$, since $2, 3 \ldots, n-2$ are invariant, we have

$$\tau_{n\sigma 1} = \tau_{1n\sigma} = (-)^{n-2}\tau_{1\sigma^T n}, \quad n_{n\sigma 1} = (-)^{n-2}n_{1\sigma^T n} \tag{5.17}$$

and especially

$$G(1\sigma^T n|1\gamma^T n) = G(1\sigma n|1\gamma n), \tag{5.18}$$

thus the relabeling property for permutation $\mathcal{P}_n$ is proved. To finish the proof, we need to check that the last permutation $\mathcal{P}_{1i}$ or $\mathcal{P}_{ni}$ satisfies the relabeling property (5.6). Since the proof is very complicated, we leave it to appendix.

[3]To avoid confusion of matrix P and the generators of permutation group, we will use $\mathcal{P}$ for later.

5.2 An application

Having shown that solution from our algorithm has the relabeling property, it is natural to ask if we can use the relabeling property to fix the solution. In this subsection, we will show that it can be done. There are two different approaches and we discuss them one by one.

For the first approach, we need to use the relabeling property and only one equation of the form (1.12). To demonstrate the idea, let us start with four-point example. First we expand τ_{1234} into the KK-basis $n_{1\sigma n}$ as

$$\tau_{1234} = \alpha n_{1234} + \beta n_{1324} \tag{5.19}$$

Using the relabeling property other τ's in the KK-basis will have similar expansion. For $n = 4$, thing is simple and we have only one

$$\tau_{1324} = \alpha n_{1324} + \beta n_{1234} \tag{5.20}$$

To completely fix α, β we need to use one relation

$$n_{1234} = 4\tau_{1234} + 2\tau_{1324} = (4\alpha + 2\beta) n_{1234} + (4\beta + 2\alpha) n_{1324} \tag{5.21}$$

Since each BCJ numerator n_α is independent, we get two equations

$$4\alpha + 2\beta = 1, \quad 4\beta + 2\alpha = 0 \Longrightarrow \beta = -\frac{1}{6}, \quad \alpha = \frac{1}{3} \tag{5.22}$$

For general n-points, we expand, for example, $\tau_{123\ldots(n-1)n}$ into the KK-basis $n_{1\sigma n}$ of BCJ numerators with $(n-2)!$ unknown variables α_i. Then we use the relabeling property to express all other τ's in KK-basis using the same set of variables α_i. Next we put it back to just one equation $n_{123\ldots n} = \tau_{1[2,[3,\ldots[n-1,n]]]}$. Identifying both sides gives $(n-2)!$ linear equations for coefficients α_i. From these equations we can solve α_i and determine expressions of τ's.

Now we want to compare the first approach with the algorithm given in section 3. The algorithm given in section 3 requires calculating a big matrix G and invert it. However, calculating matrix G by formula (3.5) is not easy and there are a lot of combinations to bookkeep. The first approach presented here does not require calculating G. All information of G is automatically included in the above procedures (see (5.21) and (5.22)). In other words, the first approach has bypassed the calculation of matrix G although solving linear equations of $(n-2)!$ variables can still be a difficult problem.

The first approach has not used the full potential of relabeling property. Now we present the second approach. To demonstrate, we use the five-point case as an example. Expanding τ_{12345} into the BCJ basis we have

$$\tau_{12345} = \alpha_1 n_{12345} + \alpha_2 n_{12435} + \alpha_3 n_{13245} + \alpha_4 n_{13425} + \alpha_5 n_{14235} + \alpha_6 n_{14325} \tag{5.23}$$

with six unknown variables. Using the relabeling property we can write down the expansion of other five τ's in KK-basis using the same six variables α_i. Up to this step, it is the same as in the first approach. However, we have not used all generators of permutation group S_5, i.e., there are still two relabelings $\mathcal{P}(1,5)$ and $\mathcal{P}(5,4)$ not being used. Using the relabeling property coming from $\mathcal{P}(1,5)$, we have on the one hand

$$\tau_{12345} \to \tau_{52341} = \alpha_1 n_{52341} + \alpha_2 n_{52431} + \alpha_3 n_{53241} + \alpha_4 n_{53421} + \alpha_5 n_{54231} + \alpha_6 n_{54321} \tag{5.24}$$

by relabeling property, but on the other hand

$$\begin{aligned}\tau_{12345} \to \tau_{52341} &= \tau_{15234} = -\tau_{14325} \\ &= -(\alpha_1 n_{14325} + \alpha_2 n_{14235} + \alpha_3 n_{13425} + \alpha_4 n_{13245} + \alpha_5 n_{12435} + \alpha_6 n_{12345})\end{aligned} \tag{5.25}$$

by cyclic symmetry and KK-relation of τ. Comparing these two results using $n_{1\sigma n} = (-)^n n_{n\sigma^T 1}$, we immediately get the following equations

$$\alpha_1 = \alpha_1, \quad \alpha_6 = \alpha_6, \quad \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_5 \tag{5.26}$$

In other words, using the relabeling property of $\mathcal{P}(1,5)$ we have reduced six unknown variables to four unknown variables.

Now we discuss the implication of relabeling property coming from permutation $\mathcal{P}(4,5)$. On the one hand we have

$$\begin{aligned}\tau_{12345} \to \tau_{12354} &= \alpha_1 n_{12354} + \alpha_2 n_{12534} + \alpha_3 n_{13254} + \alpha_4 n_{13524} + \alpha_5 n_{15234} + \alpha_6 n_{15324} \\ &= n_{12345}(-\alpha_1 - \alpha_2 - \alpha_5) + n_{12435}(\alpha_2 + \alpha_5 + \alpha_6) + n_{13245}(-\alpha_3 - \alpha_4 - \alpha_6) \\ &\quad + n_{13425}(\alpha_4 + \alpha_5 + \alpha_6) + n_{14235}(-\alpha_6) + n_{14325}(-\alpha_5)\end{aligned} \tag{5.27}$$

The first line of the equation derives from relabeling, which subsequently produces the second line using (2.8). On the other hand we have

$$\begin{aligned}\tau_{12345} \to \tau_{12354} &= -\tau_{12345} - \tau_{12435} - \tau_{14235} \\ &= n_{12345}(-\alpha_1 - \alpha_2 - \alpha_4) + n_{12435}(-\alpha_2 - \alpha_1 - \alpha_3) + n_{13245}(-\alpha_3 - \alpha_5 - \alpha_6) \\ &\quad + n_{13425}(-\alpha_4 - \alpha_6 - \alpha_5) + n_{14235}(-\alpha_5 - \alpha_3 - \alpha_1) + n_{14325}(-\alpha_4 - \alpha_6 - \alpha_2)\end{aligned} \tag{5.28}$$

where in the first line we have used KK-relation and in the second line we have used the expansion of τ's into n_α. Comparing the above two results, we obtain the following equations

$$\begin{aligned}(-\alpha_1 - \alpha_2 - \alpha_5) &= (-\alpha_1 - \alpha_2 - \alpha_4), & \alpha_2 + \alpha_5 + \alpha_6 &= (-\alpha_2 - \alpha_1 - \alpha_3), \\ (-\alpha_3 - \alpha_4 - \alpha_6) &= (-\alpha_3 - \alpha_5 - \alpha_6), & \alpha_4 + \alpha_5 + \alpha_6 &= (-\alpha_4 - \alpha_6 - \alpha_5) \\ -\alpha_6 &= (-\alpha_5 - \alpha_3 - \alpha_1), & -\alpha_5 &= (-\alpha_4 - \alpha_6 - \alpha_2)\end{aligned} \tag{5.29}$$

Combining (5.26) and (5.29) we find

$$\alpha_2 = \alpha_3 = -\frac{2}{5}\alpha_1, \quad \alpha_6 = \frac{2}{5}\alpha_1, \quad \alpha_4 = \alpha_5 = -\frac{1}{5}\alpha_1 \tag{5.30}$$

If we put $\alpha_1 = \frac{1}{4}$ back, we do reproduce the result (4.14).

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The above example can be generalized to arbitrary number of legs. The point of this second approach is that if we fully use the potential of relabeling property, i.e, the relabeling properties of all generators of permutation group S_n, we will be able to determine expressions of all τ's over $(n-2)!$ BCJ numerator $n_{1\sigma n}$ up to an overall factor (for example, α_1 in above example). It is crucial to notice that in the second approach, we have used only cyclic symmetry (1.10) and KK-relations (1.11) among τ's, but not the relation (1.12), which relates τ with BCJ numerators n_α. In other words, relabeling property, cyclic symmetry plus KK-relations for τ's *have uniquely determined* the expression of τ in terms of BCJ numerators n_α up to an overall constant!

To determine the overall factor, relation such as (1.12) enters the game. However, based on our examples in section 4, we found the following pattern of the expansion of $\tau_{123\ldots(n-1)n}$:

$$\begin{aligned}
\tau_{123} &= \frac{1}{2}n_{123} \\
\tau_{1234} &= \frac{1}{3}n_{1234} + \ldots \\
\tau_{12345} &= \frac{1}{4}n_{12345} + \ldots \\
\tau_{123456} &= \frac{1}{5}n_{123456} + \ldots \\
\tau_{1234567} &= \frac{1}{6}n_{1234567} + \ldots
\end{aligned} \tag{5.31}$$

We believe that this pattern is right although we can not prove it at the moment. If we accept this as an assumption, we find that (1.12) is not needed anymore.

Now we can see the difference between the second approach and the algorithm presented in section 3. For algorithm in section 3, the equation (1.12) is crucial. However, in the second approach, the relabeling property is crucial. In fact, using the relabeling property, plus cyclic symmetry and KK-relation, the equation (1.12) can be derived if we use our observation (5.31).

5.3 The implication of permutation $\mathcal{P}(1, n)$

From our previous discussions for the second approach, we see that all nontrivial equations for $(n-2)!$ expansion coefficients of $\tau_{123\ldots(n-1)n}$ are given by relabeling properties coming from permutations $\mathcal{P}(1, n)$ and $\mathcal{P}(n-1, n)$. These equations coming from permutation $\mathcal{P}(n-1, n)$ will be complicated to write down. However, these equations coming from permutation $\mathcal{P}(1, n)$ are very simple and we will present them in this subsection.

Let us start with the expansion

$$\tau_{123\ldots(n-1)n} = \sum_{\sigma\in S_{n-2}} c_\sigma n_{1\sigma n} \tag{5.32}$$

with $(n-2)!$ coefficients c_σ. The relabeling by permutation $\mathcal{P}(1, n)$, i.e, $1 \leftrightarrow n$, leads to the expression

$$\tau_{n23\ldots(n-1)1} = \sum_{\sigma} c_\sigma n_{n\sigma 1} \tag{5.33}$$

Using the cyclic symmetry and KK-relation for $\tau_{n23\ldots(n-1)1}$ we arrive another expression of $\tau_{n23\ldots(n-1)1}$

$$\tau_{n23\ldots(n-1)1} = \tau_{1n23\ldots(n-1)} = (-)^n \tau_{1(n-1)(n-2)\ldots 32n} = (-)^n \sum_{\sigma} c_\sigma n_{1\widetilde{\mathcal{P}}(\sigma)n} \tag{5.34}$$

where at the last equation we have used the fact that the expansion of $\tau_{1(n-1)(n-2)\ldots 32n}$ can be obtained from the expansion of $\tau_{123\ldots(n-1)n}$ by following permutation

$$\widetilde{\mathcal{P}} \equiv \begin{cases} (2,n-1)(3,n-2)\ldots(n/2,n/2+1) & n = even \\ (2,n-1)(3,n-2)\ldots((n-1)/2,(n+1)/2+1) & n = odd \end{cases} \tag{5.35}$$

Identifying two different expressions (5.33) and (5.34) we have

$$(-)^n \sum_{\sigma} c_\sigma n_{1\widetilde{\mathcal{P}}(\sigma)n} = \sum_{\widetilde{\sigma}} c_{\widetilde{\sigma}} n_{n\widetilde{\sigma}1} = (-)^n \sum_{\widetilde{\sigma}} c_{\widetilde{\sigma}} n_{1(\widetilde{\sigma})^T n} \tag{5.36}$$

where we have used $(\widetilde{\sigma})^T$ to denote reversing the ordering of the list $\widetilde{\sigma}$. Since each BCJ numerator n_α is independent, to have identity (5.36), coefficient of each n_α must be same at both sides. Identifying $n_{1\widetilde{\mathcal{P}}(\sigma)n} = n_{1(\widetilde{\sigma})^T n}$ for given pair of $\sigma, \widetilde{\sigma}$, we find following result: when two orderings σ and $\widetilde{\sigma}$ are related to each other by $\widetilde{\mathcal{P}}(\sigma) = (\widetilde{\sigma})^T$, their coefficients must be same, i.e., we will have

$$c_{\widetilde{\sigma}} = c_\sigma \tag{5.37}$$

or more explicitly

$$c_{1\sigma n} = c_{1(\widetilde{\mathcal{P}}(\sigma))^T n}, \qquad \forall \sigma \in S_{n-2} \tag{5.38}$$

Results (5.38) are equations coming from relabeling property of permutation $\mathcal{P}(1n)$. It is easy to check it with explicit results given in section 4.

From (5.38) we see that there are two special orderings which are singlet under about transformation. They are $c_{1234\ldots(n-1)n}$ and $c_{1(n-1)(n-2)\ldots 32n}$. In fact, all coefficients will organize themselves to orbits with one or two elements under the mapping (5.38).

6 Conclusion

In this paper, we have constructed the dual-trace decomposition for Yang-Mills tree amplitudes from kinematic numerators. By imposing cyclic symmetry and KK relation, and the relation between τ's and BCJ numerators in KK-basis, we find solutions of τ's as linear combinations of BCJ numerators. The dual-trace factors solved in this way are related to each other by relabeling. Thus we can get any dual trace factor by relabeling a single τ with given permutation. We find that we can also turn things around and start with the relabeling property to fix the dual-trace factors τ.

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A The relabeling of permutation P_{1i}

To complete the proof of natural relabeling property, we need to prove that (5.6) is satisfied for the permutation $\mathcal{P}_{1i} = (1, i)$. In this appendix we provide a detailed proof. We note that equation (5.6) can be rewritten as

$$\begin{aligned} &\sum_{\{\widetilde{\rho}\},\{\widetilde{\sigma}\}} P(i,\{\rho'\},1,\{\sigma'\},n|1,\{\widetilde{\rho}\},i,\{\widetilde{\sigma}\},n)G(1,\{\widetilde{\rho}\},i,\{\widetilde{\sigma}\},n|1,\{\rho\},i,\{\sigma\},n) \\ =&\sum_{\{\rho''\},\{\sigma''\}} G(i,\{\rho'\},1,\{\sigma'\},n|i,\{\rho''\},1,\{\sigma''\},n)Q(i,\{\rho''\},1,\{\sigma''\},n|1,\{\rho\},i,\{\sigma\},n) \end{aligned} \tag{A.1}$$

We introduce curly brackets to emphasize that, for example, $\{\rho\}$ stands for a list of elements $\rho_1,\ \rho_2,\ \ldots$ (which can be empty as well), and that the ordering of the list matters. We break the proof into three steps discussed below:

Step-1: finding explicit expressions of elements of P and Q matrices. Under the permutation $\mathcal{P}(1, i)$, a KK-basis numerator characterized by the fixed pair $(1, n)$ changes to a KK-basis numerator characterized by (i, n), and we need the collaboration of Jacobi identity, antisymmetry to derive the matrix of P, and KK relation together with cyclic symmetry to derive the matrix of Q in $(1, n)$ basis.

Let us consider the matrix P first. Using the Jacobi identity and antisymmetry, we have

$$n_{i,\{\rho\},1,\{\sigma\},n} = \sum_{\{\rho\}\to\{\alpha\}\{\beta\}} (-1)^{n_\alpha+1} n_{1,\{\alpha\}^T,i,\{\beta\},\{\sigma\},n}. \tag{A.2}$$

where $\sum_{\{\rho\}\to\{\alpha\}\{\beta\}}$ means summing over all possible splittings of set $\{\rho\}$ to two subsets $\{\alpha\},\{\beta\}$ with their relative orderings kept. As remarked at the beginning of section 3, this is equivalent to summing over all the ordered sets $\{\alpha\},\{\beta\}$ satisfying the condition $\{\rho\} \in OP(\{\alpha\}\bigcup\{\beta\})$, yet the advantage of regarding this process as a splitting instead of as imposing a constraint will become obvious shortly as the complexity increases. It is straightforward then, to read off the elements of matrix P from the above equation

$$P(i,\{\rho\},1,\{\sigma\},n|1,\{\alpha\}^T,i,\{\beta\},\{\sigma\},n) = \begin{cases} (-1)^{n_\alpha+1} & (\text{if } \{\rho\} \text{ can splits into } \{\alpha\},\{\beta\}) \\ 0 & \text{Otherwise} \end{cases}. \tag{A.3}$$

Next we consider the matrix Q. For $\tau_{i,\{\rho\},1,\{\sigma\},n}$ we can use cyclic symmetry and KK relation

$$\tau_{i,\{\rho\},1,\{\sigma\},n} = \tau_{1,\{\sigma\},n,i,\{\rho\}} = \sum_{\{\delta\}\in OP(\{\sigma\}\bigcup\{\rho^T,i\})} (-1)^{n_\rho+1}\tau_{1,\{\delta\},n}. \tag{A.4}$$

thus elements of Q can be read off

$$Q(i,\{\rho\},1,\{\sigma\},n|1,\{\delta\},n) = \begin{cases} (-1)^{n_\rho+1} \text{ if } \{\delta\} \in OP(\{\sigma\}\bigcup\{\rho^T,i\}) \\ 0 \qquad\qquad \text{Otherwise} \end{cases}. \tag{A.5}$$

Because the ordering $\{\rho^T, i\}$ in (A.5), nonzero elements of Q must have the form

$$Q(i,\{\alpha^T\},1,\{\beta\},\{\sigma\},n|1,\{\rho\},i,\{\sigma\},n) = (-1)^{n_\alpha+1}, \quad \text{if } \{\rho\} \in OP(\{\alpha\}\bigcup\{\beta\}). \tag{A.6}$$

Plugging the newly obtained explicit expression of matrix elements of P, the matrix PG in (A.1) can be expressed as

$$\sum_{\{\rho'\}\to\{\widetilde{\alpha}\},\{\widetilde{\beta}\}} (-1)^{n_{\widetilde{\alpha}}+1} G_{1,n}(1,\{\widetilde{\alpha}^T\},i,\{\widetilde{\beta}\},\{\sigma'\},n|1,\{\rho\},i,\{\sigma\},n). \tag{A.7}$$

Similarly, the matrix GQ in (A.1) is given by

$$\sum_{\{\rho\}\to\{\alpha\},\{\beta\}} G_{i,n}(i,\{\rho'\},1,\{\sigma'\},n|i,\{\alpha^T\},1,\{\beta\},\{\sigma\},n)(-1)^{n_\alpha+1}. \tag{A.8}$$

To compare (A.7) and (A.8), using the relabeling invariant property of matrix G, we can exchange the positions of 1 and i in (A.8) and this expression becomes

$$\sum_{\{\rho\}\to\{\alpha\},\{\beta\}} G_{1,n}(1,\{\rho'\},i,\{\sigma'\},n|1,\{\alpha^T\},i,\{\beta\},\{\sigma\},n)(-1)^{n_\alpha+1}. \tag{A.9}$$

Finally the consistency condition (A.1) can be rewritten as

$$\begin{aligned}&\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}} (-1)^{n_{\alpha'}+1} G_{1,n}(1,\{\alpha'^T\},i,\{\beta'\},\{\sigma'\},n|1,\{\rho\},i,\{\sigma\},n)\\ =&\sum_{\{\rho\}\to\{\alpha\},\{\beta\}} (-1)^{n_\alpha+1} G_{1,n}(1,\{\rho'\},i,\{\sigma'\},n|1,\{\alpha^T\},i,\{\beta\},\{\sigma\},n).\end{aligned} \tag{A.10}$$

We need to show the sums at both sides of (A.10) give the same result.

Step-2: further simplification by properties of G Before finally deriving a proof let us further simplify the condition (A.1). We notice that $G_{1,n}(\alpha|\beta)$ can be written by the property (3.5) as

$$G_{1,n}(\sigma'|\sigma) = \sum_{s'\in\{\mathcal{S}(\sigma')\}} \sum_{s\in\{\mathcal{S}(\sigma)\}} \delta(s'|s), \tag{A.11}$$

where to simplify the notation, we have used $\mathcal{S}(\sigma')$ to denote the set of all possible splittings of the set $\{\sigma'\}$ into two subsets, for example $\{\alpha'\},\{\beta'\}$, with relative ordering kept, and the sum is taken over all elements of the set $\mathcal{S}(\sigma')$. The delta-function is defined as

$$\delta(s'|s) = \begin{cases} 1 & (s'=s) \\ 0 & \text{Otherwise} \end{cases}. \tag{A.12}$$

where $s'=s$ means that both $\{\alpha'\}=\{\alpha\}$ and $\{\beta'\}=\{\beta\}$ if $\{\sigma'\}$ is split to $\{\alpha'\},\{\beta'\}$ and $\{\sigma\}$ is split to $\{\alpha\},\{\beta\}$.

Substituting (A.11) into (A.10), we get

$$\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}} (-1)^{n_{\alpha'}} \left[\sum_{s',s} \delta(s'|s)\right] = \sum_{\{\rho\}\to\{\alpha\},\{\beta\}} (-1)^{n_\alpha} \left[\sum_{s',s} \delta(s'|s)\right], \tag{A.13}$$

where we have summed over $s' \in \{\mathcal{S}(\{\alpha'^T\}, i, \{\beta'\}, \{\sigma'\})\}$ and $s \in \{\mathcal{S}(\{\rho\}, i, \{\sigma\})\}$ on the L.H.S, while we have summed over $s' \in \{\mathcal{S}(\{\rho'\}, i, \{\sigma'\})\}$ and $s \in \{\mathcal{S}(\{\alpha^T\}, i, \{\beta\}, \{\sigma\})\}$ on the R.H.S. The above equation can be further rearranged into

$$\begin{aligned}
&\sum_{s\in\{\mathcal{S}(\{\rho\},i,\{\sigma\})\}} \left[\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}} (-1)^{n_{\alpha'}} \sum_{s'\in\{\mathcal{S}(\{\alpha'^T\},i,\{\beta'\},\{\sigma'\})\}} \delta(s'|s)\right] \\
&= \sum_{s'\in\{\mathcal{S}(\{\rho'\},i,\{\sigma'\})\}} \left[\sum_{\{\rho\}\to\{\alpha\},\{\beta\}} (-1)^{n_{\alpha}} \sum_{s\in\{\mathcal{S}(\{\alpha^T\},i,\{\beta\},\{\sigma\})\}} \delta(s'|s)\right]. \qquad \text{(A.14)}
\end{aligned}$$

For a given splitting s the sum in the brackets on the L.H.S. has a useful property

$$\begin{aligned}
&\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}} (-1)^{n_{\alpha'}} \sum_{s'\in\{\mathcal{S}(\{\alpha'^T\},i,\{\beta'\},\{\sigma'\})\}} \delta(s'|s) \\
&= \sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}} (-1)^{n_{\alpha'}} \sum_{s'\in\{\mathcal{S}(\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\})\}} \delta(\{\{\alpha'^T\}, i, \{\beta'\}, \{\sigma'_L\}\}, \{\sigma'_R\}|s). \qquad \text{(A.15)}
\end{aligned}$$

For a given splitting s' the sum in the brackets on the R.H.S. has a similar property. The meaning of (A.15) is that for all possible splittings only those with $\{\alpha'^T\}, i, \{\beta'\}$ belonging to the same subset contribute.[4]

Before giving a general proof of the above property (A.15), let us have a look at some examples.

- (1) For the case $n_{\rho'} = 1$, i.e., there is only one element in the set ρ', there are two possible splittings: $\{\alpha', \beta'\} = \{\{\ \}, \{\rho'_1\}\}/\{\{\rho'_1\}, \{\ \}\}$. For the case $\alpha' = \{\rho'_1\}$, the splitting of $\{\rho'_1, i, \sigma'\}$ contains two possibilities: either ρ'_1 and i belong to the same subset or to different subsets. For the case $\beta' = \{\rho'_1\}$, the splitting of $\{i, \rho'_1, \sigma'\}$ also contains two possibilities: either ρ'_1, i belong to the same subset or to different subsets. Putting all these together, the L.H.S. of (A.15) reads

$$\begin{aligned}
&\sum_{\{\sigma'\}\to\{\sigma_L\},\{\sigma_R\}} \Big[-\delta\left(\{\rho'_1, i, \sigma'_L\}, \{\sigma'_R\}|s\right) - \delta\left(\{\rho'_1, \sigma'_L\}, \{i, \sigma'_R\}|s\right) \\
&+\delta\left(\{i, \rho'_1, \sigma'_L\}, \{\sigma'_R\}|s\right) + \delta\left(\{i, \sigma'_L\}, \{\rho'_1, \sigma'_R\}|s\right)\Big] \qquad \text{(A.16)}
\end{aligned}$$

  After summing over all splittings $\{\sigma'\} \to \{\sigma'\}_L, \{\sigma'\}_R$, the second term and the fourth term cancel each other and only two terms are left

$$\sum_{\{\sigma'\}\to\{\sigma_L\},\{\sigma_R\}} \left[-\delta\left(\{\rho'_1, i, \sigma'_L\}, \{\sigma'_R\}|s\right) + \delta\left(\{i, \rho'_1, \sigma'_L\}, \{\sigma'_R\}|s\right)\right]. \qquad \text{(A.17)}$$

  which is just the R.H.S. of (A.15) when $n_{\rho'} = 1$.

[4] One may notice the splitting can be either $\{\sigma_1, i, \sigma_2\}, \{\sigma_3\}$ or $\{\sigma_3\}, \{\sigma_1, i, \sigma_2\}$. Since there is no difference for following discussions between the two kinds of splittings, we just need to deal with only the first kind.

- (2) For the case $n_{\rho'} = 2$, similar consideration as the case $n_{\rho'} = 1$ gives the L.H.S.:

$$\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}} \Bigg\{ \Bigg[\delta\left(\{\rho'_2,\rho'_1,i,\sigma'_L\},\{\sigma'_R\}|s\right) + \delta\left(\{\rho'_2,i,\sigma'_L\},\{\rho'_1,\sigma'_R\}|s\right)$$
$$+\delta\left(\{\rho'_1,i,\sigma'_L\},\{\rho'_2,\sigma'_R\}|s\right) + \delta\left(\{\rho'_2,\rho'_1,\sigma'_L\},\{i,\sigma'_R\}|s\right)\Bigg]$$
$$-\Bigg[\delta\left(\{\rho'_2,i,\rho'_1,\sigma'_L\},\{\sigma'_R\}|s\right) + \delta\left(\{\rho'_2,i,\sigma'_L\},\{\rho'_1,\sigma'_R\}|s\right)$$
$$+\delta\left(\{\rho'_2,\rho'_1,\sigma'_L\},\{i,\sigma'_R\}|s\right) + \delta\left(\{\rho'_1,i,\sigma'_L\},\{\rho'_2,\sigma'_R\}|s\right)\Bigg]$$
$$-\Bigg[\delta\left(\{\rho'_1,i,\rho'_2,\sigma'_L\},\{\sigma'_R\}|s\right) + \delta\left(\{\rho'_1,i,\sigma'_L\},\{\rho'_2,\sigma'_R\}|s\right)$$
$$+\delta\left(\{\rho'_1,\rho'_2,\sigma'_L\},\{i,\sigma'_R\}|s\right) + \delta\left(\{\rho'_2,i,\sigma'_L\},\{\rho'_1,\sigma'_R\}|s\right)\Bigg]$$
$$+\Bigg[\delta\left(\{i,\rho'_1,\rho'_2,\sigma'_L\},\{\sigma'_R\}|s\right) + \delta\left(\{i,\rho'_1,\sigma'_L\},\{\rho'_2,\sigma'_R\}|s\right)$$
$$+\delta\left(\{i,\rho'_2,\sigma'_L\},\{\rho'_1,\sigma'_R\}|s\right) + \delta\left(\{\rho'_1,\rho'_2,\sigma'_L\},\{i,\sigma'_R\}|s\right)\Bigg]\Bigg\} \tag{A.18}$$

  Again, after summing over all splittings of $\{\sigma'\}$, terms cancel each other and we are left with

$$\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}} \Bigg[\delta\left(\{\rho'_2,\rho'_1,i,\sigma'_L\},\{\sigma'_R\}|s\right) - \delta\left(\{\rho'_2,i,\rho'_1,\sigma'_L\},\{\sigma'_R\}|s\right)$$
$$-\delta\left(\{\rho'_1,i,\rho'_2,\sigma'_L\},\{\sigma'_R\}|s\right) + \delta\left(\{i,\rho'_1,\rho'_2,\sigma'_L\},\{\sigma'_R\}|s\right)\Bigg], \tag{A.19}$$

  which is just the R.H.S. of (A.15) in the case of $n_{\rho'} = 2$.

- (3) Similar calculation has been done for the case of $n_{\rho'} = 3$, which we neglect here, since the manipulations are quite similar to the examples shown.

Above examples give the idea of proof. For the case with $n_{\rho'} = r$ after the splitting $\{\rho'\} \to \{\alpha'\},\{\beta'\}$ with $n_{\alpha'} = s$ and $n_{\beta'} = r - s$, we need to sum over all splittings of $\{\alpha', i, \beta', \sigma'\}$. In general, the splitting will be $\{\alpha'_1, i, \beta'_1, \sigma'_1\}, \{\alpha'_2, \beta'_2, \sigma'_2\}$. Among these two subsets, unlike the subset $\{\alpha'_1, i, \beta'_1, \sigma'_1\}$ where i seperates the α' part from β' part, the subset $\{\alpha'_2, \beta'_2, \sigma'_2\}$ can come from different splittings of $\{\rho'\} \to \{\alpha'\},\{\beta'\}$. More explicitly, the last element of α'_2 can be considered as the first element of β''_2. Thus when we put the factor $(-)^{n_{\alpha'}}$ back, the term coming from $n_{\alpha'} = s$ will cancel with the term coming from $n_{\alpha''} = s + 1$. Because this kind of cancelations, only splittings with all elements of $\{\alpha^T\}, i, \{\beta\}$ in the same ordered subset contribute.

Having established (A.15), (A.14) can be rewritten as

$$\sum_{s\in\{\mathcal{S}(\{\rho\},i,\{\sigma\})\}}\left[\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}}(-1)^{n_{\alpha'}}\sum_{\mathcal{S}(\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\})}\delta(\{\{\alpha'^T\},i,\{\beta'\},\{\sigma'_L\}\},\{\sigma'_R\}|s)\right]$$
$$=\sum_{s'\in\{\mathcal{S}(\{\{\rho'\},i,\{\sigma'\}\})\}}\left[\sum_{\{\rho\}\to\{\alpha\},\{\beta\}}(-1)^{n_{\alpha}}\sum_{\mathcal{S}(\{\sigma\}\to\ \{\sigma_L\},\{\sigma_R\})}\delta(\{\{\alpha^T\},i,\{\beta\},\{\sigma_L\}\},\{\sigma_R\}|s')\right]. \tag{A.20}$$

Step-3: proving the relabeling properties $PG = GQ$ via (A.20) From step-1 and step-2, we have written the relabeling property $PG = GQ$ to the form (A.20). Now let us prove (A.20) by considering various configurations:

- (1) *Both* $\{\rho\}$ *and* $\{\rho'\}$ *are empty:* In this case, (A.20) becomes

$$\sum_{\mathcal{S}(\{\sigma\}\to\{\sigma_L\},\{\sigma_R\})}\left[\sum_{\mathcal{S}(\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\})}\delta(\{\sigma'_L\},\{\sigma'_R\}|\{\sigma_L\},\{\sigma_R\})\right]$$
$$=\sum_{\mathcal{S}(\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\})}\left[\sum_{\mathcal{S}(\{\sigma\}\to\ \{\sigma_L\},\{\sigma_R\})}\delta(\{\sigma_L\},\{\sigma_R\}|\{\sigma'_L\},\{\sigma'_R\})\right], \tag{A.21}$$

  which is trivially true.

- (2) *Only one of* $\{\rho\}$ *and* $\{\rho'\}$ *is empty:* assuming $\{\rho\}$ is empty, possible nontrivial terms on the L.H.S. of (A.20) is given as

$$\sum_{\{i,\sigma\}\to\{i,\sigma_L\},\{\sigma_R\}}\ \sum_{\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\}}\delta(\{i,\rho',\sigma'_L\},\{\sigma'_R\}|\{i,\sigma_L\},\{\sigma_R\})$$
$$=\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\ \sum_{\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\}}\delta(\{\rho',\sigma'_L\}|\{\sigma_L\})\delta(\{\sigma'_R\}|,\{\sigma_R\}). \tag{A.22}$$

  where we have used the fact that since the splitting of s has element i at the first position, nonzero contribution requires the splitting of $\{\rho'\}$ to be $\{\alpha'\}$ empty.

  For the R.H.S. contribution is

$$\sum_{s'\in\mathcal{S}(\{\rho',i,\sigma'\})}\ \sum_{\{\sigma\}\to\ \{\sigma_L\},\{\sigma_R\}}\delta(\{i,\sigma_L\},\{\sigma_R\}|s')$$
$$=\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}}\ \sum_{\{\sigma\}\to\ \{\sigma_L\},\{\sigma_R\}}\delta(\{i,\sigma_L\},\{\sigma_R\}|\{i,\sigma'_L\},\{\rho',\sigma'_R\})$$
$$=\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\ \sum_{\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\}}\delta(\{\rho',\sigma'_L\}|\{\sigma_L\})\delta(\{\sigma'_R\}|,\{\sigma_R\}). \tag{A.23}$$

  where in the second line, to have nonzero result, the splitting of s' is that $\{\rho'\}$ and i belong to different subsets. Thus in this case, the relabeling property is also satisfied.

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- (3) *Both $\{\rho'\}$ and $\{\rho\}$ are nonempty and they have no element in common:* let us consider the L.H.S. of (A.20) first. Since $\{\rho\}$ and $\{\rho'\}$ have no element in common, to have nonzero contribution, we must have $\{\alpha'\}$ empty in the splitting of $\{\rho'\}$ and $\{\rho, i\}$ belong to different subsets in the splitting of s. Thus the L.H.S. of (A.20) is given as

$$\begin{aligned}&\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}}\delta(\{i,\rho',\sigma'_L\},\{\sigma'_R\}|\{i,\sigma_L\},\{\rho,\sigma_R\})\\ =&\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}}\delta(\{i,\rho',\sigma'_L\}|\{i,\sigma_L\})\delta(\{\sigma'_R\}|\{\rho,\sigma_R\}).\end{aligned}\tag{A.24}$$

For the R.H.S., we need the $\{\alpha\}$ to be empty in the splitting of $\{\rho\}$ and $\{\rho', i\}$ belong to different subsets in the splitting of s', thus we get

$$\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}}\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\delta(\{i,\rho,\sigma_L\}|\{i,\sigma'_L\})\delta(\{\sigma_R\}|\{\rho,\sigma'_R\})\tag{A.25}$$

Thus the relabeling property in this case is satisfied.

- (4) *Both $\{\rho\}$ and $\{\rho'\}$ are nonempty and they share common elements:* this is a most general case. The L.H.S. of (A.20) is

$$\begin{aligned}&\sum_{\{\rho\}\to\{\alpha\},\{\beta\}}\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\Bigg[\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}}(-1)^{n_{\alpha'}}\\ &\times\sum_{\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\}}\delta(\{\alpha'^T,i,\beta',\sigma'_L\},\{\sigma'_R\}|\{\alpha,i,\sigma_L\},\{\beta,\sigma_R\})\Bigg]\\ =&\sum_{\{\rho\}\to\{\alpha\},\{\beta\}}\sum_{\{\sigma\}\to\{\sigma_L\},\{\sigma_R\}}\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}}\sum_{\{\sigma'\}\to\ \{\sigma'_L\},\{\sigma'_R\}}\Bigg[(-1)^{n_{\alpha'}}\\ &\times\delta(\{\alpha'^T\}|\{\alpha\})\delta(\{\beta',\sigma'_L\}|\{\sigma_L\})\delta(\{\sigma'_R\}|\{\beta,\sigma_R\})\Bigg],\end{aligned}\tag{A.26}$$

Similarly, the R.H.S. of (A.20) can be written as

$$\begin{aligned}&\sum_{\{\rho'\}\to\{\alpha'\},\{\beta'\}}\sum_{\{\sigma'\}\to\{\sigma'_L\},\{\sigma'_R\}}\sum_{\{\rho\}\to\{\alpha\},\{\beta\}}\sum_{\{\sigma\}\to\ \{\sigma_L\},\{\sigma_R\}}\Bigg[(-1)^{n_{\alpha}}\\ &\times\delta(\{\alpha^T\}|\{\alpha'\})\delta(\{\beta,\sigma_L\}|\{\sigma'_L\})\delta(\{\sigma_R\}|\{\beta',\sigma'_R\})\Bigg].\end{aligned}\tag{A.27}$$

Changing $\alpha\to\alpha'$ etc, and the L.H.S. equals the R.H.S. in this case.

Taking all the above possibilities into account, the property (A.20) is satisfied for all cases, and we see that the relabeling condition (A.1) is satisfied for permutation $\mathcal{P}(1, i)$.

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