An Approach to the Identifiable Parameters of a Manipulator

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This article deals with the minimal parameters of a manipulator in the least squares sense, so that the minimal parameters are equivalent to the identifiable parameters. The least squares concept is used to introduce terminology for the minimal linear combinations (MLCs) of the system parameters that define a set of linear combinations of the system parameters. The number of elements of the set is minimal, yet the set still completely determines the system. Furthermore, it is shown that the problem of finding a set of MLCs of a manipulator can be simplified to that of finding two individual sets of MLCs that determine the entries of the inertia matrix and the gravity load. Although the approach is applied to the inertia constants of composite bodies to obtain a set of MLCs identical to an earlier one, the result is newly interpreted in the least squares sense. The approach itself is a new method for finding the identifiable parameters of a manipulator, and it yields some new insight into the manipulator dynamics. The crucial feature is that a set of MLCs found by using the present approach is guaranteed to be identifiable. The earlier approaches always require an identification method to verify the results. An equivalence theorem is also presented that rigorously states the equivalence between the different sets of minimal parameters.

この発表では、最小二乗ベクトルにおけるマニピュレータの最小パラメータを取り上げる。この最小パラメータは、理想パラメータと同じものである。最小二乗法の概念を使って、システム・パラメータのリニア・コンビネーションのセットを定義するシステム・パラメータの最小リニア・コンビネーション（MLCs）の概念を導入する。このセットの要素数は最小になっているが、セットはシステムを完全な形で定義している。さらに、マニピュレータの MLCs のセットを探す問題は、慣性マトリックスと重力負荷のエントリを定義する２つの独立した MLCs のセットを見つけるだけで済むように、簡略化できることを示す。今回の試みは、前出のものに等しい MLCs のセットを得るために、

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複合体の慣性定数に適応されるが、結果は新たに最小二乗ベクトルに変換される。この試み自体は、マニピュレータの確認可能なパラメータを探すためのまったく新しい方法であり、新しい概念をマニピュレータの力学に付け加えるものである。重要な点は、今回の試みを使って見つけられる MLCs のセットが確認可能であることを保証できることである。初期の試みでは、常に結果を検証するための確認方法が求められる。　また同一性理論を説明することで、最小パラメータの異なるセット間の同一性について正確に述べる。

1. INTRODUCTION

System analysis involves investigating the dynamic behavior of a system to understand the system or to design regulators to control it. Two methodologies used in system analysis are mathematical modeling and system identification. However, the compound approach is always recommended for a deterministic system. An analytic approach can provide a dynamic model with wide-range validity and some physical insights, while parameter identification improves the accuracy of the model parameters. If the model of a nonlinear system such as a manipulator can be written in the form of linear equations with respect to the parameters, the parameter identification is then a least squares problem. The difficulty of this parameter identification is that not all parameters are identifiable, because some parameters determine the system not independently, but in combination. Essentially, the system is uniquely determined by a set of minimal parameters that are linear combinations of the modeling parameters and are linearly independent. These parameters are termed the *minimal linear combinations* (MLCs) of the system parameters in the context. Finding a set of MLCs will facilitate solving the parameter identification problem.

The dynamic model of a manipulator is now well-known. It is highly nonlinear and requires knowledge of the kinematic parameters (relations between two adjacent links) and the inertia parameters (mass, center of mass and inertia tensor of each link). The kinematic parameters are always provided by the manufacturer or can be precisely calibrated, whereas the inertia parameters of industrial robots are almost all unavailable from the manufacturer because these values are not needed for commercial controllers. However, most modern precision control schemes for manipulators incorporate the inverse dynamics of the manipulator, which requires the values of the inertia parameters. To evaluate the inertia parameters of the manipulator dynamics, Armstrong et al.[1] disassembled a PUMA 560 robot and used a mechanical method to measure the inertia parameters. This approach is tedious and does not yield precise results. Fortunately, Atkeson et al.[2] have found that the actuator forces of a manipulator are linear functions of the inertia parameters, i.e., the dynamics of a manipulator can be expressed as linear equations with respect to the inertia parameters. Previous attempts to identify the inertia parameters have tried to formulate the linear equations either explicitly[3–9] or implicitly.[2,9–12] As mentioned above, without knowledge of the MLCs of the inertia parameters, identifying the parameters is difficult. Khosla and Kanade[7] intuitively regrouped the closed-form dynamic equations, and other researchers[4–6] developed regrouping rules to minimize the number of inertia parameters appearing in the linear equations. These approaches are not practical for a manipulator with six or more joints because the closed-form dynamic equations of a six-joint manipulator are too large and too complicated to analyze.

The MLCs of the inertia parameters of manipulators have drawn the attention of many researchers. Some researchers[2,13–15] have presented numerical approaches such as the singular value decomposition method and the QR method. Gautier et al.[16–18] developed a regrouping rule to symbolically form a set of MLCs. Mayeda et al.[19–22] found an explicit set of MLCs of the inertia parameters. Although these two sets of results are substantially equivalent, the explicit form is more attractive.

It should be remarked that, instead of MLCs, minimum inertia parameters,[16–18] base parameters,[19–22] and the basis set of the essential parameter space[13,14] have all been used in the literature. This article will show that the identifiable parameters of a linear deterministic system belong to a set of MLCs of the system parameters in the least squares sense. Of course, a set of MLCs is also the necessary (i.e., minimal) parameters required to determine the system dynamics. The name of MLCs illuminates the role of the system parameters in the dynamic model and in the problem of parameter identification. In this article, the problem of finding the MLCs for the full manipulator dynamics is divided into a search for two individual sets of MLCs, one for determining the inertia matrix and the other for determining the gravity load.

In an earlier article,[23] we showed that some inertia constants of composite bodies form the minimal knowledge of the inertia parameters needed to determine the manipulator dynamics. This article applies a new approach to the inertia constants of composite bodies and shows directly that the minimal knowledge of the inertia parameters in the earlier paper[23] is a set of MLCs. The advantage of the present approach is that it is not necessary to verify the identifiability of MLCs by an identification method. The approach also appears to provide a systematic method for analyzing the identifiable parameters of other mechanical systems. At the end of the article, an equivalence theorem is presented that rigorously states the equivalence between the different sets of minimal linear combinations of parameters. This theorem could also be a tool for exploring other possible sets of MLCs in the future.

This article is organized as follows. Section 2 introduces the concept of minimal linear combinations (MLCs) of the system parameters. Several theorems are established to simplify the problem of finding the MLCs for a manipulator. To make the idea easier to comprehend, all proofs are presented in the Appendix. In section 3, we review the inertia constants of composite bodies, which are used to construct a set of MLCs in section 4. The rigorous derivation of the set of MLCs in section 4 is the main effort of this article. The approach is systematic and comprehensible, although some of the proofs are tediously long.

2. PRELIMINARIES

Motivated by the fact that the dynamics of a manipulator can be formulated as linear equations with respect to the inertia parameters,[2] we consider a dynamic system to be identified that has the linear deterministic form of

$$\mathbf{y} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} \tag{1}$$

where $\mathbf{y} \in R^n$, $\boldsymbol{\theta} \in R^m$ are observable signals, $\mathbf{x} \in R^p$ consists of the system parameters, $p > n$, and $\mathbf{A}(\boldsymbol{\theta})$: $R^m \rightarrow R^{n\times p}$. We are concerned with the identfiability of the system parameters.

Definition 1. *A set of columns* $\mathbf{a}_i(\boldsymbol{\theta})$: $R^m \rightarrow R^n$ *is said to be linearly dependent over* R^m *if there exist constants* α_i, $i = 1, \ldots, n$, *not all zero such that*

$$\sum_{i=1}^{n} \alpha_i \mathbf{a}_i(\boldsymbol{\theta}) = 0, \qquad \forall\, \boldsymbol{\theta} \in R^m. \tag{2}$$

If α_i *are all zero, the set is said to be linearly independent over* R^m. ■

Theorem 1. *The number of linearly independent columns of* $\mathbf{A}(\boldsymbol{\theta})$ *over* R^m *is* $k \leq p$ *if and only if there exist* $\overline{\mathbf{A}}(\boldsymbol{\theta})$: $R^m \rightarrow R^{n\times k}$ *whose columns are linearly independent over* R^m *and* $\mathbf{w}(\mathbf{x})$: $R^p \rightarrow R^k$ *whose components are linear combinations of* $\mathbf{x}$ *and are linearly independent over* R^p, *such that*

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{x} = \overline{\mathbf{A}}(\boldsymbol{\theta})\mathbf{w}(\mathbf{x}), \qquad \forall\, \boldsymbol{\theta} \in R^m \text{ and } \mathbf{x} \in R^p. \tag{3}$$

■

The proof of this theorem is presented in the Appendix. According to the least squares theory,[24] not all system parameters $\mathbf{x}$ are identifiable if the columns of $\mathbf{A}(\boldsymbol{\theta})$ are linearly dependent over R^m. However, the linear combinations $\mathbf{w}(\mathbf{x})$ in Theorem 1 are identifiable because the matrix $(\overline{\mathbf{A}}^T\overline{\mathbf{A}})$ in the normal equation of the least squares problem is nonsingular for a persistently exciting trajectory. $\mathbf{A}(\boldsymbol{\theta})\mathbf{x}$ fully determines $\mathbf{y}$, as does $\overline{\mathbf{A}}(\boldsymbol{\theta})\mathbf{w}(\mathbf{x})$. If not all values of $\mathbf{x}$ are available, the knowledge of $\mathbf{w}(\mathbf{x})$ is a necessary condition for determining $\mathbf{y}$. In the sense of identification, we are interested in finding $\mathbf{w}(\mathbf{x})$ for the system in Eq. (1). Therefore, we introduce the following definition.

Definition 2. *A set* $\mathbf{w}(\mathbf{x})$ *is a set of* minimal linear combinations *of the system parameters for the system in Eq. (1) if the elements of the set are linear combinations of* $\mathbf{x}$ *and linearly independent over the domain of* $\mathbf{x}$ *and there exists* $\overline{\mathbf{A}}(\boldsymbol{\theta})$ *whose columns are linearly independent over the domain* $\overline{\mathbf{A}}$ *such that Eq. (3) holds.* ■

Before turning to the dynamics of manipulators, we briefly review the literature. Ha et al.[6] showed that the dynamic model of a manipulator can be formulated in a form like Eq. (3) by using some intuitive regrouping rules. Theorem 1, however, gives a necessary condition for the number of linearly independent columns of $\mathbf{A}(\theta)$ and rigorously shows that $\mathbf{w}(\mathbf{x})$ in Eq. (3) is a set of minimal linear combinations of the system parameters for the system in Eq. (1). Because there are numerous methods for selecting $\overline{\mathbf{A}}(\boldsymbol{\theta})$ from $\mathbf{A}(\boldsymbol{\theta})$, the set of minimal linear combinations is not unique.

We recognize that the dynamic equations of a manipulator with n joints are

$$\mathbf{H}(\mathbf{q}, \mathbf{x})\ddot{\mathbf{q}} + \boldsymbol{\tau}^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x}) + \boldsymbol{\tau}^g(\mathbf{q}, \mathbf{x}) = \boldsymbol{\tau} \quad (4)$$

where $\mathbf{q} \in R^n$ consists of the joint displacements, $\mathbf{x} \in R^p$ consists of the inertia parameters, $\boldsymbol{\tau} \in R^n$ consists of the actuator forces, $\mathbf{H}(\mathbf{q}, \mathbf{x})$: $R^{n+p} \to R^{n \times n}$ is the symmetric inertia matrix, $\boldsymbol{\tau}^g(\mathbf{q}, \mathbf{x})$: $R^{n+p} \to R^n$ consists of the gravitational forces, $\boldsymbol{\tau} \in R^n$ consists of the actuator forces, and $\boldsymbol{\tau}^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x})$: $R^{2n+p} \to R^{n \times n}$ consists of the Coriolis and centrifugal forces, which can also be related to the inertia matrix with Christoffel symbols (c_{ijk})[25,26]

$$\tau_i^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x}) = \sum_{j=1}^{n} \sum_{k=j}^{n} c_{ijk}\dot{q}_j\dot{q}_k \quad (5)$$

$$c_{ijk} = \left(\frac{\partial h_{ij}}{\partial q_k} + \frac{\partial h_{ik}}{\partial q_j} - \frac{\partial h_{jk}}{\partial q_i}\right), \quad \text{for } j \neq k \quad (6)$$

$$c_{ijj} = \left(\frac{\partial h_{ij}}{\partial q_j} - \frac{1}{2}\frac{\partial h_{jj}}{\partial q_i}\right) \quad (7)$$

where τ_i^C is the ith element of $\boldsymbol{\tau}^C$, q_i is that of $\mathbf{q}$, and h_{ij} is the (i, j)th entry of $\mathbf{H}$.

Our present problem is to find a set of MLCs of the inertia parameters (simply MLCs in the following context) for determining $\boldsymbol{\tau}$, provided that the geometrical parameters of the manipulator are known. The strategy is to separate Eq. (4) into two parts: $\boldsymbol{\tau}^g$ and

$$\boldsymbol{\tau}^I \equiv \mathbf{H}(\mathbf{q}, \mathbf{x})\ddot{\mathbf{q}} + \boldsymbol{\tau}^C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{x}) \quad (8)$$

Once the MLCs for $\boldsymbol{\tau}^I$ and $\boldsymbol{\tau}^g$ are obtained, we can exclude the linear dependent elements and then obtain a set of MLCs for $\boldsymbol{\tau}$. This strategy is supported by the following two theorems.

Theorem 2. *A set is a set of MLCs for determining $\boldsymbol{\tau}$ if it is the set consisting of the linearly independent elements of $[\mathbf{w}^{IT}(\mathbf{x}), \mathbf{w}^{gT}(\mathbf{x})]^T$, where $\mathbf{w}^I$ and $\mathbf{w}^g$ are two sets of MLCs for determining $\boldsymbol{\tau}^I$ and $\boldsymbol{\tau}^g$, respectively. Furthermore, when $\mathbf{w}^I$ is partitioned as*

$$\mathbf{w}^I = \begin{bmatrix} \mathbf{w}^J \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \mathbf{w}^g \quad (9)$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are some constant matrices and the components of $\mathbf{w}^J$ are linearly independent of one another and those of $\mathbf{w}^g$, then the components of $\mathbf{w}^J$ *and* $\mathbf{w}^g$ *form a set of MLCs for determining $\boldsymbol{\tau}$.* ∎

Theorem 3. *A set is a set of MLCs for determining $\boldsymbol{\tau}^I$ if and only if it is also a set of MLCs for determining the entries of* $\mathbf{H}(\mathbf{q}, \mathbf{x})$. ∎

Theorem 3 follows from Christoffel symbols in Eqs. (5)–(7). The detailed proofs of these theorems can also be found in the Appendix. By Theorems 2 and 3, the problem of finding the MLCs for $\boldsymbol{\tau}$ turns out to be a search for two individual sets of MLCs that determine the entries of the inertia matrix and the gravity load, respectively.

3. INERTIA CONSTANTS OF COMPOSITE BODIES

This section revisits the inertia constants of composite bodies[23] and Renaud's formulation for the inertia matrix.[26,27] The structure of the inertia constants in Renaud's formulation will reveal that these constants are consistent with the MLCs of the inertia parameters.

We consider a manipulator with n low-pair joints, which are labeled joint 1 to n outward from the base. Assign a body-fixed frame on each joint (i.e., frame E_i is fixed on joint i) in accord with the normal driving-axis coordinate system[28,29] (known also as modified Denavit-Hartenberg notation). The distance from the origin of E_i to that of E_j is designated as ${}_i^j\mathbf{s}$, and that to the center of mass of link i as $\mathbf{c}_i$.

In the normal driving-axis coordinate system (see Fig. 1), the z-axis of a body-fixed frame is the driving axis of the corresponding link, i.e., the unit vector along joint i is $\mathbf{u}_i^{\langle i \rangle} = [0, 0, 1]^T$, where superscript "$\langle i \rangle$" denotes the representation of a vector with respect to frame E_i. The distance from the origin of frame E_{i-1} to frame E_i is

$${}_{i-1}^{\;\;i}\mathbf{s}^{\langle i-1 \rangle} = \begin{bmatrix} b_i \\ -d_iS\beta_i \\ d_iC\beta_i \end{bmatrix}, \quad \text{or } {}_{i-1}^{\;\;i}\mathbf{s}^{\langle i \rangle} = \begin{bmatrix} b_iC\theta_i \\ -b_iS\theta_i \\ d_i \end{bmatrix} \quad (10)$$

where $S\theta_i \equiv \sin\theta_i$, $C\theta_i \equiv \cos\theta_i$, and b_i, d_i, and θ_i are the geometrical parameters of the coordinate system. The coordinate transformation matrix from E_{i-1} to E_i is well known as

$${}_{i-1}^{\;\;i}\mathbf{R} = \begin{bmatrix} C\theta_i & -S\theta_i & 0 \\ C\beta_iS\theta_i & C\beta_iC\theta_i & -S\beta_i \\ S\beta_iS\theta_i & S\beta_iC\theta_i & C\beta_i \end{bmatrix} \quad (11)$$

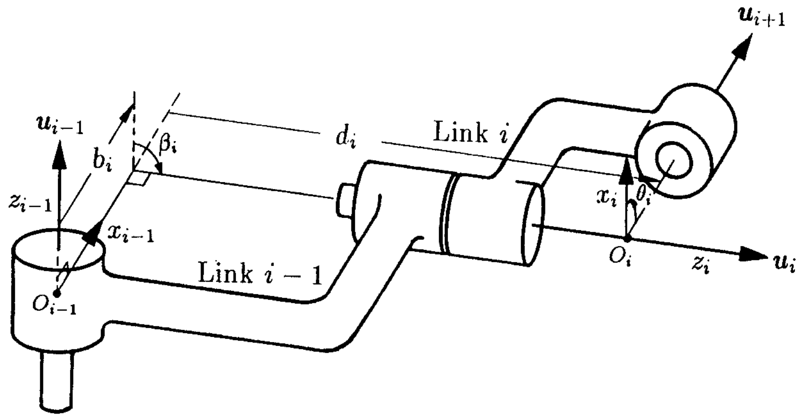

Figure 1. Illustration of the normal driving-axis coordinate system.

The *composite body* i is defined as the union of link i to link n. Let the mass of the composite body i and the first moment of the composite body about the origin of E_i be denoted by $\hat{m}_i$ and $\hat{\mathbf{c}}_i$, respectively, to obtain

$$\hat{m}_i = \sum_{j=i}^{n} m_j \tag{12}$$

$$\hat{\mathbf{c}}_i^{\langle i \rangle} = \sum_{j=i}^{n} m_j({}^{j}_{i}\mathbf{s}^{\langle i \rangle} + \mathbf{c}_j^{\langle i \rangle}) \tag{13}$$

where m_j is the mass of link j. The inertia tensor of the composite body about the origin of frame E_i (denoted by $\hat{\mathbf{J}}_i$) results from using the Huygeno-Steiner formula[30] to obtain

$$\hat{\mathbf{J}}_i^{\langle i \rangle} = \sum_{j=i}^{n} {}^{i}_{j}\mathbf{R}\,\mathbf{I}_j^{\langle j \rangle}\,{}^{i}_{j}\mathbf{R}^T - m_j[({}^{j}_{i}\mathbf{s}^{\langle i \rangle} + \mathbf{c}_j^{\langle i \rangle})\times][({}^{j}_{i}\mathbf{s}^{\langle i \rangle} + \mathbf{c}_j^{\langle i \rangle})\times] \tag{14}$$

where $\mathbf{I}_j^{\langle j \rangle}$ is the representation of the inertia tensor of link j with respect to frame E_j and $[\mathbf{a}\times]$ denotes a skew-symmetric matrix representing vector multiplication, i.e., $[\mathbf{a}\times]\mathbf{b} = \mathbf{a} \times \mathbf{b}$. In the context, the overhead symbol "^" is used to denote the inertia parameters (mass, first moment and inertia tensor) of a composite body.

We introduce the notation of

$$K_i^* \equiv (1 - K_i) \equiv \begin{cases} 1, & \text{for rotational joint } i, \\ 0, & \text{for translational joint } i. \end{cases} \tag{15}$$

Renaud's formulation[25–27,31] for the entries of the inertia matrix is

$$\begin{aligned} h_{mi} = {} & K_m^* K_i^* \left(\mathbf{u}_m^{\langle i \rangle} \cdot \begin{bmatrix} (\hat{\mathbf{J}}_i^{\langle i \rangle})_{13} \\ (\hat{\mathbf{J}}_i^{\langle i \rangle})_{23} \\ (\hat{\mathbf{J}}_i^{\langle i \rangle})_{33} \end{bmatrix} + \begin{bmatrix} (\mathbf{a}_{i,m}^{\langle i \rangle})_y \\ -(\mathbf{a}_{i,m}^{\langle i \rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i \rangle} \right) \\ & + K_m K_i^* \begin{bmatrix} (\mathbf{u}_m^{\langle i \rangle})_y \\ -(\mathbf{u}_m^{\langle i \rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i \rangle} \\ & + K_m^* K_i \left(\hat{m}_i (\mathbf{a}_{i,m}^{\langle i \rangle})_z - \begin{bmatrix} (\mathbf{u}_m^{\langle i \rangle})_y \\ -(\mathbf{u}_m^{\langle i \rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i \rangle} \right) \\ & + K_m K_i \hat{m}_i (\mathbf{u}_m^{\langle i \rangle})_z, \qquad m \le i. \end{aligned} \tag{16}$$

where $(\cdot)_{ij}$ denotes the (i, j)th entry of a matrix, $(\cdot)_x$ the x-component of a vector, and

$$\mathbf{a}_{i,m}^{\langle i \rangle} \equiv \mathbf{u}_m^{\langle i \rangle} \times {}^{i}_{m}\mathbf{s}^{\langle i \rangle} \tag{17}$$

which is, physically, the part of the acceleration of the origin of frame E_i due to a unit angular acceleration of joint m(i.e., $\ddot{q}_m = 1$). We can also obtain the gravity term of the actuator force applied on joint i in the form of[23]

$$\begin{aligned} \tau_i^g &= -\mathbf{u}_i^{\langle i \rangle} \cdot (K_i^* \hat{\mathbf{c}}_i^{\langle i \rangle} \times \mathbf{g}^{\langle i \rangle} + K_i \hat{m}_i \mathbf{g}^{\langle i \rangle}) \\ &= K_i^* \begin{bmatrix} -(\mathbf{g}^{\langle i \rangle})_y \\ (\mathbf{g}^{\langle i \rangle})_x \\ 0 \end{bmatrix} \cdot \hat{\mathbf{c}}_i^{\langle i \rangle} - K_i \hat{m}_i (\mathbf{g}^{\langle i \rangle})_z \end{aligned} \tag{18}$$

where τ_i^g is the ith element of $\boldsymbol{\tau}^g(\mathbf{q}, \mathbf{x})$ and $\mathbf{g}$ is the gravitational acceleration.

By the principle of mathematical induction, it has been shown[23,32] that the first moment and the inertia tensor of the composite body i can be formed as the sum of a constant vector ($\mathbf{k}_i$ or $\mathbf{U}_i$) and a varying vector (ℓ_i or $\mathbf{U}_i$), such as

$$\hat{\mathbf{c}}_i^{\langle i\rangle} = \mathbf{k}_i + \ell_i \tag{19}$$

$$\hat{\mathbf{J}}_i^{\langle i\rangle} = \mathbf{U}_i + \mathbf{V}_i \tag{20}$$

where $\mathbf{k}_n = m_n\mathbf{c}_n^{\langle n\rangle}$, $\ell_n = \mathbf{0}$, $\mathbf{U}_n = \mathbf{I}_n^{\langle n\rangle} - m_n[\mathbf{c}_n^{\langle n\rangle}\times][\mathbf{c}_n^{\langle n\rangle}\times]$, $\mathbf{V}_n = \mathbf{0}$ and

$$\mathbf{k}_i = m_i\mathbf{c}_i^{\langle i\rangle} + K_{i+1}^*\left(\hat{m}_{i+1}\,{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle} + {}^{i+1}_{\ i}\mathbf{R}\begin{bmatrix}0\\0\\(\mathbf{k}_{i+1})_z\end{bmatrix}\right) + K_{i+1}\left(\hat{m}_{i+1}\begin{bmatrix}({}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle})_x\\0\\0\end{bmatrix} + {}^{i+1}_{\ i}\mathbf{R}\mathbf{k}_{i+1}\right) \tag{21}$$

$$\ell_i = K_{i+1}^*\,{}^{i+1}_{\ i}\mathbf{R}\begin{bmatrix}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x\\(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y\\(\ell_{i+1})_z\end{bmatrix} + K_{i+1}\,{}^{i+1}_{\ i}\mathbf{R}\left(\ell_{i+1} + \hat{m}_{i+1}\begin{bmatrix}0\\0\\({}^{i+1}_{\ i}\mathbf{s}^{\langle i+1\rangle})_z\end{bmatrix}\right) \tag{22}$$

If joint $i + 1$ is a rotational joint, then

$$\begin{aligned}\mathbf{U}_i = {} & \mathbf{I}_i^{\langle i\rangle} - m_i[\mathbf{c}_i^{\langle i\rangle}\times][\mathbf{c}_i^{\langle i\rangle}\times] \\ & - \hat{m}_{i+1}[{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle}\times][{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle}\times] \\ & + {}^{i+1}_{\ i}\mathbf{R}\begin{bmatrix}(\mathbf{U}_{i+1})_{22} & 0 & 0\\0 & (\mathbf{U}_{i+1})_{22} & 0\\0 & 0 & (\mathbf{U}_{i+1})_{33}\end{bmatrix}{}^{i+1}_{\ i}\mathbf{R}^T \\ & - [{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle}\times]({}^{i+1}_{\ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z)\times] \\ & - [({}^{i+1}_{\ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z)\times][{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle}\times]\end{aligned} \tag{23}$$

$$\begin{aligned}\mathbf{V}_i = {} & {}^{i+1}_{\ i}\mathbf{R} \\ & \left(\mathbf{V}_{i+1} + \begin{bmatrix}(\mathbf{U}_{i+1})_{11} - (\mathbf{U}_{i+1})_{22} & (\mathbf{U}_{i+1})_{12} & (\mathbf{U}_{i+1})_{13}\\(\mathbf{U}_{i+1})_{12} & 0 & (\mathbf{U}_{i+1})_{23}\\(\mathbf{U}_{i+1})_{13} & (\mathbf{U}_{i+1})_{23} & 0\end{bmatrix}\right) \\ & {}^{i+1}_{\ i}\mathbf{R}^T - [{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle}\times][\ell_i\times] - [\ell_i\times][{}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle}\times]\end{aligned} \tag{24}$$

whereas for translational joint $i + 1$

$$\begin{aligned}\mathbf{U}_i = {} & \mathbf{I}_i^{\langle i\rangle} - m_i[\mathbf{c}_i^{\langle i\rangle}\times][\mathbf{c}_i^{\langle i\rangle}\times] \\ & + {}^{i+1}_{\ i}\mathbf{R}\mathbf{U}_{i+1}\,{}^{i+1}_{\ i}\mathbf{R}^T - \hat{m}_{i+1}[\mathbf{b}_{i+1}^{\langle i\rangle}\times][\mathbf{b}_{i+1}^{\langle i\rangle}\times] \\ & - [\mathbf{b}_{i+1}^{\langle i\rangle}\times][({}^{i+1}_{\ i}\mathbf{R}\mathbf{k}_{i+1})\times] \\ & - [({}^{i+1}_{\ i}\mathbf{R}\mathbf{k}_{i+1})\times][\mathbf{b}_{i+1}^{\langle i\rangle}\times]\end{aligned} \tag{25}$$

$$\begin{aligned}\mathbf{V}_i = {} & {}^{i+1}_{\ i}\mathbf{R}(\mathbf{V}_{i+1} - \hat{m}_{i+1}[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times][\mathbf{d}_{i+1}^{\langle i+1\rangle}\times] \\ & - \hat{m}_{i+1}[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times][\mathbf{b}_{i+1}^{\langle i+1\rangle}\times] \\ & - \hat{m}_{i+1}[\mathbf{b}_{i+1}^{\langle i+1\rangle}\times][\mathbf{d}_{i+1}^{\langle i+1\rangle}\times] - [\mathbf{d}_{i+1}^{\langle i+1\rangle}\times][\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}\times] \\ & - [\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}\times][\mathbf{d}_{i+1}^{\langle i+1\rangle}\times] - [\mathbf{b}_{i+1}^{\langle i+1\rangle}\times][\ell_{i+1}\times] \\ & - [\ell_{i+1}\times][\mathbf{b}_{i+1}^{\langle i+1\rangle}\times])\,{}^{i+1}_{\ i}\mathbf{R}^T\end{aligned} \tag{26}$$

Note that ${}^{i+1}_{\ i}\mathbf{R}_b$ in Eq. (23) is the third column of ${}^{i+1}_{\ i}\mathbf{R}$, i.e.,

$${}^{i+1}_{\ i}\mathbf{R}_b = \begin{bmatrix}0\\-S\beta_{i+1}\\C\beta_{i+1}\end{bmatrix} \tag{27}$$

which is a constant vector, and

$$\mathbf{b}_{i+1}^{\langle i\rangle} = \begin{bmatrix}({}^{i+1}_{\ i}\mathbf{s}^{\langle i\rangle})_x\\0\\0\end{bmatrix} = \begin{bmatrix}b_{i+1}\\0\\0\end{bmatrix}, \quad \mathbf{b}_{i+1}^{\langle i+1\rangle} = \begin{bmatrix}b_{i+1}C\theta_{i+1}\\-b_{i+1}S\theta_{i+1}\\0\end{bmatrix} \tag{28}$$

$$\mathbf{d}_{i+1}^{\langle i+1\rangle} = \begin{bmatrix}0\\0\\({}^{i+1}_{\ i}\mathbf{s}^{\langle i+1\rangle})_z\end{bmatrix} = \begin{bmatrix}0\\0\\d_{i+1}\end{bmatrix} \tag{29}$$

We shall call $\hat{m}_i$, the components of $\mathbf{k}_i$, and the entries of $\mathbf{U}_i$, $i = 1, \ldots, n$, *inertia constants of composite bodies*. The salient feature of these constants is that the varying terms in $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and $\hat{\mathbf{J}}_i^{\langle i\rangle}$ can be calculated with only some (not all) of the inertia constants of composite bodies. Namely, ℓ_i in Eq. (22) are calculated with some $\hat{m}_j$ and the x- and y-compounds of $\mathbf{k}_j$, $j > i$, and the recursive form for computing $\mathbf{V}_i$ in Eqs. (24) and (26) requires only the (1, 2)th, (1, 3)th, (2, 3)th entries and the difference of the (1, 1)th and (2, 2)th entries of $\mathbf{U}_{i+1}$ and some components of $\mathbf{k}_{i+1}$.

This property is consistent with the minimal linear combinations of the inertia parameters. Indeed, some elements of $\mathbf{k}_i$ and $\mathbf{U}_i$ and $\hat{m}_i$ constitute a set of MLCs for determining the actuator forces of a manipulator.

For instance, the gravity load $\boldsymbol{\tau}^g$ in Eq. (18) requires $K_i^*\hat{\mathbf{c}}_i^{(i)}$ and $K_i\hat{m}_i$. Because $K_i^*\hat{\mathbf{c}}_i^{(i)}$ can be expressed in terms of $K_j\hat{m}_j$ and the x- and y-components of $K_j^*\mathbf{k}_j$, $j > i$ (see Eqs. (19) and (22)), we obtain the following theorem.

Theorem 4. *Consider a manipulator with n low-pair joints in which joint r is the first rotational joint counting from the base and joint s is the nearest rotational joint not parallel to joint r. A set of MLCs for determining the gravity load* $\boldsymbol{\tau}^g$ *is the set* $\mathcal{S}^g = \{\delta_i K_i^*(\mathbf{k}_i)_x, \delta_i K_i^*(\mathbf{k}_i)_y, \sigma_i K_i \hat{m}_i, i = 1, \ldots, n\} - \{0\}$.

Note that δ_i *and* σ_i *are either 1 or 0 to denote the redundancy of the parameters, which are defined as follows.* $\delta_i = 0$ *for* $r \le i < s$ *and* $\mathbf{u}_r /\!/ \mathbf{g}$, *otherwise* $\delta_i = 1$. *On the other hand,* $\sigma_i = 0$ *for* $\mathbf{u}_i \perp \mathbf{g}$, $\forall \mathbf{q} \in R^n$ *(if* $r < i < s$ *for translational joint i,* $\mathbf{u}_i$ *is always perpendicular to* $\mathbf{g}$ *only when* $\mathbf{u}_r /\!/ \mathbf{g}$ *or when* $\mathbf{u}_r \perp \mathbf{g}$ *and* $\mathbf{u}_i /\!/ \mathbf{u}_r$*; while this can happen for* $i > s$ *only if* $\mathbf{u}_r /\!/ \mathbf{g}$, $\mathbf{u}_s \perp \mathbf{u}_r$ *and* $\mathbf{u}_i /\!/ \mathbf{u}_j /\!/ \mathbf{u}_s$ *for any rotational joint j,* $s \le j < i$*), otherwise* $\sigma_i = 1$. ∎

The proof of this theorem is similar to that for Theorem 2 in a previous work[33] and is thus omitted.

According to Theorems 2 and 3, the rest of the MLCs for determining the actuator forces $\boldsymbol{\tau}$ are those for determining $\mathbf{h} = [h_{11}, \ldots, h_{1n}, h_{22}, \ldots, h_{nn}]^T$, which contains all upper triangular entries of the inertia matrix $\mathbf{H}(\mathbf{q}, \mathbf{x})$ in Eq. (4) because the inertia matrix is symmetric. By Theorem 2, the problem can be temporarily formulated as the problem of finding the components of $\mathbf{v}$ such that there is a matrix $\overline{\mathbf{H}}(\mathbf{q})$ with linearly independent columns satisfying

$$\mathbf{h} = \overline{\mathbf{H}}(\mathbf{q})\mathbf{v}(\mathbf{q}, \mathbf{x}) + \mathbf{T}(\mathbf{q})\mathbf{v}_g = \overline{\mathbf{H}}(\mathbf{q})\mathbf{v}(\mathbf{q}, \mathbf{x}) + \overline{\mathbf{T}}(\mathbf{q})\mathbf{w}_g \tag{30}$$

where $\mathbf{T}$ and $\overline{\mathbf{T}}$ are some matrices, $\mathbf{v}$ is composed of $\hat{m}_i$, $\hat{\mathbf{c}}_i^{(i)}$ and $\mathbf{U}_i$, $i = 1, \ldots, n$, other than those in $\mathbf{v}_g$, and

$$\mathbf{v}_g \equiv \begin{bmatrix} K_s^*(\hat{\mathbf{c}}_s^{(s)})_x \\ K_s^*(\hat{\mathbf{c}}_s^{(s)})_y \\ \sigma_s K_s \hat{m}_s \\ \vdots \\ K_n^*(\hat{\mathbf{c}}_n^{(n)})_x \\ K_n^*(\hat{\mathbf{c}}_n^{(n)})_y \\ \sigma_n K_n \hat{m}_n \end{bmatrix}, \quad \mathbf{w}_g \equiv \begin{bmatrix} K_s^*(\mathbf{k}_s)_x \\ K_s^*(\mathbf{k}_s)_y \\ \sigma_s K_s \hat{m}_s \\ \vdots \\ K_n^*(\mathbf{k}_n)_x \\ K_n^*(\mathbf{k}_n)_y \\ \sigma_n K_n \hat{m}_n \end{bmatrix} \tag{31}$$

in which σ_i is defined as in Theorem 4. Note that the components of $\mathbf{w}_g$ belong to $\mathcal{S}^g$ in Theorem 4, and $\mathbf{v}_g = \mathbf{W}_g\mathbf{w}_g$, where $\mathbf{W}_g$ is a nonsingular upper triangular matrix with the diagonal entries of 1 (which can be seen from Eqs. (19) and (22)). After $\mathbf{v}$ in Eq. (30) is found, the terms associated with $\hat{\mathbf{c}}_i^{(i)}$ can be rewritten in terms of the x- and y-components of $\mathbf{k}_j$ and $\hat{m}_j$, $j > i$, i.e., the component of $\mathbf{w}_g$. The set of MLCs for determining $\mathbf{h}$ then has a form like Eq. (9).

4. MINIMAL LINEAR COMBINATIONS OF THE INERTIA PARAMETERS

To deduce the components of $\mathbf{v}$ in Eq. (30), we require the following property, which directly follows from the definition.

Property 1. *If all nonzero elements of a row in* $\overline{\mathbf{H}}$ *are linearly independent over* R^n, *the columns of* $\overline{\mathbf{H}}$ *containing the nonzero elements of this row are linearly independent of one another and of the other columns* $\overline{\mathbf{H}}$ *over* R^n. ∎

We treat h_{mi} of Renaud's formulation given in Eq. (16) separately for the different types of joint i. First, suppose joint i is a translational joint; it follows from Eq. (16) that

$$h_{ii} = [0 \ldots 0\ \underset{\substack{\uparrow \\ \hat{m}_i}}{1}\ 0 \ldots 0]\hat{\mathbf{x}} \tag{32}$$

$$h_{mi} = [0 \ldots 0\ \underset{\substack{\uparrow \\ \hat{m}_i}}{(\mathbf{u}_m^{(i)})_z}\ 0 \ldots 0]\hat{\mathbf{x}} \text{ for } K_m = 1,\ m < i \tag{33}$$

$$h_{mi} = [0 \ldots 0\ \underset{\substack{\uparrow \\ \hat{m}_i}}{(\mathbf{a}_{i,m}^{(i)})_z}\ 0 \ldots 0\ \underset{\substack{\uparrow \\ (\hat{\mathbf{c}}_i^{(i)})_x}}{-(\mathbf{u}_m^{(i)})_y}\ \underset{\substack{\uparrow \\ (\hat{\mathbf{c}}_i^{(i)})_y}}{(\mathbf{u}_m^{(i)})_x}\ 0 \ldots 0]\hat{\mathbf{x}}, \text{ for } K_m^* = 1,\ m < i \tag{34}$$

where $\hat{\mathbf{x}}$ consists of the masses, the first moments, and the inertia tensors of the composite bodies. By Property 1, Eq. (32) shows that the column of $\overline{\mathbf{H}}$ corresponding to $K_i\hat{m}_i$ is linearly independent of the other columns. This also reveals that the columns of $\overline{\mathbf{H}}$ corresponding to the x- and y-components of $K_i\hat{\mathbf{c}}_i^{(i)}$ must be linearly independent of that corresponding to $K_i\hat{m}_i$ regardless of whether $(\mathbf{a}_{i,m}^{(i)})_z$ in Eq. (34) is independent of $-(\mathbf{u}_m^{(i)})_y$ and $(\mathbf{u}_m^{(i)})_x$. For translational joint i, $s < i \le n$, there are at least two elements of $\mathbf{h}$ having the form of Eq. (34), e.g.,

$$\begin{bmatrix} h_{ri} \\ h_{si} \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_{i,r}^{\langle i \rangle})_z & -(\mathbf{u}_r^{\langle i \rangle})_y & (\mathbf{u}_r^{\langle i \rangle})_x \\ (\mathbf{a}_{i,s}^{\langle i \rangle} & -(\mathbf{u}_s^{\langle i \rangle})_y & (\mathbf{u}_s^{\langle i \rangle})_x \end{bmatrix} \begin{bmatrix} \hat{m}_i \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_x \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_y \end{bmatrix}, \; i > s \tag{35}$$

Because $\mathbf{u}_s$ is not parallel to $\mathbf{u}_r$, the x- and y-components of $\mathbf{u}_s^{\langle r \rangle}$ vary independently with q_s, so that the second and third columns in Eq. (35) are linearly independent. This implies that the columns of $\overline{\mathbf{H}}$ corresponding to $K_i(\hat{\mathbf{c}}_i^{\langle i \rangle})_x$ and $K_i(\hat{\mathbf{c}}_i^{\langle i \rangle})_y$ for $s < i \le n$ are linearly independent of the other columns. For the case of $r < i < s$, $\mathbf{u}_m^{\langle i \rangle}$ is constant and $\mathbf{u}_m /\!/ \mathbf{u}_r$ for any rotational joint m, $m < i$, so that the combination of $(-(\mathbf{u}_r^{\langle i \rangle})_y(\hat{\mathbf{c}}_i^{\langle i \rangle})_x + (\mathbf{u}_r^{\langle i \rangle})_x(\hat{\mathbf{c}}_i^{\langle i \rangle})_y)$ is constant and is a component of $\mathbf{v}$ in Eq. (30) if it is nonzero. It is zero when $\mathbf{u}_i /\!/ \mathbf{u}_r$.

It follows from Eq. (33) that $\mathbf{U}_i$ and $\hat{\mathbf{c}}_i^{\langle i \rangle}$ for $i < r$ are unnecessary for determining the inertia matrix. This result is summarized as follows.

Property 2. *$\mathbf{v}$ in Eq. (30) has the following components:*

1. $K_i \hat{m}_i$, $i = 1, \ldots, n$,
2. $K_i(\hat{\mathbf{c}}_i^{\langle i \rangle})_x$ *and* $K_i(\hat{\mathbf{c}}_i^{\langle i \rangle})_y$, $s < i \le n$, *and*
3. $K_i \hat{\kappa}_{1i}$ *for* $r < i < s$ *and* $\mathbf{u}_i \not\parallel \mathbf{u}_r$, *where*

$$\hat{\kappa}_{1i} = -(\mathbf{u}_r^{\langle i \rangle})_y(\hat{\mathbf{c}}_i^{\langle i \rangle})_x + (\mathbf{u}_r^{\langle i \rangle})_x(\hat{\mathbf{c}}_i^{\langle i \rangle})_y \tag{36}$$

However, $K_i\mathbf{U}_i$ *and* $K_i\hat{\mathbf{c}}_i^{\langle i \rangle}$, $i < r$, *are not components of* $\mathbf{v}$. ■

If joint i is a rotational joint, we obtain from Eq. (16) that

$$h_{ii} = [0 \ldots 0 \quad \underset{\substack{\uparrow \\ (\hat{\mathbf{J}}_i^{\langle i \rangle})_{33}}}{1} \quad 0 \ldots 0]\hat{\mathbf{x}} \tag{37}$$

$$h_{mi} = [0 \ldots 0 \; \underset{\substack{\uparrow \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_x}}{(\mathbf{u}_m^{\langle i \rangle})_y} \; \underset{\substack{\uparrow \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_y}}{-(\mathbf{u}_m^{\langle i \rangle})_x} \; 0 \ldots 0]\hat{\mathbf{x}}, \quad \text{for } K_m = 1,\, m < i \tag{38}$$

$$h_{mi} = [0 \ldots 0 \; \underset{\substack{\uparrow \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_x}}{(\mathbf{a}_{i,m}^{\langle i \rangle})_y} \; \underset{\substack{\uparrow \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_y}}{-(\mathbf{a}_{i,m}^{\langle i \rangle})_x} \; 0 \ldots 0 \; \underset{\substack{\uparrow \\ \text{Third Column of } \hat{\mathbf{J}}_i^{\langle i \rangle}}}{(\mathbf{u}_m^{\langle i \rangle})^T} \; 0 \ldots 0]\hat{\mathbf{x}}, \quad \text{for } K_m^* = 1,\, m < i \tag{39}$$

Note that $\hat{\mathbf{J}}_i^{\langle i \rangle} = \mathbf{U}_i + \mathbf{V}_i$. To obtain a general form for $\mathbf{V}_i$, we suppose that joints i and j, $i < j$, are rotational joints, but the joints between them are all translational joints. It follows from Eq. (22) that

$$\ell_k = {}^{j-1}_{k}\mathbf{R}\,\ell_{j-1} + \mathbf{e}_{k+1}^{\langle k \rangle}, \text{ for } i < k < j-1 \tag{40}$$

where

$$\mathbf{e}_{k+1}^{\langle k+1 \rangle} \equiv \sum_{m=k+1}^{j-1} \hat{m}_m \, {}^{m}_{k+1}\mathbf{R}\, \mathbf{d}_m^{\langle m \rangle} \tag{41}$$

Applying Eqs. (24), (26), and (40), we obtain

$$\begin{aligned} \mathbf{V}_i = {}&{}^{j}_{i}\mathbf{R}\,\mathbf{V}_j\,{}^{j}_{i}\mathbf{R}^T \\ &+ {}^{j}_{i}\mathbf{R} \begin{bmatrix} (\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22} & (\mathbf{U}_j)_{12} & (\mathbf{U}_j)_{13} \\ (\mathbf{U}_j)_{12} & 0 & (\mathbf{U}_j)_{23} \\ (\mathbf{U}_j)_{13} & (\mathbf{U}_j)_{23} & 0 \end{bmatrix} {}^{j}_{i}\mathbf{R}^T \\ &- [{}^{j}_{i}\mathbf{s}^{*\langle i \rangle}\times][({}^{j-1}_{i}\mathbf{R}\,\ell_{j-1})\times] - [({}^{j-1}_{i}\mathbf{R}\,\ell_{j-1})\times][{}^{j}_{i}\mathbf{s}^{*\langle i \rangle}\times] \\ &- \sum_{k=i+1}^{j-1} {}^{k}_{i}\mathbf{R}(\hat{m}_k[\mathbf{d}_k^{\langle k \rangle}\times][\mathbf{d}_k^{\langle k \rangle}\times] + [\mathbf{b}_k^{\langle k \rangle}\times][\mathbf{e}_k^{\langle k \rangle}\times] \\ &+ [\mathbf{e}_k^{\langle k \rangle}\times][\mathbf{b}_k^{\langle k \rangle}\times] + [\mathbf{d}_k^{\langle k \rangle}\times][\hat{\mathbf{c}}_k^{\langle k \rangle}\times] \\ &+ [\hat{\mathbf{c}}_k^{\langle k \rangle}\times][\mathbf{d}_k^{\langle k \rangle}\times])\,{}^{k}_{i}\mathbf{R}^T \end{aligned} \tag{42}$$

where ${}^{j}_{i}\mathbf{s}^{*\langle i \rangle}$ is constant in the form of

$${}^{j}_{i}\mathbf{s}^{*\langle i \rangle} \equiv {}_{j-1}{}^{j}_{i}\mathbf{s}^{\langle i \rangle} + \sum_{k=i+1}^{j-1} \mathbf{b}_k^{\langle i \rangle} \tag{43}$$

For convenience of analysis, we relate frames E_i and E_j directly through a common normal of $\mathbf{u}_i$ and $\mathbf{u}_j$, which is in accord with the normal driving-axis coordinate system[28,29] (see Fig. 2). The x-axis of frame E_i will be in alignment with the common normal by a rotation of angle ϕ_{ij} about $\mathbf{u}_i$ (i.e., the z-axis of frame E_i). Then, the rotation of β_j^* about the normal followed by the rotation of θ_j^* about the z-axis of frame E_j transforms frame E_i so that it is in alignment with frame E_j. In mathematical form, the coordinate transformation matrix from frame E_i to E_j can be written as[28,29]

$${}^{j}_{i}\mathbf{R} = \begin{bmatrix} C\phi_{ij} & -S\phi_{ij} & 0 \\ S\phi_{ij} & C\phi_{ij} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_j^* & -S\theta_j^* & 0 \\ C\beta_j^* S\theta_j^* & C\beta_j^* C\theta_j^* & -S\beta_j^* \\ S\beta_j^* S\theta_j^* & S\beta_j^* C\theta_j^* & C\beta_j^* \end{bmatrix} \tag{44}$$

Note that ϕ_{ij} and β_j^* are constants since joints $i + 1, \ldots, j - 1$ are translational joints, whereas θ_j^* is a variable because it is the sum of q_j and some constant. It can be shown that

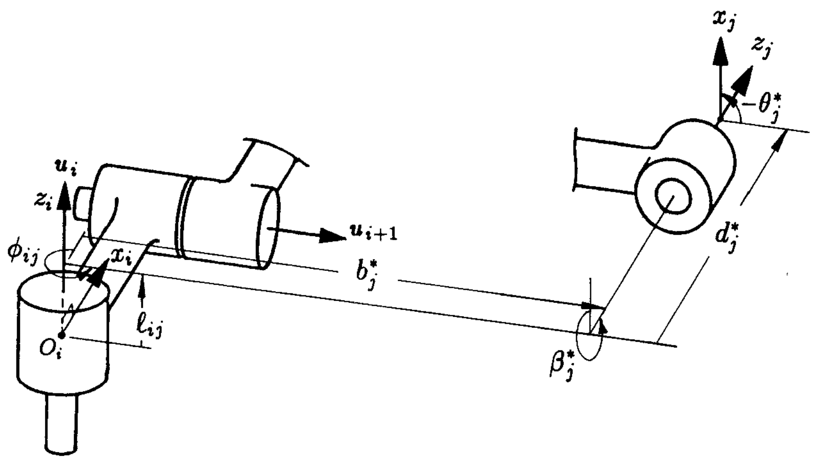

Figure 2. Geometrical relation of any two joints.

$$\mathbf{u}_i^{\langle j\rangle} = \begin{bmatrix} S\beta_j^* S\theta_j^* \\ S\beta_j^* C\theta_j^* \\ C\beta_j^* \end{bmatrix} \tag{45}$$

This equation also reveals that $\mathbf{u}_i /\!/ \mathbf{u}_j$ if and only if $S\beta_j^* = 0$.

We are concerned with the third column of $\mathbf{V}_i$. Expanding Eq. (42) yields

$$\begin{bmatrix} (\mathbf{V}_i)_{13} \\ (\mathbf{V}_i)_{23} \\ (\mathbf{V}_i)_{33} \end{bmatrix} = \begin{bmatrix} (\mathbf{V}_i^{(1)})_{33} \\ (\mathbf{V}_i^{(1)})_{23} \\ (\mathbf{V}_i^{(1)})_{33} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & C\phi_{ij} & -S\phi_{ij} \\ 0 & S\phi_{ij} & C\phi_{ij} \\ 1 & 0 & 0 \end{bmatrix} \mathbf{B}(\beta_j^*, \theta_j^*) \begin{bmatrix} (\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22} \\ (\mathbf{U}_j)_{12} \\ (\mathbf{U}_j)_{13} \\ (\mathbf{U}_j)_{23} \end{bmatrix}$$

$$+ \begin{bmatrix} -({}^j_i\mathbf{s}^{*\langle i\rangle})_z & 0 & -({}^j_i\mathbf{s}^{*\langle i\rangle})_x \\ 0 & -({}^j_i\mathbf{s}^{*\langle i\rangle})_z & -({}^j_i\mathbf{s}^{*\langle i\rangle})_y \\ 2({}^j_i\mathbf{s}^{*\langle i\rangle})_x & 2({}^j_i\mathbf{s}^{*\langle i\rangle})_y & 0 \end{bmatrix} {}^j_i\mathbf{R} \begin{bmatrix} (\hat{\mathbf{c}}_j^{\langle j\rangle})_x \\ (\hat{\mathbf{c}}_j^{\langle j\rangle})_y \\ (\ell_j)_z \end{bmatrix}$$

$$+ \sum_{k=i+1}^{j-1} \mathbf{f}_k(q_{i+1}, \ldots, q_k)\hat{m}_k$$

$$+ \sum_{k=i+1}^{j-1} d_k \begin{bmatrix} 0 & C\phi_{ik} & -S\phi_{ik} \\ 0 & S\phi_{ik} & C\phi_{ik} \\ 1 & 0 & 0 \end{bmatrix} \mathbf{D}(\beta_k^*, \theta_k^*) \begin{bmatrix} -(\hat{\mathbf{c}}_k^{\langle k\rangle})_x \\ -(\hat{\mathbf{c}}_k^{\langle k\rangle})_y \\ (\hat{\mathbf{c}}_k^{\langle k\rangle})_z \end{bmatrix} \tag{46}$$

where $\mathbf{V}_i^{(1)} \equiv {}^j_i\mathbf{R}\,\mathbf{V}_j\;{}^j_i\mathbf{R}^T$, $\mathbf{f}_k$ are some appropriate vector functions and

$$\mathbf{B}(\beta_j^*, \theta_j^*) \equiv \begin{bmatrix} S^2\beta_j^* S^2\theta_j^* & 2S^2\beta_j^* C\theta_j^* S\theta_j^* & 2S\beta_j^* C\beta_j^* S\theta_j^* & 2S\beta_j^* C\beta_j^* C\theta_j^* \\ S\beta_j^* S\theta_j^* C\theta_j^* & S\beta_j^*(C^2\theta_j^* - S^2\theta_j^*) & C\beta_j^* C\theta_j^* & -C\beta_j^* S\theta_j^* \\ C\beta_j^* S\beta_j^* S^2\theta_j^* & 2S\beta_j^* C\beta_j^* C\theta_j^* S\theta_j^* & (C^2\beta_j^* - S^2\beta_j^*)S\theta_j^* & (C^2\beta_j^* - S^2\beta_j^*)C\theta_j^* \end{bmatrix} \tag{47}$$

$$\mathbf{D}(\beta_k^*, \theta_k^*) \equiv \begin{bmatrix} 2S\beta_k^* C\beta_k^* S\theta_k^* & 2S\beta_k^* C\beta_k^* C\theta_k^* & 2S^2\beta_k^* \\ C\beta_k^* C\theta_k^* & -C\beta_k^* S\theta_k^* & 2C\beta_k^* S\beta_k^* \\ (C^2\beta_k^* - S^2\beta_k^*)S\theta_k^* & (C^2\beta_k^* - S^2\beta_k^*)C\theta_k^* & 0 \end{bmatrix} \tag{48}$$

It should be remarked that $\mathbf{V}_i^{(1)}$ is a function of q_j, $q_{j+1}, \ldots, q_n$ and $\mathbf{B}(\beta_j^*, \theta_j^*)$ and ${}_i^j\mathbf{R}$ in the second and third terms of Eq. (46) are functions of q_j only, while $\mathbf{D}(\beta_k^*, \theta_k^*)$ in the last term is a constant matrix. The terms in Eq. (46) can be divided into three groups: (1) the first term, (2) the second and third terms, and (3) the last two terms. The columns of the coefficient matrices in any one group are linearly independent of those in the other groups because of the different variables.

$K_k\mathbf{U}_k$, $k = 1, \ldots n$ do not appear in Eq. (46), so they are not needed to compute $K_i^*\mathbf{V}_i$, $i = 1, \ldots, n$, and are then redundant for determining $\mathbf{h}$.

By Property 2, the columns of $\overline{\mathbf{H}}$ corresponding to $\hat{m}_k$ for translational joint k are linearly independent of the other columns, and the role of the terms associated with $\hat{m}_k$ for computing $\mathbf{J}_i^{(i)}$ in Eqs. (37) and (39) can be ignored, as can those with $(\hat{\mathbf{c}}_k^{(k)})_x$ and $(\hat{\mathbf{c}}_k^{(k)})_y$, $k > s$. In Eq. (46), only the coefficient matrices in front of $\hat{\mathbf{c}}_k^{(k)}$ (i.e., the last term) vary with d_k, the displacements of the translational joints. In row $\mathbf{r}^T$ of $h_{ii} = \mathbf{r}^T\mathbf{v}$, as in Eq. (37), the component corresponding to $(\hat{\mathbf{c}}_k^{(k)})_z$ is then linearly independent of the components other than those corresponding to $(\hat{\mathbf{c}}_k^{(k)})_x$, $(\hat{\mathbf{c}}_k^{(k)})_y$, and $\hat{m}_k$ if it is nonzero. It is zero when $\mathbf{u}_i /\!/ \mathbf{u}_k$ (i.e., $S\beta_k^* = 0$), because it is $2S^2\beta_k^* d_k$ according to Eqs. (46) and (48). For $k > s$, $\mathbf{u}_k$ may be parallel to $\mathbf{u}_i$, $i \geq s$, but it is never parallel to $\mathbf{u}_r$. Because Eq. (42) is in a recursive form, there must be a nonzero term associated with $(\hat{\mathbf{c}}_k^{(k)})_z$ for computing h_{rr}, which is also one of the terms varying with d_k (the other terms are those associated with $(\hat{\mathbf{c}}_k^{(k)})_x$, $(\hat{\mathbf{c}}_k^{(k)})_y$ and $\hat{m}_k$). We then conclude that $K_k(\hat{\mathbf{c}}_k^{(k)})_z$, $k > s$, is a component of $\mathbf{v}$ in Eq. (30).

For the case of $i < k < s$, h_{mi} in Eq. (39) requires only the (3, 3)th entry of $\hat{\mathbf{J}}_i^{(i)}$ because $(\mathbf{u}_m^{(i)}) = [0, 0, \pm 1]^T$. Combining Eqs. (37), (39), (46), and (48) yields

$$\begin{bmatrix} h_{ii} \\ K_m^* h_{mi} \end{bmatrix} = \mathbf{p}(\mathbf{q}, \mathbf{v}) + \begin{bmatrix} 2d_k & (\mathbf{f}_k(d_k))_{33} \\ 2(\mathbf{u}_m^{(i)})_z d_k & (\mathbf{u}_m^{(i)})_z(\mathbf{f}_k(d_k))_{33} \end{bmatrix} \begin{bmatrix} \hat{\kappa}_{2k} \\ \hat{m}_k \end{bmatrix}, m < i < k < s \quad (49)$$

where $\mathbf{p}$ is some appropriate vector that is not a function of $\hat{\mathbf{c}}_k^{(k)}$ and d_k, and

$$\begin{aligned} \hat{\kappa}_{2k} &= -S\beta_k^* C\beta_k^* S\theta_k^* (\hat{\mathbf{c}}_k^{(k)})_x \\ &\quad - S\beta_k^* C\beta_k^* C\theta_k^* (\hat{\mathbf{c}}_k^{(k)})_y + S^2\beta_k^* (\hat{\mathbf{c}}_k^{(k)})_z \\ &= -(\mathbf{u}_r^{(k)})_z((\mathbf{u}_r^{(k)})_x(\hat{\mathbf{c}}_k^{(k)})_x \\ &\quad + (\mathbf{u}_r^{(k)})_y(\hat{\mathbf{c}}_k^{(k)})_y) + (1 - (\mathbf{u}_r^{(k)})_z^2)(\hat{\mathbf{c}}_k^{(k)})_z \end{aligned} \quad (50)$$

which follows from Eq. (45) and $\mathbf{u}_m^{(k)} = \mathbf{u}_r^{(k)}$ for $r \leq m < k < s$. If $\mathbf{u}_i /\!/ \mathbf{u}_k$, then $S\beta_k^*$ is zero, as is $\hat{\kappa}_{2k}$. It should be remarked that $\hat{\kappa}_{1k}$ in (36) is also zero when $S\beta_k^* = 0$. This entails that $K_k\hat{\mathbf{c}}_k^{(k)}$, $r < k < s$, is redundant for computing $\mathbf{h}$ if $\mathbf{u}_r /\!/ \mathbf{u}_k$. Applying Property 1 to Eq. (49), we may state the following property.

Property 3. $\mathbf{v}$ *in Eq. (30) has the following components:*

1. $K_i(\hat{\mathbf{c}}_i^{(i)})_z$, *$s < i \leq n$, and*
2. $K_i\hat{\kappa}_{2i}$ *(see Eq. (50)) for $r < i < s$ and $\mathbf{u}_r \not/\!/ \mathbf{u}_i$.*

However, $K_i\hat{\mathbf{c}}_i^{(i)}$ for $r < i < s$ and $\mathbf{u}_r \not/\!/ \mathbf{u}_i$ are not components of $\mathbf{v}$, nor are $K_i\mathbf{U}_i$, $i = 1, \ldots, n$. ∎

Let us return again to Eqs. (37)–(39). It is now apparent that $K_j^*\hat{m}_j$ and $K_j^*(\hat{\mathbf{c}}_i^{(i)})_z$, $j = 1, \ldots, n$, are redundant for computing $\mathbf{h}$.

We still assume that joints i and j are rotational joints and joints k, $i < k < j$, are translational joints. Because the components of $\mathbf{V}_i$ are all functions of q_{i+1} or q_j, Eq. (37) indicates that the column of $\overline{\mathbf{H}}$ in Eq. (30) corresponding to $K_i^*(\mathbf{U}_i)_{33}$, $i = 1, \ldots, n$, is linearly independent of the other columns. On the other hand, the rotational joints in front of joint s are parallel to one another, so $S\beta_j^*$ in Eq. (44) for $j < s$ is zero and Eq. (47) is reduced to

$$\mathbf{B}(\beta_j^*, \theta_j^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & C\beta_j^* C\theta_j^* & -C\beta_j^* S\theta_j^* \\ 0 & 0 & C^2\beta_j^* S\theta_j^* & C^2\beta_j^* C\theta_j^* \end{bmatrix}, \quad j < s \quad (51)$$

Therefore, the contribution of $\mathbf{U}_j$ to $(\mathbf{V}_i)_{33}$, $i < j < s$, is zero. By Eqs. (37) and (39) and the parallelism of the rotational joints, the entries (other than the (3, 3)th) of $K_i^*\mathbf{V}_i$, $i < s$, are redundant for computing $\mathbf{h}$. This entails that the entries, other than the (3, 3)th, of $K_i^*\mathbf{U}_i$ for $r \leq i < s$ are not components of $\mathbf{v}$ in Eq. (30).

Consider rotational joint j, $j \geq s$, and reformulate Eq. (39) as

$$\begin{aligned} h_{rj} &= (\mathbf{a}_{j,r}^{(j)})_y(\hat{\mathbf{c}}_j^{(j)})_x - (\mathbf{a}_{j,r}^{(j)})_x(\hat{\mathbf{c}}_j^{(j)})_y \\ &\quad + [(\hat{\mathbf{J}}_j^{(j)})_{13} \; (\hat{\mathbf{J}}_j^{(j)})_{23} \; (\hat{\mathbf{J}}_j^{(j)})_{33}]\mathbf{u}_r^{(j)}, j \geq s \end{aligned} \quad (52)$$

Because the terms associated with $K^*(\hat{\mathbf{c}}_j^{(j)})_x$ and $K^*(\hat{\mathbf{c}}_j^{(j)})_y$, $j \geq s$, for computing $\mathbf{h}$ in Eq. (30) are partitioned into $\mathbf{T}\mathbf{v}_g$, the third term associated with $K^*\hat{\mathbf{c}}_j^{(j)}$ in Eq. (46) and the first two terms in Eq. (52) for $j \geq s$ can be ignored in the analysis of $\mathbf{v}$. Keeping in mind that the x- and y-components of $\mathbf{u}_r^{(j)}$, $j \geq s$, vary independently (because joint s is not parallel to joint r), and $\mathbf{U}_j$ is constant while $\mathbf{V}_j$ varies, it follows from Eq. (52) that $(\mathbf{U}_j)_{13}$ and $(\mathbf{U}_j)_{23}$, $j \geq s$, in addition to $(\mathbf{U}_j)_{33}$, are components of $\mathbf{v}$ in Eq. (30).

Now let $i < j = s$. h_{ii} in Eq. (37) requires $(\mathbf{V}_i)_{33}$. The contribution of $\mathbf{U}_s$ to $(\mathbf{V}_i)_{33}$ is the first row of $\mathbf{B}(\beta_s^*, \theta_s^*)$ in Eq. (47), whose components are either zero or linearly independent of one another and of the other terms in $(\mathbf{V}_i)_{33}$ because the terms associated with $\hat{\mathbf{c}}_j^{(j)}$ are ignored. The fact that rotational joint i is not parallel to joint s implies that the first two components of the first row of $\mathbf{B}$ must be nonzero. Thus, $(\mathbf{U}_s)_{11} - (\mathbf{U}_s)_{22}$ and $(\mathbf{U}_s)_{12}$ are components of $\mathbf{v}$. Fix joint s such that rotational joint j, $j > s$, is not parallel to joint r. Let β_j^* and θ_j^* be the transformation parameters between joint r and joint j when joint s is held stationary; then the above result applies to $j > s$. In summary, $(\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22}$, $(\mathbf{U}_j)_{12}$, $(\mathbf{U}_j)_{13}$, and $(\mathbf{U}_j)_{23}$ for rotational joint j, $j \geq s$, as well as $K_i^*(\mathbf{U}_i)_{33}$, $i = 1, \ldots, n$, are components of $\mathbf{v}$ in Eq. (30).

The x- and y-components of $K_i^*\hat{\mathbf{c}}_i^{(i)}$ for $r \leq i < s$ are still left to be considered.

For any two rotational joints m and i, $m < i < s$, we define Case 1 to be ${}_m^i\mathbf{s} = 0$ or ${}_m^i\mathbf{s} /\!/ \mathbf{u}_m$ (i.e., the origins of frames E_i and E_m are coincident or the distance between them is collinear with $\mathbf{u}_m$), and Case 2 to be all other situations. In Case 1, $\mathbf{a}_{i,m}^{(i)} = \mathbf{0}$ according to Eq. (17). In addition, if there are no translational joints in front of joint i that are not parallel to joint i, the x- and y-components of $K_k\mathbf{u}_k^{(i)}$ for all translational joints k, $k < i$, are zero. Thus, Eqs. (38) and (39) are, respectively, reduced to $K_k h_{ki} = 0$, $k < i$, and $h_{mi} = \pm(\hat{\mathbf{J}}_i^{(i)})_{33}$, because $\mathbf{u}_m^{(i)} = [0, 0, \pm 1]^T$. In addition, $({}_m^i\mathbf{s}^{*(m)})_x = ({}_m^i\mathbf{s}^{*(m)})_y = 0$ in Case 1, so that $\hat{\mathbf{c}}_i^{(i)}$ has no contribution to $(\mathbf{V}_m)_{33}$ according to Eq. (46). We then conclude that $(\hat{\mathbf{c}}_i^{(i)})_x$ and $(\hat{\mathbf{c}}_i^{(i)})_y$ of rotational joint i, $r < i < s$, are redundant for computing $\mathbf{h}$ if there are no translational joints in front of joint i that are not parallel to joint i and if the distance from any rotational joint in front of joint i to joint i is either zero or parallel to joint i.

On the other hand, in Case 2, $\mathbf{a}_{i,m}$ is not zero and perpendicular to $\mathbf{u}_m$. Although $\mathbf{a}_{i,m}^{(m)}$ may be constant (when ${}_m^i\mathbf{s}^{(m)}$ is constant), the x- and y-components of $\mathbf{a}_{i,m}^{(i)}$ vary with the rotation of joint i. They are linearly independent of each other and of the nonzero component of $\mathbf{u}_m^{(i)}$ (i.e., $(\mathbf{u}_m^{(i)})_z$, which is ± 1). Applying Property 1 to Eq. (39) yields the result that $(\hat{\mathbf{c}}_i^{(i)})_x$ and $(\hat{\mathbf{c}}_i^{(i)})_y$ of rotational joint i, $r < i < s$, in Case 2, are components of $\mathbf{v}$ in Eq. (30).

In front of joint r, if there are no translational joints not parallel to joint r, then $(\hat{\mathbf{c}}_r^{(r)})_x$ and $(\hat{\mathbf{c}}_r^{(r)})_y$ are redundant for computing $\mathbf{h}$. Otherwise, they are components of $\mathbf{v}$ in Eq. (30) according to Eq. (38). This also applies to rotational joint i, $r < i < s$.

Property 4. *$\mathbf{v}$ in Eq. (30) has the following components:*

1. $K_i^*((\mathbf{U}_i)_{11} - (\mathbf{U}_i)_{22})$, $K_i^*(\mathbf{U}_i)_{12}$, $K_i^*(\mathbf{U}_i)_{13}$, $K_i^*(\mathbf{U}_i)_{23}$, $s \leq i \leq n$,
2. $K_i^*(\mathbf{U}_i)_{33}$, $r \leq i \leq n$,
3. $K_i^*(\hat{\mathbf{c}}_i^{(i)})_x$, $K_i^*(\hat{\mathbf{c}}_i^{(i)})_y$ *for* $r \leq i < s$ *if there is a translational joint* k, $k < i$, *such that* $\mathbf{u}_k \not\parallel \mathbf{u}_i$ *or if there is a rotational joint* m, $m < i$, *such that* ${}_m^i\mathbf{s} \neq \mathbf{0}$ *and* ${}_m^i\mathbf{s} \not\parallel \mathbf{u}_r$ *(otherwise, they are not).*

However, $K_i^*\hat{m}_i$ *and* $K_i^*(\hat{\mathbf{c}}_i^{(i)})_z$, $r \leq i \leq n$, *are redundant for computing* $\mathbf{h}$, *as are all entries, other than the (3, 3)th, of* $K_i^*\mathbf{U}_i$, $r \leq i < s$. ■

All components of $\mathbf{v}$ in Eq. (30) are included in Properties 2 to 4. Let $\mathbf{v}_A$ consist of all $K_i\hat{\kappa}_{1i}$, $K_i\hat{\kappa}_{2i}$, $K_i^*(\hat{\mathbf{c}}_i^{(i)})_x$ and $K_i^*(\hat{\mathbf{c}}_i^{(i)})_y$, $i < s$, in $\mathbf{v}$, $\mathbf{v}_B$ consist of the three components of all $K_j\hat{\mathbf{c}}_j^{(j)}$, $j > s$, and $\mathbf{v}_C$ consist of all other components of $\mathbf{v}$. Furthermore, the components of $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ are all assumed to be presented in the order from joint 1 to joint n. On the other hand, $\mathbf{w}_A$ and $\mathbf{w}_B$ are, respectively, defined the same as $\mathbf{v}_A$ and $\mathbf{v}_B$ except that $\hat{\mathbf{c}}_i^{(i)}$, $\hat{\kappa}_{1i}$ and $\hat{\kappa}_{2i}$ are, respectively, replaced with $\mathbf{k}_i$ and

$$\kappa_{1i} = -(\mathbf{u}_r^{(i)})_y(\mathbf{k}_i)_x + (\mathbf{u}_r^{(i)})_x(\mathbf{k}_i)_y \tag{53}$$

$$\kappa_{2i} = -(\mathbf{u}_r^{(i)})_z((\mathbf{u}_r^{(i)})_x(\mathbf{k}_i)_x + (\mathbf{u}_r^{(i)})_y(\mathbf{k}_i)_y) + (1 - (\mathbf{u}_r^{(i)})_z^2)(\mathbf{k}_i)_z \tag{54}$$

The last two equations are different from Eqs. (36) and (50) only in the terms of $\mathbf{k}_i$.

Suppose that there are translational joint k and rotational joint i, $k < i < s$. By Properties 2 to 4, if $\mathbf{u}_k \not\parallel \mathbf{u}_i$, then $\hat{\kappa}_{1k}$, $\hat{\kappa}_{2k}$, $(\hat{\mathbf{c}}_i^{(i)})_x$, and $(\hat{\mathbf{c}}_i^{(i)})_y$ are all components of $\mathbf{v}$ in Eq. (30). This means that if $K_k\hat{\kappa}_{1k}$ and $K_k\hat{\kappa}_{2k}$ are components of $\mathbf{v}$, then so are $K_i^*(\hat{\mathbf{c}}_i^{(i)})_x$ and $K_i^*(\hat{\mathbf{c}}_i^{(i)})_y$ for all i, $k < i < s$. In the other situation, when there are only parallel translational joints in front of joint i, if there is a rotational joint m in front of joint i such that ${}_m^i\mathbf{s} \not\parallel \mathbf{u}_i$, then for any rotational joint j behind joint i, $i < j < s$, neither ${}_m^j\mathbf{s}$ nor ${}_m^i\mathbf{s}$ is parallel to $\mathbf{u}_m$ because ${}_m^j\mathbf{s} = {}_m^i\mathbf{s} + {}_i^j\mathbf{s}$. Applying Property 4 yields that if $K_i^*(\hat{\mathbf{c}}_i^{(i)})_x$ and $K_i^*(\hat{\mathbf{c}}_i^{(i)})_y$ are components of $\mathbf{v}$, then so are $K_j^*(\hat{\mathbf{c}}_j^{(j)})_x$ and $K_j^*(\hat{\mathbf{c}}_j^{(j)})_y$ for all j, $i < j < s$. Consequently, if either $K_i\hat{\kappa}_{1i}$ and $K_i\hat{\kappa}_{2i}$ or $K_i^*(\hat{\mathbf{c}}_i^{(i)})_x$ and $K_i^*(\hat{\mathbf{c}}_i^{(i)})_y$, $i < s$, are in $\mathbf{v}$, then $K_j^*(\hat{\mathbf{c}}_j^{(j)})_x$ and $K_j^*(\hat{\mathbf{c}}_j^{(j)})_y$, $i < j < s$, are also in $\mathbf{v}$. Note that $\hat{\kappa}_{1i}$ and $\hat{\kappa}_{2i}$ are linear combinations of the components of $\hat{\mathbf{c}}_i^{(i)}$, and that $\hat{\mathbf{c}}_i^{(i)}$ for translational or rotational joint i can be expressed in terms of $K_j\hat{m}_j$ and the x- and y-components of $K_j^*\mathbf{k}_j$, $j > i$. Thus it can be shown that

$$\mathbf{v}_A = \mathbf{W}_A\mathbf{w}_A + \mathbf{P}_A\mathbf{w}_g \tag{55}$$

$$\mathbf{v}_B = \mathbf{I}\mathbf{w}_B + \mathbf{P}_B\mathbf{w}_g \tag{56}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{W}_A$ is an upper triangular square matrix with the diagonal entries of 1, $\mathbf{P}_A$ and $\mathbf{P}_B$ are some appropriate matrices, and $\mathbf{w}_g$ is defined as in Eq. (31).

Therefore, Eq. (30) can be rewritten as

$$\begin{aligned} \mathbf{h} &= [\overline{\mathbf{H}}_A \vdots \overline{\mathbf{H}}_B \vdots \overline{\mathbf{H}}_C]\begin{bmatrix} \mathbf{v}_A \\ \mathbf{v}_B \\ \mathbf{v}_C \end{bmatrix} + \overline{\mathbf{T}}\mathbf{w}_g \\ &= \mathbf{C}\begin{bmatrix} \mathbf{w}_A \\ \mathbf{w}_B \\ \mathbf{v}_C \end{bmatrix} + \mathbf{D}\mathbf{w}_g \end{aligned} \tag{57}$$

where

$$\mathbf{C} = [(\overline{\mathbf{H}}_A\mathbf{W}_A) \vdots \overline{\mathbf{H}}_B \vdots \overline{\mathbf{H}}_C] \tag{58}$$

$$\mathbf{D} = \overline{\mathbf{H}}_A\mathbf{P}_A + \overline{\mathbf{H}}_B\mathbf{P}_B + \overline{\mathbf{T}} \tag{59}$$

Because the columns of $\overline{\mathbf{H}}_A$, $\overline{\mathbf{H}}_B$, and $\overline{\mathbf{H}}_C$ are linearly independent to one another and $\mathbf{W}_A$ is nonsingular, the columns of $\mathbf{C}$ are also linearly independent according to Lemma A2. Note that some columns of $\mathbf{D}$ may be linearly dependent on those of $\mathbf{C}$. In this case, there exists a matrix $\overline{\mathbf{H}}_D$ whose columns are linearly independent to one another and to those of $\mathbf{C}$, such that the second term in Eq. (57) can be decoupled into

$$\mathbf{D}\mathbf{w}_g = \mathbf{C}\mathbf{P}_1\mathbf{w}_g + \overline{\mathbf{H}}_D\mathbf{P}_2\mathbf{w}_g \tag{60}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are constant matrices. Finally, we conclude that a set of MLCs for determining $\mathbf{h}$ is

$$\mathbf{w}^I = \begin{bmatrix} \begin{bmatrix} \mathbf{w}_A \\ \mathbf{w}_B \\ \mathbf{v}_C \end{bmatrix} + \mathbf{P}_1\mathbf{w}_g \\ \cdots\cdots\cdots\cdots \\ \mathbf{P}_2\mathbf{w}_g \end{bmatrix} \tag{61}$$

which is also the MLCs for determining $\boldsymbol{\tau}^I$ according to Theorem 3. By Theorem 2, $\mathbf{w}_A$, $\mathbf{w}_B$, $\mathbf{v}_C$, and $\mathbf{w}_g$ constitute a set of MLCs for determining $\boldsymbol{\tau}$, as stated in the following theorem.

Theorem 5. *For the manipulator considered in Theorem 4, a set of MLCs for determining the actuator forces $\boldsymbol{\tau}$ is the set $\mathcal{S}$ consisting of all nonzero elements of*

1. $K_j^*(\mathbf{U}_j)_{33}$, $\delta_j K_j^*(\mathbf{k}_j)_x$, $\delta_j K_j^*(\mathbf{k}_j)_y$ *for* $r \le j < s$,
2. $K_j^*((\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22})$, $K_j^*(\mathbf{U}_j)_{33}$, $K_j^*(\mathbf{U}_j)_{12}$, $K_j^*(\mathbf{U}_j)_{13}$, $K_j^*(\mathbf{U}_j)_{23}$, $K_j^*(\mathbf{k}_j)_x$, $K_j^*(\mathbf{k}_j)_y$ *for* $s \le j \le n$,
3. $K_i\hat{m}_i$ *for* $i = 1, \ldots, n$,
4. $K_i(\mathbf{k}_i)_x$, $K_i(\mathbf{k}_i)_y$, $K_i(\mathbf{k}_i)_z$ *for* $s < i \le n$, *and*
5. $\sigma_i K_i \kappa_{1i}$, $\sigma_i K_i \kappa_{2i}$ *for* $r < i < s$,

where $\delta_j = 0$ *for the case that* $\mathbf{u}_r /\!/ \mathbf{u}_k /\!/ \mathbf{g}$, $\forall k \le j < S$, *and* ${}^j_m\mathbf{s}$ *(when* $j > r$*) is zero or parallel to* $\mathbf{u}_r$ *for every rotational joint* m, $r \le m < j$, *otherwise* $\delta_j = 1$, *and where* $\sigma_i = 0$ *for the case of* $\mathbf{u}_i /\!/ \mathbf{u}_r$, $r < i < s$, *otherwise* $\sigma_i = 1$. ■

An example for a set of MLCs of the Stanford arm can be found in previous work.[23] We are now ready to state and prove the equivalence theorem.

Corollary 6. *Suppose that two sets $\mathcal{S}_a$ and $\mathcal{S}_b$, formed by the linear combinations of the inertia parameters of a manipulator, have the same number of elements and $\mathcal{S}_a$ is a set of MLCs for determining $\boldsymbol{\tau}$. Then $\mathcal{S}_b$ is also a set of MLCs for $\boldsymbol{\tau}$ if and only if the values of the elements of $\mathcal{S}_b$ can be obtained from those of $\mathcal{S}_a$ and vice versa, i.e., there is a nonsingular constant matrix $\mathbf{M}$ such that*

$$\mathbf{b} = \mathbf{M}\mathbf{a} \tag{62}$$

where the components of $\mathbf{a}$ and $\mathbf{b}$ are the elements of $\mathcal{S}_a$ and $\mathcal{S}_b$, respectively.

Proof: According to the assumption, there are matrix $\mathbf{A}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ with linearly independent columns and matrix $\mathbf{P}$ with linearly independent rows, such that $\boldsymbol{\tau} = \mathbf{A}\mathbf{a}$ and $\mathbf{a} = \mathbf{P}\mathbf{x}$, where $\mathbf{x}$ is the vector of all inertia parameters.

When Eq. (62) holds, applying Lemma A2 to $\mathbf{P}^T\mathbf{M}^T$ yields the result that the rows of $(\mathbf{M}\mathbf{P})$ are linearly independent, so the components of $\mathbf{b}$ are also linearly independent. Moreover, we also obtain $\boldsymbol{\tau} = \mathbf{A}\mathbf{M}^{-1}\mathbf{b}$, where the columns of $(\mathbf{A}\mathbf{M}^{-1})$ are linearly independent by Lemma A2. This completes the proof of the sufficiency.

If $\mathcal{S}_b$ is also a set of MLCs for $\boldsymbol{\tau}$, there are matrix $\mathbf{B}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ with linearly independent columns and matrix $\mathbf{Q}$ with linearly independent rows, such that $\boldsymbol{\tau} = \mathbf{B}\mathbf{b}$ and $\mathbf{b} = \mathbf{Q}\mathbf{x}$. Suppose that $\mathbf{x} \in R^p$ and $\mathbf{a}$ and $\mathbf{b}$ are k-tuples. There are $(p - k)$ linear combinations (denoted by $\mathbf{c}$) of $\mathbf{x}$, such that the components of $\mathbf{a}$ and $\mathbf{c}$ are linearly independent and form a basis of the domain of $\mathbf{x}$. Thus $\mathbf{x}$ can be expressed as

$$\mathbf{x} = [\mathbf{W}_1 \vdots \mathbf{W}_2]\begin{bmatrix} \mathbf{a} \\ \mathbf{c} \end{bmatrix} \tag{63}$$

which yields the results that

$$\mathbf{b} = \mathbf{QW}_1\mathbf{a} + \mathbf{QW}_2\mathbf{c} \quad (64)$$

$$\boldsymbol{\tau} = \mathbf{BQW}_1\mathbf{a} + \mathbf{BQW}_2\mathbf{c} \quad (65)$$

Note that $\mathcal{S}_a$ is a set of MLCs. By Theorem 1, $(\mathbf{BQW}_2)$ must be zero and the columns of $(\mathbf{BQW}_1)$ are linearly independent. This leads to the conclusion that $\mathbf{QW}_2 = \mathbf{0}$ because the columns of $\mathbf{B}$ are linearly independent, and that $(\mathbf{QW}_1)$ is a nonsingular square matrix according to Lemma A2. The claim of necessity then holds. Q.E.D. ■

5. CONCLUSION

The minimal linear combinations of the inertia parameters (MLCs) introduced in this article illuminate the role of the identifiable parameters in the manipulator dynamics. This article has presented a systematic approach to finding a set of MLCs. The problem is first divided into two searches for two individual sets of MLCs that determine the entries of the inertia matrix and the gravity load. The MLCs for the gravity load are easier to find and have been addressed in an earlier work.[33] To find a set of MLCs for the inertia matrix, we begin by decoupling the inertia parameters of composite bodies into two parts, so that the varying part of each parameter can be calculated from the constant parts of the other parameters. The rest of our task is then to carefully inspect the roles of the constant parts of the parameters in the inertia matrix. The technique for this is presented in section 4, where it is used to formulate Properties 2 to 4. Applying Theorems 2 and 3 converts the elements in Properties 2 to 4 to those of the MLCs for the full manipulator dynamics. This step-by-step approach is different from that in the author's earlier work,[23] although the results are identical. However, this set of MLCs is slightly different from others in the literature[16,17,19–22] in some minor terms because a set of MLCs is not unique. The equivalence of these different results can be proved using Corollary 6 in the last section. The crucial feature of the present approach is that all derivations are in accordance with the least squares theory, so the identifiability of the present set of MLCs is assured.

The present set of MLCs also has some other advantages: a systematic off-line identification method for it has already been proposed,[23] and a recursive formulation of the manipulator inverse dynamics in terms of the set of MLCs has been derived and shown to be more efficient than most other formulations of the inverse dynamics in the literature.[34] The main emphasis of the present article is that the minimal parameters of a manipulator should be treated in the least squares sense, so that the minimal parameters are equivalent to the identifiable parameters.

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APPENDIX

Lemma A1. *Suppose that*

$$\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{x}) = \sum_{j=1}^{n} \mathbf{b}_j(\mathbf{q}, \mathbf{x})\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=j}^{n} \mathbf{c}_{jk}(\mathbf{q}, \mathbf{x})\dot{q}_j\dot{q}_k \quad (\text{A1})$$

where $\mathbf{x} \in R^p$ *are the system parameters,* $\mathbf{q} \in R^n$, $\mathbf{a} : R^{n\times n\times n\times p} \rightarrow R^m$, $\mathbf{b}_j : R^{n\times p} \rightarrow R^m$, $\mathbf{c}_{jk} : R^{n\times p} \rightarrow R^m$. *Then, a set is a set of MLCs for determining* $\mathbf{a}$ *if and only if it is also a set of MLCs for determining* $[\mathbf{b}_1^T, \ldots, \mathbf{b}_n^T, \mathbf{c}_{11}^T, \ldots, \mathbf{c}_{1n}^T, \mathbf{c}_{21}^T, \ldots, \mathbf{c}_{nn}^T]^T$. ■

Lemma A2. *Suppose that the columns of matrix* $\mathbf{A}(\theta) : R^m \rightarrow R^{n\times k}$ *are linearly independent over* R^m. *The constant square matrix* $\mathbf{B} \in R^{k\times k}$ *is nonsingular if and only if the columns of the product* $\mathbf{A}(\theta)\mathbf{B}$ *(or* $\mathbf{BA}(\theta)$*) are also linearly independent over* R^m. ■

Lemmas A1 and A2 are used in the proofs of Theorem 3 and Corollary 6, respectively. The proofs of these two lemmas are very simple and can be found in a previous work.[32] In the following, we prove the theorems in section 2.

Proof of Theorem 1: Sufficiency (←): According to the assumption that the elements of $\mathbf{w}$ are linear combinations of $\mathbf{x}$, there exists a constant matrix $\mathbf{B} \in R^{k\times p}$ such that

$$\mathbf{w}(\mathbf{x}) = \mathbf{Bx} \quad (\text{A2})$$

The rank of $\mathbf{B}$ is k since the rows of $\mathbf{B}$ are linearly independent. It follows from Eqs. (3) and (A2) that

$$\mathbf{a}_i(\boldsymbol{\theta}) = \overline{\mathbf{A}}(\boldsymbol{\theta})\mathbf{b}_i \quad (\text{A3})$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the ith columns of $\mathbf{A}$ and $\mathbf{B}$, respectively.

We select k linearly independent columns $\mathbf{b}_i$ and then reorder them and the corresponding $\mathbf{a}_i(\boldsymbol{\theta})$ to be the first to kth columns of $\mathbf{B}$ and $\mathbf{A}(\boldsymbol{\theta})$, respectively. If we let

$$\alpha_1\mathbf{a}_1(\boldsymbol{\theta}) + \cdots + \alpha_k\mathbf{a}_k(\boldsymbol{\theta}) = \bar{\mathbf{A}}(\boldsymbol{\theta})(\mathbf{b}_1\alpha_1 + \cdots + \mathbf{b}_k\alpha_k) = \mathbf{0} \quad \text{(A4)}$$

then $\alpha_1 = \ldots = \alpha_k = 0$ because the linear independence of the columns of $\bar{\mathbf{A}}(\boldsymbol{\theta})$ over R^m implies $(\mathbf{b}_1\alpha_1 + \cdots + \mathbf{b}_k\alpha_k) = \mathbf{0}$. We conclude that $\mathbf{a}_1(\boldsymbol{\theta}), \ldots, \mathbf{a}_k(\boldsymbol{\theta})$ are linearly independent over R^m. Because the rank of $\mathbf{B}$ is k, there exist not all zero α_i such that $(\mathbf{b}_1\alpha_1 + \ldots + \mathbf{b}_{k+1}\alpha_{k+1}) = \mathbf{0}$. Because

$$\sum_i \mathbf{a}_i(\boldsymbol{\theta})\alpha_i = \bar{\mathbf{A}}(\boldsymbol{\theta})\left(\sum_i \mathbf{b}_i\alpha_i\right), \forall\, \boldsymbol{\theta} \in R^m \quad \text{(A5)}$$

any $k + 1$ or more columns of $\mathbf{A}(\boldsymbol{\theta})$ are then linearly dependent over R^m. Consequently, the number of linearly independent columns of $\mathbf{A}$ is k.

Necessity ($\rightarrow$): Because the number of linearly independent columns of $\mathbf{A}(\boldsymbol{\theta})$ is k, we choose k linearly independent columns to construct $\bar{\mathbf{A}}(\theta): R^m \rightarrow R^{n\times k}$, and let the other $(m - k)$ columns form $\tilde{\mathbf{A}}(\theta): R^m \rightarrow R^{n\times(m-k)}$. We can partition $\mathbf{x}$ into $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ such that

$$\mathbf{A}(\theta)\mathbf{x} = \bar{\mathbf{A}}(\boldsymbol{\theta})\bar{\mathbf{x}} + \tilde{\mathbf{A}}(\boldsymbol{\theta})\tilde{\mathbf{x}} \quad \text{(A6)}$$

Every column ($\tilde{\mathbf{a}}_i$) of $\tilde{\mathbf{A}}$ can be expressed as a linear combination of the columns ($\bar{\mathbf{a}}_i$) of $\bar{\mathbf{A}}$ in the form of

$$\tilde{\mathbf{a}}_j = \sum_{i=1}^{k} \alpha_{ij}\bar{\mathbf{a}}_i \quad \text{(A7)}$$

where α_{ij} are some constants. Substituting Eq. (A7) into Eq. (A6), we get

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{x} = \bar{\mathbf{A}}(\boldsymbol{\theta})\mathbf{w}(\mathbf{x}) \quad \text{(A8)}$$

where

$$\mathbf{w}(\mathbf{x}) = \bar{\mathbf{x}} + \begin{bmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1k} \\ \vdots & \ddots & & \vdots \\ \vdots & & \alpha_{ij} & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{k1} & \cdots & \cdots & \alpha_{kk} \end{bmatrix} \tilde{\mathbf{x}} \quad \text{(A9)}$$

whose elements are linearly independent over R^p. Q.E.D. ■

Proof of Theorem 2: Suppose that there are $\bar{\mathbf{H}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = [\bar{\mathbf{H}}_1 : \bar{\mathbf{H}}_2]$ and $\bar{\mathbf{G}}(\mathbf{q})$ whose columns are linearly independent over R^{3n} and R^n, respectively, such that

$$\boldsymbol{\tau}^I = \bar{\mathbf{H}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\mathbf{w}^I(\mathbf{x}) = \bar{\mathbf{H}}_1\mathbf{w}^I + (\bar{\mathbf{H}}_1\mathbf{P}_1 + \bar{\mathbf{H}}_2\mathbf{P}_2)\mathbf{w}^g \quad \text{(A10)}$$

$$\boldsymbol{\tau}^g = \bar{\mathbf{G}}(\mathbf{q})\mathbf{w}^g(\mathbf{x}) \quad \text{(A11)}$$

Thus,

$$\boldsymbol{\tau}^I + \boldsymbol{\tau}^g = [\bar{\mathbf{H}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) : \bar{\mathbf{G}}(\mathbf{q})]\begin{bmatrix} \mathbf{w}^I(\mathbf{x}) \\ \mathbf{w}^g(\mathbf{x}) \end{bmatrix} = [\bar{\mathbf{H}}_1 : \bar{\mathbf{H}}_2 : \bar{\mathbf{G}}]\mathbf{P}\begin{bmatrix} \mathbf{w}^I(\mathbf{x}) \\ \mathbf{w}^g(\mathbf{x}) \end{bmatrix} \quad \text{(A12)}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{P}_1 \\ \mathbf{0} & \mathbf{P}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \text{(A13)}$$

and $\mathbf{I}$ is the identity matrix. The fact that each entry of $\bar{\mathbf{H}}$ is associated with $\dot{\mathbf{q}}$ and/or $\ddot{\mathbf{q}}$ (see Eqs. (4) and (5)) implies that the columns of $[\bar{\mathbf{H}} : \bar{\mathbf{G}}]$ are linearly independent over R^{3n}. This completes the proof of the first part. Let $([\bar{\mathbf{H}}_1 : \bar{\mathbf{H}}_2 : \bar{\mathbf{G}}]\mathbf{P})\boldsymbol{\alpha} = \mathbf{0}$. Because the columns of $[\bar{\mathbf{H}} : \bar{\mathbf{G}}]$ are linearly independent, $\mathbf{P}\boldsymbol{\alpha} = \mathbf{0}$. This is only possible when $\boldsymbol{\alpha} = \mathbf{0}$. The claim of the second part is then true. Q.E.D. ■

Proof of Theorem 3: Let $\mathbf{h} = [h_{11}, \ldots, h_{1n}, h_{21}, \ldots, h_{nn}]^T$ and $\mathbf{c} = [c_{111}, c_{112}, \ldots, c_{11n}, c_{122}, \ldots, c_{nnn}]^T$. Suppose that $\mathbf{w}(\mathbf{x}): R^p \rightarrow R^k$ is a set of MLCs for determining $\mathbf{h}$, and there exists $\bar{\mathbf{H}}(\mathbf{q}): R^n \rightarrow R^{n^2\times k}$ such that

$$\mathbf{h} = \bar{\mathbf{H}}(\mathbf{q})\mathbf{w}(\mathbf{x}) \quad \text{(A14)}$$

By Christoffel symbols in Eqs. (6) and (7), we get

$$\mathbf{c} = \bar{\mathbf{C}}(\mathbf{q})\mathbf{w}(\mathbf{x}) \quad \text{(A15)}$$

where $\bar{\mathbf{C}}(\mathbf{q})$ is some appropriate matrix, whose columns may be linearly dependent over R^n. Therefore, $\mathbf{w}(\mathbf{x})$ is also a set of MLCs for $[\mathbf{h}^T, \mathbf{c}^T]^T$ because the columns of $[\bar{\mathbf{H}}^T, \bar{\mathbf{C}}^T]^T$ are linearly independent over R^n.

Conversely, we assume that the columns of $[\bar{\mathbf{H}}^T, \bar{\mathbf{C}}^T]^T$ are linearly independent, but the columns of $\bar{\mathbf{H}}$ are linearly dependent over R^n. We linearly combine the dependent columns of $\overline{\mathbf{H}}$ to form $\tilde{\mathbf{H}}$, whose columns are linearly independent over R^n, such that

$$\mathbf{h} = \tilde{\mathbf{H}}(\mathbf{q})\tilde{\mathbf{w}}(\mathbf{x}) \tag{A16}$$

where the elements of $\tilde{\mathbf{w}}(\mathbf{x})$ are linear combinations of $\mathbf{w}(\mathbf{x})$ and are linearly independent over R^p. Using Christoffel symbols again, we obtain the result that $\tilde{\mathbf{w}}(\mathbf{x})$ is also a set of MLCs for $[\mathbf{h}^T, \mathbf{c}^T]^T$. This contradicts Theorem 1 since the dimensions of $\mathbf{w}$ and $\tilde{\mathbf{w}}$ are not the same. Consequently, the columns of $\overline{\mathbf{H}}$ are linearly independent over R^n if and only if the columns of $[\overline{\mathbf{H}}^T, \overline{\mathbf{C}}^T]^T$ are linearly independent over R^n. This implies that $\mathbf{w}(\mathbf{x})$ is a set of MLCs for $\mathbf{h}$ if and only if it is also a set of MLCs for $[\mathbf{h}^T, \mathbf{c}^T]^T$.

The rest of the proof consists of showing that a set is a set of MLCs for $\boldsymbol{\tau}^I$ if and only if it is also one for $[\mathbf{h}^T, \mathbf{c}^T]^T$, which follows directly from Lemma A1 above. Q.E.D. ∎

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