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Disjoint cycles in hypercubes with prescribed vertices in each cycle☆

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ABSTRACT

A graph G is spanning r-cyclable of order t if for any r nonempty mutually disjoint vertex subsets $A_1, A_2, \ldots, A_r$ of G with $|A_1 \cup A_2 \cup \cdots \cup A_r| \leq t$, there exist r disjoint cycles $C_1, C_2, \ldots, C_r$ of G such that $C_1 \cup C_2 \cup \cdots \cup C_r$ spans G, and C_i contains A_i for every i. In this paper, we prove that the n-dimensional hypercube Q_n is spanning 2-cyclable of order $n-1$ for $n \geq 3$. Moreover, Q_n is spanning k-cyclable of order k if $k \leq n-1$ for $n \geq 2$. The spanning r-cyclability of a graph G is the maximum integer t such that G is spanning r-cyclable of order k for $k = r, r+1, \ldots, t$ but is not spanning r-cyclable of order $t+1$. We also show that the spanning 2-cyclability of Q_n is $n-1$ for $n \geq 3$.

1. Introduction

For those graph definitions and notations not defined here, we follow the standard terminology given in [12]. A pair of two sets $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of elements of $V\}$. We say that $V = V(G)$ is the *vertex set*, and $E = E(G)$ is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. The *neighborhood* of vertex u in G, denoted by $Nbd_G(u)$, is the set $\{v \in V | (u, v) \in E\}$. The *degree* of u in G, denoted by $deg_G(u)$, is $|Nbd_G(u)|$. A *path* is a sequence of adjacent vertices, written as $\langle v_0, v_1, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except that possibly $v_0 = v_m$.

A *cycle* of a graph G is a path with at least three vertices such that the first vertex is the same as the last one. A hamiltonian cycle is a spanning cycle in a graph. Until the 1970s, the interest in hamiltonian cycles had long been centered on their relationship to the 4-color problem. Recently, some refined conditions for a graph to be hamiltonian were proposed by researchers [8,17,18], and the study of hamiltonian cycles in general graphs has been fueled by the issue of computational complexity and practical applications. Furthermore, a number of variations were developed and research efforts have been dedicated to pancyclicity [4,9], super spanning connectivity [1,6,19,20], k-ordered hamiltonicity [17], and hamiltonian decomposition [2,21,22] among many other areas. In particular, hamiltonian cycles are a major requirement to design effective interconnection networks [12,14,25,26].

There are several directions of research based on the hamiltonian property. One direction involves the spanning property of cycles. For example, a 2-*factor* of a graph G is a spanning 2-regular subgraph of G; that is, G has a 2-factor if it can be

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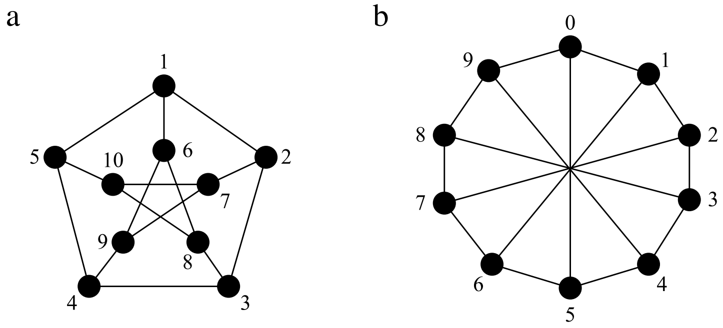

Fig. 1. Illustration for Examples 1 and 2.

decomposed into several disjoint cycles. This notion can be applied to identify faulty units in a multiprocessor system. In particular, Fujita and Araki [7] proposed a three-round adaptive diagnosis algorithm by decomposing the hypercube into a fixed number of disjoint cycles such that the length of each cycle is not too small. The other direction addresses the cyclability of a graph G. Let S be a subset of $V(G)$. Then, S is *cyclable* in G if there exists a cycle C of G such that $S \subseteq V(C)$. Many results of cyclability are known [3,5,11,13,23]. In this paper, we study a new property which is a mixture of these two directions.

Now, we extend the concept behind hamiltonian graphs and consider two or more cycles spanning a whole graph. Let $A_1, A_2, \ldots, A_r$ be mutually disjoint nonempty vertex subsets of a graph G. Then G is *cyclable* with respect to $A_1, A_2, \ldots, A_r$ if there exist mutually disjoint cycles $C_1, C_2, \ldots, C_r$ of G such that C_i contains A_i for every i. Obviously, a graph is unlikely to be cyclable with respect to any r mutually disjoint vertex subsets if $r \geq 2$. For example, G cannot be cyclable with respect to $A_1 = \{u, v\}$ and $A_2 = V(G) - \{u, v\}$ for any two vertices u, v of G. To make this notion more reasonable, we impose one restriction on the order of $A_1 \cup A_2 \cdots \cup A_r$. To be precise, G is r-cyclable of order t if it is cyclable with respect to $A_1, A_2, \ldots, A_r$ for any r nonempty mutually disjoint subsets $A_1, A_2, \ldots, A_r$ of $V(G)$ such that $|A_1 \cup A_2 \cup \cdots A_r| \leq t$. In addition, if $C_1 \cup C_2 \cup \cdots \cup C_r$ spans G, then G is *spanning r-cyclable of order t*. Here we have two parameters r and t. We can fix one of them and find the optimal value for the other. The (spanning) r-cyclability of G is t if G is (spanning) r-cyclable of order k for $k = r, r+1, \ldots, t$ but is not (spanning) r-cyclable of order $t+1$. On the other hand, the (spanning) cyclability of G of order t is r if G is (spanning) k-cyclable of order t for $k = 1, 2, \ldots, r$ but is not (spanning) $(r+1)$-cyclable of order t. According to the presented notion, the problem of finding hamiltonian cycles focuses on $r = 1$. It is also noticed that not only is the set of disjoint spanning cycles of G a 2-factor, but also each cycle contains a designated vertex subset. Rather than 2-factors, the number of disjoint cycles is controlled. We give two examples to clarify the proposed notion.

Example 1. Fig. 1(a) depicts the Petersen graph. Since the Petersen graph is not hamiltonian, it is not spanning 1-cyclable of any order. However, it is 1-cyclable of order 9. To see that the Petersen graph is spanning 2-cyclable of order 2, we assume that $A_1 = \{1\}$ and $A_2 = \{i\}$ for $i \neq 1$. We set $C_1 = \langle 1, 2, 3, 4, 5, 1\rangle$ and $C_2 = \langle 6, 8, 10, 7, 9, 6\rangle$ if $i \in \{6, 7, 8, 9, 10\}$; we set $C_1 = \langle 1, 5, 4, 9, 6, 1\rangle$ and $C_2 = \langle 2, 3, 8, 10, 7, 2\rangle$ if $i \in \{2, 3\}$; we set $C_1 = \langle 1, 2, 3, 8, 6, 1\rangle$ and $C_2 = \langle 4, 5, 10, 7, 9, 4\rangle$ if $i \in \{4, 5\}$. Then C_1 and C_2 are two disjoint spanning cycles with $A_1 \subset V(C_1)$ and $A_2 \subset V(C_2)$, respectively.

Example 2. Let G be the graph shown in Fig. 1(b). Obviously, G is hamiltonian. Thus, it is spanning 1-cyclable of order 10. However, as an example, it is not 2-cyclable with respect to $A_1 = \{i\}$ and $A_2 = \{i+5\}$ for $i = 0, 1, 2, 3, 4$. As a result, G is not spanning 2-cyclable of order 2.

In this paper, we limit ourself by considering the n-dimensional hypercube Q_n as the underlying graph and study its spanning 2-cyclability. We have the following results: (1) for $n \geq 3$, Q_n is spanning 2-cyclable of order $n-1$; (2) Q_n is spanning k-cyclable of order k if $k \leq n-1$ for $n \geq 2$.

2. Properties of hypercubes

Let $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$ be an n-bit binary string. The *Hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is the number of indices i, $1 \leq i \leq n$, such that $u_i = 1$. Let $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$ and $\mathbf{v} = v_n v_{n-1} \ldots v_2 v_1$ be two n-bit binary strings. The *Hamming distance* $h(\mathbf{u}, \mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$ is the number of different bits in the corresponding strings. The *n-dimensional hypercube*, denoted by Q_n for $n \geq 1$, consists of all n-bit binary strings as its vertices, and two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. Obviously, Q_n is a bipartite graph with bipartition $W = \{\mathbf{u} \in V(Q_n) \mid w(\mathbf{u}) \text{ is even}\}$ and $B = \{\mathbf{u} \in V(Q_n) \mid w(\mathbf{u}) \text{ is odd}\}$. For $i = 0, 1$, let Q_n^i denote the subgraph of Q_n induced by $\{\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1 \mid u_n = i\}$. Obviously, Q_n^i is isomorphic to Q_{n-1} with $n \geq 2$. For any vertex $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$ of Q_n, we use $(\mathbf{u})_j$ to denote the bit u_j, where $1 \leq j \leq n$. Moreover, we use $(\mathbf{u})^k$ to denote the vertex $\mathbf{v} = v_n v_{n-1} \ldots v_2 v_1$ with $u_i = v_i$ for $1 \leq i \neq k \leq n$ and $v_k = 1 - u_k$.

The hypercube Q_n is one of the most popular interconnection networks for parallel computer/communication systems [16]. In the following, we discuss some properties of the hypercube that will be used in this paper.

First, Theorem 1 states that Q_n is hamiltonian laceable and hyper-hamiltonian laceable.

Theorem 1 (*[10,25]*). *Assume that n is any positive integer with $n \geq 2$. Then there exists a hamiltonian path of Q_n joining any two vertices from different partite sets. Moreover, there exists a hamiltonian path of $Q_n - \{\mathbf{x}\}$ joining $\mathbf{y}$ to $\mathbf{z}$ if $\mathbf{x}$ is in one partite set whereas $\mathbf{y}$ and $\mathbf{z}$ are in the other partite set.*

In particular, Lemmas 1 and 2 indicate that $Q_n - \{\mathbf{w}, \mathbf{b}\}$ remains hamiltonian laceable whenever $\mathbf{w}$ and $\mathbf{b}$ are vertices in different partite sets.

Lemma 1 (*[24]*). *Let n be any positive integer with $n \geq 4$. Let W and B form the bipartition of Q_n. Assume that $\mathbf{x}$ and $\mathbf{w}$ are any two different vertices in W, whereas $\mathbf{y}$ and $\mathbf{b}$ are any two different vertices in B. Then there exists a hamiltonian path of $Q_n - \{\mathbf{w}, \mathbf{b}\}$ joining $\mathbf{x}$ and $\mathbf{y}$.*

Lemma 2 (*[14]*). *Let n be any positive integer with $n \geq 4$. Assume that $\mathbf{w}$ and $\mathbf{b}$ are any two adjacent vertices of Q_n, and F is any edge subset of $Q_n - \{\mathbf{w}, \mathbf{b}\}$ with $|F| \leq n - 3$. Then there exists a hamiltonian path of $(Q_n - \{\mathbf{w}, \mathbf{b}\}) - F$ joining any two vertices from different partite sets.*

Theorem 2 generalizes the fault-tolerance of hamiltonian laceability for Q_n, and Theorem 3 gives two types of 2-disjoint-path cover in Q_n.

Theorem 2 (*[24]*). *Assume that $n \geq 3$. Let F_v be a union of f_v disjoint pairs of adjacent vertices in Q_n, and let F_e be a set consisting of f_e edges in Q_n with $f_v + f_e \leq n - 3$. Then there exists a hamiltonian path of $Q_n - (F_v \cup F_e)$ joining any two vertices from different partite sets. Moreover, there exists a hamiltonian path of $Q_n - (F_v \cup F_e \cup \{\mathbf{x}\})$ joining $\mathbf{y}$ and $\mathbf{z}$ if $\mathbf{x}$ is in one partite set, and $\mathbf{y}, \mathbf{z}$ are in the other partite set.*

Theorem 3 (*[15]*). *Let n be any positive integer with $n \geq 4$. Let W and B form the bipartition of Q_n. Assume that $\mathbf{x}$ and $\mathbf{w}$ are any two different vertices in W, $\mathbf{y}$ and $\mathbf{b}$ are any two different vertices in B. There are two disjoint paths P_1 and P_2 in Q_n such that (1) P_1 is a path of length $2^{n-1} - 1$ joining $\mathbf{x}$ and $\mathbf{y}$, (2) P_2 is a path of length $2^{n-1} - 1$ joining $\mathbf{w}$ and $\mathbf{b}$, and (3) $P_1 \cup P_2$ spans Q_n. Moreover, there are two disjoint paths P_3 and P_4 in Q_n such that (1) P_3 is a path joining $\mathbf{x}$ and $\mathbf{w}$, (2) P_4 is a path joining $\mathbf{y}$ and $\mathbf{b}$, and (3) $P_3 \cup P_4$ spans Q_n.*

In the rest of this section, we apply the results introduced above to prove Lemmas 3 and 4, which specify 2-disjoint-path covers in Q_n that are able to contain the prescribed vertices. The two lemmas will be used in the proof of Lemma 5, which is a key result presented in the next section for deriving the spanning 2-cyclability of Q_n.

Lemma 3. *Let W and B form the bipartition of Q_n with $n \geq 4$. Suppose that $\mathbf{x}$ and $\mathbf{u}$ are two different vertices in W, whereas $\mathbf{y}$ and $\mathbf{v}$ are two different vertices in B. Let S be any nonempty subset of $V(Q_n) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ with $|S| \leq n - 3$. Then there are two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{x}$ to $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $S \subseteq P_1$, and (4) $P_1 \cup P_2$ spans Q_n.*

Proof. We prove this lemma by induction on n. We describe in Appendix A that this lemma holds for $n = 4$. Since Q_n is vertex-transitive and edge-transitive, we assume, without loss of generality, that $\mathbf{x}$ is in Q_n^0, and $\mathbf{y}$ is in Q_n^1. For $i \in \{0, 1\}$, we set $W_i = W \cap V(Q_n^i)$, $B_i = B \cap V(Q_n^i)$, and $S_i = S \cap V(Q_n^i)$. We have the following cases.

Case 1: $|S_0| \geq 1$ and $|S_1| \geq 1$. Thus, $|S_0| \leq n - 4$ and $|S_1| \leq n - 4$.

Subcase 1.1: Both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^i for some $i \in \{0, 1\}$. Without loss of generality, we assume that both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^0. Since $|B_0| = 2^{n-2} > (n-3) \geq |S_0 \cup \{\mathbf{v}\}|$ for $n \geq 5$, we can choose any vertex $\mathbf{b}$ from $B_0 - (S_0 \cup \{\mathbf{v}\})$. By induction, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $S_0 \subseteq R_1$, and (4) $R_1 \cup R_2$ spans Q_n^0. By Theorem 1, there is a hamiltonian path H of Q_n^1 joining $(\mathbf{b})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H, \mathbf{y} \rangle$ and $P_2 = R_2$. Obviously, P_1 and P_2 form the desired paths. See Fig. 2(a).

Subcase 1.2: $\mathbf{u}$ is in Q_n^0, and $\mathbf{v}$ is in Q_n^1. We set $T = \{\mathbf{p} \in V(Q_n^0) \mid (\mathbf{p})^n \in S_1\}$. Obviously, $|S_0 \cup T| \leq |S_0| + |T| = |S_0| + |S_1| = |S| \leq n - 3$. Since $|B_0 - (S_0 \cup T)| \geq |B_0| - |S_0 \cup T| \geq 2^{n-2} - (n-3) \geq 2$ for $n \geq 5$, we can choose two distinct vertices $\mathbf{b_1}$ and $\mathbf{b_2}$ in $B_0 - (S_0 \cup T)$. By induction, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b_1}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{b_2}$, (3) $S_0 \subseteq R_1$, and (4) $R_1 \cup R_2$ spans Q_n^0. Moreover, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b_1})^n$ to $\mathbf{y}$, (2) H_2 joins $(\mathbf{b_2})^n$ to $\mathbf{v}$, (3) $S_1 \subseteq H_1$, and (4) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b_1}, (\mathbf{b_1})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_2, \mathbf{b_2}, (\mathbf{b_2})^n, H_2, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 2(b).

Subcase 1.3: $\mathbf{u}$ is in Q_n^1, and $\mathbf{v}$ is in Q_n^0. We set $T = \{\mathbf{p} \in V(Q_n^0) \mid (\mathbf{p})^n \in S_1\}$. Similar to that shown in Subcase 1.2, we have $|B_0 - (S_0 \cup T \cup \{(\mathbf{u})^n\})| \geq 1$ and $|W_0 - (S_0 \cup T \cup \{\mathbf{x}, (\mathbf{y})^n\})| ge1$. Thus, there exists at least one vertex $\mathbf{b}$ in $B_0 - (S_0 \cup T \cup \{(\mathbf{u})^n\})$, and there exists at least one vertex $\mathbf{w}$ in $W_0 - (S_0 \cup T \cup \{\mathbf{x}, (\mathbf{y})^n\})$. By induction, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{w}$ to $\mathbf{v}$, (3) $S_0 \subseteq R_1$, and (4) $R_1 \cup R_2$ spans Q_n^0. Moreover, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b})^n$ to $\mathbf{y}$, (2) H_2 joins $\mathbf{u}$ to $(\mathbf{w})^n$, (3) $S_1 \subseteq H_1$, and (4) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, H_2, (\mathbf{w})^n, \mathbf{w}, R_2, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 2(c).

Case 2: Either $|S_0| = 0$ or $|S_1| = 0$. Without loss of generality, we assume that $|S_0| = 0$.

Subcase 2.1: Both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^0. Let $\mathbf{b}$ be any vertex in $B_0 - \{\mathbf{v}\}$. By Theorem 3, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{v}$, and (3) $R_1 \cup R_2$ spans Q_n^0. By Theorem 1, there is a hamiltonian path

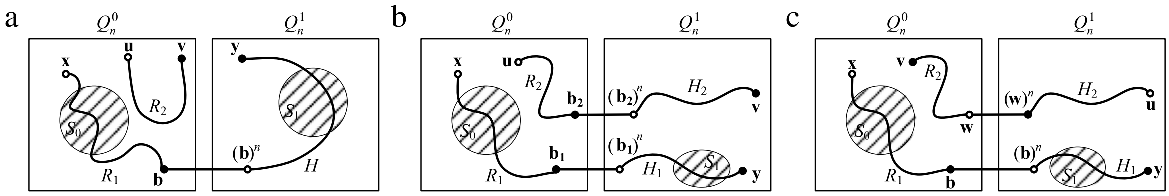

Fig. 2. Illustration for Case 1 of Lemma 3.

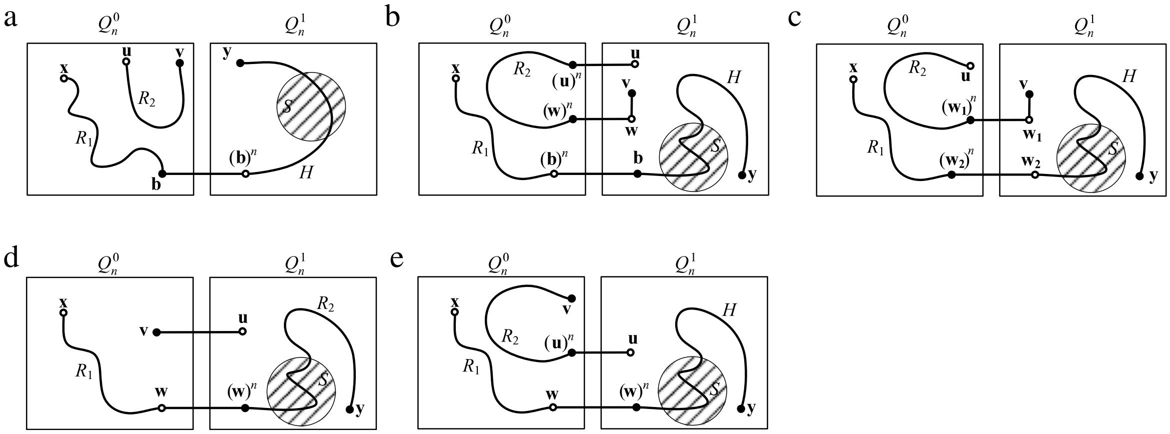

Fig. 3. Illustration for Case 2 of Lemma 3.

H of Q_n^1 joining $(\mathbf{b})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H, \mathbf{y}\rangle$ and $P_2 = R_2$. Obviously, P_1 and P_2 form the desired paths. See Fig. 3(a).

Subcase 2.2: Both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^1. Since $|W_1| > deg_{Q_n^1}(\mathbf{v}) = n-1 > n-2 \geq |S \cup \{\mathbf{u}\}|$, there exists a vertex $\mathbf{w}$ in $W_1 - (S \cup \{\mathbf{u}\})$ such that $(\mathbf{v}, \mathbf{w}) \in E(Q_n)$. Since $|B_1| = 2^{n-2} > n-3 \geq |S_1 \cup \{(\mathbf{x})^n\}|$ for $n \geq 5$, there exists a vertex $\mathbf{b}$ in $B_1 - (S_1 \cup \{(\mathbf{x})^n\})$. By Theorem 2, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ joining $\mathbf{b}$ to $\mathbf{y}$. By Theorem 3, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $(\mathbf{b})^n$, (2) R_2 joins $(\mathbf{u})^n$ to $(\mathbf{w})^n$, and (3) $R_1 \cup R_2$ spans Q_n^0. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{b})^n, \mathbf{b}, H, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, (\mathbf{u})^n, R_2, (\mathbf{w})^n, \mathbf{w}, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 3(b).

Subcase 2.3: $\mathbf{u}$ is in Q_n^0, and $\mathbf{v}$ is in Q_n^1. Obviously, there exists a vertex $\mathbf{w_1}$ in $W_1 - S_1$ such that $(\mathbf{v}, \mathbf{w_1}) \in E(Q_n^1)$. Let $\mathbf{w_2}$ be a vertex in $W_1 - \{\mathbf{w_1}\}$. By Theorem 2, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{v}, \mathbf{w_1}\}$ joining $\mathbf{w_2}$ to $\mathbf{y}$. By Theorem 3, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $(\mathbf{w_2})^n$, (2) R_2 joins $\mathbf{u}$ to $(\mathbf{w_1})^n$, and (3) $R_1 \cup R_2$ spans Q_n^0. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w_2})^n, \mathbf{w_2}, H, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, R_2, (\mathbf{w_1})^n, \mathbf{w_1}, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 3(c).

Subcase 2.4: $\mathbf{u}$ is in Q_n^1, and $\mathbf{v}$ is in Q_n^0.

Suppose that $(\mathbf{u}, \mathbf{v}) \in E(Q_n)$. Let $\mathbf{w}$ be any vertex in W_0. By Theorem 1, there exists a hamiltonian path R_1 of $Q_n^0 - \{\mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{w}$, and there exists a hamiltonian path R_2 of $Q_n^1 - \{\mathbf{u}\}$ joining $(\mathbf{w})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{w}, (\mathbf{w})^n, R_2, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 3(d).

Suppose that $(\mathbf{u}, \mathbf{v}) \notin E(Q_n)$. Let $\mathbf{w}$ be any vertex in $W_0 - \{\mathbf{x}, (\mathbf{y})^n\}$. By Theorem 3, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{w}$, (2) R_2 joins $(\mathbf{u})^n$ to $\mathbf{v}$, and (3) $R_1 \cup R_2$ spans Q_n^0. By Theorem 1, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{u}\}$ joining $(\mathbf{w})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{w}, (\mathbf{w})^n, H, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, (\mathbf{u})^n, R_2, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 3(e). □

Lemma 4. *Let W and B form the bipartition of Q_n with $n \geq 5$. Let $\mathbf{p}$, $\mathbf{x}$, and $\mathbf{y}$ be three different vertices in W, and let $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$ be three different vertices in B such that $\{(\mathbf{p}, \mathbf{q}), (\mathbf{x}, \mathbf{u}), (\mathbf{x}, \mathbf{v})\} \subset E(Q_n)$. Then there exist two disjoint paths P_1 and P_2 in $Q_n - \{\mathbf{p}, \mathbf{q}\}$ such that (1) P_1 joins $\mathbf{x}$ to $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ to $\mathbf{v}$, and (3) $P_1 \cup P_2$ spans $Q_n - \{\mathbf{p}, \mathbf{q}\}$.*

Proof. Since $n \geq 5$, there exists an integer $1 \leq k \leq n$ such that $\mathbf{q} \neq (\mathbf{p})^k$, $\mathbf{u} \neq (\mathbf{x})^k$, and $\mathbf{v} \neq (\mathbf{x})^k$. By the symmetric property of Q_n, we can assume $k = n$. Without loss of generality, we consider that both $\mathbf{p}$ and $\mathbf{q}$ are in Q_n^0. For $i \in \{0, 1\}$, we set $W_i = W \cap V(Q_n^i)$ and $B_i = B \cap V(Q_n^i)$. Note that $\{\mathbf{x}, \mathbf{u}, \mathbf{v}\} \subset V(Q_n^i)$ for some $i \in \{0, 1\}$. We have the following cases.

Case 1: $\{\mathbf{x}, \mathbf{u}, \mathbf{v}\} \subset V(Q_n^0)$ and $\mathbf{y} \in V(Q_n^1)$. By Theorem 2, there exists a hamiltonian path R of $Q_n^0 - \{\mathbf{p}, \mathbf{q}, \mathbf{x}\}$ joining $\mathbf{u}$ and $\mathbf{v}$. By Theorem 1, there exists a hamiltonian path H of Q_n^1 joining $(\mathbf{x})^n$ and $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, (\mathbf{x})^n, H, \mathbf{y}\rangle$ and $P_2 = R$. Obviously, P_1 and P_2 form the required paths. See Fig. 4(a).

Case 2: $\mathbf{y} \in V(Q_n^0)$ and $\{\mathbf{x}, \mathbf{u}, \mathbf{v}\} \subset V(Q_n^1)$. Since $|B_0| = 2^{n-2} > 2$, there exists a vertex $\mathbf{b}$ in $B_0 - \{\mathbf{q}, (\mathbf{x})^n\}$. By Theorem 2, there exists a hamiltonian path R of $Q_n^0 - \{\mathbf{p}, \mathbf{q}\}$ joining $\mathbf{b}$ and $\mathbf{y}$. By Theorem 3, there exist two disjoint paths H_1 and H_2 in

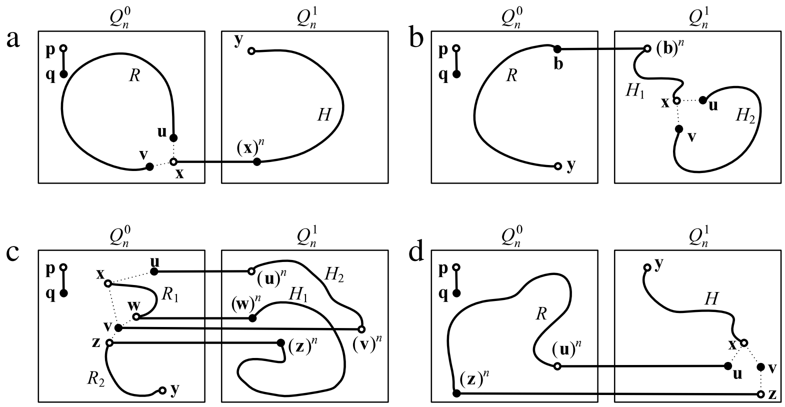

Fig. 4. Illustration for Lemma 4.

Q_n^1 such that (1) H_1 joins $\mathbf{x}$ and $(\mathbf{b})^n$, (2) H_2 joins $\mathbf{u}$ to $\mathbf{v}$, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, H_1, (\mathbf{b})^n, \mathbf{b}, R, \mathbf{y} \rangle$ and $P_2 = H_2$. Obviously, P_1 and P_2 form the required paths. See Fig. 4(b).

Case 3: $\{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\} \subset V(Q_n^0)$. By Theorem 2, there exists a hamiltonian path R of $Q_n^0 - \{\mathbf{p}, \mathbf{q}, \mathbf{u}\}$ joining $\mathbf{x}$ and $\mathbf{y}$. Without loss of generality, we write $R = \langle \mathbf{x}, R_1, \mathbf{w}, \mathbf{v}, \mathbf{z}, R_2, \mathbf{y} \rangle$. By Theorem 1, there exist two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{w})^n$ and $(\mathbf{z})^n$, (2) H_2 joins $(\mathbf{u})^n$ to $(\mathbf{v})^n$, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{w}, (\mathbf{w})^n, H_1, (\mathbf{z})^n, \mathbf{z}, R_2, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, (\mathbf{u})^n, H_2, (\mathbf{v})^n, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the required paths. See Fig. 4(c).

Case 4: $\{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\} \subset V(Q_n^1)$. Obviously, either $\mathbf{u} \neq (\mathbf{p})^n$ or $\mathbf{v} \neq (\mathbf{p})^n$. Without loss of generality, we assume that $\mathbf{u} \neq (\mathbf{p})^n$. Since $deg_{Q_n^1}(\mathbf{v}) > 3$, there exists a vertex $\mathbf{z}$ in $W_1 - \{\mathbf{x}, \mathbf{y}, (\mathbf{q})^n\}$ such that $(\mathbf{z}, \mathbf{v}) \in E(Q_n)$. By Theorem 2, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{u}, \mathbf{v}, \mathbf{z}\}$ joining $\mathbf{x}$ and $\mathbf{y}$, and there exists a hamiltonian R of $Q_n^0 - \{\mathbf{p}, \mathbf{q}\}$ joining $(\mathbf{u})^n$ and $(\mathbf{z})^n$. We set $P_1 = H$ and $P_2 = \langle \mathbf{u}, (\mathbf{u})^n, R, (\mathbf{z})^n, \mathbf{z}, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the required paths. See Fig. 4(d). □

3. Two disjoint cycles span hypercubes

A bipartite graph G, with bipartition W and B, is called 2-disjoint-path-coverable of order t if for any $\{x, u\} \subset W$, $\{y, v\} \subset B$, and any two disjoint subsets A_1, A_2 of $V(G) - \{x, y, u, v\}$ with $|A_1 \cup A_2| \le t$, there exists two disjoint paths P_1 and P_2 of G such that (1) P_1 joins x and y, (2) P_2 joins u and v, (3) $A_1 \subseteq P_1$, (4) $A_2 \subseteq P_2$, and (5) $P_1 \cup P_2$ spans G. The following lemma is the key result to derive a tight lower bound of spanning 2-cyclability of Q_n. Our proof idea is based on constructing two disjoint paths that can span Q_n and cover any two disjoint vertex subsets with the sum of orders not exceeding $n - 3$. The proof will be divided into various cases, each of which may consist of a number of subcases. To stress the main contribution of this paper, we thus defer those tedious details to Appendix B for the sake of clarity.

Lemma 5. *Suppose that $n \ge 3$. Then, Q_n is 2-disjoint-path-coverable of order $n - 3$.*

The following theorem holds directly from Lemma 5.

Theorem 4. *Assume that $n \ge 4$. Let A_1 and A_2 be any two disjoint vertex subsets of Q_n with $|A_1 \cup A_2| \le n - 1$. Then there exist two disjoint cycles C_1 and C_2 of Q_n such that (1) $A_1 \subseteq C_1$ (2) $A_2 \subseteq C_2$, and (3) $C_1 \cup C_2$ spans Q_n.*

Proof. Without loss of generality, we consider $|A_1 \cup A_2| = n - 1$. There are two cases as follows.

Case 1: Both A_1 and A_2 are nonempty. Thus, $|A_1| \le n - 2$ and $|A_2| \le n - 2$. Since $|A_1| + |A_2| = n - 1 \ge 3$, we may assume, without loss of generality, that $|A_1| \ge 2$. Let $\mathbf{u}$ be a vertex in A_2. Since $deg_{Q_n}(\mathbf{u}) = n > n - 2 \ge |A_1|$, there exists a vertex $\mathbf{v}$ in $Nbd_{Q_n}(\mathbf{u}) - A_1$. (Note that it is possible that $\mathbf{v}$ is in A_2.) Let $\mathbf{x}$ and $\mathbf{x}'$ be any two distinct vertices in A_1. Since $|(Nbd_{Q_n}(\mathbf{x}) \cup Nbd_{Q_n}(\mathbf{x}')) - \{\mathbf{x}, \mathbf{x}'\}| \ge 2n - 2 > n \ge |A_1 \cup A_2 \cup \{\mathbf{v}\}|$ for $n \ge 4$, there exists a vertex $\mathbf{y}$ in $(Nbd_{Q_n}(\mathbf{x}) \cup Nbd_{Q_n}(\mathbf{x}')) - (A_1 \cup A_2 \cup \{\mathbf{v}\})$. Without loss of generality, we assume that $\mathbf{y} \in Nbd_{Q_n}(\mathbf{x})$. Let $A_1' = A_1 - \{\mathbf{x}\}$ and $A_2' = A_2 - \{\mathbf{u}, \mathbf{v}\}$. Obviously, $|A_1' \cup A_2'| \le n - 3$. By Lemma 5, there exist two disjoint paths P_1 and P_2 in Q_n such that (1) P_1 joins $\mathbf{x}$ and $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ and $\mathbf{v}$, (3) $A_1 \subseteq V(P_1)$, (4) $A_2 \subseteq V(P_2)$, and (5) $P_1 \cup P_2$ spans Q_n. We set $C_1 = \langle \mathbf{x}, P_1, \mathbf{y}, \mathbf{x} \rangle$ and $C_2 = \langle \mathbf{u}, P_2, \mathbf{v}, \mathbf{u} \rangle$. Obviously, C_1 and C_2 form the required cycles in Q_n.

Case 2: A_1 or A_2 is empty. We can assume that A_1 is empty. First, we consider $n \ge 5$. Obviously, there exists a cycle C_1 of length 4 in Q_n such that $V(C_1) \cap A_2 = \emptyset$. By Theorem 2, there exists a hamiltonian cycle C_2 of $Q_n - V(C_1)$. Then, we have $A_2 \subseteq C_2$.

On the other hand, we consider $n = 4$. Since Q_4 is both vertex-symmetric and edge-symmetric, we assume that $|A_2 \cap V(Q_4^i)| = 1$ and $|A_2 \cap V(Q_4^{1-i})| = 2$ with $i \in \{0, 1\}$. For convenience, let $A_2 \cap V(Q_4^i) = \{\mathbf{s}\}$. Obviously, there exists a cycle C_1 of length 4 in Q_4^i not containing $\mathbf{s}$. Moreover, $Q_4^i - V(C_1)$ is a cycle of length 4, denoted by $\langle \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{s} \rangle$. Then, we can find a hamiltonian path P of Q_4^{1-i} joining $(\mathbf{s})^4$ and $(\mathbf{t})^4$. As a result, $C_2 = \langle \mathbf{s}, (\mathbf{s})^4, P, (\mathbf{t})^4, \mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{s} \rangle$ and C_1 form the requested cycles. □

According to Theorem 4, Q_n is spanning 2-cyclable of order $n-1$ for $n \geq 4$. For Q_3, let $A_1 = \{\mathbf{x}\}$ and $A_2 = \{\mathbf{u}\}$, where $\mathbf{x}$ and $\mathbf{u}$ are different vertices of Q_3. Since Q_3 is vertex-symmetric and edge-symmetric, we assume that $\mathbf{x}$ is in Q_3^0, and $\mathbf{u}$ is in Q_3^1. Clearly, both Q_3^0 and Q_3^1 are isomorphic to Q_2, which is a cycle of length 4. Thus, Q_3 is spanning 2-cyclable of order 2. We summarize the first main result of this paper as follows.

Corollary 1. *The n-cube Q_n is spanning 2-cyclable of order $n-1$ for $n \geq 3$.*

To study the generalized spanning k-cyclability of Q_n for $k \geq 3$, we argue by induction that Q_n is spanning k-cyclable of order k if $k \leq n-1$. Trivially, Q_2 is spanning 1-cyclable of order 1. As the inductive hypothesis, we assume that Q_{n-1} is spanning r-cyclable of order r for $r \leq n-2$ with $n \geq 3$. Let $A = \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_k}\}$ consist of any k vertices of Q_n with $k \leq n-1$. By the symmetric property of Q_n, we may assume that $\mathbf{u_1}$ is in Q_n^0, and $\mathbf{u_k}$ is in Q_n^1. We set $A_i = A \cap V(Q_n^i)$ for $i \in \{0, 1\}$. Then, A is partitioned into two nonempty subsets A_0 and A_1. Let $t = |A_0|$. Without loss of generality, we may assume that $\mathbf{u_i} \in A_0$ if $1 \leq i \leq t$, and $\mathbf{u_i} \in A_1$ if $t < i \leq k$. Note that Q_n^i is isomorphic to Q_{n-1} for $i = 0, 1$. By induction, there exist t disjoint cycles $C_1, C_2, \dots, C_t$ of Q_n^0 such that $\mathbf{u_i}$ is in C_i for $1 \leq i \leq t$ and $C_1 \cup C_2 \cup \cdots \cup C_t$ spans Q_n^0, and there exist $k-t$ disjoint cycles $C_{t+1}, C_{t+2}, \dots, C_k$ of Q_n^1 such that $\mathbf{u_i}$ is in C_i for $t+1 \leq i \leq k$ and $C_{t+1} \cup C_{t+2} \cup \cdots \cup C_k$ spans Q_n^1. As a result, $C_1, C_2, \dots, C_k$ form k disjoint cycles of Q_n such that $\mathbf{u_i}$ is in C_i for $1 \leq i \leq k$ and $C_1 \cup C_2 \cup \cdots \cup C_k$ spans Q_n. For clarity, this result is summarized below.

Theorem 5. *The n-cube Q_n is spanning k-cyclable of order k if $k \leq n-1$ for $n \geq 2$.*

We give an example to indicate that Q_n is not spanning n-cyclable of order n. Let $\mathbf{u}$ be any vertex of Q_n, and let $\{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}\}$ be the set of vertices adjacent to $\mathbf{u}$. We set $A = \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_{n-1}}\} \cup \{\mathbf{u}\}$. Obviously, $|A| = n$. Since $deg_{Q_n - \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_{n-1}}\}}(\mathbf{u}) = 1$, there is no cycle of $G - \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_{n-1}}\}$ containing $\mathbf{u}$. Thus, we cannot find n cycles $C_1, C_2, \dots, C_n$ of Q_n such that $\mathbf{u_i}$ is in C_i for $1 \leq i \leq n-1$, and $\mathbf{u}$ is in C_n.

4. Concluding remarks

In this paper we proved that Q_n is spanning 2-cyclable of order $n-1$ for $n \geq 3$. Now we show an example to indicate that Q_n is not 2-cyclable of order n. Let $\mathbf{u}$ and $\mathbf{v}$ be any two adjacent vertices of Q_n. We set $A_1 = Nbd_{Q_n}(\mathbf{u}) - \{\mathbf{v}\}$ and $A_2 = \{\mathbf{u}\}$. Obviously, $|A_1| + |A_2| = n$. Since $deg_{Q_n - A_1}(\mathbf{u}) = 1$, there is no cycle of $G - A_1$ containing A_2. Thus, the spanning 2-cyclability of Q_n is $n-1$ for $n \geq 3$, and this result is optimal. Furthermore, we proved that Q_n is spanning k-cyclable of order k if $k \leq n-1$ for $n \geq 2$.

For possible future directions with our result, we first conjecture that Q_n is spanning k-cyclable of order $n-1$ for every $k \leq n-1$ and $n \geq 3$. As we allowed A_1 or A_2 to be empty set in the statement of Theorem 4, we indeed have a stronger conjecture: assume that $n \geq 4$. Let $A_1, A_2, \dots, A_k$ be k disjoint vertex subsets of Q_n with $|A_1 \cup A_2 \cup \cdots \cup A_k| \leq n-1$ and $k \leq n-1$, there exist k disjoint cycles $C_1, C_2, \dots, C_k$ of Q_n such that (1) A_i is in C_i for $1 \leq i \leq k$, and (2) $C_1 \cup C_2 \cup \cdots \cup C_k$ spans Q_n. Notice that the statement is not always true for $n = 3$. For counterexample, let $A_1 = \{000, 111\}$ and $A_2 = \emptyset$. Then the length of any cycle containing A_1 is at least 6. Thus, we cannot find two disjoint cycles C_1 and C_2 of Q_3 such that (1) A_i is in C_i for $1 \leq i \leq 2$, and (2) $C_1 \cup C_2$ spans Q_3.

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Appendix A. Q_4 is 2-disjoint-path-coverable of order one

We prepare the following lemma in advance.

Lemma 6. *Let $\mathbf{p}$ and $\mathbf{q}$ be any two adjacent vertices of Q_3. Let $\mathbf{u}$ and $\mathbf{v}$ be any two nonadjacent vertices of $Q_3 - \{\mathbf{p}, \mathbf{q}\}$ such that they are in different partite sets. Then there exists a hamiltonian path of $Q_3 - \{\mathbf{p}, \mathbf{q}\}$ joining $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Since Q_3 is vertex-symmetric and edge-symmetric, we assume that $\mathbf{p} = 000$ and $\mathbf{q} = 001$. We have $\{\mathbf{u}, \mathbf{v}\} \in \big\{\{011, 100\}, \{101, 010\}\big\}$. Clearly, both $\langle 011, 010, 110, 111, 101, 100\rangle$ and $\langle 101, 100, 110, 111, 011, 010\rangle$ are hamiltonian paths of $Q_3 - \{\mathbf{p}, \mathbf{q}\}$. □

Recall that W and B form the bipartition of Q_4. Let $A_1 = \{\mathbf{z}\}$ and $A_2 = \emptyset$, where $\mathbf{z}$ is any vertex of $Q_4 - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$. Since Q_4 is vertex-symmetric and edge-symmetric, we assume that $\mathbf{u} = 0000$ and $\mathbf{v} \in \{0001, 0111\}$.

Case 1: $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \subset V(Q_4^1)$. By Theorem 1, there exists a hamiltonian path P_1 of Q_4^1 joining $\mathbf{x}$ and $\mathbf{y}$, and there exists a hamiltonian path P_2 of Q_4^0 joining $\mathbf{u}$ and $\mathbf{v}$.

Table 1
The vertex **b** and paths R_1 and R_2.

	R_1	R_2
$\mathbf{x} = 0011, \mathbf{z} = 0101$	$\langle 0011, 0001, 0101, 0100 = \mathbf{b}\rangle$	$\langle 0000, 0010, 0110, 0111\rangle$
$\mathbf{x} = 0011, \mathbf{z} = 0110$	$\langle 0011, 0010, 0110, 0100 = \mathbf{b}\rangle$	$\langle 0000, 0001, 0101, 0111\rangle$
$\mathbf{x} = 0101, \mathbf{z} = 0011$	$\langle 0101, 0001, 0011, 0010 = \mathbf{b}\rangle$	$\langle 0000, 0100, 0110, 0111\rangle$
$\mathbf{x} = 0101, \mathbf{z} = 0110$	$\langle 0101, 0100, 0110, 0010 = \mathbf{b}\rangle$	$\langle 0000, 0001, 0011, 0111\rangle$
$\mathbf{x} = 0110, \mathbf{z} = 0011$	$\langle 0110, 0010, 0011, 0001 = \mathbf{b}\rangle$	$\langle 0000, 0100, 0101, 0111\rangle$
$\mathbf{x} = 0110, \mathbf{z} = 0101$	$\langle 0110, 0100, 0101, 0001 = \mathbf{b}\rangle$	$\langle 0000, 0010, 0011, 0111\rangle$

Table 2
The path P_1.

x	**y**	P_1
0011	0001	$\langle 0011, 0010, 0110, 0100, 0101, 0001\rangle$
0011	0010	$\langle 0011, 0001, 0101, 0100, 0110, 0010\rangle$
0101	0001	$\langle 0101, 0100, 0110, 0010, 0011, 0001\rangle$
0101	0100	$\langle 0101, 0001, 0011, 0010, 0110, 0100\rangle$
0110	0010	$\langle 0110, 0100, 0101, 0001, 0011, 0010\rangle$
0110	0100	$\langle 0110, 0010, 0011, 0001, 0101, 0100\rangle$

Case 2: Either $\{\mathbf{x}\} \subset V(Q_4^0)$, $\{\mathbf{y}, \mathbf{z}\} \subset V(Q_4^1)$ or $\{\mathbf{y}\} \subset V(Q_4^0)$, $\{\mathbf{x}, \mathbf{z}\} \subset V(Q_4^1)$. Without loss of generality, we only consider that $\{\mathbf{x}\} \subset V(Q_4^0)$ and $\{\mathbf{y}, \mathbf{z}\} \subset V(Q_4^1)$. Let $\mathbf{b} \in B \cap V(Q_4^0) - \{\mathbf{v}\}$. By Theorem 3, there exist two disjoint paths R_1 and R_2 of Q_4^0 such that (1) R_1 joins **x** and **b**, (2) R_2 joins **u** and **v**, and (3) $R_1 \cup R_2$ spans Q_4^0. By Theorem 1, there exists a hamiltonian path H of Q_4^1 joining $(\mathbf{b})^4$ and **y**. Then, we set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^4, H, \mathbf{y}\rangle$ and $P_2 = R_2$.

Case 3: $\{\mathbf{z}\} \subset V(Q_4^0)$, $\{\mathbf{x}, \mathbf{y}\} \subset V(Q_4^1)$. Since $deg_{Q_4^0}(\mathbf{z}) = 3 > 2$, we can choose a vertex **s** of $Q_4^0 - \{(\mathbf{x})^4, (\mathbf{y})^4, \mathbf{u}, \mathbf{v}\}$ such that $(\mathbf{s}, \mathbf{z}) \in E(Q_4)$. Note that both $(\mathbf{x})^4$ and **v** are in B, and both $(\mathbf{y})^4$ and **u** are in W. Let $\{\mathbf{w}, \mathbf{b}\} = \{\mathbf{s}, \mathbf{z}\}$ such that $\mathbf{w} \in W$ and $\mathbf{b} \in B$. By Theorem 3, there exist two disjoint paths R_1 and R_2 of Q_4^1 such that (1) R_1 joins **x** and $(\mathbf{w})^4$, (2) R_2 joins $(\mathbf{b})^4$ and **y**, and (3) $R_1 \cup R_2$ spans Q_4^1. Then, P_1 is set to be $\langle \mathbf{x}, R_1, (\mathbf{w})^4, \mathbf{w}, \mathbf{b}, (\mathbf{b})^4, R_2, \mathbf{y}\rangle$. By Lemma 6, there exists a hamiltonian path P_2 of $Q_4^0 - \{\mathbf{w}, \mathbf{b}\}$ joining **u** and **v**.

Case 4: $\{\mathbf{x}, \mathbf{y}\} \subset V(Q_4^0)$, $\{\mathbf{z}\} \subset V(Q_4^1)$. By Theorem 3, there exist two disjoint paths R_1 and R_2 of Q_4^0 such that (1) R_1 joins **x** and **y**, (2) R_2 joins **u** and **v**, and (3) $R_1 \cup R_2$ spans Q_4^0. We write R_1 as $\langle \mathbf{x}, H_1, \mathbf{w}, \mathbf{y}\rangle$. By Theorem 1, there exists a hamiltonian path H_2 of Q_4^1 joins $(\mathbf{w})^4$ and $(\mathbf{y})^4$. We set $P_1 = \langle \mathbf{x}, H_1, \mathbf{w}, (\mathbf{w})^4, H_2, (\mathbf{y})^4, \mathbf{y}\rangle$ and $P_2 = R_2$.

Case 5: $\{\mathbf{x}, \mathbf{z}\} \subset V(Q_4^0)$, $\{\mathbf{y}\} \subset V(Q_4^1)$.

Subcase 5.1: Suppose that $\mathbf{z} \in B$. By Theorem 3, there exist two disjoint paths R_1 and R_2 of Q_4^0 such that (1) R_1 joins **x** and **z**, (2) R_2 joins **u** and **v**, and (3) $R_1 \cup R_2$ spans Q_4^0. By Theorem 1, there exists a hamiltonian path H of Q_4^1 joining $(\mathbf{z})^4$ and **y**. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{z}, (\mathbf{z})^4, H, \mathbf{y}\rangle$ and $P_2 = R_2$.

Subcase 5.2: Suppose that $\mathbf{z} \in W$ and $\mathbf{v} = 0001$. By Theorem 1, there exists a hamiltonian path R of $Q_4^0 - \{\mathbf{v}\}$ joining **x** and **u**. We write R as $\langle \mathbf{x}, R', \mathbf{b}, \mathbf{u}\rangle$. Similarly, there exists a hamiltonian path H of Q_4^1 joining $(\mathbf{b})^4$ and **y**. Then we set $P_1 = \langle \mathbf{x}, R', \mathbf{b}, (\mathbf{b})^4, H, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{v}\rangle$.

Subcase 5.3: Suppose that $\mathbf{z} \in W$ and $\mathbf{v} = 0111$. We have $\{\mathbf{x}, \mathbf{z}\} \subset \{0011, 0101, 0110\}$. We set a vertex **b** and paths R_1 and R_2 according to Table 1. By Theorem 1, there exists a hamiltonian path H of Q_4^1 joining $(\mathbf{b})^4$ and **y**. Then, $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^4, H, \mathbf{y}\rangle$ and $P_2 = R_2$ are the requested paths.

Case 6: $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \subset V(Q_4^0)$.

Subcase 6.1: $\mathbf{v} = 0001$. By Theorem 1, there exists a hamiltonian path R of $Q_4^0 - \{\mathbf{v}\}$. We write R as $\langle \mathbf{x}, R_1, \mathbf{w}, \mathbf{y}, R_2, \mathbf{b}, \mathbf{u}\rangle$. Similarly, there exists a hamiltonian path H of Q_4^1 joining $(\mathbf{w})^4$ and $(\mathbf{b})^4$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{w}, (\mathbf{w})^4, H, (\mathbf{b})^4, \mathbf{b}, rev(R_2), \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{v}\rangle$, where $rev(R_2)$ is the reverse path of R_2.

Subcase 6.2: $\mathbf{v} = 0111$.

(i) $(\mathbf{x}, \mathbf{y}) \notin \{(0011, 0100), (0101, 0010), (0110, 0101)\}$. We set P_1 according to Table 2. Obviously, P_1 is a hamiltonian path of $Q_4^0 - \{\mathbf{u}, \mathbf{v}\}$. By Theorem 1, there exists a hamiltonian path H of Q_4^1 joining $(\mathbf{u})^4$ and $(\mathbf{v})^4$. Then, we set P_2 as $\langle \mathbf{u}, (\mathbf{u})^4, H, (\mathbf{v})^4, \mathbf{v}\rangle$.

(ii) $(\mathbf{x}, \mathbf{y}) \in \{(0011, 0100), (0101, 0010), (0110, 0101)\}$. We set R_1 and R_2 according to Table 3. Clearly, $R_1 \cup R_2$ spans Q_4^0, and we can write R_2 as $\langle \mathbf{u}, R_2', \mathbf{w}, \mathbf{v}\rangle$. By Theorem 1, there exists a hamiltonian path H of Q_4^1 joins $(\mathbf{w})^4$ and $(\mathbf{v})^4$. Then we set $P_1 = R_1$ and $P_2 = \langle \mathbf{u}, R_2', \mathbf{w}, (\mathbf{w})^4, H, (\mathbf{v})^4, \mathbf{v}\rangle$.

Appendix B. Proof of Lemma 5

To prove that Q_n is 2-disjoint-path-coverable of order $n - 3$, we prepare four propositions as follows. In the rest of this paper, we continue using W and B to denote the bipartition of Q_n. For convenience, we also call W and B partite sets of *white* and *black* vertices, respectively.

Table 3
The paths R_1 and R_2.

	R_1	R_2
$\mathbf{x} = 0011, \mathbf{y} = 0100, \mathbf{z} \in \{0001, 0101\}$	$\langle 0011, 0001, 0101, 0100 \rangle$	$\langle 0000, 0010, 0110, 0111 \rangle$
$\mathbf{x} = 0011, \mathbf{y} = 0100, \mathbf{z} \in \{0010, 0110\}$	$\langle 0011, 0010, 0110, 0100 \rangle$	$\langle 0000, 0001, 0101, 0111 \rangle$
$\mathbf{x} = 0101, \mathbf{y} = 0010, \mathbf{z} \in \{0001, 0011\}$	$\langle 0101, 0001, 0011, 0010 \rangle$	$\langle 0000, 0100, 0110, 0111 \rangle$
$\mathbf{x} = 0101, \mathbf{y} = 0010, \mathbf{z} \in \{0100, 0110\}$	$\langle 0101, 0100, 0110, 0010 \rangle$	$\langle 0000, 0001, 0011, 0111 \rangle$
$\mathbf{x} = 0110, \mathbf{y} = 0101, \mathbf{z} \in \{0100, 0101\}$	$\langle 0110, 0100, 0101, 0001 \rangle$	$\langle 0000, 0010, 0011, 0111 \rangle$
$\mathbf{x} = 0110, \mathbf{y} = 0101, \mathbf{z} \in \{0010, 0011\}$	$\langle 0110, 0010, 0011, 0001 \rangle$	$\langle 0000, 0100, 0101, 0111 \rangle$

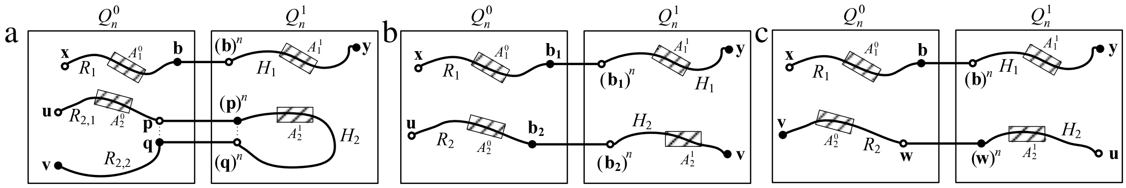

Fig. 5. Illustration for Proposition 1.

Proposition 1. *Let W and B form the bipartition of Q_n with $n \geq 7$. Suppose that* $\mathbf{x}$ *and* $\mathbf{u}$ *are any two different vertices in W, whereas* $\mathbf{y}$ *and* $\mathbf{v}$ *are any two different vertices in B. Furthermore, suppose that $\mathbf{x} \in V(Q_n^0)$, $\mathbf{y} \in V(Q_n^1)$, and $\mathbf{y} \neq (\mathbf{u})^n$. Let A_1^0 and A_2^0 be any two disjoint nonempty subsets of $V(Q_n^0) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$, and let A_1^1 and A_2^1 be any two disjoint nonempty subsets of $V(Q_n^1) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ such that $|A_1^0| + |A_1^1| + |A_2^0| + |A_2^1| = n - 3$. Assume that Q_{n-1} is 2-disjoint-path-coverable of order $n - 4$. Then, there exist two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{x}$ to $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $A_1^0 \cup A_1^1 \subseteq P_1$, (4) $A_2^0 \cup A_2^1 \subseteq P_2$, and (5) $P_1 \cup P_2$ spans Q_n.*

Proof. Obviously, $|A_i^j| \leq n - 6$ for $i \in \{1, 2\}$ and $j \in \{0, 1\}$, and $|A_1^1| + |A_2^1| + |\{\mathbf{y}\}| \leq n - 4$. We have the following two cases.

Case 1: Both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^j for some $j \in \{0, 1\}$. Without loss of generality, we assume that $j = 0$. Since $|V(Q_n^0)| = 2^{n-1} > n(n-4) + (n-3) = n^2 - 3n - 3 \geq n|A_1^1 \cup A_2^1 \cup \{\mathbf{y}\}| + |A_1^0 \cup \{\mathbf{x}, \mathbf{u}, \mathbf{v}\}|$ and $2^{n-2} > n - 3$ for $n \geq 7$, there exists a vertex $\mathbf{p}$ in $V(Q_n^0) - (A_1^0 \cup \{\mathbf{x}, \mathbf{u}, \mathbf{v}\})$ such that $(\mathbf{t})^n \notin A_1^1 \cup A_2^1 \cup \{\mathbf{y}\}$ for every $\mathbf{t} \in Nbd_{Q_n^0}(\mathbf{p}) \cup \{\mathbf{p}\}$, and there exists a black vertex $\mathbf{b}$ in $V(Q_n^0) - (A_2^0 \cup \{\mathbf{v}, \mathbf{p}\})$ such that $(\mathbf{b})^n \notin A_2^1$. Since Q_{n-1} is 2-disjoint-path-coverable of order $n - 4$, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $A_1^0 \subseteq R_1$, (4) $A_2^0 \cup \{\mathbf{p}\} \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. Without loss of generality, we write R_2 as $\langle \mathbf{u}, R_{2,1}, \mathbf{p}, \mathbf{q}, R_{2,2}, \mathbf{v} \rangle$. Again, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b})^n$ to $\mathbf{y}$, (2) H_2 joins $(\mathbf{p})^n$ to $(\mathbf{q})^n$, (3) $A_1^1 \subseteq H_1$, (4) $A_2^1 \subseteq H_2$, and (5) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_{2,1}, \mathbf{p}, (\mathbf{p})^n, H_2, (\mathbf{q})^n, \mathbf{q}, R_{2,2}, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 5(a).

Case 2: $\mathbf{u}$ is in Q_n^j, and $\mathbf{v}$ is in Q_n^{1-j} for $j \in \{0, 1\}$. On the one hand, we assume that $j = 0$; that is, $\mathbf{u}$ is in Q_n^0, and $\mathbf{v}$ is in Q_n^1. Since $2^{n-2} > n - 4$ for $n \geq 7$, there exists a black vertex $\mathbf{b_1}$ in $V(Q_n^0) - A_2^0$ such that $(\mathbf{b_1})^n \notin A_2^1$, and there exists a black vertex $\mathbf{b_2}$ in $V(Q_n^0) - (A_1^0 \cup \{\mathbf{b_1}\})$ such that $(\mathbf{b_2})^n \notin A_1^1$. Since Q_{n-1} is 2-disjoint-path-coverable of order $n - 4$, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b_1}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{b_2}$, (3) $A_1^0 \subseteq R_1$, (4) $A_2^0 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0; and there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b_1})^n$ to $\mathbf{y}$, (2) H_2 joins $(\mathbf{b_2})^n$ to $\mathbf{v}$, (3) $A_1^1 \subseteq H_1$, (4) $A_2^1 \subseteq H_2$, and (5) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b_1}, (\mathbf{b_1})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_2, \mathbf{b_2}, (\mathbf{b_2})^n, H_2, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 5(b).

On the other hand, if $j = 1$, then $\mathbf{u}$ is in Q_n^1, and $\mathbf{v}$ is in Q_n^0. Since $2^{n-2} > n - 3$ for $n \geq 7$, there exists a black vertex $\mathbf{b}$ in $V(Q_n^0) - (A_2^0 \cup \{(\mathbf{u})^n, \mathbf{v}\})$ such that $(\mathbf{b})^n \notin A_2^1$, and there exists a white vertex $\mathbf{w}$ in $V(Q_n^0) - (A_1^0 \cup \{\mathbf{x}, (\mathbf{y})^n\})$ such that $(\mathbf{w})^n \notin A_1^1$. Similarly, there exist disjoint paths R_1, R_2, H_1, H_2 joining $\mathbf{x}$ to $\mathbf{b}$, $\mathbf{w}$ to $\mathbf{v}$, $(\mathbf{b})^n$ to $\mathbf{y}$, and $\mathbf{u}$ to $(\mathbf{w})^n$, respectively, such that (1) $A_1^0 \subseteq R_1, A_2^0 \subseteq R_2, A_1^1 \subseteq H_1, A_2^1 \subseteq H_2$, (2) $R_1 \cup R_2$ spans Q_n^0, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, H_2, (\mathbf{w})^n, \mathbf{w}, R_2, \mathbf{v} \rangle$. See Fig. 5(c). □

Proposition 2. *Let W and B form the bipartition of Q_n with $n \geq 6$. Suppose that* $\mathbf{x}$ *and* $\mathbf{u}$ *are any two different vertices in W, whereas* $\mathbf{y}$ *and* $\mathbf{v}$ *are any two different vertices in B. Furthermore, suppose that $\mathbf{x} \in V(Q_n^0)$, $\mathbf{y} \in V(Q_n^1)$, and $\mathbf{y} \neq (\mathbf{u})^n$. Let A_1^0 and A_2^0 be any two disjoint nonempty subsets of $V(Q_n^0) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$, and let A_1^1 be any nonempty subset of $V(Q_n^1) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ such that $|A_1^0| + |A_1^1| + |A_2^0| = n - 3$. Assume that Q_{n-1} is 2-disjoint-path-coverable of order $n - 4$. Then, there exist two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{x}$ to $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $A_1^0 \cup A_1^1 \subseteq P_1$, (4) $A_2^0 \subseteq P_2$, and (5) $P_1 \cup P_2$ spans Q_n.*

Proof. We consider the following three cases.

Case 1: Both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^0. Since $2^{n-2} > n - 4 \geq |A_2^0| + |\{\mathbf{v}\}|$ for $n \geq 6$, there exists a black vertex $\mathbf{b}$ in $Q_n^0 - (A_2^0 \cup \{\mathbf{v}\})$. Since Q_{n-1} is 2-disjoint-path-coverable of order $n - 4$, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$

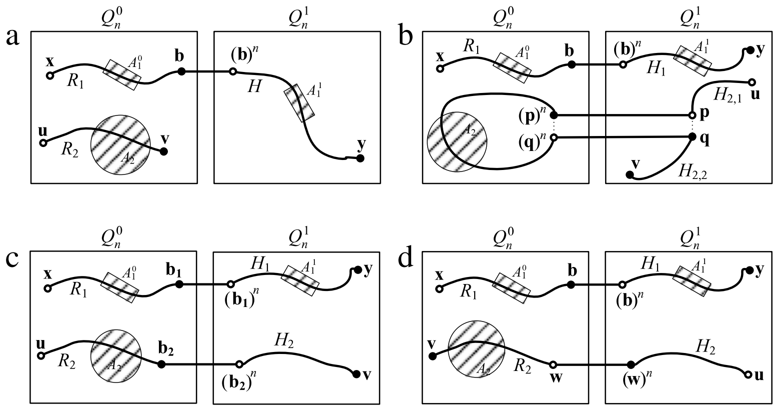

Fig. 6. Illustration for Proposition 2.

to **b**, (2) R_2 joins **u** to **v**, (3) $A_1^0 \subseteq R_1$, (4) $A_2^0 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^1. By Theorem 1, there is a hamiltonian path H of Q_n^1 joining $(\mathbf{b})^n$ to **y**. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, R_2, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 6(a).

Case 2: Both **u** and **v** are in Q_n^1. Since $|V(Q_n^1)| = 2^{n-1} > n(n-4) + n \geq n|A_1^0 \cup \{\mathbf{x}\}| + |A_1^1 \cup \{\mathbf{y}, \mathbf{u}, \mathbf{v}\}|$ for $n \geq 6$, there exists a vertex $\mathbf{p} \in V(Q_n^1) - (A_1^1 \cup \{\mathbf{y}, \mathbf{u}, \mathbf{v}\})$ such that $(\mathbf{t})^n \notin A_1^0 \cup \{\mathbf{x}\}$ for every $\mathbf{t} \in Nbd_{Q_n^1}(\mathbf{p}) \cup \{\mathbf{p}\}$. Since $2^{n-2} > (n-4) + n \geq |A_2 \cup \{(\mathbf{u})^n\}| + |Nbd_{Q_n^1}(\mathbf{p}) \cup \{\mathbf{p}\}|$, there exists a black vertex **b** in $V(Q_n^0) - (A_2 \cup \{(\mathbf{u})^n\})$ such that $(\mathbf{b})^n \notin Nbd_{Q_n^1}(\mathbf{p}) \cup \{\mathbf{p}\}$. Since Q_{n-1} is 2-disjoint-path-coverable of order $n-4$, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b})^n$ to **y**, (2) H_2 joins **u** to **v**, (3) $A_1^1 \subseteq H_1$, (4) $\{\mathbf{p}\} \subseteq H_2$, and (5) $H_1 \cup H_2$ spans Q_n^1. We can write H_2 as $\langle \mathbf{u}, H_{2,1}, \mathbf{p}, \mathbf{q}, H_{2,2}, \mathbf{v}\rangle$. Again, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins **x** to **b**, (2) R_2 joins $(\mathbf{p})^n$ to $(\mathbf{q})^n$, (3) $A_1^0 \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{b})^n, \mathbf{b}, H_1, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, H_{2,1}, \mathbf{p}, (\mathbf{p})^n, R_2, (\mathbf{q})^n, \mathbf{q}, H_{2,2}, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 6(b).

Case 3: **u** is in $V(Q_n^j)$, and **v** is in $V(Q_n^{1-j})$ for $j \in \{0, 1\}$. On the one hand, we assume that $j = 0$. Hence, **u** is in $V(Q_n^0)$, and **v** is in $V(Q_n^1)$. Since $2^{n-2} > n - 4$, there exists a black vertex $\mathbf{b_1}$ in $V(Q_n^0) - A_2^0$, and there exists a black vertex $\mathbf{b_2}$ in $V(Q_n^0) - (A_1^0 \cup \{\mathbf{b_1}\})$ such that $(\mathbf{b_2})^n \notin A_1^1$. Since Q_{n-1} is 2-disjoint-path-coverable of order $n-4$, there are two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins **x** to $\mathbf{b_1}$, (2) R_2 joins **u** to **z**, (3) $A_1^0 \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0, and there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b_1})^n$ to **y**, (2) H_2 joins $(\mathbf{b_2})^n$ to **v**, (3) $A_1^1 \subseteq H_1$, and (4) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b_1}, (\mathbf{b_1})^n, H_1, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, R_2, \mathbf{b_2}, (\mathbf{b_2})^n, H_2, \mathbf{v}\rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 6(c).

On the other hand, if $j = 1$, then **u** is in $V(Q_n^1)$, and **v** is in $V(Q_n^0)$. Since $2^{n-2} > n - 2$, there exists a black vertex **b** in $V(Q_n^0) - (A_2 \cup \{\mathbf{v}, (\mathbf{u})^n\})$, and there exists a white vertex **w** in $V(Q_n^0) - (A_1^0 \cup \{\mathbf{x}\})$ such that $(\mathbf{w})^n \notin A_1^1 \cup \{(\mathbf{y})^n\}$. Similarly, there exist disjoint paths R_1, R_2, H_1, H_2 joining **x** to **b**, **w** to **v**, $(\mathbf{b})^n$ to **y**, and **u** to $(\mathbf{w})^n$, respectively, such that (1) $A_1^0 \subseteq R_1, A_2 \subseteq R_2$, $A_1^1 \subseteq H_1$, (2) $R_1 \cup R_2$ spans Q_n^0, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H_1, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, H_2, (\mathbf{w})^n, \mathbf{w}, R_2, \mathbf{v}\rangle$. See Fig. 6(d). □

Proposition 3. *Let W and B form the bipartition of Q_n with $n \geq 5$. Suppose that* **x** *and* **u** *are any two different vertices in W, whereas* **y** *and* **v** *are any two different vertices in B. Furthermore, suppose that $\mathbf{x} \in V(Q_n^0)$, $\mathbf{y} \in V(Q_n^1)$, and $\mathbf{y} \neq (\mathbf{u})^n$. Let A_1 be any nonempty subset of $V(Q_n^0) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$, and let A_2 be any nonempty subset of $V(Q_n^1) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ such that $|A_1| + |A_2| = n - 3$. Then there exist two disjoint paths P_1 and P_2 such that (1) P_1 joins* **x** *to* **y**, *(2) P_2 joins* **u** *to* **v**, *(3) $A_1 \subseteq P_1$, (4) $A_2 \subseteq P_2$, and (5) $P_1 \cup P_2$ spans Q_n.*

Proof. We consider the following three cases.

Case 1: Both **u** and **v** are in $V(Q_n^0)$. Since $(\mathbf{u})^n \neq \mathbf{y}$ and $|Nbd_{Q_n^1}(\mathbf{y})| = n - 1 > |A_2 \cup \{(\mathbf{v})^n\}|$, there exists a vertex $\mathbf{w} \in Nbd_{Q_n^1}(\mathbf{y}) - (A_2 \cup \{(\mathbf{v})^n\})$. By Lemma 1, there exists a hamiltonian path R_1 of $Q_n^0 - \{\mathbf{u}, \mathbf{v}\}$ joining **x** and $(\mathbf{w})^n$. By Theorem 2, there exists a hamiltonian path R_2 of $Q_n^1 - \{\mathbf{y}, \mathbf{w}\}$ joining $(\mathbf{u})^n$ and $(\mathbf{v})^n$. Obviously, $A_1 \subseteq V(R_1)$ and $A_2 \subseteq V(R_2)$. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w})^n, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, (\mathbf{u})^n, R_2, (\mathbf{v})^n, \mathbf{v}\rangle$. It is apparent that P_1 and P_2 form the desired paths. See Fig. 7(a).

Case 2: Both **u** and **v** are in $V(Q_n^1)$. Since $|Nbd_{Q_n^1}(\mathbf{y})| = n - 1 > |A_2 \cup \{\mathbf{u}\}|$, there exists a vertex $\mathbf{w} \in Nbd_{Q_n^1}(\mathbf{y}) - (A_2 \cup \{\mathbf{u}\})$. By Theorem 1, there exists a hamiltonian path R_1 of Q_n^0 joining **x** and $(\mathbf{w})^n$. By Theorem 2, there exists a hamiltonian path R_2 of $Q_n^1 - \{\mathbf{y}, \mathbf{w}\}$ joining **u** and **v**. Obviously, $A_1 \subseteq V(R_1)$ and $A_2 \subseteq V(R_2)$. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w})^n, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = R_2$. Obviously, P_1 and P_2 form the desired paths. See Fig. 7(b).

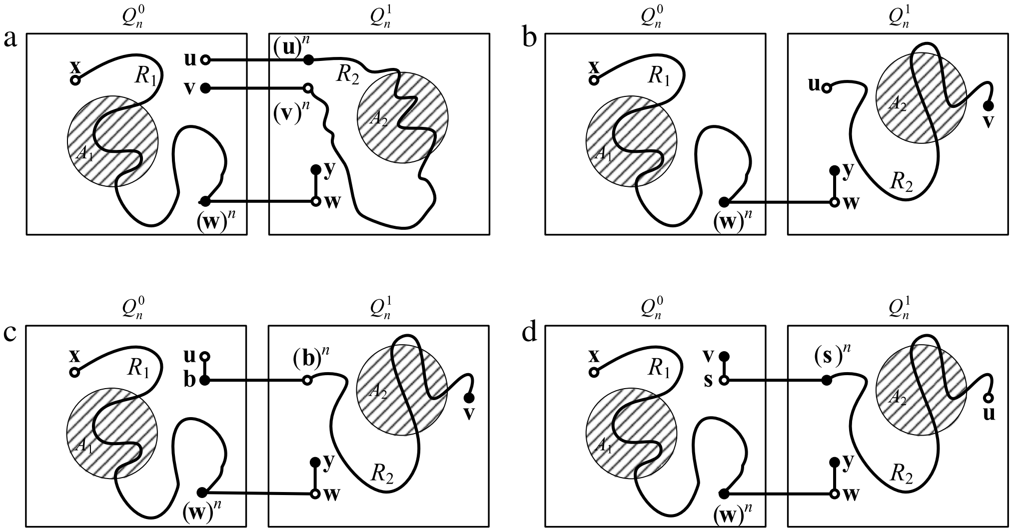

Fig. 7. Illustration for Proposition 3.

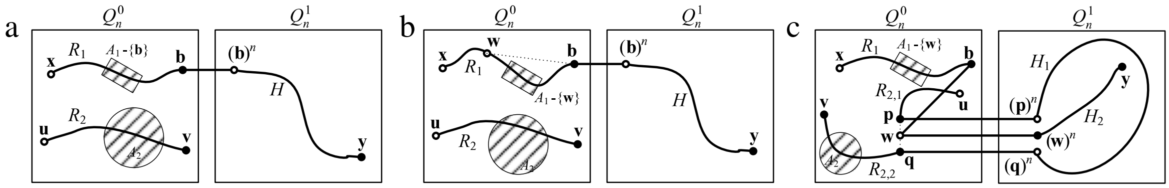

Fig. 8. Illustration for Case 1 of Proposition 4.

Case 3: $\mathbf{u}$ is in $V(Q_n^j)$, and $\mathbf{v}$ is in $V(Q_n^{1-j})$ for $j \in \{0, 1\}$. On the one hand, we assume that $j = 0$; i.e., $\mathbf{u}$ is in $V(Q_n^0)$, and $\mathbf{v}$ is in $V(Q_n^1)$. Since $|Nbd_{Q_n^1}(\mathbf{y})| = n-1 > |A_2|$, there exists a vertex $\mathbf{w} \in Nbd_{Q_n^1}(\mathbf{y}) - A_2$. Since $|Nbd_{Q_n^0}(\mathbf{u})| = n-1 > |A_1 \cup \{(\mathbf{w})^n\}|$, there exists a vertex $\mathbf{b} \in Nbd_{Q_n^0}(\mathbf{u}) - (A_1 \cup \{(\mathbf{w})^n\})$. By Theorem 2, there exists a hamiltonian path R_1 of $Q_n^0 - \{\mathbf{u}, \mathbf{b}\}$ joining $\mathbf{x}$ and $(\mathbf{w})^n$. Similarly, there exists a hamiltonian path R_2 of $Q_n^1 - \{\mathbf{y}, \mathbf{w}\}$ joining $(\mathbf{b})^n$ and $\mathbf{v}$. Clearly, $A_1 \subseteq V(R_1)$ and $A_2 \subseteq V(R_2)$. Now, we set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w})^n, \mathbf{w}, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{b}, (\mathbf{b})^n, R_2 \mathbf{v} \rangle$. Again, P_1 and P_2 form the desired paths. See Fig. 7(c).

On the other hand, we consider $j = 1$; i.e., $\mathbf{u}$ is in $V(Q_n^1)$, and $\mathbf{v}$ is in $V(Q_n^0)$. Since $|Nbd_{Q_n^1}(\mathbf{y})| = n-1 > n-2 \geq |A_2 \cup \{\mathbf{u}\}| + |\{\mathbf{v}\}|$, there exists a vertex $\mathbf{w} \in Nbd_{Q_n^1}(\mathbf{y}) - (A_2 \cup \{\mathbf{u}\})$ with $(\mathbf{w})^n \neq \mathbf{v}$. Since $|Nbd_{Q_n^0}(\mathbf{u})| = n-1 > n-2 \geq |A_1 \cup \{\mathbf{x}\}| + |\{\mathbf{y}\}|$, there exists a vertex $\mathbf{s} \in Nbd_{Q_n^0}(\mathbf{v}) - (A_1 \cup \{\mathbf{x}\})$ with $(\mathbf{s})^n \neq \mathbf{y}$. Again, there exists a hamiltonian path R_1 of $Q_n^0 - \{\mathbf{s}, \mathbf{v}\}$ joining $\mathbf{x}$ and $(\mathbf{w})^n$, and there exists a hamiltonian path R_2 of $Q_n^1 - \{\mathbf{y}, \mathbf{w}\}$ joining $\mathbf{u}$ and $(\mathbf{s})^n$. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w})^n, \mathbf{w}, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_2, (\mathbf{s})^n, \mathbf{s}, \mathbf{v} \rangle$. See Fig. 7(d). □

Proposition 4. *Let W and B form the bipartition of Q_n with $n \geq 5$. Suppose that $\mathbf{x}$ and $\mathbf{u}$ are any two different vertices in W, whereas $\mathbf{y}$ and $\mathbf{v}$ are any two different vertices in B. Furthermore, suppose that $\mathbf{x} \in V(Q_n^0)$, $\mathbf{y} \in V(Q_n^1)$, and $\mathbf{y} \neq (\mathbf{u})^n$. Let A_1 and A_2 be any two disjoint nonempty subsets of $V(Q_n^0) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ such that $|A_1| + |A_2| = n-3$. Assume that Q_{n-1} is 2-disjoint-path-coverable of order $n-4$. Then, there exist two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{x}$ to $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $A_1 \subseteq P_1$, (4) $A_2 \subseteq P_2$, and (5) $P_1 \cup P_2$ spans Q_n.*

Proof. We consider the following cases.

Case 1: Both $\mathbf{u}$ and $\mathbf{v}$ are in $V(Q_n^0)$. We have the following two subcases, (a) and (b).

(a) There is a black vertex, say $\mathbf{b}$, in A_1. Since Q_{n-1} is 2-disjoint-path-coverable of order $n-4$, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $A_1 - \{\mathbf{b}\} \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. By Theorem 1, there is a hamiltonian path H of Q_n^1 joining $(\mathbf{b})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_2, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 8(a).

(b) Every vertex in A_1 is white. Let $\mathbf{w}$ be any vertex in A_1. Since $deg_{Q_n^0}(\mathbf{w}) = n-1 > n-2 \geq |A_2| + |\{\mathbf{v}, (\mathbf{y})^n\}|$, there exists a vertex $\mathbf{b}$ in $Nbd_{Q_n^0}(\mathbf{w}) - (A_2 \cup \{\mathbf{v}, (\mathbf{y})^n\})$. By the premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{v}$, (3) $A_1 - \{\mathbf{w}\} \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0.

(b.1) $\mathbf{w}$ is in R_1. By Theorem 1, there exists a hamiltonian path H of Q_n^1 joining $(\mathbf{b})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H, \mathbf{y} \rangle$ and $P_2 = R_2$. Obviously, P_1 and P_2 form the desired paths. See Fig. 8(b).

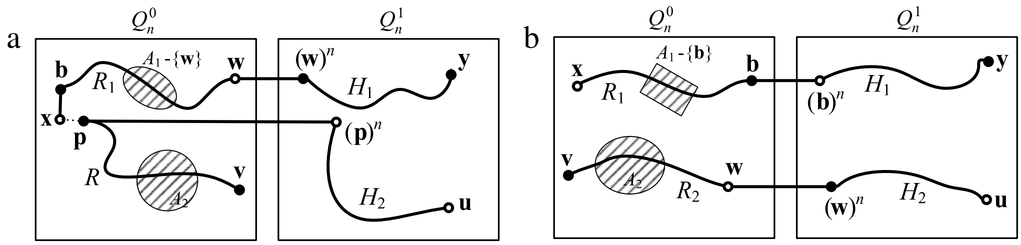

Fig. 9. Illustration for Case 2 of Proposition 4.

(b.2) $\mathbf{w}$ is in R_2. Without loss of generality, we can write R_2 as $\langle \mathbf{u}, R_{2,1}, \mathbf{p}, \mathbf{w}, \mathbf{q}, R_{2,2}, \mathbf{v}\rangle$. Suppose that $(\mathbf{w})^n \neq \mathbf{y}$. By Theorem 3, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{w})^n$ to $\mathbf{y}$, (2) H_2 joins $(\mathbf{p})^n$ to $(\mathbf{q})^n$, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, \mathbf{w}, (\mathbf{w})^n, H_1, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, R_{2,1}, \mathbf{p}, (\mathbf{p})^n, H_2, (\mathbf{q})^n, \mathbf{q}, R_{2,2}, \mathbf{v}\rangle$ to form the desired paths. See Fig. 8(c). On the other hand, we consider the case that $(\mathbf{w})^n = \mathbf{y}$. By Theorem 1, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{y}\}$ joining $(\mathbf{p})^n$ to $(\mathbf{q})^n$. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, R_{2,1}, \mathbf{p}, (\mathbf{p})^n, H, (\mathbf{q})^n, \mathbf{q}, R_{2,2}, \mathbf{v}\rangle$ to form the desired paths.

Case 2: $\mathbf{u}$ is in $V(Q_n^1)$, and $\mathbf{v}$ is in $V(Q_n^0)$. We have the following three subcases, (c), (d), and (e).

(c) Every vertex in A_1 is white, and every vertex in A_2 is black. Let $\mathbf{w}$ be a vertex in A_1. Since $deg_{Q_n^0}(\mathbf{x}) = n-1 > |A_2 \cup \{\mathbf{v}\}|$, we can choose a black vertex $\mathbf{b}$ in $Nbd_{Q_n^0}(\mathbf{x}) - (A_2 \cup \{\mathbf{v}\})$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{b}$ to $\mathbf{w}$, (2) R_2 joins $\mathbf{x}$ to $\mathbf{v}$, (3) $(A_1 - \{\mathbf{w}\}) \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. Without loss of generality, we write $R_2 = \langle \mathbf{x}, \mathbf{p}, R, \mathbf{v}\rangle$.

(c.1) $\mathbf{y} \neq (\mathbf{w})^n$ and $\mathbf{p} \neq (\mathbf{u})^n$. By Theorem 3, there are two disjoint paths H_1 and H_2 of Q_n^1 such that (1) H_1 joins $(\mathbf{w})^n$ to $\mathbf{y}$, (2) H_2 joins $\mathbf{u}$ to $(\mathbf{p})^n$, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, \mathbf{b}, R_1, \mathbf{w}, (\mathbf{w})^n, H_1, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, H_2, (\mathbf{p})^n, \mathbf{p}, R, \mathbf{v}\rangle$ to form the desired paths. See Fig. 9(a).

(c.2) $\mathbf{y} \neq (\mathbf{w})^n$ and $\mathbf{p} = (\mathbf{u})^n$. By Theorem 2, there is a hamiltonian path H of $Q_n^1 - \{\mathbf{u}\}$ joining $(\mathbf{w})^n$ to $\mathbf{y}$. We set $P_1 = \langle \mathbf{x}, \mathbf{b}, R_1, \mathbf{w}, (\mathbf{w})^n, H, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{p}, R, \mathbf{v}\rangle$ to form the desired paths.

(c.3) $\mathbf{y} = (\mathbf{w})^n$ and $\mathbf{p} \neq (\mathbf{u})^n$. By Theorem 2, there is a hamiltonian path H of $Q_n^1 - \{\mathbf{y}\}$ joining $\mathbf{u}$ to $(\mathbf{p})^n$. We set $P_1 = \langle \mathbf{x}, \mathbf{b}, R_1, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, H, (\mathbf{p})^n, \mathbf{p}, R, \mathbf{v}\rangle$ to form the desired paths.

(c.4) $\mathbf{y} = (\mathbf{w})^n$ and $\mathbf{p} = (\mathbf{u})^n$. Obviously, the length of R_1 or the length of R_2 is greater than 3. On the one hand, assume that the length of R_1 is greater than 3. We write $R_1 = \langle \mathbf{b}, \mathbf{z}, R', \mathbf{w}\rangle$. By Lemma 1, there exists a hamiltonian path H' of $Q_n^1 - \{\mathbf{u}, \mathbf{y}\}$ joining $(\mathbf{b})^n$ to $(\mathbf{z})^n$. We set $P_1 = \langle \mathbf{x}, \mathbf{b}, (\mathbf{b})^n, H', (\mathbf{z})^n, \mathbf{z}, R', \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{p}, R, \mathbf{v}\rangle$ to form the desired paths. On the other hand, we consider the length of R_2 is greater than 3. We write $R_2 = \langle \mathbf{x}, \mathbf{p}, R'', \mathbf{q}, \mathbf{v}\rangle$. By Lemma 1, there exists a hamiltonian path H'' of $Q_n^1 - \{\mathbf{u}, \mathbf{y}\}$ joining $(\mathbf{q})^n$ to $(\mathbf{v})^n$. We set $P_1 = \langle \mathbf{x}, \mathbf{b}, R_1, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, \mathbf{p}, R'', \mathbf{q}, (\mathbf{q})^n, H'', (\mathbf{v})^n, \mathbf{v}\rangle$ to form the desired paths.

(d) There is a black vertex in $A_1 - \{(\mathbf{u})^n\}$, or there is a white vertex in $A_2 - \{(\mathbf{y})^n\}$. Without loss of generality, we assume that there is a black vertex $\mathbf{b}$ in $A_1 - \{(\mathbf{u})^n\}$. Since $2^{n-2} > n-3$, we can choose a white vertex $\mathbf{w}$ in $V(Q_n^0) - (A_1 \cup \{(\mathbf{y})^n\})$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{w}$ to $\mathbf{v}$, (3) $(A_1 - \{\mathbf{b}\}) \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. By Theorem 3, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b})^n$ to $\mathbf{y}$, (2) H_2 joins $\mathbf{u}$ to $(\mathbf{w})^n$, and (3) $R_1 \cup R_2$ spans Q_n^0. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H_1, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, H_2, (\mathbf{w})^n, \mathbf{w}, R_2, \mathbf{v}\rangle$ to form the desired paths. See Fig. 9(b).

(e) $A_1 = \{(\mathbf{u})^n\}$ and $A_2 = \{(\mathbf{y})^n\}$. Since $h(\mathbf{x}, \mathbf{y}) \geq 3$, there exists an integer i with $1 \leq i \leq n-1$ to divide Q_n into two subcubes so that the following properties are satisfied: (1) $\mathbf{x}$ and $\mathbf{y}$ are in different subcubes, and (2) $\mathbf{y} \neq (\mathbf{u})^i$. To construct the required paths, we can use the same approach described in part (c) and Case 1 of this proposition, or in Cases 1 and 3 of Proposition 3.

Case 3: Both $\mathbf{u}$ and $\mathbf{v}$ are in $V(Q_n^1)$. Since $deg_{Q_n^1}(\mathbf{y}) = n-1 > n-3 \geq |A_2| + |\{\mathbf{u}\}|$, there exists a vertex $\mathbf{w}$ in $Nbd_{Q_n^1}(\mathbf{y}) - \{\mathbf{u}\}$ such that $(\mathbf{w})^n \notin A_2$. We have the following subcases, (f) and (g).

(f) $A_2 \neq \{(\mathbf{y})^n\}$. Obviously, there exists a vertex $\mathbf{p}$ in $A_2 - \{(\mathbf{y})^n\}$.

(f.1) $\mathbf{p} \neq (\mathbf{u})^n$. Let $F = \{((\mathbf{p})^n, (\mathbf{t})^n) \mid \mathbf{t} \in A_1, (\mathbf{p}, \mathbf{t}) \in E(Q_n^0)\}$. Obviously, $|F| \leq |A_1| \leq n-4$. By Lemma 2, there exists a hamiltonian path H of $(Q_n^1 - \{\mathbf{w}, \mathbf{y}\}) - F$ joining $\mathbf{u}$ and $\mathbf{v}$. Apparently, $(\mathbf{p})^n$ is in $V(H)$. Without loss of generality, we write H as $\langle \mathbf{u}, H_1, (\mathbf{p})^n, (\mathbf{q})^n, H_2, \mathbf{v}\rangle$ such that $\mathbf{q} \in V(Q_n^0) - (A_1 \cup \{\mathbf{x}\})$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $(\mathbf{w})^n$, (2) R_2 joins $\mathbf{p}$ to $\mathbf{q}$, (3) $A_1 \subseteq R_1$, (4) $A_2 - \{\mathbf{p}\} \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. We set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w})^n, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, H_1, (\mathbf{p})^n, \mathbf{p}, R_2, \mathbf{q}, (\mathbf{q})^n, H_2, \mathbf{v}\rangle$ to form the desired paths. See Fig. 10(a).

(f.2) $\mathbf{p} = (\mathbf{u})^n$. Since $2^{n-2} > n-1 \geq |\{\mathbf{v}, \mathbf{y}\}| + |A_1 \cup \{\mathbf{x}\}|$, there exists a black vertex $\mathbf{b}$ in $V(Q_n^1) - \{\mathbf{v}, \mathbf{y}\}$ such that $(\mathbf{b})^n \notin A_1 \cup \{\mathbf{x}\}$. By Theorem 2, there exists a hamiltonian path H of $(Q_n^1 - \{\mathbf{w}, \mathbf{y}\}) - \{\mathbf{u}\}$ joining $\mathbf{b}$ and $\mathbf{v}$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $(\mathbf{w})^n$, (2) R_2 joins $(\mathbf{u})^n$ to $(\mathbf{b})^n$, (3) $A_1 \subseteq R_1$, (4) $A_2 - \{(\mathbf{u})^n\} \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. Thus, we can set $P_1 = \langle \mathbf{x}, R_1, (\mathbf{w})^n, \mathbf{w}, \mathbf{y}\rangle$ and $P_2 = \langle \mathbf{u}, (\mathbf{u})^n, R_2, (\mathbf{b})^n, \mathbf{b}, H, \mathbf{v}\rangle$ to form the desired paths. See Fig. 10(b).

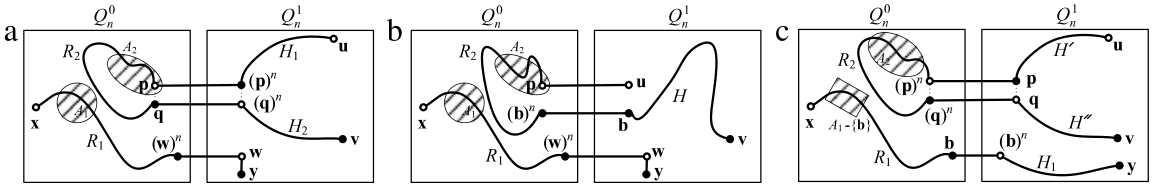

Fig. 10. Illustration for Case 3 of Proposition 4.

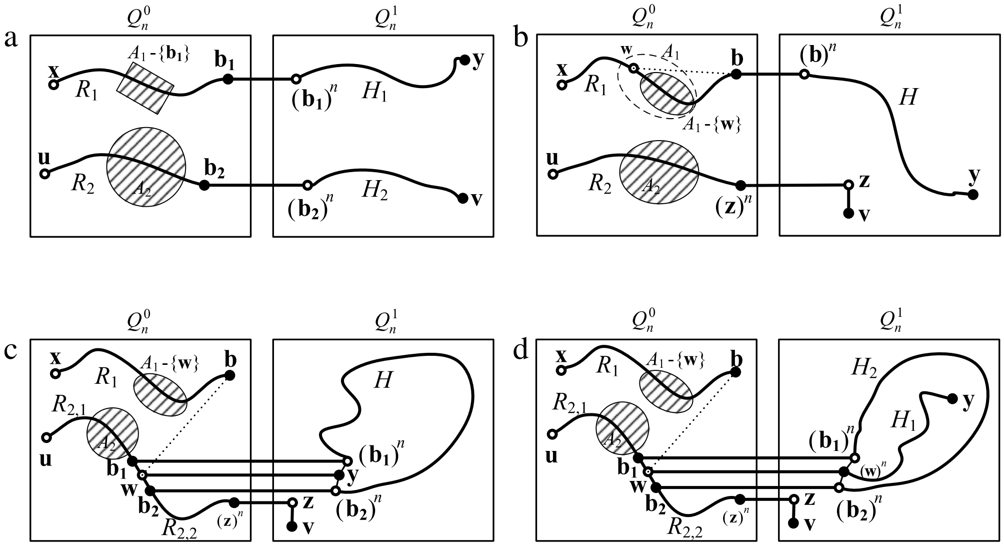

Fig. 11. Illustration for Case 4 of Proposition 4.

(g) $A_2 = \{(\mathbf{y})^n\}$. We have the following three possibilities.

(g.1) There exists a black vertex $\mathbf{b}$ in $A_1 - \{(\mathbf{u})^n\}$. By Theorem 3, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b})^n$ to $\mathbf{y}$ with length $2^{n-2}-1$, and (2) H_2 joins $\mathbf{u}$ to $\mathbf{v}$ with length $2^{n-2}-1$. Since $\lceil \frac{2^{n-2}-1}{2} \rceil > n-3 \geq |A_1 - \{\mathbf{b}\}| + |\{\mathbf{x}\}|$, there exists an edge $(\mathbf{p}, \mathbf{q})$ in H_2 such that $\{(\mathbf{p})^n, (\mathbf{q})^n\} \cap (A_1 \cup \{\mathbf{x}\}) = \emptyset$. Without loss of generality, we write H_2 as $\langle \mathbf{u}, H', \mathbf{p}, \mathbf{q}, H'', \mathbf{v} \rangle$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $(\mathbf{p})^n$ to $(\mathbf{q})^n$, (3) $A_1 - \{\mathbf{b}\} \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. Hence, we set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, H', \mathbf{p}, (\mathbf{p})^n, R_2, (\mathbf{q})^n, \mathbf{q}, H'', \mathbf{v} \rangle$ to form the required paths. See Fig. 10(c).

(g.2) $A_1 = \{(\mathbf{u})^n\}$. Since $h(\mathbf{x}, \mathbf{y}) \geq 3$, there exists an integer i, $1 \leq i \leq n-1$, to re-partition Q_n so that (1) $\mathbf{x}$ and $\mathbf{y}$ are in different subcubes, and (2) $\mathbf{y} \neq (\mathbf{u})^i$. To construct the required paths, we can use the same approach described in part (c) and Case 1 of this proposition, or in Cases 1 and 3 of Proposition 3.

(g.3) Every vertex of A_1 is white vertex. Since $h(\mathbf{x}, \mathbf{y}) \geq 3$, there exists an integer i, $1 \leq i \leq n-1$, to re-partition Q_n such that (1) $\mathbf{x}$ and $\mathbf{y}$ are in different subcubes, and (2) $\mathbf{y} \neq (\mathbf{u})^i$. To construct the required paths, we can use the same approach described in part (f), or in Propositions 2 and 3.

Case 4: $\mathbf{u}$ is in $V(Q_n^0)$, and $\mathbf{v}$ is in $V(Q_n^1)$. We have the following subcases, (h) and (i).

(h) There is a black vertex $\mathbf{b_1}$ in $A_1 \cup A_2$. Without loss of generality, we assume that $\mathbf{b_1} \in A_1$. Since $2^{n-2} > n-3 = |A_1 \cup A_2|$, we can choose a black vertex $\mathbf{b_2}$ in $V(Q_n^0) - (A_1 \cup A_2)$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b_1}$, (2) R_2 joins $\mathbf{u}$ to $\mathbf{b_2}$, (3) $(A_1 - \{\mathbf{b_1}\}) \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0. By Theorem 3, there are two disjoint paths H_1 and H_2 in Q_n^1 such that (1) H_1 joins $(\mathbf{b_1})^n$ to $\mathbf{y}$, (2) H_2 joins $(\mathbf{b_2})^n$ to $\mathbf{v}$, and (3) $H_1 \cup H_2$ spans Q_n^1. We set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b_1}, (\mathbf{b_1})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_2, \mathbf{b_2}, (\mathbf{b_2})^n, H_2, \mathbf{v} \rangle$. Obviously, P_1 and P_2 form the desired paths. See Fig. 11(a).

(i) Every node in $A_1 \cup A_2$ is white.

(i.1) $|A_1 - \{(\mathbf{v})^n\}| \geq 1$ or $|A_2 - \{(\mathbf{y})^n\}| \geq 1$. Without loss of generality, there exists a white vertex $\mathbf{w}$ in A_1 such that $(\mathbf{w})^n \neq \mathbf{v}$. Let $\mathbf{b}$ be a black vertex in $Nbd_{Q_n^0}(\mathbf{w})$, and let $\mathbf{z}$ be a white vertex in $Nbd_{Q_n^1}(\mathbf{v}) - \{(\mathbf{b})^n\}$ such that $(\mathbf{z})^n \notin A_1$. With this premise, there exist two disjoint paths R_1 and R_2 in Q_n^0 such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{b}$, (2) R_2 joins $\mathbf{u}$ to $(\mathbf{z})^n$, (3) $(A_1 - \{\mathbf{w}\}) \subseteq R_1$, (4) $A_2 \subseteq R_2$, and (5) $R_1 \cup R_2$ spans Q_n^0.

(i.1.1) $\mathbf{w}$ is in R_1. By Lemma 1, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{z}, \mathbf{v}\}$ joining $(\mathbf{b})^n$ to $\mathbf{y}$. Then we set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, (\mathbf{b})^n, H, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_2, (\mathbf{z})^n, \mathbf{z}, \mathbf{v} \rangle$ to form the desired paths. See Fig. 11(b).

(i.1.2) $\mathbf{w}$ is in R_2. Without loss of generality, we write $R_2 = \langle \mathbf{u}, R_{2,1}, \mathbf{b}_1, \mathbf{w}, \mathbf{b}_2, R_{2,2}, (\mathbf{z})^n \rangle$. We have the following two possibilities.

Suppose that $\mathbf{w} = (\mathbf{y})^n$. By Theorem 2, there exists a hamiltonian path H of $Q_n^1 - \{\mathbf{y}, \mathbf{v}, \mathbf{z}\}$ joining $(\mathbf{b}_1)^n$ to $(\mathbf{b}_2)^n$. Then we set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, \mathbf{w}, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_{2,1}, \mathbf{b}_1, (\mathbf{b}_1)^n, H, (\mathbf{b}_2)^n, \mathbf{b}_2, R_{2,2}, (\mathbf{z})^n, \mathbf{z}, \mathbf{v} \rangle$ to form the desired paths. See Fig. 11(c).

Suppose that $\mathbf{w} \neq (\mathbf{y})^n$. By Lemma 4, there exist two disjoint paths H_1 and H_2 of $Q_n^1 - \{\mathbf{v}, \mathbf{z}\}$ such that (1) H_1 joins $(\mathbf{w})^n$ to $\mathbf{y}$, (2) H_2 joins $(\mathbf{b_1})^n$ to $(\mathbf{b_2})^n$, and (3) $H_1 \cup H_2$ spans $Q_n^1 - \{\mathbf{v}, \mathbf{z}\}$. Then we set $P_1 = \langle \mathbf{x}, R_1, \mathbf{b}, \mathbf{w}, (\mathbf{w})^n, H_1, \mathbf{y} \rangle$ and $P_2 = \langle \mathbf{u}, R_{2,1}, \mathbf{b}_1, (\mathbf{b}_1)^n, H_2, (\mathbf{b}_2)^n, \mathbf{b}_2, R_{2,2}, (\mathbf{z})^n, \mathbf{z}, \mathbf{v} \rangle$ to form the desired paths. See Fig. 11(d).

(i.2) $|A_1 - \{(\mathbf{v})^n\}| = 0$ and $|A_2 - \{(\mathbf{y})^n\}| = 0$. That is, $A_1 = \{(\mathbf{v})^n\}$ and $A_2 = \{(\mathbf{y})^n\}$. Since $h(\mathbf{x}, \mathbf{y}) \geq 3$, there exists an integer i, $1 \leq i \leq n-1$, to re-partition Q_n so that (1) $\mathbf{x}$ and $\mathbf{y}$ are in different subcubes, and (2) $\mathbf{y} \neq (\mathbf{u})^i$. To construct the required paths, we can use the same approach described in part (h) and Case 1 of this proposition, or in Cases 1 and 4 of Proposition 3. □

Below is the proof of Lemma 5: let W and B form the bipartition of Q_n with $n \geq 3$. Suppose that $\mathbf{x}$ and $\mathbf{u}$ are any two different vertices in W, whereas $\mathbf{y}$ and $\mathbf{v}$ are any two different vertices in B. Let A_1 and A_2 be any two disjoint vertex subsets of $Q_n - \{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ such that $|A_1| + |A_2| = n - 3$. The proof proceeds by induction. Obviously, the lemma holds for $n = 3$. By Lemma 3, this lemma holds for $n = 4$. As the inductive hypothesis, we assume that the lemma holds for Q_{n-1} for $n \geq 5$. Lemma 3 also implies that this lemma holds if A_1 or A_2 is empty. Thus, we consider that $n \geq 5$, $|A_1| \geq 1$, and $|A_2| \geq 1$.

Since $\mathbf{x}$ and $\mathbf{y}$ are in different partite sets of Q_n, there exists an integer k, $1 \leq k \leq n$, to partition Q_n so that $\mathbf{x}$ and $\mathbf{y}$ belong to different subcubes and $\mathbf{y} \neq (\mathbf{u})^k$. By the symmetry of Q_n, we assume that $k = n$; that is, $\mathbf{x} \in V(Q_n^0)$, $\mathbf{y} \in V(Q_n^1)$, and $\mathbf{y} \neq (\mathbf{u})^n$. For $i \in \{1, 2\}$ and $j \in \{0, 1\}$, we set $A_i^j = A_i \cap V(Q_n^j)$. Then, we have the following four cases.

Case 1: $|\{(i, j) \mid A_i^j = \emptyset\}| = 0$. Obviously, $n - 3 = |A_1| + |A_2| = |A_1^0| + |A_1^1| + |A_2^0| + |A_2^1| \geq 4$. Thus, $n \geq 7$. Moreover, $|A_i^j| \leq n - 6$ for $i \in \{1, 2\}$ and $j \in \{0, 1\}$, and $|A_1^1| + |A_2^1| + |\{\mathbf{y}\}| \leq n - 4$. By Proposition 1, this case follows.

Case 2: $|\{(i, j) \mid A_i^j = \emptyset\}| = 1$. Without loss of generality, we assume that $|A_2^1| = 0$. Obviously, $n - 3 = |A_1| + |A_2| = |A_1^0| + |A_1^1| + |A_2^0| \geq 3$. Thus, $n \geq 6$. By Proposition 2, this case follows.

Case 3: Either $|A_1^0| = |A_2^1| = 0$ or $|A_1^1| = |A_2^0| = 0$. Without loss of generality, we assume that $|A_1^1| = |A_2^0| = 0$. That is, $A_1 \subset V(Q_n^0)$ and $A_2 \subset V(Q_n^1)$. By Proposition 3, this case follows.

Case 4: Either $|A_1^0| = |A_2^0| = 0$ or $|A_1^1| = |A_2^1| = 0$. Without loss of generality, we assume that $|A_1^1| = |A_2^1| = 0$. Obviously, $n - 3 = |A_1| + |A_2| = |A_1^0| + |A_2^0| \geq 2$. Thus, $n \geq 5$. By Proposition 4, this case follows.

These enumerated cases have addressed all possibilities and complete the proof.

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