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Edge domination in complete partite graphs☆

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Abstract

An *edge dominating set* in a graph G is a set of edges D such that every edge not in D is adjacent to an edge of D. An *edge domatic partition* of a graph $G=(V,E)$ is a collection of pairwise-disjoint edge dominating sets of G whose union is E. The maximum size of an edge domatic partition of G is called the *edge domatic number*. In this paper, we study the edge domatic number of the complete partite graphs and give the answers for *balanced complete partite graphs* and *complete split graphs*.

1. Introduction

In this paper all graphs are finite, undirected, loopless, and without multiple edges. A *balanced* complete t-partite graph is a complete t-partite graph $K(m_1, m_2, \ldots, m_t)$ where $m_1 = m_2 = \cdots = m_t = r$. Such a graph is denoted by O_r^t. It is also known as a regular complete partite graph. A *complete split graph* is the join of a complete graph K_n and an *independent* set O_r which we shall denote by $S(n,r)$. ($S(n,r)$ can be viewed as a complete $(n+1)$-partite graph $K(r,1,1,\ldots,1)$.)

An edge dominating set D of a graph G is a set of edges such that every edge of G not in D is adjacent to an edge in D. An *edge domatic partition* of a graph $G=(V,E)$ is a collection of pairwise-disjoint edge dominating sets of G whose union is E. The *edge domatic number problem* is to determine the *edge domatic number* $\text{ed}(G)$ of G, which is the maximum size of an edge domatic partition of G. Zelinka [7] showed that $\delta(G) \leqslant \text{ed}(G) \leqslant \delta_e(G)+1$ where $\delta(G)$ is the minimum degree of G and $\delta_e(G)$ is the minimum degree of the line graph of G, i.e., $\delta(L(G))$. He also determined the values of $\text{ed}(G)$ when G is a circuit, a complete graph, a complete bipartite graph or a tree.

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In [4], it was proved that $\mathrm{ed}(O_r^t)\leqslant\lfloor r^2\binom{t}{2}/\lceil r(t-1)/2\rceil\rfloor$, and the equality holds for (i) t is odd, and (ii) $t=4$ and r is even. Subsequently, Hwang [3] conjectured that the equality holds for t even. In this paper, we solve the edge domination problem of O_r^t by showing that for $t\geqslant 3$ and $r\geqslant 2$, $\mathrm{ed}(O_r^t)=rt-2$ if t is even and r is odd, and $\mathrm{ed}(O_r^t)=rt$ otherwise. Moreover, we consider the complete split graph $S(n,r)$ and we prove that $\mathrm{ed}(S(n,r))=n+r$ if n is even and $n-r$ is a positive odd integer, and $\mathrm{ed}(S(n,r))=n+r-1$ otherwise.

2. The edge domatic number of O_r^t

We start with some definitions. Let $S=\{1,2,\ldots,v\}$. A *Latin square* of order v based on S is a $v\times v$ array with entries from S such that in each row and each column, every element of S occurs exactly once. A Latin square $L=[l_{ij}]$ is said to be commutative provided that $l_{ij}=l_{ji}$ for every $1\leqslant i,j\leqslant v$. L is idempotent if $l_{ii}=i$ for each $i\in S$. It is well known that a commutative Latin square exists for all orders and an idempotent commutative Latin square of order v exists if and only if v is odd. For $v=2k$, let $H=\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}$. The elements in H are called holes. A *Latin square with holes H* is a Latin square such that for each hole $h\in H$, the subarray formed by $h\times h$ is a subsquare based on h. Since all the holes are of size two, we also refer to such a Latin square as a *Latin square with* 2×2 *holes H*. It was shown by Fu [2] that a commutative Latin square of order $2k$ with 2×2 holes H (briefly CLSH $(2k)$) exists for each $k\geqslant 3$. In what follows, we will use these Latin squares to obtain the edge domatic number of O_r^t.

Since $O_1^t=K_t$, $O_r^2=K_{r,r}$ and $O_r^1=O_r$, their edge domatic numbers are either known or trivial; hence, we will consider $t>2$ and $r>1$.

It is not difficult to see that if D is an edge dominating set of O_r^t, then the edges in D must be incident with at least all the vertices in $t-1$ partite sets.

Lemma 2.1 [Hwang and Chang [4]]. $\mathrm{ed}(O_r^t)\leqslant\lfloor r^2\binom{t}{2}/\lceil r(t-1)/2\rceil\rfloor$ *for* $t\geqslant 3$.

Proof. Since every edge dominating set of O_r^t must cover at least $t-1$ parts of O_r^t, we have that every edge dominating set of O_r^t has at least $\lceil r(t-1)/2\rceil$ edges. Then

$$\mathrm{ed}(O_r^t)\leqslant\left\lfloor |E(O_r^t)|\Big/\left\lceil\frac{r(t-1)}{2}\right\rceil\right\rfloor\leqslant\left\lfloor r^2\binom{t}{2}\Big/\left\lceil\frac{r(t-1)}{2}\right\rceil\right\rfloor. \qquad\square$$

Corollary 2.2. $\mathrm{ed}(O_r^t)\leqslant\begin{cases} rt-2 & \textit{if } t \textit{ is even, and } r \textit{ is odd,}\\ rt & \textit{otherwise.}\end{cases}$

Now it is clear that if we can obtain an edge domatic partition with the size mentioned in Corollary 2.2, then we have found $\mathrm{ed}(O_r^t)$.

It is easier to solve the case when t is odd. We note here that this case was solved in [4]. For completeness, we give a different proof by using special Latin squares.

Proposition 2.3. *If t is odd, then* $\mathrm{ed}(O_r^t)=rt$.

Proof. Let $V(O_r^t)$ be the disjoint union of t partite sets $V_1, V_2, \ldots, V_t$ such that $V_i=\{v_{j+(i-1)r} \mid j=1,2,\ldots,r\}$, $1 \leqslant i \leqslant t$. Then $\{v_h, v_k\}$ is an edge of O_r^t if and only if v_h and v_k are in two different partite sets. We first find a proper edge coloring for O_r^t which uses rt colors. Let $M=[m_{ij}]$ be an idempotent commutative Latin square of order t based on $\{1,2,\ldots,t\}$ and $A=[a_{xy}]$ be a commutative Latin square of order r based on $\{1,2,\ldots,r\}$. Define an $rt \times rt$ array L in block form $[B_{ij}]_{t\times t}$ such that $B_{ij}=[a_{xy}+(m_{ij}-1)r]$. It is easy to check that L is a commutative Latin square of order rt. (L is also referred as the direct product of A and M.) Now let $L'=[l'_{hk}]_{rt\times rt}$ be the array obtained from L by deleting B_{ii}, $1 \leqslant i \leqslant t$. By coloring the edge $\{v_h, v_k\}$ with l'_{hk}, we obtain a proper edge coloring of O_r^t which uses rt colors. Since M is idempotent, for each color $1 \leqslant i \leqslant rt$, the edges colored i form an edge dominating set. We conclude that $\mathrm{ed}(O_r^t) \geqslant rt$ and by Corollary 2.2, we have the proof. □

In [4] they showed that $\mathrm{ed}(O_r^4)=4r$ if r is even. Here we obtain a more general result.

Proposition 2.4. *If r is even, then* $\mathrm{ed}(O_r^t)=rt$.

Proof. Let the t partite sets of O_r^t be $B_1, B_2, \ldots, B_t$, $B_1=A_1 \cup A_2, \ldots, B_t=A_{2t-1} \cup A_{2t}$ such that $|A_i|=r/2$ for each $i=1,2,\ldots,2t$. Then the proof is similar to the idea of the proof of Proposition 2.3. The Latin square M is replaced by a CLHS ($2t$) and the Latin square A is replaced with a commutative Latin square of order $r/2$ based on $\{1,2,\ldots,r/2\}$. Furthermore, we delete the entries obtained from the holes of M. □

Finally, we deal with the case when t is even and r is odd. For each commutative Latin square M, we let the upper triangular part, diagonal, and lower triangular part be denoted by $U(M)$, $D(M)$, and $L(M)$ respectively. The array obtained by using $U(M_1)$, $D(M_2)$ and $L(M_3)$ is denoted by $\langle U(M_1), D(M_2), L(M_3)\rangle$ (briefly $\langle 1,2,3\rangle$) where M_1, M_2, M_3 are three commutative Latin squares (or commutative arrays) of the same order (of the same side). Now we are ready to show that $\mathrm{ed}(O_r^t) \geqslant rt-2$ whenever t is even and r is odd.

Proposition 2.5. *If t is even and r is odd, then* $\mathrm{ed}(O_r^t)=rt-2$.

Proof. Let $M_1=[m_{ij}^{(1)}]$ be an idempotent Latin square of order t based on $\{1,2,\ldots,t\}$ and $M_2=[m_{ij}^{(2)}]$ be a unipotent commutative Latin square of order t based on $\{0,1,2,\ldots,t-1\}$ such that $c=0$. (A Latin square $L=[l_{ij}]$ is unipotent is $l_{ii}=c$ for each i.) Let $A^{(k)}$ be an idempotent commutative Latin square of order r based on $A_k=\{(k-1)r+1,(k-1)r+2,\ldots,kr\}$, $1 \leqslant k \leqslant t-1$, and $A^{(t)}$ be an $r\times r$ array obtained by adding one more row $\boldsymbol{b}=\langle r(t-1)+1, r(t-1)+2, \ldots, rt-2, rt-2, 0\rangle$ and symmetrically one more column $\boldsymbol{b}^{\mathrm{T}}$ to a unipotent commutative Latin square of order

$r-1$ based on $\{0, r(t-1)+1, r(t-1)+2, \ldots, rt-2\}$. Similar to the idea of the proof of Proposition 2.3, construct an array L in block form $[B_{ij}]_{t\times t}$ such that $B_{ij}=\langle U(A^{(k)}), D(A^{(h)}), L(A^{(d)})\rangle$ where $m_{ij}^{(1)}=k$, $m_{ij}^{(2)}=h$ and $m_{ji}^{(1)}=d$, $1\leqslant i\leqslant j\leqslant t$. Now we claim that the array L' obtained from L by deleting the diagonal blocks corresponds to an edge domatic partition of O_r^t with size $rt-2$. Since, we use $rt-2$ entries in L', it suffices to show that for each i, $1\leqslant i\leqslant rt-2$, the edges colored i form an edge dominating set. (It will be helpful to look at the example O_3^4 in Fig. 1.) By the construction of L', for each $i\in A_k$, $1\leqslant k\leqslant t-1$, i occurs in each row and each column except possibly the jth row (column) where $j\in A_k$. This implies that for each edge $\{u,v\}$ in $E(O_r^t)$ not colored i, there exists an edge colored i which dominates $\{u,v\}$. Hence, the set of edges colored i form an edge dominating set. As to the entry in $\{r(t-1)+1, r(t-1)+2, \ldots, rt-2\}$, the proof is similar. Thus, we have the proof. □

By combining Propositions 2.3–2.5 and the known results we have the following theorem.

	M_1:				M_2:				$A^{(1)}$			$A^{(2)}$			$A^{(3)}$			$A^{(4)}$		
1	4	2	3		0	2	3	1												
3	2	4	1		2	0	1	3	1	3	2	4	6	5	7	9	8	0	10	10
4	1	3	2		3	1	0	2	3	2	1	6	5	4	9	8	7	10	0	10
2	3	1	4		1	3	2	0	2	1	3	5	4	6	8	7	9	10	10	0

<1,0,1>	<4,2,3>	<2,3,4>	<3,1,2>
<3,2,4>	<2,0,2>	<4,1,1>	<1,3,3>
<4,3,2>	<1,1,4>	<3,0,3>	<2,2,1>
<2,1,3>	<3,3,1>	<1,2,2>	<4,0,4>

			4	10	10	7	6	5	1	9	8
			9	5	10	10	8	4	6	2	7
			8	7	6	10	10	9	5	4	3
4	9	8				1	10	10	7	3	2
10	5	7				3	2	10	9	8	1
10	10	6				2	1	3	8	7	9
7	10	10	1	3	2				4	6	5
6	8	10	10	2	1				3	5	4
5	4	9	10	10	3				2	1	6
1	6	5	7	9	8	4	3	2			
9	2	4	3	8	7	6	5	1			
8	7	3	2	1	9	5	4	6			

$\mathrm{ed}(O_3^4) \geq 10$

Fig. 1.

Theorem 2.6.

$$\mathrm{ed}(O_r^t)=\begin{cases}0 & \textit{if } t=1,\\ r & \textit{if } t=2,\\ t-1 & \textit{if } r=1 \textit{ and } t \textit{ is even},\\ rt-2 & \textit{if } t>3,\ t \textit{ is even},\\ & \quad r>2 \textit{ and } r \textit{ is odd},\\ rt & \textit{otherwise}.\end{cases}$$

3. The edge domatic number of $S(n,r)$

In what follows, we will use $V(D)$ to denote the set of vertices which are incident with the edges of an edge dominating set D. The following result is easy to see.

Proposition 3.1. *D is an edge dominating set of $S(n,r)$ if and only if either $V(K_n)\subseteq V(D)$ or $V(S(n,r)\setminus v)\subseteq V(D)$ for some $v\in V(K_n)$.*

By a direct counting, we have the following proposition.

Proposition 3.2. $n+r\geqslant \mathrm{ed}(S(n,r))\geqslant n+r-1$.

Proof. Let $\{D_1,D_2,\ldots,D_l\}$ be an edge domatic partition of $S(n,r)$. By Proposition 3.1, the degree sum of all vertices in K_n on the edge-induced subgraph $\langle D_i\rangle$ is at least $n-1$, i.e.,

$$\sum_{v\in V(K_n)} \deg_{\langle D_i\rangle} v\geqslant n-1,\quad 1\leqslant i\leqslant l.$$

However, in at most n of the l edge dominating sets the degree sum equals to $n-1$. This implies that

$$n(n-1)+(l-n)n\leqslant \sum_{i=1}^{l}\sum_{v\in V(K_n)} \deg_{\langle D_i\rangle} v\leqslant n(n+r-1). \tag{1}$$

Hence $l\leqslant n+r$. To prove the other inequality, assume $V(O_r)=\{u_1,u_2,\ldots,u_r\}$. If n is even, let $\{D_1',D_2',\ldots,D_{n-1}'\}$ be a 1-factorization of K_n and let $D_{i+(n-1)}'=\{(v,u_i)\colon v\in V(K_n)\}$ for each i, $1\leqslant i\leqslant r$, so $\{D_1',D_2',\ldots,D_{n+r-1}'\}$ is an edge domatic partition of $S(n,r)$. If n is odd, let $\{D_1',D_2',\ldots,D_n'\}$ be a 1-factorization of $K_n+\{v_r\}$ and let

$$D_{i+n}'=\{(v,u_i)\colon v\in V(K_n)\}\quad \text{for each } i,\ 1\leqslant i\leqslant r-1.$$

Then $\{D_1',D_2',\ldots,D_{n+r-1}'\}$ is an edge domatic partition of $S(n,r)$. Hence, $\mathrm{ed}(S(n,r))\geqslant n+r-1$. □

Since the cases $n=1$ or $r=1$ are known, we consider only $n>1$ and $r>1$.

Proposition 3.3. *If* $\mathrm{ed}(S(n,r))=n+r$, *then there exists an edge domatic partition* $\{D_1, D_2, \ldots, D_{n+r}\}$ *such that*

$$\sum_{v\in V(K_n)} \deg_{\langle D_i\rangle} v = n-1 \text{ if } 1\leqslant i\leqslant n \quad \textit{and} \quad \sum_{v\in V(K_n)} \deg_{\langle D_j\rangle} v = n \textit{ if } n+1\leqslant j\leqslant n+r.$$

Proof. This is a direct result of inequality (1) which turns to be an equality. □

Proposition 3.4. *If* $r\geqslant n$ *or* $n-r$ *is a positive even integer, then* $\mathrm{ed}(S(n,r))=n+r-1$.

Proof. First, we consider the case $r\geqslant n$. Let D be an edge dominating set of $S(n,r)$ such that $\sum_{v\in V(K_n)}\deg_{\langle D\rangle} v$ is minimum. Since $\sum_{v\in V(K_n)}\deg_{\langle D\rangle} v\geqslant n$, there exists

$$\mathrm{ed}(S(n,r))\leqslant \sum_{v\in V(K_n)} \deg_{S(n,r)} v \Big/ \sum_{v\in V(K_n)} \deg_{\langle D\rangle} v\leqslant n+r-1.$$

Now if $n>r$ and $n-r$ is even, for each dominating set D either $\sum_{v\in V(K_n)}\deg_{\langle D\rangle} v\geqslant n$, or $\sum_{v\in V(K_n)}\deg_{\langle D\rangle} v=n-1$ and $\sum_{v\in V(O_r)}\deg_{\langle D\rangle} v\geqslant r+1$. Let $\{D_1, D_2, \ldots, D_t\}$ be an edge domatic partition of $S(n,r)$. By counting the number of edges in $E(S(n,r))-E(K_n)$, we conclude that there are at most $n-1$ edge dominating sets D_i such that $\sum_{v\in V(K_n)}\deg_{\langle D_i\rangle} v=n-1$ assume that there are $k\leqslant n-1$ such edge dominating sets. Thus,

$$\begin{aligned} t &\leqslant \left\lfloor \left\{ \sum_{v\in V(K_n)} \deg_{S(n,r)} v-(n-1)k \right\} \Big/ n \right\rfloor + k \\ &\leqslant \left\lfloor \left\{ \sum_{v\in V(K_n)} \deg_{S(n,r)} v-(n-1)^2 \right\} \Big/ n \right\rfloor + (n-1) = n+r-1, \end{aligned}$$

By Proposition 3.2, we conclude the proof. □

Proposition 3.5. *If* n *is odd,* r *is even and* $r<n$, *then* $\mathrm{ed}(S(n,r))=n+r-1$.

Proof. If $\mathrm{ed}(S(n,r))=n+r$, then by Proposition 3.3, there exists an edge domatic partition $\{D_1, D_2, \ldots, D_{n+r}\}$ such that $\sum_{v\in V(K_n)}\deg_{\langle D_i\rangle} v=n-1$ if $1\leqslant i\leqslant n$, and $\sum_{v\in V(K_n)}\deg_{\langle D_i\rangle} v=n$, $n+1\leqslant i\leqslant n+r$. By Proposition 3.1, D_i, $1\leqslant i\leqslant n$, is incident to every vertex in O_r. Moreover, $r<n$; thus all the edges joining the vertices in $V(K_n)$ and O_r are in $\bigcup_{i=1}^{n} D_i$. Hence, for each $n<j\leqslant n+r$, D_j is a set of edges in K_n which is an edge dominating set of $S(n,r)$. By the fact that n is odd, we must have $\sum_{v\in V(K_n)}\deg_{\langle D_j\rangle} v>n$. This contradicts Proposition 3.3. Therefore, $\mathrm{ed}(S(n,r))=n+r-1$. □

Proposition 3.6. *If* n *is even and* $n-r$ *is a positive odd integer, then* $\mathrm{ed}(S(n,r))=n+r$.

Proof. It is well known that an idempotent commutative Latin square of order v can be embedded in an idempotent commutative Latin square of odd order $t\geqslant 2v+1$ [1].

Hence, we can embed an idempotent commutative Latin square A of order r into an idempotent commutative Latin square $L=[l_{ij}]$ of order $n+r\geqslant 2r+1$. Deleting the entries of A and l_{ii}, $r+1\leqslant i\leqslant n+r$, we obtain an array which corresponds to an edge domatic partition of $S(n,r)$. By the fact that for each color $1\leqslant i\leqslant n+r$, the edges colored i form an edge dominating set, this implies that $\mathrm{ed}(S(n,r))\geqslant n+r$. By Proposition 3.2, we conclude the proof. □

Combining Propositions 3.4–3.6, and the known results, we have the following theorem.

Theorem 3.7.

$$\mathrm{ed}(S(n,r))=\begin{cases} n & \text{if } r=0 \text{ and } n \text{ is odd,}\\ n+r & \text{if } n \text{ is even and } n-r \text{ is an odd positive integer,}\\ n+r-1 & \text{otherwise.}\end{cases}$$

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References

[1] A.B. Cruse, On embedding incomplete symmetric latin squares, J. Combin. Theory Ser. A 16 (1974) 18–20.
[2] C.-M. Fu, The intersection problem of pentagon systems, Ph.D. Thesis, Auburn University, USA, 1987.
[3] S.F. Hwang, private communication.
[4] S.F. Hwang and G.J. Chang, The edge domination problem and the edge domatic number problem, submitted.
[5] S. Mitchell and S.T. Hedetniemi, Edge domination in trees, in: Proc. 8th S–E Conf. Combin.; Graph theory and computing, Congr. Numer. 19 (1977) 489–509.
[6] M. Yannakakis and F. Gavril, Edge dominating sets in graphs, SIAM J. Appl. Math. 38 (1980) 364–372.
[7] B. Zelinka, Edge-domatic number of a graph, Czechoslovak Math. J. 33 (1983) 107–110.