

ALGEBRAIC TRANSFORMATIONS OF HYPERGEOMETRIC FUNCTIONS AND AUTOMORPHIC FORMS ON SHIMURA CURVES

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ABSTRACT. In this paper, we will obtain new algebraic transformations of the ${}_2F_1$ -hypergeometric functions. The main novelty in our approach is the interpretation of identities among ${}_2F_1$ -hypergeometric functions as identities among automorphic forms on different Shimura curves.

1. INTRODUCTION

For a real number a and a nonnegative integer n , let

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)\dots(a+n-1), & \text{if } n \geq 1, \end{cases}$$

be the Pochhammer symbol. Recall that, for real numbers a, b, c with $c \neq 0, -1, -2, \dots$, the ${}_2F_1$ -hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

for $z \in \mathbb{C}$ with $|z| < 1$. The hypergeometric function is a solution of the differential equation

$$\theta(\theta + c - 1)F - z(\theta + a)(\theta + b)F = 0, \quad \theta = z \frac{d}{dz}.$$

Hypergeometric functions arise naturally in many branches of mathematics. For example, the periods

$$\int_1^{\infty} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

of the Legendre family of elliptic curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ can be expressed as

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda\right).$$

Also, it is well-known that

$$E_4(\tau) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(\tau)}\right)^4$$

where $E_4(\tau)$ is the Eisenstein series of weight 4 on $\mathrm{SL}(2, \mathbb{Z})$ and $j(\tau)$ is the elliptic j -function.

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In this paper, we are concerned with algebraic transformations of hypergeometric functions, that is, identities of the form

$$(1) \quad {}_2F_1(a, b; c; z) = R(z) {}_2F_1(a', b'; c'; S(z))$$

with suitable parameters a, b, c, a', b', c' and algebraic functions $R(z)$ and $S(z)$. If $w = R(z)$ is of degree m over the field $\mathbb{C}(z)$ or if z is of degree m over the field $\mathbb{C}(w)$, we say the algebraic transformation has *degree* m . Two examples of algebraic transformations of degree 1 are given by

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right).$$

These identities can be easily proved using the well-known result in the classical analysis that a Fuchsian differential equation with precisely 3 singularities at 0, 1, and ∞ is completely determined by the local exponents at these three points.

Beyond transformations of degree 1, one of the simplest examples is Kummer's quadratic transformation

$${}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; z\right) = {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4z(1-z)\right),$$

valid for any real numbers a, b with $a+b+1/2 \neq 0, -1, -2, \dots$. In [3], Goursat gave more than 100 algebraic transformations of degrees 2, 3, 4, 6. One such example is

$${}_2F_1\left(a, a+\frac{1}{3}; \frac{1}{2}; \frac{z(9-8z)^2}{(4z-3)^3}\right) = \left(1+\frac{z}{3}\right)^{3a} {}_2F_1\left(3a, a+\frac{1}{6}; \frac{1}{2}; z\right)$$

of degree 3. (See Entry (96) on Page 132 of [3].) More recently, Vidūnas [10] gave dozens of new algebraic transformations of degrees 6, 8, 9, 10, 12. For example, he showed that if we set $\beta = \pm\sqrt{-2}$,

$$(2) \quad S(z) = \frac{4z(z-1)(8\beta z+7-4\beta)^8}{(2048\beta z^3-3072\beta z^2-3264z^2+912\beta z+3264z+56\beta-17)^3},$$

and

$$R(z) = \left(1 + \frac{16}{9}(4-17\beta)z - \frac{64}{243}(167-136\beta)z^2 + \frac{2048}{6561}(112-17\beta)z^3\right)^{-1/16},$$

then

$${}_2F_1\left(\frac{5}{24}, \frac{13}{24}; \frac{7}{8}; z\right) = R(z) {}_2F_1\left(\frac{1}{48}, \frac{17}{48}; \frac{7}{8}; S(z)\right),$$

which is a transformation of degree 10. (See (32) of [10].) Vidūnas' examples usually involve Gröbner-basis computation. This is perhaps one of the reasons why Goursat could not find these transformations.

In this paper, we will present several new algebraic transformations. For example, one of our favorite identities is

$$(3) \quad \begin{aligned} & {}_2F_1\left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; \frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5}\right) \\ & = (1-z^2)^{1/20}(1+4z-z^2)^{1/4} {}_2F_1\left(\frac{3}{10}, \frac{2}{5}; \frac{9}{10}; z^2\right). \end{aligned}$$

The main novelty in our approach is the interpretation of ${}_2F_1$ -hypergeometric functions as automorphic forms on Shimura curves. Then proving identities such as the one above amounts to showing two certain automorphic forms on two Shimura curves are equal. This point of view is especially useful in determining the function $R(z)$ in (1). We will review

the basic definitions regarding Shimura curves and their automorphic forms in the next section.

2. PRELIMINARIES

The materials in this section are mostly taken from [11].

Quaternion algebras. A *quaternion algebra* B over a field K is a central simple algebra of dimension 4 over K . (Here *central* means that the center of the algebra is K and *simple* means that B has no proper nontrivial two-sided ideals.) If the characteristic of K is not 2, then one can show that there are elements i and j in B and $a, b \in K^\times$ such that

$$i^2 = a, \quad j^2 = b, \quad ij = -ji,$$

and $B = K + Ki + Kj + Kij$. In this case, we denote this algebra by $\left(\frac{a,b}{K}\right)$. For example, we have $M(2, K) \simeq \left(\frac{1,1}{K}\right)$ and $\left(\frac{-1,-1}{\mathbb{R}}\right)$ is the set of Hamilton's quaternions. Moreover, for $\alpha = a_0 + a_1i + a_2j + a_3ij \in B$, we set $\bar{\alpha} = a_0 - a_1i - a_2j - a_3ij$. Then the *reduced trace* $tr(\alpha)$ is defined to be $\alpha + \bar{\alpha} = 2a_0 \in K$ and the *reduced norm* $n(\alpha)$ is defined to be $\alpha\bar{\alpha} = a_0^2 - a_1^2a - a_2^2b + a_3^2ab \in K$.

If $K = \mathbb{C}$, then up to isomorphisms, there is only one quaternion algebra over \mathbb{C} , which is $M(2, \mathbb{C})$. If $K = \mathbb{R}$ or a non-Archimedean local field, then up to isomorphism, there are only two quaternion algebras. One is $M(2, K)$ and the other is a division algebra.

Now assume that K is a number field. Let v be a place of K and K_v be the completion of K with respect to v . If the localization $B \otimes_K K_v$ is isomorphic to $M(2, K_v)$, we say B *splits* at v . If $B \otimes_K K_v$ is isomorphic to a division algebra, we say B *ramifies* at v . It is known that the number of ramified places is finite and in fact an even integer. The product of finite ramified places is called the *discriminant* of the quaternion algebra.

Still assume that K is a number field. Let R be its ring of integers. An *order* in B is a finite generated R -module that is also a ring with unity containing a basis for B . An order is *maximal* if it is not properly contained in another order. An *Eichler order* is the intersection of two maximal orders. For example, if $B = M(2, \mathbb{Q})$, then $M(2, \mathbb{Z})$ is a maximal order and $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right) = M(2, \mathbb{Z}) \cap \left(\begin{smallmatrix} 1 & 0 \\ 0 & N \end{smallmatrix}\right) M(2, \mathbb{Z}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & N \end{smallmatrix}\right)^{-1}$ is an Eichler order.

Shimura curves. To define a Shimura curve, we assume that K is a totally real number field and take a quaternion algebra B over K that splits at exactly one infinite place, that is,

$$B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M(2, \mathbb{R}) \times \mathbb{H}^{[K:\mathbb{Q}]-1},$$

where \mathbb{H} is Hamilton's quaternion algebra $\left(\frac{-1,-1}{\mathbb{R}}\right)$. Then, up to conjugation, there is a unique embedding ι of B into $M(2, \mathbb{R})$. Note that we have $n(\alpha) = \det \iota(\alpha)$ for all $\alpha \in B$.

Let \mathcal{O} be an order and $\mathcal{O}_1^* = \{\alpha \in \mathcal{O} : n(\alpha) = 1\}$ be the norm-one group of \mathcal{O} . Then the image $\Gamma(\mathcal{O})$ of \mathcal{O}_1^* under the embedding ι is a discrete subgroup of $SL(2, \mathbb{R})$. Let $\Gamma(\mathcal{O})$ act on the upper half-plane \mathfrak{H} in the usual manner

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

Then the quotient space $\Gamma(\mathcal{O}) \backslash \mathfrak{H}$ is called the *Shimura curve* associated to \mathcal{O} . For example, if $B = M(2, \mathbb{Q})$ and $\mathcal{O} = M(2, \mathbb{Z})$, then $\Gamma(\mathcal{O}) = SL(2, \mathbb{Z})$ and $\Gamma(\mathcal{O}) \backslash \mathfrak{H}$ is just the usual modular curve $Y_0(1)$. Thus, Shimura curves are generalizations of classical modular curves and they are moduli spaces of certain abelian varieties with quaternionic multiplication [5]. In a broader setting, if Γ is any discrete subgroup of $SL(2, \mathbb{R})$ commensurable with $\Gamma(\mathcal{O})$, then the quotient space $\Gamma \backslash \mathfrak{H}$ will also be called a Shimura curve.

An element γ of $\Gamma(\mathcal{O})$ is *parabolic*, *elliptic*, or *hyperbolic*, according to whether $|tr(\gamma)| = 2$, $|tr(\gamma)| < 2$, or $|tr(\gamma)| > 2$. The fixed point of a parabolic element is called a *cusp*. This can appear only when $B = M(2, \mathbb{Q})$. The fixed point τ of an elliptic element in \mathfrak{H} is called an *elliptic point* of order n , where n is the order of the isotropy subgroup of τ in $\Gamma(\mathcal{O})/\pm 1$.

Note that if $B \neq M(2, \mathbb{Q})$, then the quotient space $\Gamma(\mathcal{O})\backslash\mathfrak{H}$ is a compact Riemann surface, which we denote by $X(\mathcal{O})$. If $B = M(2, \mathbb{Q})$, we compactify the Riemann surface $\Gamma(\mathcal{O})\backslash\mathfrak{H}$ by adding cusps and the resulting compact surface will also be denoted by $X(\mathcal{O})$.

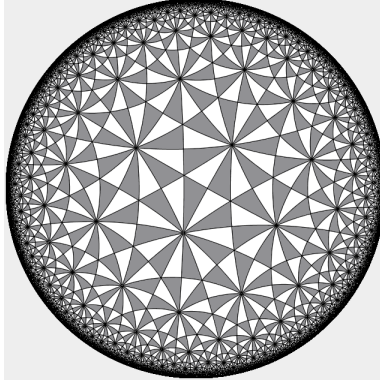
Now suppose that the compact Riemann surface $X(\mathcal{O})$ has genus g . Then a classical result says that there exist hyperbolic elements $A_1, \dots, A_g, B_1, \dots, B_g$, and elliptic or parabolic elements C_1, \dots, C_r that generate $\Gamma(\mathcal{O})/\pm 1$ with the single relation

$$[A_1, B_1] \dots [A_g, B_g] C_1 \dots C_r = \text{Id},$$

where $[A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1}$ is the commutator of A_i and B_i . (See [4, Chapter 4].) We let $(g; e_1, \dots, e_r)$ be the *signature* of $X(\mathcal{O})$.

Triangle groups. Suppose that a Shimura curve $X(\mathcal{O})$ has signature $(0; e_1, e_2, e_3)$. Then we say $\Gamma(\mathcal{O})$ is an *arithmetic triangle group*. The complete lists of all arithmetic triangle groups and their commensurability classes were determined by Takeuchi [7, 8].

If we cut each fundamental domain of an arithmetic triangle group $\Gamma(\mathcal{O})$ into 2 halves in a suitable way, then the fundamental half-domains give a tessellation of the upper half-plane \mathfrak{H} by congruent triangles with internal angles $\pi/e_1, \pi/e_2$, and π/e_3 . The following figure shows the tessellation of the unit disc, which is conformally equivalent to \mathfrak{H} through $\tau \rightarrow (\tau - i)/(\tau + i)$, by fundamental half-domains of the arithmetic triangle group $(0; 2, 3, 7)$.



Here each triangle represents a fundamental half-domain. Any combination of a grey triangle with a neighboring white triangle will be a fundamental domain for the triangle group $(0; 2, 3, 7)$. The triangle group $(0; 2, 3, 7)$ and its associated Shimura curve have been studied in details in [2].

In general, for any discrete subgroup Γ of $SL(\mathbb{R})$ such that $\Gamma\backslash\mathfrak{H}$ has finite volume, we can define its signature in the same way. If the signature is $(0; e_1, e_2, e_3)$, then we say Γ is a (hyperbolic) triangle group. (There are also notions of parabolic and elliptic triangle groups, corresponding to tessellation of \mathbb{C} and $\mathbb{P}^1(\mathbb{C})$, respectively.)

Automorphic forms on Shimura curves. The definition of an automorphic form on Shimura curves is the same as that of a modular form on classical modular curves.

For simplicity, we assume that $B \neq M(2, \mathbb{Q})$ so that we do not need to consider cusps. Then an *automorphic form* of weight k on $\Gamma(\mathcal{O})$ is a holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$

such that

$$(4) \quad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathcal{O})$ and all $\tau \in \mathfrak{H}$. The space of automorphic forms of weight k on $\Gamma(\mathcal{O})$ will be denoted by $S_k(\mathcal{O})$. Also, if a meromorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$ satisfies (4) with $k = 0$, we say f is an *automorphic function*. If $X(\mathcal{O})$ has genus 0, we call an automorphic function a *Hauptmodul* if it generates the field of automorphic functions on $\Gamma(\mathcal{O})$.

Using the Riemann-Roch formula, one can calculate the dimension of $S_k(\mathcal{O})$.

Proposition 1 ([6, Theorem 2.23]). *Assume that $B \neq M(2, \mathbb{Q})$. Suppose that the Shimura curve $X(\mathcal{O})$ associated to an order \mathcal{O} in B has signature $(g; e_1, \dots, e_r)$. Then for even integers k , we have*

$$\dim S_k(\mathcal{O}) = \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k = 0, \\ g, & \text{if } k = 2, \\ (k-1)(g-1) + \sum_{j=1}^r \left\lfloor \frac{k}{2} \left(1 - \frac{1}{e_j}\right) \right\rfloor, & \text{if } k \geq 4. \end{cases}$$

The dimension formula for the case $B = M(2, \mathbb{Q})$ is slightly different.

In the case $B = M(2, \mathbb{Q})$, there are many methods to construct modular forms, such as Eisenstein series, theta series, the Dedekind eta function, and etc. In practice, most explicit methods for modular curves rely on the Fourier expansions of modular forms and modular functions, i.e., the expansions with respect to the local parameter at the cusp ∞ . However, in the case $B \neq M(2, \mathbb{Q})$, because of the lack of cusps on Shimura curves, very few explicit methods are available for Shimura curves. One of the few methods is given by the second author of the present paper.

One of the key ideas in [12] is the following characterization of $S_k(\mathcal{O})$. Here we assume that the quaternion algebra is not $M(2, \mathbb{Q})$.

Proposition 2 ([12, Theorem 4, Propositions 1 and 6]). *Assume that a Shimura curve X has genus zero with elliptic points τ_1, \dots, τ_r of order e_1, \dots, e_r , respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$, $i = 1, \dots, r$. For a positive even integer $k \geq 4$, let*

$$d_k = \dim S_k(\mathcal{O}) = 1 - k + \sum_{j=1}^r \left\lfloor \frac{k}{2} \left(1 - \frac{1}{e_j}\right) \right\rfloor.$$

Then a basis for the space of automorphic forms of weight k on X is

$$t'(\tau)^{k/2} t(\tau)^j \prod_{i=1, a_i \neq \infty}^r (t(\tau) - a_i)^{-\lfloor k(1-1/e_i)/2 \rfloor}, \quad j = 0, \dots, d_k - 1.$$

Moreover, the functions $t'(\tau)^{1/2}$ and $\tau t'(\tau)^{1/2}$, as functions of t , satisfy the differential equation

$$f'' + Q(t)f = 0,$$

where

$$Q(t) = \frac{1}{4} \sum_{j=1, a_j \neq \infty}^r \frac{1 - 1/e_j^2}{(t - a_j)^2} + \sum_{j=1, a_j \neq \infty}^r \frac{B_j}{t - a_j}$$

for some constants B_j . Moreover, if $a_j \neq \infty$ for all j , then the constants B_j satisfy

$$\sum_{j=1}^r B_j = \sum_{j=1}^r \left(a_j B_j + \frac{1}{4}(1 - 1/e_j^2) \right) = \sum_{j=1}^r \left(a_j^2 B_j + \frac{1}{2} a_j (1 - 1/e_j^2) \right) = 0.$$

Also, if $a_r = \infty$, then B_j satisfy

$$\sum_{j=1}^{r-1} B_j = 0, \quad \sum_{j=1}^{r-1} \left(a_j B_j + \frac{1}{4}(1 - 1/e_j^2) \right) = \frac{1}{4}(1 - 1/e_r^2).$$

Remark 3. In [12], the differential equation $f'' + Q(t)f = 0$ is called the *Schwarzian differential equation* associated to t because $Q(t)$ is related to the Schwarzian derivative by the relation

$$2Q(t)t'(\tau)^2 + \{t, \tau\} = 0,$$

where

$$\{t, \tau\} = \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2} \left(\frac{t''(\tau)}{t'(\tau)} \right)^2$$

is the *Schwarzian derivative*. In general, in literature [1], if f is a thrice-differentiable function of z , then

$$D(f, z) := -\frac{\{f, z\}}{2f'(z)^2}$$

is called the *automorphic derivative* associated to f and z . In the case f is an automorphic function on a Shimura curve, then $D(f, \tau)$ is also an automorphic function. In particular, if t is a Hauptmodul on a Shimura curve of genus 0, then $Q(t) = D(t, \tau)$ is a rational function of t .

The upshot of this result is that it is often possible to determine the differential equation without explicitly constructing a Hauptmodul. For example, if $\Gamma(\mathcal{O})$ is a triangle group with signature $(0; e_1, e_2, e_3)$, then there always exists a (unique) Hauptmodul t with $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Then the relations between B_j uniquely determine the differential equation. In general, one can usually use coverings between Shimura curves of genus 0 to determine the differential equation. This is done by the first author [9] of the present paper for many Shimura curves of genus 0 associated to orders in quaternion algebras over \mathbb{Q} . Once the differential equation is determined, one can express automorphic forms in terms of t -series and then study properties of automorphic forms using these t -series. For example, in [12] the second author devised a method to compute Hecke operators on these t -series.

In the case of triangle groups, since the number of singularities of the differential equation is 3, the differential equation is essentially a hypergeometric differential equation.

Proposition 4 ([12, Theorem 9]). *Assume that a Shimura curve X has signature $(0; e_1, e_2, e_3)$. Let $t(\tau)$ be the Hauptmodul of X with values 0, 1, and ∞ at the elliptic points of order e_1 , e_2 , and e_3 , respectively. Let $k \geq 4$ be an even integer. Then a basis for the space of automorphic forms of weight k on X is given by*

$$t^{\{k(1-1/e_1)/2\}} (1-t)^{\{k(1-1/e_2)/2\}} t^j \left({}_2F_1(a, b; c; t) + C t^{1/e_1} {}_2F_1(a', b', c'; t) \right)^k,$$

$j = 0, \dots, \lfloor k(1-1/e_1)/2 \rfloor + \lfloor k(1-1/e_2)/2 \rfloor + \lfloor k(1-1/e_3)/2 \rfloor - k$, for some constant C , where for a rational number x , we let $\{x\}$ denote the fractional part of x ,

$$a = \frac{1}{2} \left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3} \right), \quad b = a + \frac{1}{e_3}, \quad c = 1 - \frac{1}{e_1}$$

and

$$a' = a + \frac{1}{e_1}, \quad b' = b + \frac{1}{e_1}, \quad c' = c + \frac{2}{e_1}.$$

For general Shimura curves of genus 0, the following properties of automorphic derivatives are very useful in determining the Schwarzian differential equation associated to a Hauptmodul.

Proposition 5. *Automorphic derivatives have the following properties.*

- (1) $D((az + b)/(cz + d), z) = 0$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$.
- (2) $D(g \circ f, z) = D(g, f(z)) + D(f, z)/(dg/df)^2$.

Proposition 6. *Let $z(\tau)$ be a Hauptmodul for a Shimura curve $X(\mathcal{O})$ of genus 0. Let $R(x) \in \mathbb{C}(x)$ be the rational function such that the automorphic derivative $Q(z) = D(z, \tau)$ is equal to $R(z)$. Assume that γ is an element of $\mathrm{SL}(2, \mathbb{R})$ normalizing the norm-one group of \mathcal{O} and let σ be the automorphism of $X(\mathcal{O})$ induced by γ . If $\sigma : z \mapsto (az + b)/(cz + d)$, then $R(x)$ satisfies*

$$\frac{(ad - bc)^2}{(cx + d)^4} R\left(\frac{ax + b}{cx + d}\right) = R(x).$$

Proof. We shall compute $D(z(\gamma\tau), \tau)$ in two ways. By Proposition 5, we have

$$D(z(\gamma\tau), \tau) = D\left(\frac{az(\tau) + b}{cz(\tau) + d}, z(\tau)\right) + \frac{D(z(\tau), \tau)}{(dz(\gamma\tau)/dz(\tau))^2} = 0 + \frac{(cz + d)^4 R(z)}{(ad - bc)^2}.$$

On the other hand, by the same proposition, we also have

$$D(z(\gamma\tau), \tau) = D(z(\gamma\tau), \gamma\tau) + \frac{D(\gamma\tau, \tau)}{(dz(\gamma\tau)/d\gamma\tau)^2} = R(z(\gamma\tau)) = R\left(\frac{az + b}{cz + d}\right).$$

Comparing the two expressions, we get the formula. \square

Algebraic transformations of hypergeometric functions. Consider the following situation. Suppose that $\Gamma_1 < \Gamma_2$ are two arithmetic triangle groups with Hauptmoduls z_1 and z_2 , respectively. Since any automorphic function on Γ_2 is also an automorphic function on Γ_1 , we have $z_2 = S(z_1)$ for some $S(x) \in \mathbb{C}(x)$. Likewise, if f_1 and f_2 are two automorphic forms of the same weight k on Γ_1 and Γ_2 , respectively, then the ratio f_1/f_2 is an automorphic function on Γ_1 and hence is equal to $R(z_1)$ for some $R(x) \in \mathbb{C}(x)$. In view of Proposition 4, after taking the k th roots of the two sides of $f_1/f_2 = R(z_1)$, we obtain an algebraic transformation of hypergeometric function. This explains the existence of Kummer's, Goursat's and Vidūnas' transformations. (Of course, the triangle groups appearing in their transformations may not be arithmetic, but the argument above is still valid.)

More generally, if Γ_1 and Γ_2 are two commensurable arithmetic triangle groups such that the Shimura curve associated to $\Gamma = \Gamma_1 \cap \Gamma_2$ has genus 0. Let z be a Hauptmodul on Γ . Then each of z_1 and z_2 is a rational function of z . Similarly, the ratio f_1/f_2 is also a rational function of z . Again, Proposition 4 yields an algebraic transformation of the form

$${}_2F_1(a_1, b_1; c_1; S_1(z)) = R(z) {}_2F_1(a_2, b_2; c_2; S_2(z))$$

for some rational functions $S_1(z)$ and $S_2(z)$ and some algebraic function $R(z)$. This is the theory behind (3) and other algebraic transformations given in the paper.

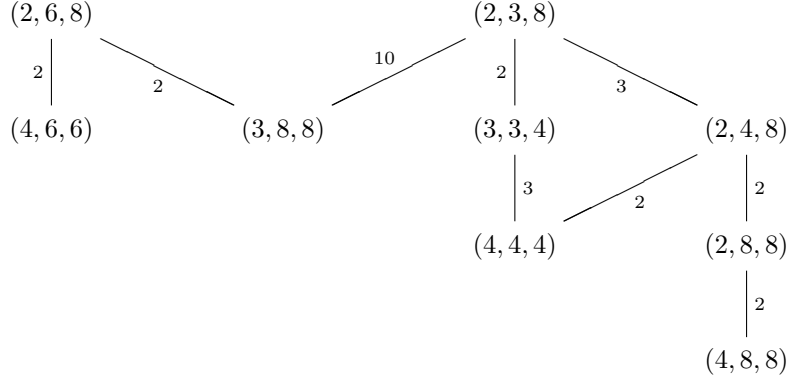
Definition 7. Let $S(z) \in \mathbb{C}(z)$ be a rational function. If the finite covering $\mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ defined by $S : z \rightarrow S(z)$ is ramified at most at three points 0, 1, and ∞ , then S is called a *Belyi function*.

In practice, the Belyi functions $S_1(z)$ and $S_2(z)$ can be determined by the ramification data of the coverings of Shimura curves. The function $R(z)$ can be determined by Propositions 2 and 4.

We now obtain algebraic transformations of hypergeometric functions using the above idea. Note that according to [8], arithmetic triangle groups fall in 19 commensurability classes. The first class in his list corresponds to classical modular curves. In this case, it is easier to use classical modular forms to derive identities. We will not discuss this case here. Identities arising from Classes II, V, and XII are special cases of a family of identities, and so are identities from Classes IV, VIII, XI, XIII, XV, and XVII. These cases will be treated in a later section. Here we first consider Class III in Section 3 and Class VI in Section 4. (There are no identities from Classes IX and XIX since these classes consist of a single group. Also, identities from Classes VII, XIV, XVI, and XVIII are just Kummer's quadratic transformations.)

3. ALGEBRAIC TRANSFORMATIONS ASSOCIATED TO CLASS III

According to [8], Takeuchi's Class III of commensurable arithmetic triangle groups has the following subgroup diagram. Here because all groups involved have genus zero, we omit the genus information in the signatures of the groups.



The main goal in this section is to prove an algebraic transformation associated to the pair of triangle groups $(4, 6, 6)$ and $(4, 4, 4)$.

Theorem 1. *Let α be a root of $x^2 + 3 = 0$ and β a root of $x^2 + 2 = 0$. We have*

$$\begin{aligned}
 (5) \quad & \frac{(1+z)^{1/8}(1-3z)^{1/8}}{(1+\alpha z)^{5/4}} {}_2F_1\left(\frac{5}{24}, \frac{3}{8}; \frac{3}{4}; \frac{12\alpha z(1-z^2)(1-9z^2)}{(1+\alpha z)^6}\right) \\
 & = \frac{1}{(1+(4+2\beta)z - (1+2\beta)z^2)^{1/2}} {}_2F_1\left(\frac{1}{8}, \frac{3}{8}; \frac{3}{4}; R(z)\right),
 \end{aligned}$$

and

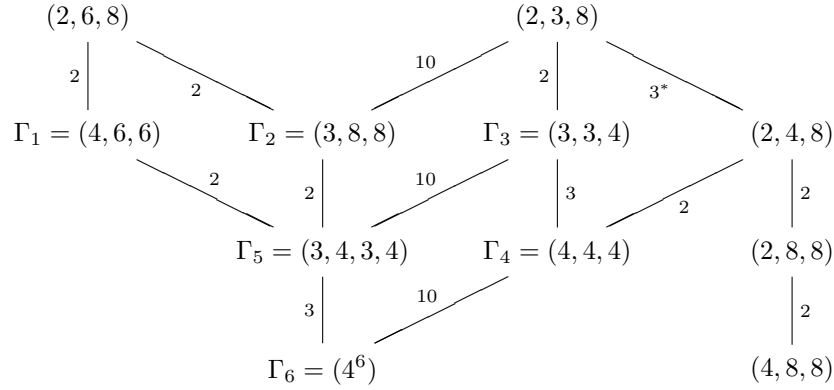
$$\begin{aligned}
 (6) \quad & \frac{(1-z)^{1/4}(1+z)^{5/8}(1-3z)^{1/4}(1+3z)^{5/8}}{(1+\alpha z)^{11/4}} {}_2F_1\left(\frac{11}{24}, \frac{5}{8}; \frac{5}{4}; \frac{12\alpha z(1-z^2)(1-9z^2)}{(1+\alpha z)^6}\right) \\
 & = \frac{(1+(-7+4\beta)z^2/3)}{(1+(4+2\beta)z - (1+2\beta)z^2)^{3/2}} {}_2F_1\left(\frac{3}{8}, \frac{5}{8}; \frac{5}{4}; R(z)\right)
 \end{aligned}$$

where

$$R(z) = -\frac{4(1+\beta)^4 z(1+(-7+4\beta)z^2/3)^4}{(1+z)(1-3z)(1+(4+2\beta)z-(1+2\beta)z^2)^4}.$$

We first determine the signatures of the intersections.

Lemma 8. *We have*



Moreover, the group of signature (4^6) is a normal subgroup of the group of signature $(3, 4, 3, 4)$. (Here (4^6) is a shorthand for $(4, 4, 4, 4, 4, 4)$.)

Proof. Let $\Gamma_2 = (3, 8, 8)$ and Γ'_2 be its commutator subgroup. From the group presentation

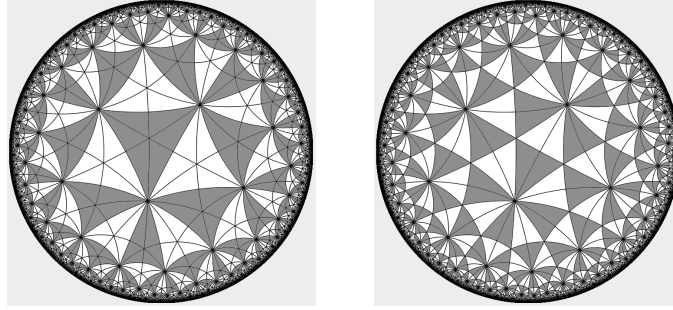
$$\Gamma_2 \simeq \langle \gamma_1, \gamma_2 : \gamma_1^3, \gamma_2^8, (\gamma_1 \gamma_2)^8 \rangle$$

for Γ_2 , we know that Γ_2/Γ'_2 is cyclic of order 8. Thus, Γ_2 has exactly one subgroup of index 2, which must be the common intersection of the groups $(4, 6, 6)$, $(3, 8, 8)$ and $(3, 3, 4)$. The signature of this subgroup can be easily determined by observing that a covering of degree 2 from a Shimura curve to the Shimura curve associated to $(3, 8, 8)$ can only ramify at the two elliptic points of order 8. We find that the signature must be $(3, 4, 3, 4)$.

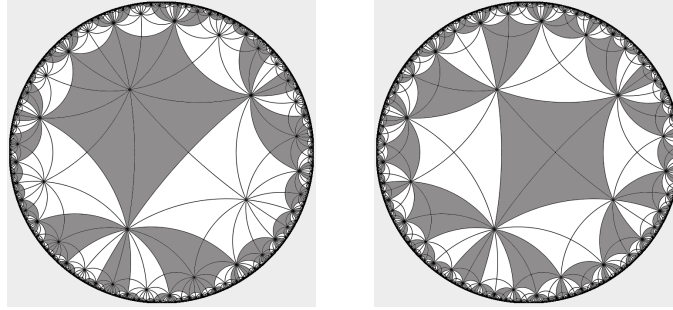
We next observe that the commutator subgroup Γ'_3 of the group $\Gamma_3 = (3, 3, 4)$ is cyclic of order 3. Thus, Γ'_3 is a normal subgroup of index 3 in Γ_3 . This Γ'_3 must be the same as the group of signature $(4, 4, 4)$. If $\Gamma'_3 \neq (4, 4, 4)$, then $\Gamma'_3 \cap (4, 4, 4)$ is a normal subgroup of $(4, 4, 4)$ of index 3, but the group $(4, 4, 4)$ cannot have a normal subgroup of index 3. We next determine the signature of the intersection of $\Gamma_4 = (4, 4, 4)$ and $\Gamma_5 = (3, 4, 3, 4)$.

Let X_j denote the Shimura curve associated to the group Γ_j . Since Γ_4 is a normal subgroup of Γ_3 of index 3, the intersection Γ_6 of Γ_4 and Γ_5 is a normal subgroup of index 3 in Γ_5 , which implies that the two elliptic points of order 4 of X_5 must split completely on X_6 . In view of the Riemann-Hurwitz formula, the two elliptic points of order 3 of X_5 must be totally ramified. We conclude that Γ_6 has signature (4^6) .

In fact, the subgroup relations mentioned above can be visualized by the following figures.

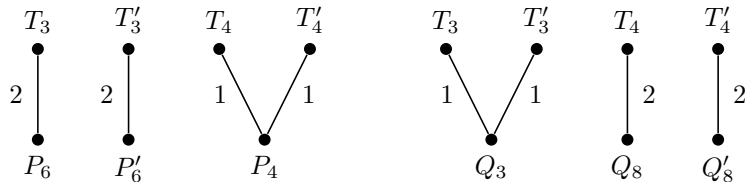


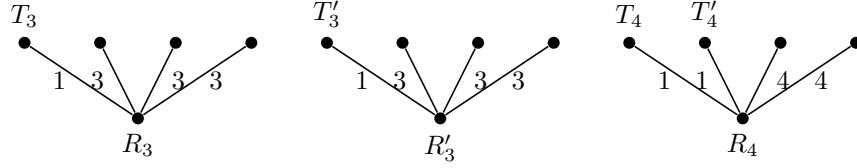
Here the small triangles are $(2, 3, 8)$ -triangles. Let G be the group of all symmetries of the tessellation of the hyperbolic plane by the $(4, 4, 4)$ -triangles and G_0 be the subgroup generated by the reflections across the edges of $(4, 4, 4)$ -triangles. Then G/G_0 is isomorphic to D_3 . The $(3, 3, 4)$ -triangle group corresponds to the cyclic subgroup of order 3 in G/G_0 , while the group $(2, 3, 8)$ corresponds the whole group G/G_0 . Similarly, if we piece 12 copies of $(2, 6, 8)$ -triangles around the vertex of inner angle $\pi/4$, we get a regular hexagon with inner angles $\pi/4$. Let H be the group of all symmetries of the tessellation by this regular hexagon and H_0 be the subgroup generated by the reflections across the edges of hexagons. Then H/H_0 is isomorphic to D_6 . The unique cyclic subgroup of order 3 in H/H_0 corresponds to the group $(3, 4, 3, 4)$. See the figures below.



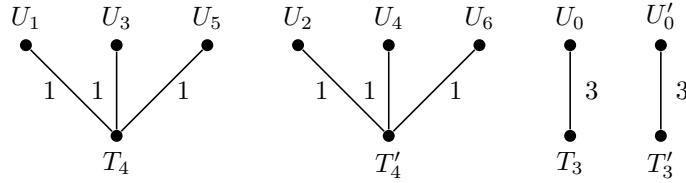
(The groups $(2, 6, 8)$, $(4, 6, 6)$, and $(3, 8, 8)$ correspond to the whole H/H_0 , the cyclic subgroup of order 6 of H/H_0 , and one of the D_3 -subgroups, respectively.) \square

Now let $\Gamma_1 = (4, 6, 6)$, $\Gamma_2 = (3, 8, 8)$, $\Gamma_3 = (3, 3, 4)$, $\Gamma_4 = (4, 4, 4)$, $\Gamma_5 = (3, 4, 3, 4)$, and $\Gamma_6 = (4^6)$. Let $X_j = X(\Gamma_j)$, $j = 1, \dots, 6$, be the corresponding Shimura curves. Label the elliptic points on X_1 by P_4, P_6 , and P'_6 , those on X_2 by Q_3, Q_8 , and Q'_8 , those on X_3 by R_3, R'_3 , and R_4 , those on X_4 by S_4, S'_4, S''_4 , and those on X_5 by T_3, T'_3, T_4 , and T'_4 (with the subscripts carrying the obvious meaning) such that the ramification data are given by



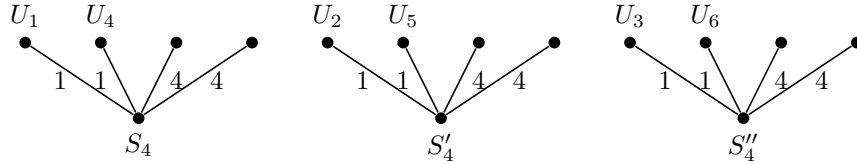


Label the elliptic points of X_6 by U_1, \dots, U_6 such that the rotation around the center of the (4^6) -polygon by the angle $\pi/3$ permutes the six points cyclically. From the figures above, we know that if we label the points such that U_1 lies above T_4 , then the ramification data for $X_6 \rightarrow X_5$ are



where U_0 and U'_0 are the centers of the (4^6) -polygons. (The reader is reminded that each (4^6) -polygon represents only half of the fundamental domain for the Shimura curve X_6 . Referring to the figure in the proof of the lemma above, a fundamental domain consists of a grey (4^6) -polygon and a neighboring white (4^6) -polygon.)

Lemma 9. *The two elliptic points of X_6 at the two ends of a diagonal of a (4^6) -polygon lie above the same elliptic point of X_4 . That is, labeling the elliptic points of X_4 suitably, we have*



Moreover, if we choose Hauptmoduls $z_j(\tau)$ for X_j , $j = 1, \dots, 6$, by requiring

$$\begin{aligned} z_1(P_4) &= 0, & z_1(P_6) &= 1, & z_1(P'_6) &= \infty, \\ z_2(Q_8) &= 0, & z_2(Q_3) &= 1, & z_2(Q'_8) &= \infty, \\ z_3(R_4) &= 0, & z_3(R_3) &= 1, & z_3(R'_3) &= \infty, \\ z_4(S_4) &= 0, & z_4(S'_4) &= 1, & z_4(S''_4) &= \infty, \\ z_5(T_4) &= 0, & z_5(T_3) &= 1, & z_5(T'_4) &= \infty, \\ z_6(U_1) &= 0, & z_6(U_3) &= 1, & z_6(U_4) &= \infty, \end{aligned}$$

then we have

$$\begin{aligned} z_1 &= \frac{4z_5}{(1+z_5)^2}, & z_2 &= z_5^2, \\ z_3 &= \frac{3(\zeta - \zeta^2)z_4(1-z_4)}{(1+\zeta z_4)^3}, & z_5 &= \frac{3(\zeta - \zeta^2)z_6(1-z_6^2)}{1-9z_6^2}, \\ z_3 &= \frac{(28+16\beta)z_5(1+(-17+56\beta)z_5^2/81)^4}{(1+z_5)(1+(13+8\beta)z_5/3 - (25+32\beta)z_5^2/9 + (17-56\beta)z_5^3/81)^3}, \end{aligned}$$

and

$$z_4 = -\frac{4(1+\beta)^4 z_6 (1 + (-7 + 4\beta)z_6^2/3)^4}{(1+z_6)(1-3z_6)(1+(4+2\beta)z_6 - (1+2\beta)z_6^2)^4},$$

where ζ is a 3rd root of unity and β is a root of $x^2 + 2 = 0$.

Proof. The ramification data for the covering $X_5 \rightarrow X_2$ and the assumption $z_2(Q_3) = z_5(T_3) = 1$ imply that $z_2 = z_5^2$ and

$$z(T_3') = -1.$$

The relation between z_1 and z_5 is easy to determine. We find $z_1 = 4z_5/(1+z_5)^2$.

To determine the relation between z_3 and z_4 , we recall from Lemma 8 that Γ_4 is a normal subgroup Γ_3 . Any element of Γ_3 not in Γ_4 induces an automorphism of order 3 on X_4 . Such an automorphism must permute the three elliptic points S_4 , S_4' , and S_4'' cyclically. In term of the Hauptmodul z_4 , such an automorphism is either

$$\sigma : z_4 \mapsto \frac{-1}{z_4 - 1}$$

or its square. Moreover, the fixed points of such an automorphism are the ramified points in the covering $X_4 \rightarrow X_3$. That is, if we let S_0 and S_0' be the points lying above R_3 and R_3' respectively, then $z_4(S_0), z_4(S_0') \in \{-\zeta, -\zeta^2\}$, where ζ is a primitive 3rd root of unity. Then from the ramification data, we easily deduce that $z_3 = (\zeta - \zeta^2)z_4(1 - z_4^2)/(1 + \zeta z_4)^3$.

To determine the relation between z_5 and z_6 , we argue similarly as above. The tessellation of the hyperbolic plane by Γ_6 has a D_6 -symmetry, in addition to the symmetries arising from the reflections across the edges of the (4^6) -polygons. This provides many useful informations. For example, if we let τ be the reflection across the diagonal joining U_1 and U_4 , then τ induces an involution on X_6 , which, in terms of z_6 , is given by

$$\tau : z_6 \mapsto -z_6,$$

which implies that

$$z_6(P_5) = -1.$$

Furthermore, let ρ denote the rotation by angle $\pi/3$ around the center of the hexagon. Then

$$\rho : z_6 \mapsto \frac{cz_6 + 1}{-cz_6 + c}$$

for some zero constant c since ρ maps 1 to ∞ and ∞ to -1 . In light of $\rho^2 : 0 \rightarrow 1$, we conclude that $c = 3$ and

$$z_6(U_2) = 1/3, \quad z_6(U_6) = -1/3.$$

It follows that $z_5 = Az_6(1 - z_6^2)/(1 - 9z_6^2)$ for some A . This constant A has the property that $Ax(1 - x^2) - (1 - 9x^2)$ has repeated roots. We find $A = \pm 3\sqrt{-3}$. The choice of the sign must be synchronized with the choice of the third root of unity in the relation between z_4 and z_5 . This will be done later.

We now come to the more complicated part of the lemma. Let $\pi : X_6 \rightarrow X_4$ be the covering of the Shimura curves. Let γ be an element of Γ_5 not in Γ_6 . Then γ normalizes both Γ_4 and Γ_6 and induces automorphisms ρ_1 and ρ_2 on X_4 and X_6 , respectively. We may assume that $\rho_2 = \rho^2$, where ρ permutes U_1, \dots, U_6 cyclically, as defined in the previous paragraph. It is easy to check that $\pi \circ \rho_1 = \rho_2 \circ \pi$. Thus, $\pi(U_1)$, $\pi(U_3)$, and $\pi(U_5)$ are three different elliptic points on X_4 . We label them by S_4 , S_4' , and S_4'' , respectively. Let V_1, V_2 be the two ramified points lying above S_4 . Now there are three possibilities

$$\pi^{-1}(S_4) = \{U_1, U_2, V_1, V_2\}, \quad \pi^{-1}(S_4) = \{U_1, U_4, V_1, V_2\}, \quad \pi^{-1}(S_4) = \{U_1, U_6, V_1, V_2\}.$$

We will show that the correct one is $\{U_1, U_4, V_1, V_2\}$.

Let $V'_j = \rho_2(V_j)$ and $V''_j = \rho_2^2(V_j)$ for $j = 1, 2$. If $\pi^{-1}(S_4) = \{U_1, U_2, V_1, V_2\}$, then we have

$$z_4 = \frac{Bz_6(1-3z_6)(z_6-z_6(V_1))^4(z_6-z_6(V_2))^4}{(1+z_6)(1+3z_6)(z_6-z_6(V'_1))^4(z_6-z_6(V''_1))^4}$$

for some constant B . The values of $z_6(V_1)$ and etc. must satisfy

$$(7) \quad \begin{aligned} & Bx(1-3x)(1-x/z_6(V_1))^4(1-x/z_6(V_2))^4 \\ & - (1+x)(1+3x)(1-x/z_6(V'_1))^4(1-x/z_6(V''_1))^4 \\ & = C(1-x)(1-x/z_6(V'_1))^4(1-x/z_6(V''_1))^4 \end{aligned}$$

for some constant C . Now if we let $p_1(x) = 1+ax+bx^2 = (1-x/z_6(V_1))(1-x/z_6(V_2))$, then $(1-x/z_6(V'_1))(1-x/z_6(V''_1))$ and $(1-x/z_6(V'_1))(1-x/z_6(V''_1))$ are scalar multiples of

$$\begin{aligned} p_2(x) &= (1+3x)^2 p_1 \left(\frac{x-1}{3x+1} \right) = (1-a+b) + (6-2a-2b)x + (9+3a+b)x^2, \\ p_3(x) &= (1-3x)^2 p_1 \left(\frac{x+1}{1-3x} \right) = (1+a+b) + (-6-2a+2b)x + (9-3a+b)x^2, \end{aligned}$$

respectively. Substituting these into (7) and equating the coefficients in the two sides, we find $A = B = 0$, $a = -2$, $b = -3$, but obviously this is invalid. This means that $\pi^{-1}(S_4) \neq \{U_1, U_2, V_1, V_2\}$. Likewise, $\pi^{-1}(S_4) \neq \{U_1, U_6, V_1, V_2\}$. Thus, we must have $\pi^{-1}(S_4) = \{U_1, U_4, V_1, V_2\}$. Now equating the coefficients in the two sides of

$$Bx(1+ax+bx^2)^4 - (1-x)(1+3x)p_2(x)^4 = C(1+x)(1-3x)p_3(x)^4$$

and excluding the invalid solutions, we get the claimed relation between z_4 and z_6 . The relation between z_3 and z_5 can be determined by the known relation between z_3 and z_4 , that between z_4 and z_6 , and that between z_5 and z_6 . This process also determines the choices of the third roots of unity in the relation between z_3 and z_4 and that between z_5 and z_6 . We omit the details. \square

Lemma 10. *The automorphic derivative $Q(z_6) = D(z_6, \tau)$ is equal to*

$$(8) \quad \begin{aligned} & \frac{15}{64} \left(\frac{1}{z_6^2} + \frac{1}{(1-z_6)^2} + \frac{1}{(1+z_6)^2} + \frac{1}{(z_6-1/3)^2} + \frac{1}{(z_6+1/3)^2} \right) \\ & + \frac{45}{128} \left(\frac{1}{1-z_6} + \frac{1}{1+z_6} + \frac{3}{1-3z_6} + \frac{3}{1+3z_6} \right). \end{aligned}$$

Proof. By Proposition 2, the rational function $R(x)$ such that automorphic $Q(z_6) = D(z_6, \tau)$ is equal to $R(z_6)$ is equal to

$$\begin{aligned} R(x) &= \frac{15}{64} \left(\frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(x-1/3)^2} + \frac{1}{(x+1/3)^2} \right) \\ & + \frac{B_1}{x} + \frac{B_2}{x-1} + \frac{B_3}{x+1} + \frac{B_4}{x-1/3} + \frac{B_5}{x+1/3} \end{aligned}$$

for some constants B_j satisfying

$$(9) \quad B_1 + B_2 + B_3 + B_4 + B_5 = 0, \quad B_2 - B_3 + \frac{1}{3}B_4 - \frac{1}{3}B_5 + \frac{15}{16} = 0.$$

Now the normalizer of Γ_6 in $\mathrm{SL}(2, \mathbb{R})$ contains at least the group of signature $(2, 6, 8)$. The factor group, in terms of the Hauptmodul z_6 , is generated by $\sigma : z_6 \mapsto (3z_6+1)/(-3z_6+3)$ and $\tau : z_6 \mapsto -z_6$. By Proposition 6, $R(x)$ satisfies

$$R(-x) = R(x), \quad \frac{144}{(-3x+3)^4} R\left(\frac{3x+1}{-3x+3}\right) = R(x).$$

Combining these informations with (9), we find

$$B_1 = 0, \quad B_2 = B_4 = -\frac{45}{128}, \quad B_3 = B_5 = \frac{45}{128}.$$

This gives us the formula. \square

We now prove the theorem.

Proof of Theorem 1. By Proposition 1, we have

$$\begin{aligned} \dim S_6(\Gamma_1) &= 1 - 6 + \left\lfloor \frac{6}{2} \left(1 - \frac{1}{4}\right) \right\rfloor + 2 \left\lfloor \frac{6}{2} \left(1 - \frac{1}{6}\right) \right\rfloor = 1, \\ \dim S_6(\Gamma_4) &= 1 - 6 + 3 \left\lfloor \frac{6}{2} \left(1 - \frac{1}{4}\right) \right\rfloor = 1, \\ \dim S_6(\Gamma_6) &= 1 - 6 + 6 \left\lfloor \frac{6}{2} \left(1 - \frac{1}{4}\right) \right\rfloor = 7. \end{aligned}$$

By Proposition 4, the one-dimensional spaces $S_6(\Gamma_1)$ and $S_6(\Gamma_4)$ are spanned by

$$(10) \quad F_1 = z_1^{1/4}(1-z_1)^{1/2} \left({}_2F_1\left(\frac{5}{24}, \frac{3}{8}; \frac{3}{4}; z_1\right) + C_1 z_1^{1/4} {}_2F_1\left(\frac{11}{24}, \frac{5}{8}; \frac{5}{4}; z_1\right) \right)^6$$

and

$$(11) \quad F_2 = z_4^{1/4}(1-z_4)^{1/4} \left({}_2F_1\left(\frac{1}{8}, \frac{3}{8}; \frac{3}{4}; z_4\right) + C_2 z_4^{1/4} {}_2F_1\left(\frac{3}{8}, \frac{5}{8}; \frac{5}{4}; z_4\right) \right)^6$$

for some complex numbers C_1 and C_2 , respectively. Furthermore, by Proposition 2, if we let

$$\begin{aligned} f_1 &= z_6^{3/8} \left(1 - \frac{15}{7} z_6^2 - \frac{111}{14} z_6^4 - \frac{2045}{46} z_6^6 - \frac{11355195}{39928} z_6^8 - \frac{77997477}{39928} z_6^{10} - \dots \right) \\ f_2 &= z_6^{5/8} \left(1 - \frac{5}{3} z_6^2 - \frac{245}{34} z_6^4 - \frac{7269}{170} z_6^6 - \frac{115223}{408} z_6^8 - \frac{55230121}{27880} z_6^{10} - \dots \right) \end{aligned}$$

be a basis for the solution space of the Schwarzian differential equation $d^2 f/dz_6^2 + Q(z_6)f = 0$, where $Q(z_6)$ is the rational function in (8), then a basis for $S_8(\Gamma_6)$ is

$$\{z_6^j g : j = 0, \dots, 6\}, \quad g = \frac{(f_1 + C_3 f_2)^8}{z_6^2(1-z_6^2)^2(1-9z_6^2)^2}.$$

Now from Lemma 9, we have

$$z_1 = \frac{12\alpha z_6(1-z_6^2)(1-9z_6^2)}{(1+\alpha z_6)^6}$$

and

$$z_4 = -\frac{4(1+\beta)^4 z_6(1+(-7+4\beta)z_6^2/3)^4}{(1+z_6)(1-3z_6)(1+(4+2\beta)z_6-(1+2\beta)z_6^2)^4},$$

where α is a root of $x^2 + 3 = 0$ and β is a root of $x^2 + 2 = 0$. Substituting these into (10) and (11) and comparing the coefficients, we find

$$F_1 = c_1(1 + 3z_6^2)^3 g$$

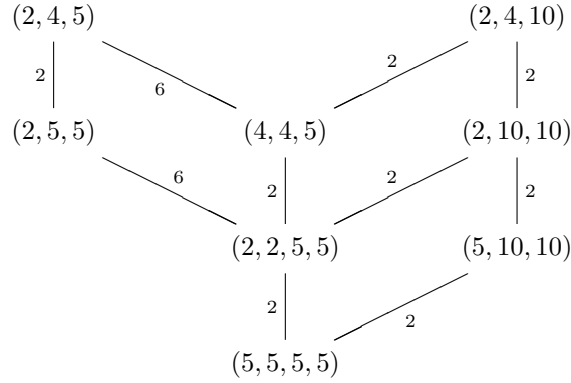
and

$$F_2 = c_2 \left(1 + \frac{-7 + 4\beta}{3} z_6^2 \right) (1 + (4 + 2\beta)z_6 - (1 + 2\beta)z_6^2) \\ \times (1 - (4 + 2\beta)z_6 - (1 + 2\beta)z_6^2) g$$

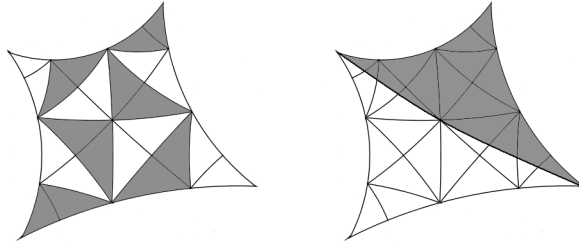
for some constants c_1 and c_2 . Taking the sixth roots of F_1 and F_2 and simplifying, we obtain the identities claimed in the theorem. \square

4. ALGEBRAIC TRANSFORMATIONS ASSOCIATED TO CLASS VI

According to Appendix A, the subgroup diagram for Takeuchi's Class VI is



Let $\Gamma_1 = (2, 5, 5)$, $\Gamma_2 = (5, 10, 10)$, $\Gamma_3 = (5, 5, 5, 5)$, and X_1, X_2, X_3 be the Shimura curves associated to these three groups. (The reader is reminded that the subgroup diagram should be read as “*there are* arithmetic Fuchsian subgroups of $SL(2, \mathbb{R})$ such that their subgroup relations are given by the diagram”.) The subgroups relations $\Gamma_3 < \Gamma_1, \Gamma_2$ admit Coxeter decompositions as the following figures show.



Here the small triangles are $(2, 4, 5)$ -triangles. Associated to this triplet of groups is the following identities.

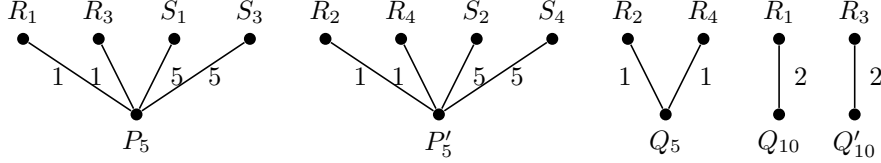
Theorem 2. *We have*

$$(12) \quad {}_2F_1 \left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; \frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5} \right) \\ = (1-z^2)^{1/20} (1+4z-z^2)^{1/4} {}_2F_1 \left(\frac{3}{10}, \frac{2}{5}; \frac{9}{10}; z^2 \right).$$

and

$$(13) \quad \begin{aligned} & (1-z-z^2)_2F_1\left(\frac{1}{4}, \frac{9}{20}; \frac{6}{5}; \frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5}\right) \\ & = (1-z^2)^{1/4}(1+4z-z^2)^{5/4} {}_2F_1\left(\frac{2}{5}, \frac{1}{2}; \frac{11}{10}; z^2\right). \end{aligned}$$

Proof. Label the elliptic points of X_j by P_2, P_5, P'_5 for $X_1, Q_5, Q_{10}, Q'_{10}$ for X_2 , and $R_i, i = 1, \dots, 4$, for X_3 such that the ramifications data are given by



Here the numbers next to the lines are the ramification indices. We have omitted P_2 from the diagram. There are 6 points lying above P_2 . Each has ramification index 2. Choose Hauptmoduls z_j for X_j by requiring

$$z_1(P_5) = 0, \quad z_1(P_2) = 1, \quad z_1(P'_5) = \infty, \quad z_2(Q_{10}) = 0, \quad z_2(Q_5) = 1, \quad z_2(Q'_{10}) = \infty$$

and

$$z_3(R_1) = 0, \quad z_3(R_2) = 1, \quad z_3(R_3) = \infty.$$

The relation between z_2 and z_3 is easy to figure out. We have

$$(14) \quad z_2 = z_3^2,$$

which implies that $z_3(R_4) = -1$. To determine the relation between z_1 and z_3 , we observe that the tessellation of the hyperbolic plane by the $(5, 5, 5, 5)$ -polygons has extra symmetries by rotation by 90 degree around the center of any $(5, 5, 5, 5)$ -polygon. In terms of groups, this means that Γ_3 has a supergroup Γ normalizing Γ_3 such that Γ/Γ_3 is cyclic of order 4. (In fact, Γ is the $(4, 4, 5)$ -triangle group in the subgroup diagram.) Therefore, the automorphism group of X_3 has an element σ of order 4 that permutes R_1, R_2, R_3, R_4 cyclically. In terms of the Hauptmodul, we have

$$\sigma : z_3 \mapsto \frac{z_3 + 1}{z_3 - 1}.$$

Thus, if the value of z_3 at S_1 is a , then we have

$$z_3(S_1) = a, \quad z_3(S_2) = \frac{a-1}{a+1}, \quad z_3(S_3) = -\frac{1}{a}, \quad z_3(S_4) = -\frac{a+1}{a-1}.$$

Therefore, the relation between z_1 and z_3 is

$$z_1 = \frac{Bz_3(z_3 - a)^5(z_3 + 1/a)^5}{(1 - z_3^2)(z_3 - (a-1)/(a+1))^5(z_3 + (a+1)/(a-1))^5}$$

for some constant B . Moreover, the automorphism σ of X_3 rotates 4 of the six points lying above P_2 cyclically and fixes the other two. (The reader is reminded that each $(5, 5, 5, 5)$ -polygon represents only half of the fundamental domain for Γ_3 . The two fixed of σ are the centers of the $(5, 5, 5, 5)$ -polygons.) In terms of the Hauptmodul z_3 , this means that the values of z_3 at the two fixed points of σ are $\pm i$ and if the value of z_3 at one of the other 4 points above P_2 is b , then the values at the other 3 points are $-1/b, (b-1)/(b+1)$, and $-(b+1)/(b-1)$. Thus, we have

$$z_1 - 1 = \frac{C(1 + z_3^2)^2(z_3 - b)^2(z_3 + 1/b)^2(z_3 - (b-1)/(b+1))^2(z_3 + (b+1)/(b-1))^2}{(1 - z_3^2)(z_3 - (a-1)/(a+1))^5(z_3 + (a+1)/(a-1))^5}$$

for some constant C . Comparing the two sides, we find $a = 0, \pm 1, \pm i, a^2 + a - 1 = 0$, or $a^2 - a - 1 = 0$. The first five solutions are invalid. The other two solutions give

$$(15) \quad z_1 = \frac{64z_3(1 - z_3 - z_3^2)^5}{(1 - z_3^2)(1 + 4z_3 - z_3^2)^5}$$

or

$$(16) \quad z_1 = -\frac{64z_3(1 + z_3 - z_3^2)^5}{(1 - z_3^2)(1 - 4z_3 - z_3^2)^5}.$$

Both are valid because of the following reason. Notice that Γ_2 normalizes Γ_3 . If we take an element γ of Γ_2 not in Γ_3 , then $\gamma^{-1}\Gamma_1\gamma$ is again a triangle of signature $(2, 5, 5)$ containing the same Γ_3 . If the relation between the Hauptmoduls of Γ_1 and Γ_3 is (15), then the relation between the Hauptmoduls of $\gamma^{-1}\Gamma_1\gamma$ and Γ_3 will be (16).

Having determining the relations among Hauptmoduls, we can composite identity (36) in [10] with Kummer's quadratic transformation several times to get the identities in the theorem. However, the procedure is very tedious. Here we provide a better proof using the theory of automorphic forms on Shimura curves.

By Proposition 1, we have

$$\begin{aligned} \dim S_8(\Gamma_1) &= 1 - 8 + \left\lfloor \frac{8}{2} \left(1 - \frac{1}{2}\right) \right\rfloor + 2 \left\lfloor \frac{8}{2} \left(1 - \frac{1}{5}\right) \right\rfloor = 1, \\ \dim S_8(\Gamma_2) &= 1 - 8 + \left\lfloor \frac{8}{2} \left(1 - \frac{1}{5}\right) \right\rfloor + 2 \left\lfloor \frac{8}{2} \left(1 - \frac{1}{10}\right) \right\rfloor = 2, \\ \dim S_8(\Gamma_3) &= 1 - 8 + 4 \left\lfloor \frac{8}{2} \left(1 - \frac{1}{5}\right) \right\rfloor = 5. \end{aligned}$$

By Proposition 4, the one-dimensional space $S_8(\Gamma_1)$ is spanned by

$$(17) \quad F_1 = z_1^{1/5} \left({}_2F_1 \left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; z_1 \right) + C_1 z_1^{1/5} {}_2F_1 \left(\frac{1}{4}, \frac{9}{20}; \frac{6}{5}; z_1 \right) \right)^8$$

for some constant C_1 , and the function

$$(18) \quad F_2 = z_2^{3/5} (1 - z_2)^{1/5} \left({}_2F_1 \left(\frac{3}{10}, \frac{2}{5}; \frac{9}{10}; z_2 \right) + C_2 z_2^{1/10} {}_2F_1 \left(\frac{2}{5}, \frac{1}{2}; \frac{11}{10}; z_2 \right) \right)^8$$

is contained in $S_8(\Gamma_2)$ for some constant C_2 . To get a basis for $S_8(\Gamma_3)$, we need to work out the Schwarzian differential equation associated to z_3 . It is actually easy in this case.

By Proposition 2, the function $z_3'(\tau)$, as a function of z_3 , satisfies

$$\frac{d^2}{dz_3^2} f + Q(z_3)f = 0,$$

where

$$(19) \quad Q(z_3) = \frac{6}{25} \left(\frac{1}{z_3^2} + \frac{1}{(1 - z_3)^2} + \frac{1}{(1 + z_3)^2} \right) + \frac{B_1}{z_3} + \frac{B_2}{z_3 - 1} + \frac{B_3}{z_3 + 1}$$

for some complex numbers satisfying

$$(20) \quad B_1 + B_2 + B_3 = 0, \quad B_2 - B_3 + \frac{12}{25} = 0.$$

To determine the values of B_j , we use the automorphism of X_3 coming from the normal subgroup relation $\Gamma_3 \triangleleft \Gamma_1$. Let γ be an element of Γ_2 not in Γ_3 . We know that

$$z_3(\gamma\tau) = -z_3(\tau).$$

Now by Proposition 5, we have

$$(21) \quad \begin{aligned} D(-z_3(\tau), \tau) &= D(z_3(\gamma\tau), \tau) = D(z_3(\gamma\tau), \gamma\tau) + D(\gamma\tau, \tau)/(d\gamma\tau/d\tau)^2 = Q(z_3(\gamma\tau)) \\ &= \frac{6}{25} \left(\frac{1}{z_3^2} + \frac{1}{(1-z_3)^2} + \frac{1}{(1+z_3)^2} \right) + \frac{B_1}{-z_3} + \frac{B_2}{-z_3-1} + \frac{B_3}{-z_3+1} \end{aligned}$$

On the other hand, we also have, by the same proposition,

$$(22) \quad D(-z_3, \tau) = D(-z_3, z_3) + D(z_3, \tau)/(-1)^2 = Q(z_3)$$

Comparing (19), (21), and (22), we find $B_1 = 0$ and $B_2 = -B_3$. Together with (20), this gives us

$$B_1 = 0, \quad B_2 = -\frac{6}{25}, \quad B_3 = \frac{6}{25}$$

and

$$Q(z_3) = \frac{6}{25} \left(\frac{1}{z_3^2} + \frac{1}{(1-z_3)^2} + \frac{1}{(1+z_3)^2} + \frac{1}{1-z_3} + \frac{1}{1+z_3} \right).$$

Now a basis for the solution space of the Schwarzian differential equation $d^2f/dz_3^2 + Q(z_3)f = 0$ is given by

$$\begin{aligned} f_1 &= z_3^{2/5} \left(1 - \frac{4}{15}z_3^2 - \frac{52}{475}z_3^4 - \frac{13436}{206625}z_3^6 - \frac{46348}{1033125}z_3^8 - \frac{2024924}{60265625}z_3^{10} - \dots \right) \\ f_2 &= z_3^{3/5} \left(1 - \frac{12}{55}z_3^2 - \frac{28}{275}z_3^4 - \frac{2708}{42625}z_3^6 - \frac{393636}{8738125}z_3^8 - \frac{7503908}{218453125}z_3^{10} - \dots \right). \end{aligned}$$

By Proposition 2,

$$g, z_3g, z_3^2g, z_3^3g, z_3^4g, \quad g = \frac{(f_1 + C_3f_2)^8}{z_3^3(1-z_3)^3(1+z_3)^3},$$

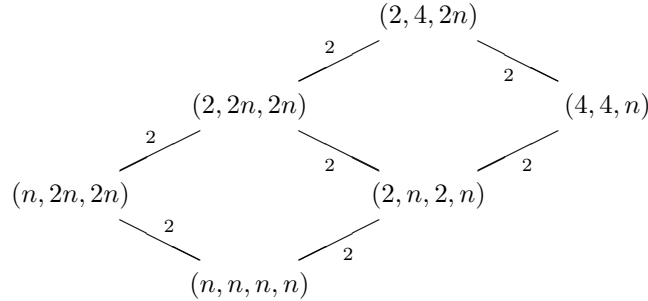
form a basis for $S_8(\Gamma_3)$ for some constant C_3 . That is, after substituting (15) and (14) into (17) and (18), respectively, we have $F_1 = h_1(z_3)g$ and $F_2 = h_2(z_3)g$ for some polynomials $h_1(x)$ and $h_2(x)$ of degree ≤ 4 . Indeed, by comparing the coefficients, we find

$$F_1 = 2^{6/5}(1-z_3-z_3^2)(1+4z_3-z_3^2)g, \quad F_2 = z_3g.$$

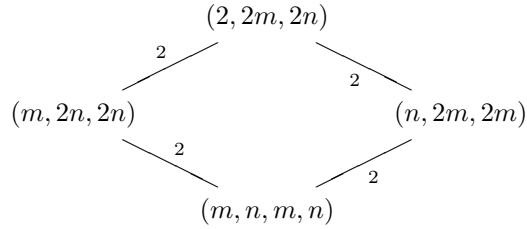
(The computation becomes easier if we take the 8th roots of the functions first.) Simplifying the relation $z_3F_1 = 2^{6/5}(1-z_3-z_3^2)(1+4z_3-z_3^2)F_2$, we get the two identities in the theorem. This completes the proof. \square

5. ALGEBRAIC TRANSFORMATIONS ASSOCIATED TO OTHER CLASSES

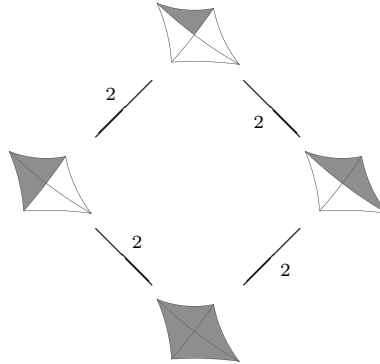
5.1. Classes II, V, and XII. The subgroup diagrams of Class II, V, and XII are all of the form



The subgroup relation $(2, 2n, 2n) \cap (4, 4, n) = (2, n, 2, n)$ is a special case of



which arises from the Coxeter decompositions of a quadrilateral polygon that is symmetric with respect to both the diagonals as shown below



Associated to this family of subgroup relations is the following identity.

Theorem 3. For real numbers a and b such that neither $b + 3/4$ nor $2b + 1/2$ is a nonpositive integer, we have

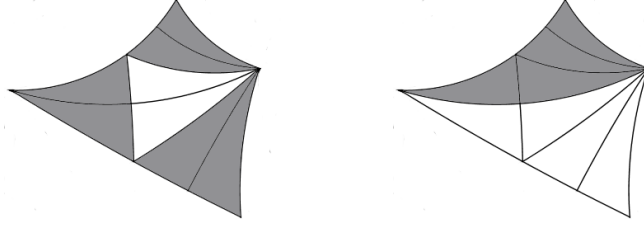
$$(1+z)^{2a+2b} {}_2F_1\left(a+b, a+\frac{1}{4}; b+\frac{3}{4}; z^2\right) = {}_2F_1\left(a+b, b+\frac{1}{4}; 2b+\frac{1}{2}; \frac{4z}{(1+z)^2}\right)$$

in a neighborhood of $z = 0$.

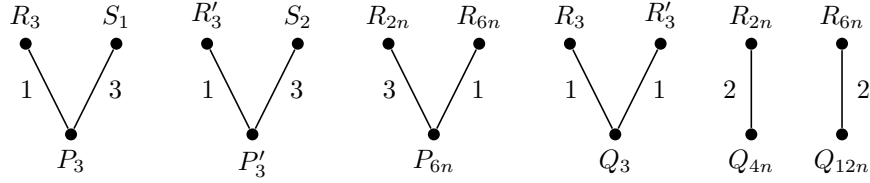
This identity can be easily proved using Kummer's quadratic transformation. Alternatively, one can verify that both sides are solutions of the differential equation

$$2z(1-z)(1+z)^2 F'' - (1+z)((3-4b)z^2 + 8(a+b)z - 4b - 1)F' - (a+b)(1+4b)(1-z)F = 0.$$

and that the local behaviors at $z = 0$ agree. We omit the details.



Here the parameter n in the figures is 1 and the smaller triangles are $(2, 3, 12)$ -triangles. Let $\Gamma_1 = (3, 3, 6n)$, $\Gamma_2 = (3, 4n, 12n)$, $\Gamma_3 = \Gamma_1 \cap \Gamma_2$, and let X_i , $i = 1, \dots, 3$ be the associated Shimura curves. Denote by P_3 , P'_3 , and P_{6n} the elliptic points of orders 3, 3, and $6n$ on X_1 , by Q_3 , Q_{4n} , and Q_{12n} the elliptic points of orders 3, $4n$, and $12n$ on X_2 , and by R_3 , R'_3 , R_{2n} , and R_{6n} the elliptic points of order 3, 3, $2n$, and $6n$ on X_3 . The points are labelled in a way such that the ramification data are given by



Choose Hauptmoduls z_j on X_j , $j = 1, 2, 3$, by requiring

$$z_1(P_{6n}) = 0, \quad z_1(P_3) = 1, \quad z_1(P'_3) = \infty, \quad z_2(Q_{4n}) = 0, \quad z_2(Q_3) = 1, \quad z_2(Q_{12n}) = \infty$$

and

$$z_3(R_{2n}) = 0, \quad z_3(R_3) = 1, \quad z_3(R_{6n}) = \infty.$$

It is easy to see from the ramification informations that

$$(23) \quad z_2 = z_3^2,$$

which implies that $z_3(R'_3) = -1$. For z_1 , we have

$$z_1 = \frac{Az_3^3}{(1+z_3)(1-az_3)^3}$$

for some complex numbers A and a , where $1/a$ is the value of z_3 at S_1 . These two numbers satisfy

$$(24) \quad 1 - \frac{Az_3^3}{(1+z_3)(1-az_3)^3} = 1 - z_1 = \frac{(1-z_3)(1-bz_3)^3}{(1+z_3)(1-az_3)^3},$$

where $1/b$ is the value of z_3 at S_2 . Now observe that Γ_3 is a normal subgroup of Γ_2 . Thus, an element of Γ_2 not in Γ_3 induces an automorphism on X_3 . In terms of the Hauptmodul z_3 , it is easy to see that this automorphism sends z_3 to $-z_3$. Since this automorphism maps S_1 to S_2 , we find $b = -a$. Then comparing the two sides of (24), we get $A = 16/27$, $a = 1/3$, and

$$(25) \quad z_1 = \frac{16z_3^3}{(1+z_3)(3-z_3)^3}.$$

Now by Proposition 1, we have

$$\dim S_6(\Gamma_1) = \dim S_6(\Gamma_2) = 1, \quad \dim S_6(\Gamma_3) = \begin{cases} 2, & \text{if } n = 1, \\ 3, & \text{if } n \geq 2. \end{cases}$$

From now on, we assume that $n \geq 2$.

By Proposition 4, the one-dimensional spaces $S_6(\Gamma_1)$ and $S_6(\Gamma_2)$ are spanned by

$$(26) \quad F_1 = z_1^{1-1/2n} \left({}_2F_1 \left(\frac{1}{6} - \frac{1}{12n}, \frac{1}{2} - \frac{1}{12n}; 1 - \frac{1}{6n}; z_1 \right) + C_1 z_1^{1/6n} {}_2F_1 \left(\frac{1}{6} + \frac{1}{12n}, \frac{1}{2} + \frac{1}{12n}; 1 + \frac{1}{6n}; z_1 \right) \right)^6$$

and

$$(27) \quad F_2 = z_2^{1-3/4n} \left({}_2F_1 \left(\frac{1}{3} - \frac{1}{6n}, \frac{1}{3} - \frac{1}{12n}; 1 - \frac{1}{4n}; z_2 \right) + C_2 z_2^{1/4n} {}_2F_1 \left(\frac{1}{3} + \frac{1}{12n}, \frac{1}{3} + \frac{1}{6n}; 1 + \frac{1}{4n}; z_2 \right) \right)^6,$$

respectively, for some constants C_1 and C_2 . Also, if we let $f_1 = z_3^{1/2-1/4n}(1+c_1z+\dots)$ and $f_2 = z_3^{1/2+1/4n}(1+d_1z+\dots)$ be a basis of the solution space of the Schwarzian differential equation $d^2f/dz_3^2 + Q(z_3)f = 0$ associated to z_3 , then by Proposition 2, $S_6(\Gamma_3)$ is spanned by g , z_3g , and z_3^2g , where

$$g = \frac{(f_1 + C_3 f_2)^6}{z_3^2(1-z_3)^2(1+z_3)^2}$$

for some constant C_3 . Now we substitute (25) and (23) into (26) and (23), respectively. We find

$$F_1 = a_1 z_3^{3-3/2n} + \dots, \quad F_2 = z_3^{2-3/2n} + \dots,$$

where $a_1 = (16/27)^{1-1/2n}$, and thus

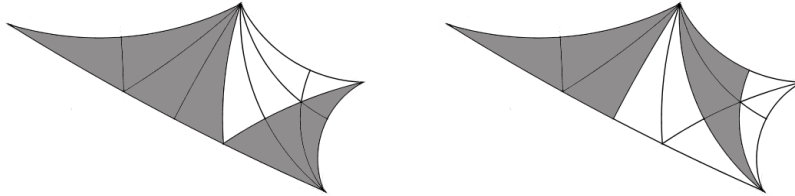
$$F_1 = a_1 z_3^2 g, \quad F_2 = (z_3 + a_2 z_3^2) g$$

for some constant a_2 . That is, $a_1 z F_2 / F_1 = 1 + a_2 z_3$. We then take the 6th roots of the two sides and compare the coefficients of $z^{3/2-1/4n}$, we find that a_2 is actually 0. After simplifying, we arrive at

$$\begin{aligned} & (1+z)^{1/6-1/12n} (1-z/3)^{1/2-1/4n} {}_2F_1 \left(\frac{1}{3} - \frac{1}{6n}, \frac{1}{3} - \frac{1}{12n}; 1 - \frac{1}{4n}; z^2 \right) \\ &= {}_2F_1 \left(\frac{1}{6} - \frac{1}{12n}, \frac{1}{2} - \frac{1}{12n}; 1 - \frac{1}{6n}; \frac{16z^3}{(1+z)(3-z)^3} \right). \end{aligned}$$

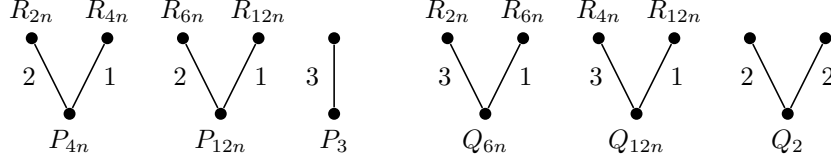
This proves Theorem 4 in the case the parameters correspond to arithmetic triangle groups. \square

Proof of Theorem 5 in the cases of Shimura curves. The subgroups $(3, 4n, 12n)$, $(2, 6n, 12n)$ and their intersection admit Coxeter decompositions as the figures below show.



Here the parameter n in the figures is 1 and the small triangles are $(2, 3, 12)$ -triangles.

Denote the groups $(3, 4n, 12n)$, $(2, 6n, 12n)$, and $(2n, 4n, 6n, 12n)$ by Γ_1, Γ_2 , and Γ_3 , respectively. Label the elliptic points of $(3, 4n, 12n)$ by P_3, P_{4n} , and P_{12n} , those of $(2, 6n, 12n)$ by Q_2, Q_{6n} , and Q_{12n} , and those of $(2n, 4n, 6n, 12n)$ by R_{2n}, R_{4n}, R_{6n} , and R_{12n} . The ramifications are shown as follows.



Choose Hauptmoduls z_j for $\Gamma_j, j = 1, \dots, 3$, by requiring that

$$\begin{aligned} z_1(P_{4n}) &= 0, & z_1(P_3) &= 1, & z_1(P_{12n}) &= \infty, \\ z_2(Q_{6n}) &= 0, & z_2(Q_2) &= 1, & z_2(Q_{12n}) &= \infty, \\ z_3(R_{2n}) &= 0, & z_3(R_{4n}) &= 1, & z_3(R_{6n}) &= \infty. \end{aligned}$$

It is easy to work out the relation between z_1 and z_3 and that between z_2 and z_3 . They are

$$(28) \quad z_1 = \frac{27z_3^2(1-z_3)}{1-9z_3}, \quad z_2 = -\frac{64z_3^3}{(1-z_3)^3(1-9z_3)}.$$

Here $1/9$ is the value of z_3 at R_{12n} . We then follow the same arguments as before to obtain the claimed identities. We omit the details. \square

APPENDIX A. LIST OF ARITHMETIC TRIANGLE GROUPS

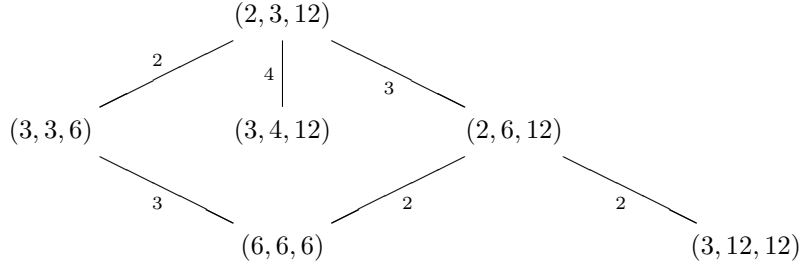
In this section, we determine the signatures of the intersections of commensurable triangle groups.

According to [7, 8], there are totally 85 arithmetic triangle groups, falling in 19 different commensurability classes. Here we give the subgroup diagrams. Note that since most groups here have genus 0, we omit the genus information from the signature, unless the group has a positive genus. Also, to save space, the notation $(g; e_1^{n_1}, \dots, e_r^{n_r})$ means that the Shimura curve has n_i elliptic points order e_i . Furthermore, for convenience, we will often call the groups by their signatures, even though this raises some ambiguity.

Remark 11. There is some ambiguity when we say “the intersections of commensurable triangle groups” because there may be more than one orders whose norm-one groups have the same signature and the intersections of these groups with another group may have different signatures. For example, in the case $B = M(2, \mathbb{Q})$, the subgroups $\Gamma_0(2)$ and $\Gamma^0(2)$ of $SL(2, \mathbb{Z})$ have the same signature $(0; 2, \infty, \infty)$ and the group $\Gamma_0(4)$ has signature $(0; \infty, \infty, \infty)$. The intersection of $\Gamma_0(2)$ and $\Gamma_0(4)$ is just $\Gamma_0(4)$, but the intersection of $\Gamma^0(2)$ and $\Gamma_0(4)$ has signature $(0; \infty, \infty, \infty, \infty)$. Thus, the subgroup diagrams described here should be read as “*there are arithmetic groups whose subgroup relations are given by the subgroup diagrams*”.

Since it is not easy to describe explicitly the orders associated to arithmetic triangle groups, here we use group theory and properties of discrete subgroups of $SL(2, \mathbb{R})$ to determine the signatures. We will work out the case of Class IV in [8] and omit the proof of the others.

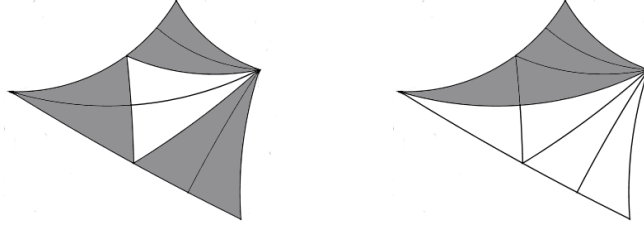
According to [8], Class IV of arithmetic triangle groups has the following subgroup diagram.



Here the numbers next to the lines are the indices. Set

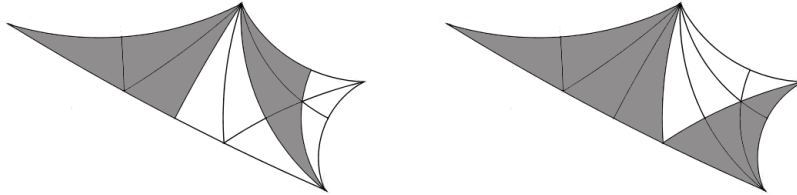
$$\begin{aligned} \Gamma_1 &= (2, 3, 12), & \Gamma_2 &= (3, 3, 6), & \Gamma_3 &= (3, 4, 12), \\ \Gamma_4 &= (2, 6, 12), & \Gamma_5 &= (6, 6, 6), & \Gamma_6 &= (3, 12, 12), \end{aligned}$$

and let X_i , $i = 1, \dots, 6$, denote the respective Shimura curves. To determine $\Gamma_2 \cap \Gamma_3$, we observe that Γ_2 is a normal subgroup of Γ_1 of index 2 and $\Gamma_1 = \Gamma_2 \Gamma_3$. Thus, $\Gamma_2 \cap \Gamma_3$ is a normal subgroup of Γ_3 of index 2. Now the elliptic point of order 3 on X_3 must split into two points in $X(\Gamma_2 \cap \Gamma_3)$ because $2 \nmid 3$. Then from the Riemann-Hurwitz formula, we see that the elliptic points of order 4 and 12 must be ramified. That is, the curve $X(\Gamma_2 \cap \Gamma_3)$ must have signature $(2, 3, 3, 6)$. In fact, this can also be seen from the following figures.

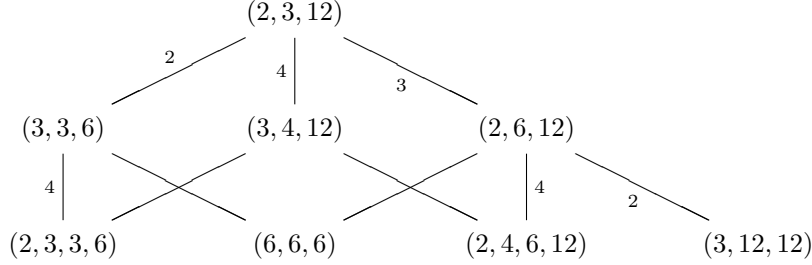


Here the smaller triangles are $(2, 3, 12)$ -triangles. The figures show that the triangle group $(2, 3, 12)$ contains two subgroups of signatures $(3, 3, 6)$ and $(3, 4, 12)$, respectively, whose intersection has signature $(2, 3, 3, 6)$. (In fact, the theoretical argument above shows that for any pair of subgroups of Γ_1 with signatures $(3, 3, 6)$ and $(3, 4, 12)$, respectively, the intersection must have signature $(2, 3, 3, 6)$.)

Likewise, the figures



show that there are two subgroups of Γ_1 of signatures $(2, 6, 12)$ and $(3, 4, 12)$ such that their intersection has signature $(2, 4, 6, 12)$. We have the following subgroup diagram.



Let $\Gamma_7 = (2, 3, 3, 6)$ and $\Gamma_8 = (2, 4, 6, 12)$ and X_7 and X_8 be their associated Shimura curves. Again, because Γ_5 is a normal subgroup of Γ_4 of index 2 and $\Gamma_5\Gamma_8 = \Gamma_4$, the intersection of Γ_5 and Γ_8 is a subgroup of index 2 of Γ_8 . Now the group $(2, 4, 6, 12)$ has many subgroups of index 2. (The structure of the quotient group of $(2, 4, 6, 12)$ over its commutator subgroup is $C_2 \times C_4 \times C_6$.) To determine which of them is contained is the group $(6, 6, 6)$, we use the following properties.

- (1) If p is an elliptic point of order e on X_8 , then its preimage in the covering $X(\Gamma_5 \cap \Gamma_8) \rightarrow X_8$ consists of either a single elliptic point of order $e/2$ or two elliptic points of order e .
- (2) The total branch number of any finite covering of compact Riemann surface is always even.
- (3) The volume of $X(\Gamma_5 \cap \Gamma_8)$ is twice of that of X_8 . Thus, if $(g; e_1, \dots, e_r)$ is the signature of $X(\Gamma_5 \cap \Gamma_8)$, then we must have

$$2g - 2 + \sum_{i=1}^r \left(1 - \frac{1}{e_j}\right) = 2 \left(2 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{12}\right) = 2.$$

From these informations, we find that possible signatures of a subgroup of index 2 of $(2, 4, 6, 12)$ are

$$(29) \quad (1; 2, 3, 6), (0; 2, 6^2, 12^2), (0; 3, 4^2, 12^2), (0; 4^2, 6^3), \\ (0; 2^3, 3, 12^2), (0; 2^3, 6^3), (0; 2^2, 3, 4^2, 6).$$

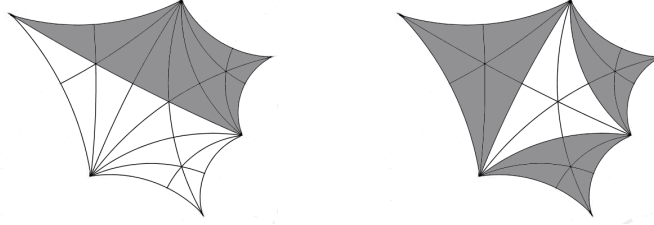
Likewise, an elliptic point of order 6 on X_5 can

- (1) splits into 4 elliptic points of order 6, or
- (2) splits into 2 elliptic points of order 3, or
- (3) splits into 1 elliptic point of order 3 and 2 elliptic point of order 6, or
- (4) splits into 1 elliptic point of order 2 and 1 elliptic point of order 6,

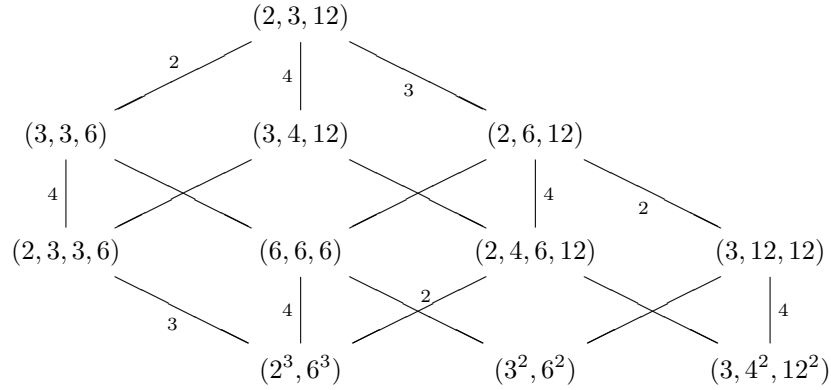
in the covering $X(\Gamma_5 \cap \Gamma_8) \rightarrow X_5$ of degree 4. Also, the total branch number of $X(\Gamma_5 \cap \Gamma_8) \rightarrow X_5$ must be a positive even integer and the volume of $X(\Gamma_5 \cap \Gamma_8)$ is 2. We find the possible signatures of a subgroup of index 4 of Γ_5 are

$$(30) \quad (0; 2^3, 6^3), (0; 2^2, 3^2, 6^2), (0; 2, 3^4, 6), (0; 3^6).$$

From (29) and (30), we conclude that the signature of $\Gamma_5 \cap \Gamma_8$ must be $(0; 2^3, 6^3)$. This can also be seen from the figures.



By the same argument, we can also show that the intersection of Γ_6 and Γ_8 must have signature $(0; 3, 4^2, 12^2)$ and the intersection of Γ_5 and Γ_6 has signature $(0; 3, 3, 6, 6)$. The subgroup diagram becomes



Finally, we can show that the only possible signatures of subgroups of index 2 in $(2^3, 6^3)$ are

$$(0; 2^6, 3^2, 6^2), (0; 2^4, 3, 6^4), (0; 2^2, 6^6), (1; 2^4, 3^3), (1; 2^2, 3^2, 6^2), (1; 3, 6^4), (2; 3^3),$$

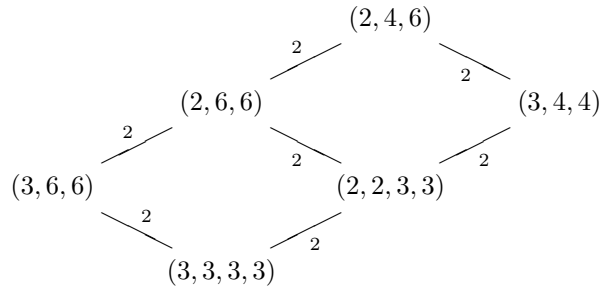
while the only possible signatures of subgroups of index 2 in $(3, 4^2, 12^2)$ are

$$(0; 3^2, 4^4, 6^2), (0; 2, 3^2, 4^2, 6, 12^2), (0; 2^2, 3^2, 12^4), (1; 2^2, 3^2, 6^2).$$

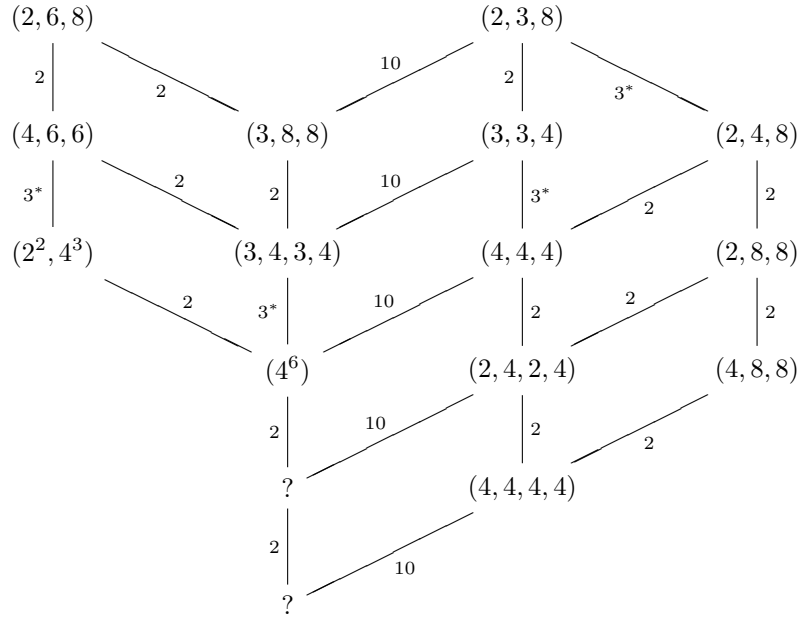
From these, we see that the common intersection of $(2^3, 6^3)$, $(3, 4^2, 12^2)$, and $(3^2, 6^2)$ has signature $(1; 2^2, 3^2, 6^2)$. This completes the proof of the case of Class IV.

Now we give the subgroup diagrams for arithmetic triangle groups.

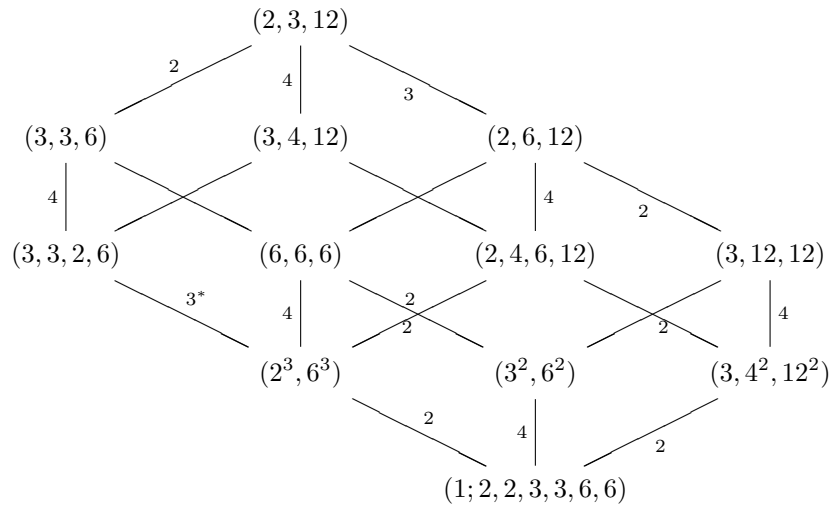
Class II.

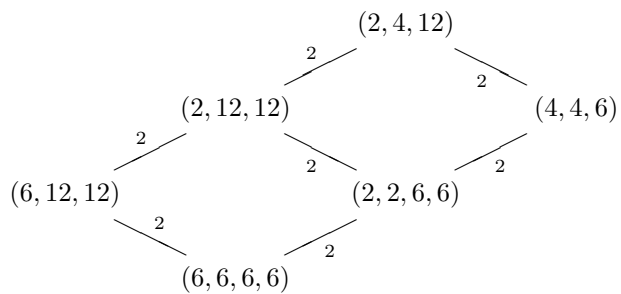
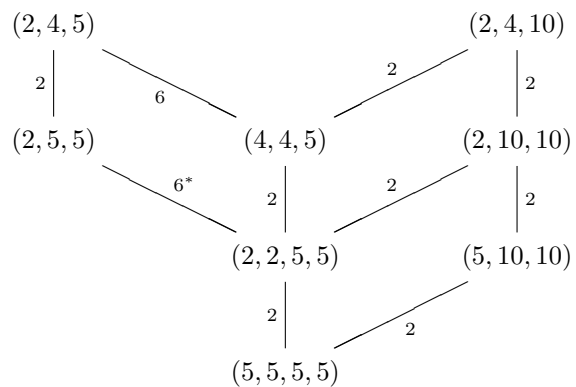
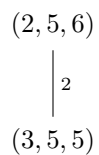
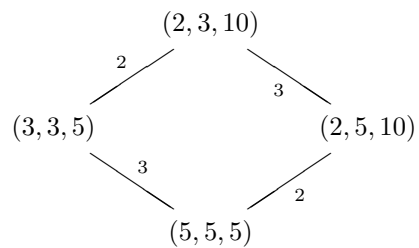


Class III.

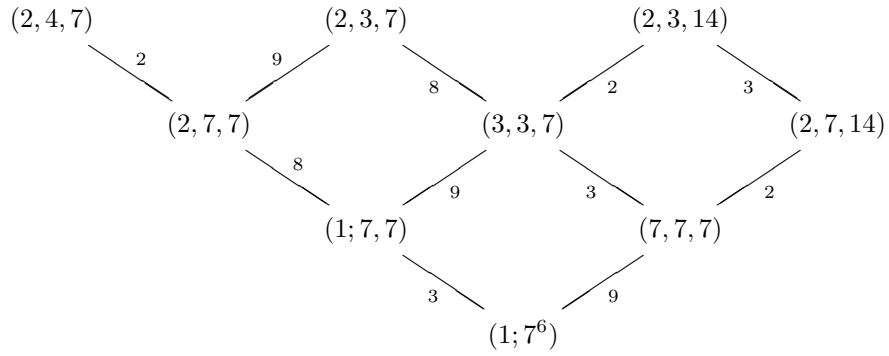


Class IV.

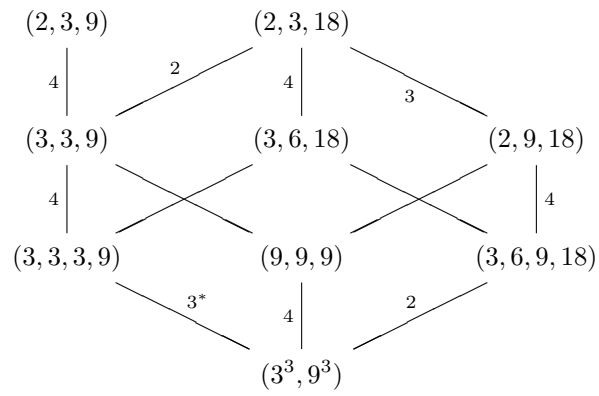


Class V.**Class VI.****Class VII.****Class VIII.****Class IX.**

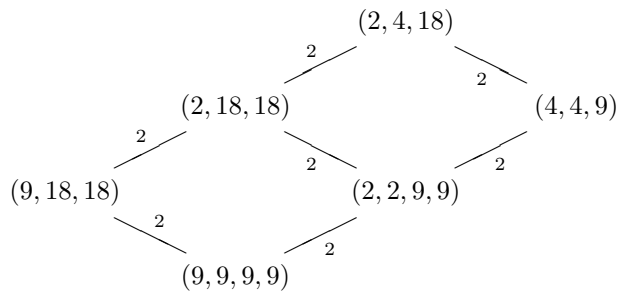
Class X.



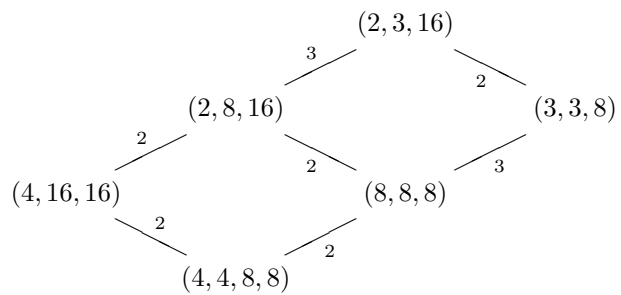
Class XI.



Class XII.

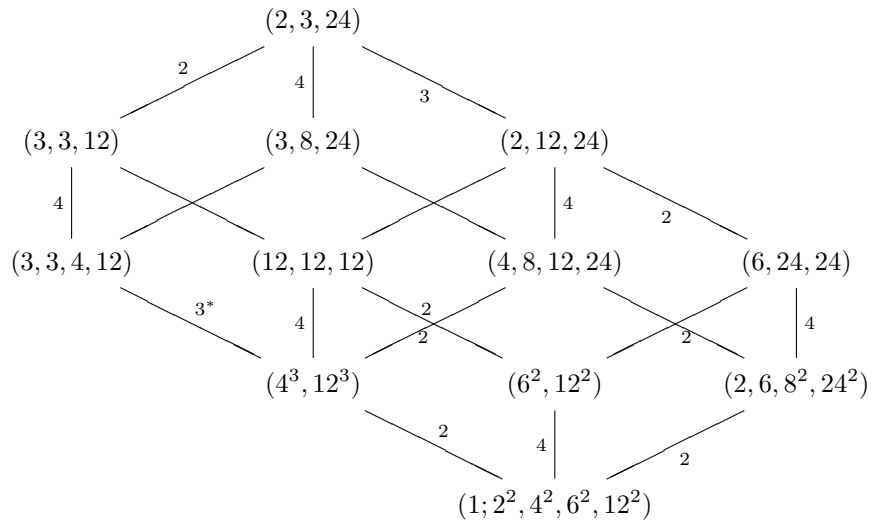


Class XIII.

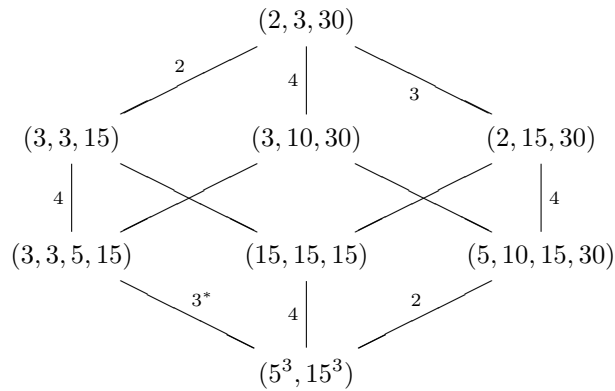


Class XIV.

$$\begin{array}{c} (2, 5, 20) \\ | \\ 2 \\ | \\ (5, 5, 10) \end{array}$$

Class XV.**Class XVI.**

$$\begin{array}{c} (2, 5, 30) \\ | \\ 2 \\ | \\ (5, 5, 15) \end{array}$$

Class XVII.

Class XVIII.

$$\begin{array}{c} (2, 5, 8) \\ | \\ 2 \\ | \\ (4, 5, 5) \end{array}$$

Class XIX.

$$(2, 3, 11)$$

REFERENCES

- [1] Pilar Bayer and Artur Travesa. Uniformizing functions for certain Shimura curves, in the case $D = 6$. *Acta Arith.*, 126(4):315–339, 2007.
- [2] Noam D. Elkies. Shimura curves for level-3 subgroups of the $(2, 3, 7)$ triangle group, and some other examples. In *Algorithmic number theory*, volume 4076 of *Lecture Notes in Comput. Sci.*, pages 302–316. Springer, Berlin, 2006.
- [3] Édouard Goursat. Sur l'équation différentielle linéaire, qui admet pour intégrale la série hypergéométrique. *Ann. Sci. École Norm. Sup.*, 10:3–142, 1881.
- [4] Svetlana Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
- [5] Goro Shimura. Construction of class fields and zeta functions of algebraic curves. *Ann. of Math. (2)*, 85:58–159, 1967.
- [6] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kano Memorial Lectures, 1.
- [7] Kisao Takeuchi. Arithmetic triangle groups. *J. Math. Soc. Japan*, 29(1):91–106, 1977.
- [8] Kisao Takeuchi. Commensurability classes of arithmetic triangle groups. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24(1):201–212, 1977.
- [9] Fang-Ting Tu. Schwarzian differential equations associated to Shimura curves of genus zero. *preprint*, 2011.
- [10] Raimundas Vidūnas. Transformations of some Gauss hypergeometric functions. *J. Comput. Appl. Math.*, 178(1-2):473–487, 2005.
- [11] Marie-France Vignéras. *Arithmétique des algèbres de quaternions*, volume 800 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [12] Yifan Yang. Schwarzian differential equations and Hecke eigenforms on Shimura curves. *arxiv:1110.6284*, 2011.

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