Difference and Similarity Models of Two Idempotent Operators

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Dedicated to Chandler Davis on his retirement

Submitted by Peter Rosenthal

ABSTRACT

In a recent work, Hartwig and Putcha obtained a complete characterization of those finite matrices which can be expressed as the difference of two idempotents. Extending this result to operators on a possibly infinite-dimensional Hilbert space seems more difficult. In this paper, we initiate its study and obtain, among other things, (1) that not every nilpotent operator is the difference of two idempotents, (2) that if T is the difference of two idempotents, then the spectra of T and $-T$ differ at most by the two points ± 1, and (3) a characterization of differences of two idempotents among normal operators. In the second part of the paper, we develop some similarity-invariant models of two idempotents. These are analogous to the known unitary-equivalence-invariant models for two orthogonal projections.

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INTRODUCTION

The study of the difference of two orthogonal projections on a Hilbert space has a long history. The pioneering work was done by C. Davis: he obtained in [1] a complete characterization of such differences. The corresponding problem for idempotents was taken up more recently. In [8], Hartwig and Putcha were able to characterize differences of two idempotents on any finite-dimensional space. The purpose of this paper is to initiate the study of this latter problem for operators on infinite-dimensional spaces.

In the following, we consider only bounded linear operators on a complex, possibly infinite-dimensional Hilbert space. (Although many results below hold on Banach spaces, we restrict ourselves to Hilbert spaces for ease of exposition.) Recall that an operator T is *idempotent* if $T^2 = T$ and is an (*orthogonal*) *projection* if $T^2 = T = T^*$. The problem of characterizing differences of idempotents in general is more difficult than its finite-dimensional counterpart. Although many assertions in [8] carry over to the more general setting, there are some which fail to follow through. In particular, we show below that, in contrast to the finite-dimensional case, there are nilpotent operators which cannot be expressed as the difference of two idempotents. On the positive side, we obtain many necessary or sufficient conditions for an operator to be so expressed. In particular, we prove that if T is the difference of two idempotents, then the spectra of T and $-T$ differ at most by the two points ± 1. We also obtain a complete characterization of such differences among normal operators. All these will be contained in Section 1 below.

In Section 2, we consider some similarity-invariant models for two idempotents. More precisely, given idempotents E and F, we seek some canonical operators E' and F' which are simultaneously similar to E and F. With certain restrictions on the spectrum of $E - F$, we can choose E' and F' to be of one of the following simple forms:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A & I-A \\ A & I-A \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} A & (A-A^2)^{1/2} \\ (A-A^2)^{1/2} & I-A \end{bmatrix}.$$

As an example, we show that if 0 and ± 1 are not in the spectrum of $E - F$, then E and F are simultaneously similar to

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I-A \\ A & I-A \end{bmatrix}$$

for some unique (up to similarity) operator A with spectrum not containing 0 and 1. Such models simplify many constructions involving two idempotents and serve to deepen our understanding of their structures. They are also expected to play a role in the characterizations of the difference, sum, and product formed by two idempotents. Their counterparts, the unitary-equivalence-invariant models, for two projections have been considered for decades (cf. [6]). In many cases, our models will lead to the latter ones.

Before starting, we fix some notation. For two operators T and S, we use $T \approx S$ to denote that T and S are *similar*, that is, there is an invertible operator X such that $XT = SX$. For pairs of operators (T_1, T_2) and (S_1, S_2), $(T_1, T_2) \approx (S_1, S_2)$ denotes that they are *simultaneously similar*, that is, the invertible X is such that $XT_1 = S_1 X$ and $XT_2 = S_2 X$. $\sigma(T)$, $\sigma_e(T)$, and $\sigma_p(T)$ denote, respectively, the spectrum, the essential spectrum, and the point spectrum of an operator T; $\operatorname{ran} T$ and $\ker T$ denote its range and kernel.

1. DIFFERENCE

In this section, we present some sufficient or necessary conditions for an operator to be expressible as the difference of two idempotents. The presentation is based on [15, Chapter 1] with modifications and improvements. The theory bears resemblance to that of sums of two square-zero operators (cf. [16, Section 2]).

We start with the following lemma. Recall that an operator X is an *involution* if $X^2 = I$.

LEMMA 1.1. *If T is an operator such that $XT = -TX$ for some involution X, then T is the difference of two idempotents.*

Proof. Let $E = \frac{1}{2}(I - X)$. Then E is an idempotent and $TE = (I - E)T$. Since E is similar to an operator of the form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, we may, for convenience, assume that

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and also

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

Carrying out the matrix multiplications on both sides of $TE = (I - E)T$, we derive that $T_1 = 0$ and $T_4 = 0$. Hence

$$T = \begin{bmatrix} I & T_2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ -T_3 & 0 \end{bmatrix}$$

is the difference of two idempotents. ■

We next extend Lemma 1.1 slightly by relaxing the restriction on X. For an operator X, we say that $\sigma(X)$ *does not surround* λ if λ belongs to the unbounded connected component of $\mathbb{C} \setminus \sigma(X)$. It is well known that, in general, an invertible operator need not have a square root (cf. [5]), but if $\sigma(X)$ does not surround 0, then X has a square root which is an analytic function of X (cf. [13, pp. 264–265]).

PROPOSITION 1.2. *If T is an operator such that either*

(1) *$XT = -TX$ for some operator X with $\sigma(X^2)$ not surrounding 0 or*
(2) *T is unitarily equivalent to $-T$,*

then T is the difference of two idempotents.

Proof. (1): Let Y be an analytic function of X^2 and satisfy $Y^2 = X^2$, and let $Z = XY^{-1}$. Since X and Y^{-1} commute, it is easily seen that Z is an involution. On the other hand, $XT = -TX$ implies that $X^2T = TX^2$, whence $YT = TY$ and therefore $ZT = -TZ$. The conclusion then follows from Lemma 1.1.

(2): Let U be a unitary operator such that $UT = -TU$. By the spectral theorem, U^2 has a square root V which commutes with every operator commuting with U^2. Let $Z = UV^{-1}$. Then, as above, Z is an involution and $ZT = -TZ$. The conclusion follows. ■

It seems to be unknown whether the similarity of T and $-T$ implies that T is the difference of two idempotents. However, the converse is certainly false, as $T = \pm I$ attests. But if $\pm 1 \notin \sigma(T)$, then T being the difference of two idempotents does imply the similarity of T and $-T$. This follows from the following

THEOREM 1.3. *Let E and F be idempotents.*

(1) *If $\pm 1 \notin \sigma(E - F)$, then $(E, F) \approx (F, E)$ and thus, in particular, $E \approx F$, $E - F \approx F - E$, and $EF \approx FE$. Moreover, if $\sigma((E - F)^2)$ does not surround 1, then the similarity of (E, F) and (F, E) can be implemented by an involution.*

(2) *If $0, \pm 1 \notin \sigma(E - F)$, then $(E, F) \approx (I - E, I - F)$.*

Proof. Let $X = E - F$ and $Y = I - E - F$. Then it is easily verified that $X^2 + Y^2 = I$.

(1): From this identity, $\pm 1 \notin \sigma(X)$ is equivalent to $0 \notin \sigma(Y)$. Hence Y is invertible. Since $YE = FY$ and $YF = EY$, we have $(E, F) \approx (F, E)$. If $\sigma(X^2)$ does not surround 1, then $\sigma(Y^2)$ does not surround 0. Hence Y^2 has a square root Y_1 which is an analytic function of Y^2. Let $Z = YY_1^{-1}$. Then, as before, Z is an involution, and $ZE = FZ$ and $ZF = EZ$.

(2): Since $0, \pm 1 \notin \sigma(X)$, both X and Y are invertible. It is easily verified that $XYE = (I - E)XY$ and $XYF = (I - F)XY$, whence $(E, F) \approx (I - E, I - F)$. ∎

In the above proof, the use of the identity $(E - F)^2 + (I - E - F)^2 = I$ is an old trick. Indeed, for arbitrary idempotents E and F, $(E - F)^2$ and $(I - E - F)^2$ are their associated separation and closeness operators; they are the operator analogues of $\sin^2\theta$ and $\cos^2\theta$, where θ is the non-obtuse angle between the ranges of E and F (cf. [1] and [11]). The implementing involution for $E \approx F$ in the preceding proof appeared before in [10, p. 35, Problem 4.11a].

Also note that, for idempotents E and F without the condition $\pm 1 \notin \sigma(E - F)$, EF and FE may not be similar. An example is

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Combining Proposition 1.2 and Theorem 1.3 yields the following

COROLLARY 1.4. *If T is an operator with $\sigma(T^2)$ not surrounding* 1, *then the following statements are equivalent*:

(1) *T is the difference of two idempotents*;
(2) *$XT = -TX$ for some involution X*;
(3) *$XT = -TX$ for some X with $\sigma(X^2)$ not surrounding* 0.

COROLLARY 1.5. *Let T be an operator with $\sigma(T) \cap \mathbb{R} = \varnothing$. Then T is the difference of two idempotents if and only if T is similar to $-T$.*

Proof. The necessity follows from Theorem 1.3. To prove the sufficiency, let $T = T_1 \oplus T_2$, where $\sigma(T_1)$ and $\sigma(T_2)$ are contained in the upper and lower (open) half planes, respectively. If

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

is an invertible operator such that $XT = -TX$, then a simple computation yields that $X_1T_1 = -T_1X_1$. Since $\sigma(T_1) \cap \sigma(-T_1) = \emptyset$, we infer that $X_1 = 0$ (cf. [12]). Similarly, we have $X_4 = 0$. Hence the invertibility of X implies that of X_2. From $X_2T_2 = -T_1X_2$, we obtain $T \approx T_1 \oplus (-T_1)$. This latter operator is similar to $(-T_1) \oplus T_1$ via the involution

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Hence T is the difference of two idempotents by Lemma 1.1. ■

The next result is another necessary condition for the difference of two idempotents.

THEOREM 1.6. *If T is the difference of two idempotents, then $\sigma(T)\setminus\{\pm 1\} = \sigma(-T)\setminus\{\pm 1\}$. If, in addition, T is acting on an infinite-dimensional separable space, then $\sigma_e(T)\setminus\{\pm 1\} = \sigma_e(-T)\setminus\{\pm 1\}$.*

Proof. Let $T = E - F$, where E and F are idempotents, and $S = I - E - F$. For any complex number λ, we can easily verify that $(T - \lambda I)(S - T - \lambda I) = (S + T - \lambda I)(-T - \lambda I)$. Since $(S - T)^2 = (S + T)^2 = I$, $S - T - \lambda I$ and $S + T - \lambda I$ are invertible for any $\lambda \neq \pm 1$. From the above identity, we deduce that $T - \lambda I$ is invertible (or is invertible modulo compact operators) if and only if $-T - \lambda I$ is. Our assertions follow immediately. ■

As we mentioned in the beginning of this paper, many considerations here are motivated by the work [8] of Hartwig and Putcha for the finite-dimensional case. The next few results are essentially due to them, some with modifications to adapt to the present situation.

LEMMA 1.7. *For an invertible operator T, the following statements are equivalent:*

(1) *T is the difference of two idempotents;*
(2) *T^{-1} is the difference of two idempotents;*
(3) *T is similar to an operator of the form*

$$\begin{bmatrix} I_H & B \\ C & -I_K \end{bmatrix} \quad \text{on } H \oplus K.$$

This is [8, Proposition 2 and Corollary 2].

Note that (3) ⇒ (1) above is always true, even without the invertibility assumption, as

$$\begin{bmatrix} I & B \\ C & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix} - \begin{bmatrix} 0 & -B \\ 0 & I \end{bmatrix}$$

shows. This is not the case of (1) ⇒ (3). Indeed,

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the difference of the idempotents

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

while, as can be easily checked, it is not similar to operators of the form

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ c_1 & c_2 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & b_1 & b_2 \\ c_1 & -1 & 0 \\ c_2 & 0 & -1 \end{bmatrix}.$$

LEMMA 1.8. *Let*

$$T = \begin{bmatrix} A & B \\ C & -A \end{bmatrix},$$

where C is invertible, $AC = CA$, and $\sigma(A^2 + BC)$ does not surround 0. Then T is similar to

$$\begin{bmatrix} (A^2 + BC)^{1/2} & 0 \\ 0 & -(A^2 + BC)^{1/2} \end{bmatrix}.$$

Proof. Let $D = (A^2 + BC)^{1/2}$, the analytic square root of $A^2 + BC$, and let

$$X = \begin{bmatrix} A + D & A - D \\ C & C \end{bmatrix}.$$

Since both C and $(A + D)C - (A - D)C = 2DC$, the "determinant" of X, are invertible, [7, Problem 71] implies that X is invertible. Using $AC = CA$, we can easily check that

$$TX = X\begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}.$$

This completes the proof. ∎

PROPOSITION 1.9. *Let T be such that $\sigma(T^2)$ does not contain 1 and does not surround 0. Then T is the difference of two idempotents if and only if T is similar to an operator of the form $D \oplus (-D)$.*

Using Lemmas 1.7 and 1.8, this can be proved essentially like [8, Proposition 4].

PROPOSITION 1.10. *Let T be an operator with $\sigma(T) \subseteq \{\pm 1\}$. Then T is the difference of two idempotents if and only if T is similar to an operator of the form $(I_H - XY) \oplus (-I_K + YX)$, where $X : K \to H$ and $Y : H \to K$ are such that both XY and YX are quasinilpotent.*

Recall that an operator is *quasinilpotent* if its spectrum is the singleton zero. The preceding proposition is [8, Proposition 5]. Note that, in the degenerate case, K or H may be the zero space, whence T is $\pm I$. Thus, in particular, if T is the difference of two idempotents with $\sigma(T) = \{1\}$, then $T = I$.

An operator T is *nilpotent* if $T^n = 0$ for some $n \geqslant 1$. It is proved in [8, Proposition 1] that on a finite-dimensional space every nilpotent operator is the difference of two idempotents. The corresponding result on infinite-dimensional spaces is not true in general.

PROPOSITION 1.11.

(1) *Every operator T with $T^2 = 0$ is the difference of two idempotents.*

(2) *There is an operator T with $T^3 = 0$ such that T and $-T$ are not similar, whence it is not the difference of two idempotents.*

Proof. (1): Since T has the representation

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $\ker T \oplus (\ker T)^{\perp}$, it is similar to $-T$ via the involution

$$\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

By Lemma 1.1, T is the difference of two idempotents.

(2): Let ϵ be such that $0 < \epsilon < 1$, and let

$$S = \begin{bmatrix} 1 & & & 0 \\ & \epsilon & & \\ & & \epsilon^2 & \\ 0 & & & \ddots \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & S & I \\ 0 & 0 & S \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, T satisfies $T^3 = 0$. Assume that there is an invertible operator $X = [X_{ij}]_{i,j=1}^3$ such that $XT = -TX$. A simple computation yields that

$$\begin{aligned} X_{31}S &= 0, \\ X_{31} + X_{32}S &= 0, \\ X_{21}S &= -SX_{32}, \\ X_{11}S &= -SX_{22} - X_{32}, \\ X_{21} + X_{22}S &= -SX_{33}, \\ X_{11} + X_{12}S &= -SX_{23} - X_{33}. \end{aligned}$$

Since S is one-to-one with dense range, we derive from the first three of the preceding equations that $X_{31} = X_{32} = X_{21} = 0$. The next two equations then reduce to $X_{11}S = -SX_{22}$ and $X_{22}S = -SX_{33}$. Thus we have $S^2X_{33} = X_{11}S^2$. Assume that the matrix representations of X_{11} and X_{33} are $[a_{ij}]_{i,j=1}^\infty$ and $[b_{ij}]_{i,j=1}^\infty$, respectively. A simple computation yields that $b_{ij} = \epsilon^{2(j-i)}a_{ij}$ for all $i, j \geqslant 1$. Therefore, $X_{11} + X_{33} = [(1 + \epsilon^{2(j-i)})a_{ij}]_{i,j=1}^\infty$. On the other hand, since S is a Hilbert-Schmidt operator, the equation $X_{11} + X_{33} = -(X_{12}S + SX_{23})$ implies the same for $X_{11} + X_{33}$. Hence from the inequalities

$$\sum_{i,j} |a_{ij}|^2 \leqslant \sum_{i,j} (1 + \epsilon^{2(j-i)})^2 |a_{ij}|^2 < \infty,$$

we obtain that X_{11} is also Hilbert-Schmidt. This contradicts the fact that it is left invertible. Hence T is not similar to $-T$. In particular, by Theorem 1.3(1), T is not the difference of two idempotents. ■

The following finite-dimensional characterization is the main result in [8].

THEOREM 1.12. *On a finite-dimensional space, T is the difference of two idempotents if and only if it is similar to an operator of the form $N \oplus (I - XY) \oplus (-I + YX) \oplus D \oplus (-D)$, where N, XY, and YX are all nilpotent and D is such that its spectrum does not contain 0 and ± 1.*

Moving to infinite-dimensional spaces, we can use the preceding result to characterize differences of idempotents among operators of the form $I + K$ with K compact.

PROPOSITION 1.13. *Let K be a compact operator on a separable space. Then $I + K$ is the difference of two idempotents if and only if it is similar to $0 \oplus N \oplus (I - XY) \oplus (-I + YX) \oplus D \oplus (-D)$, where 0 acts on an infinite-dimensional space, N, XY, and YX are nilpotent operators on finite-dimensional spaces, and D on a finite-dimensional space is such that its spectrum does not contain 0 and ± 1.*

Proof. The sufficiency follows easily from the previous propositions. To prove the necessity, assume that $I + K = E - F$, where E and F are idempotents. From $(I - E) + K = -F$ and the compactness of K, we have $\sigma_e(I - E) = -\sigma_e(F)$. Since both $\sigma_e(I - E)$ and $\sigma_e(F)$ are subsets of $\{0, 1\}$, they must equal $\{0\}$. Hence $I - E$ and F are of finite rank. We may assume that they are of the forms $(I - E') \oplus 0$ and $F' \oplus 0$, where $I - E'$ and F' are idempotents acting on the same finite-dimensional space. Thus $I + K = (E' - F') \oplus 0$, and our assertion follows from Theorem 1.12. ■

Enlarging our consideration, we next come to operators of the form $\lambda I + K$, where λ is a complex number and K is compact. It is easily seen that such an operator is the difference of two idempotents only when $\lambda = 0$ or ± 1. Indeed, if this is the case, then $\sigma_e(\lambda I + K) \cup \{\pm 1\} = \sigma_e(-\lambda I - K) \cup \{\pm 1\}$ by Theorem 1.6, which is the same as $\{\lambda, \pm 1\} = \{-\lambda, \pm 1\}$. Our assertion follows immediately. In light of Proposition 1.13, we may restrict our consideration to the case $\lambda = 0$.

PROPOSITION 1.14. *A compact operator K on a separable space is the difference of two idempotents if and only if K is similar to $(I - XY) \oplus (-I + YX) \oplus K_1$, where XY and YX are nilpotent operators on a finite-dimensional space, K_1 is compact such that $\pm 1 \notin \sigma(K_1)$, and K_1 is similar to $-K_1$ via an involution.*

Proof. The assertion follows, on taking out the eigenvalues ± 1 from $\sigma(K)$, from Corollary 1.4, Proposition 1.10, and [8, Lemma 2]. ■

This allows us to conclude that the Volterra operator V is not the difference of two idempotents, since it is known that V is not similar to $-V$

(cf. [9] or [4]). Hence not every compact quasinilpotent operator is such a difference. It is not known whether every compact nilpotent operator is.

In the remainder of this section, we give a characterization of differences of idempotents among normal operators. As we will see, the result is consistent with the one for finite-dimensional operators in Theorem 1.12. We start with the following lemma, whose proof is similar to that of [8, Lemma 2].

LEMMA 1.15. *Let* $T = 0 \oplus I_{H_2} \oplus (-I_{H_3}) \oplus T_1$ *on* $H_1 \oplus H_2 \oplus H_3 \oplus H_4$, *where* 0 *and* ± 1 *are not eigenvalues of* T_1 *and* T_1^*. *Then* T *is the difference of two idempotents if and only if* T_1 *is.*

Proof. We need only prove the necessity. Let $T_2 = I \oplus (-I) \oplus T_1$. Then T_2 is one-to-one with dense range, and

$$T = \begin{bmatrix} 0 & 0 \\ 0 & T_2 \end{bmatrix}.$$

If

$$T = E - F = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} - \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}$$

is the difference of two idempotents, then $E_j = F_j$ for $j = 1$, 2, and 3, and $T_2 = E_4 - F_4$. The idempotency of E and F yields that $E_2 = E_1 E_2 + E_2 E_4$ and $F_2 = F_1 F_2 + F_2 F_4$. Hence

$$\begin{aligned} E_2 T_2 &= E_2(E_4 - F_4) = E_2 E_4 - F_2 F_4 \\ &= (E_2 - E_1 E_2) - (F_2 - F_1 F_2) = 0. \end{aligned}$$

Since T_2 has dense range, we infer that $E_2 = F_2 = 0$. Similarly, we have $E_3 = F_3 = 0$. This shows that $T_2 = E_4 - F_4$ is the difference of two idempotents.

An analogous argument shows that if

$$T_2 = \begin{bmatrix} I \oplus (-I) & 0 \\ 0 & T_1 \end{bmatrix} = E_4 - F_4 = \begin{bmatrix} E_1' & E_2' \\ E_3' & E_4' \end{bmatrix} - \begin{bmatrix} F_1' & F_2' \\ F_3' & F_4' \end{bmatrix},$$

then $T_1 = E_4' - F_4'$ is also the difference of two idempotents. ■

LEMMA 1.16. *Let T be a normal operator with $\pm 1 \notin \sigma_p(T)$. Then T is the difference of two idempotents if and only if T is unitarily equivalent to $-T$.*

Proof. The sufficiency is a consequence of Proposition 1.2(2). To prove the necessity, assume that $T = E - F$, where E and F are idempotents. Since

$$(I - E - F)^2 = I - T^2 = (I - T)(I + T)$$

and the latter operator is one-to-one with dense range, the same is true for $I - E - F$. From $(I - E - F)T = -T(I - E - F)$ we infer that T is unitarily equivalent to $-T$ (cf. [2, Lemma 4.1]). ∎

THEOREM 1.17. *A normal operator T is the difference of two idempotents if and only if T is unitarily equivalent to $0 \oplus I_{H_2} \oplus (-I_{H_3}) \oplus D \oplus (-D)$ on $H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5$, where D is a normal operator with $0, \pm 1 \notin \sigma_p(D)$.*

Proof. The sufficiency is clear. For the necessity, let $E(\cdot)$ be the spectral measure of T, and let $\sigma_1 = \{z \in \mathbb{C} \setminus \{\pm 1\} : \operatorname{Re} z > 0$ or $\operatorname{Re} z = 0$ and $\operatorname{Im} z > 0\}$ and $\sigma_2 = \{z \in \mathbb{C} : -z \in \sigma_1\}$. Let $H_1 = \ker T$, $H_2 = \ker(T - I)$, $H_3 = \ker(T + I)$, $H_4 = \operatorname{ran} E(\sigma_1)$, and $H_5 = \operatorname{ran} E(\sigma_2)$. Then $T = 0 \oplus I \oplus (-I) \oplus T_1 \oplus T_2$ on $H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5$, where $T_1 = T|H_4$ and $T_2 = T|H_5$. Since $0, \pm 1 \notin \sigma_p(T_1 \oplus T_2)$, Lemma 1.15 implies that $T_1 \oplus T_2$ is the difference of two idempotents. By Lemma 1.16, there exists a unitary operator

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$$

such that $U(T_1 \oplus T_2) = -(T_1 \oplus T_2)U$. This yields that $U_1T_1 = -T_1U_1$. Since the spectral measures of T_1 and $-T_1$ are mutually singular, we obtain $U_1 = 0$ by [3, Proposition 2.4]. Similarly, we have $U_4 = 0$. Hence both U_2 and U_3 are unitary operators. Thus T is unitarily equivalent to $0 \oplus I \oplus (-I) \oplus T_1 \oplus (-T_1)$. ∎

A special case of the above theorem was obtained in [14, Theorem 2].

COROLLARY 1.18. *If T is the difference of two idempotents and $T \geqslant I$, then $T = I$.*

2. SIMILARITY MODEL

In this section, we derive various similarity-invariant models for certain pairs of idempotents. We start with a two-parameter model. For operators $B : H \to K$ and $C : K \to H$ with $I + CB$ invertible, let $M(B, C)$ denote the idempotent

$$\begin{bmatrix} B(I+CB)^{-1}C & -B(I+CB)^{-1} \\ -(I+CB)^{-1}C & (I+CB)^{-1} \end{bmatrix}.$$

THEOREM 2.1. *Let E and F be idempotents. Then $E - F$ is invertible if and only if*

$$(E,F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B,C) \right)$$

for some operators B and C with $I + CB$ invertible. Here B and C are unique in the following sense: if

$$(E,F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B',C') \right)$$

for another pair B' and C' with $I + C'B'$ invertible, then there are invertible operators X and Y such that $XB = B'Y$ and $YC = C'X$.

The model given above has a geometric meaning: it can be considered as the operator analogue of the following two-dimensional situation. Let L_1 and L_2 be two lines in the Euclidean plane both passing through the origin. Let E be the orthogonal projection of the plane onto its x-axis and F the projection along the direction of L_1 onto points of L_2. With respect to the x-y coordinates, E and F can be represented as

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dfrac{\dfrac{-\tan\theta_1}{\tan\theta_2}}{1-\dfrac{\tan\theta_1}{\tan\theta_2}} & \dfrac{\dfrac{1}{\tan\theta_2}}{1-\dfrac{\tan\theta_1}{\tan\theta_2}} \\ \dfrac{-\tan\theta_1}{1-\dfrac{\tan\theta_1}{\tan\theta_2}} & \dfrac{1}{1-\dfrac{\tan\theta_1}{\tan\theta_2}} \end{bmatrix},$$

where θ_1 and θ_2 are the inclinations of L_1 and L_2, respectively. Hence if we let $B = -1/\tan\theta_2$ and $C = \tan\theta_1$, then the matrices of E and F coincide with those in the model of Theorem 2.1. Note that in this case the condition of the invertibility of $I + CB$ is equivalent to $\tan\theta_1 \neq \tan\theta_2$ or, equivalently, $L_1 \neq L_2$.

Proof of Theorem 2.1. Assume that $E - F$ is invertible and

$$X^{-1}EX = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

for some invertible X. Then, as proved in [8, Proposition 2],

$$X^{-1}(E-F)^{-1}X = \begin{bmatrix} I & B \\ C & -I \end{bmatrix}.$$

Since $(E - F)^{-2}$ is similar to

$$\begin{bmatrix} I + BC & 0 \\ 0 & I + CB \end{bmatrix},$$

$I + CB$ is invertible. Moreover,

$$X^{-1}(E-F)X = \begin{bmatrix} I & B \\ C & -I \end{bmatrix}^{-1}.$$

This latter inverse matrix can be easily checked to be

$$\begin{bmatrix} I - B(I+CB)^{-1}C & B(I+CB)^{-1} \\ (I+CB)^{-1}C & -(I+CB)^{-1} \end{bmatrix}. \qquad (*)$$

Hence we obtain

$$X^{-1}FX = X^{-1}EX - X^{-1}(E-F)X = M(B,C).$$

Conversely, since the matrix $(*)$ is invertible (with inverse $\begin{bmatrix} I & B \\ C & -I \end{bmatrix}$), the invertibility of $E - F$ follows immediately.

As for the uniqueness, let

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix}$$

be an invertible operator which implements the similarity of

$$\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B,C)\right) \quad \text{and} \quad \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B',C')\right).$$

From

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X & Z \\ W & Y \end{bmatrix},$$

we derive that $Z = 0$ and $W = 0$. On the other hand, $(X \oplus Y)M(B,C) = M(B',C')(X \oplus Y)$ implies that

$$XB(I + CB)^{-1} = B'(I + C'B')^{-1}Y,$$

$$Y(I + CB)^{-1}C = (I + C'B')^{-1}C'X,$$

$$Y(I + CB)^{-1} = (I + C'B')^{-1}Y.$$

Hence

$$XB(I + CB)^{-1} = B'Y(I + CB)^{-1},$$

and it follows that $XB = B'Y$. Similarly, $YC = C'X$. Note that uniqueness in this sense is the best we can hope for, since if $XB = B'Y$ and $YC = C'X$ for some invertible X and Y, then

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

implements the similarity of

$$\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B,C)\right) \quad \text{and} \quad \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B',C')\right). \qquad \blacksquare$$

The next several corollaries give models for pairs of idempotents with various conditions on the spectrum of their difference.

COROLLARY 2.2. *Let E and F be idempotents. Then* $\pm 1 \notin \sigma(E - F)$ *if and only if*

$$(E, F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, I - M(B, C) \right)$$

for some B and C with $I + CB$ *invertible.*

Proof. This follows from Theorem 2.1 on replacing F there by $I - F$ and noting that $(E - F)^2 + (E + F - I)^2 = I$. ■

COROLLARY 2.3. *If E and F are commuting idempotents with* $\pm 1 \notin \sigma(E - F)$, *then* $E = F$.

Proof. Obviously, the commutativity of

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad I - M(B, C)$$

implies that $B = 0$ and $C = 0$, whence

$$I - M(B, C) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $E = F$. ■

The preceding corollary generalizes [13, p. 302, Exercise 11] for operators on Hilbert spaces.

COROLLARY 2.4. *Let E and F be idempotents. Then* $\sigma(E - F) \subseteq \{\pm 1\}$ *if and only if*

$$(E, F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B, C) \right)$$

for some B and C with BC and CB quasinilpotent.

Proof. In view of Theorem 2.1, we need only show that $\sigma(E - F) \subseteq \{\pm 1\}$ is equivalent to the quasinilpotency of BC and CB. This is evident from the relation

$$X^{-1}(E-F)^{-2}X = \begin{bmatrix} I+BC & 0 \\ 0 & I+CB \end{bmatrix}$$

(cf. the proof of Theorem 2.1). ■

COROLLARY 2.5. *Let E and F be idempotents. Then $E-F$ is quasinilpotent if and only if*

$$(E,F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, I - M(B,C) \right)$$

for some B and C with BC and CB quasinilpotent.

Proof. Since $(E-F)^2 + (E+F-I)^2 = I$, $E-F$ is quasinilpotent if and only if $\sigma(E+F-I) \subseteq \{\pm 1\}$. Our assertion then follows from Corollary 2.4 on replacing F there by $I-F$. ■

Combining the previous results, we obtain various similarity-invariant models for operators with "decomposable" spectrum (e.g., normal operators or operators on a finite-dimensional space). Rather than going into details, we next proceed to consider a one-parameter model which is more adapted to problems involving products of two idempotents.

THEOREM 2.6. *For idempotents E and F, the following statements are equivalent*:

(1) $0, \pm 1 \notin \sigma(E-F)$;
(2) $(E,F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & I-A \\ A & I-A \end{bmatrix} \right)$ *for some operator A with $0,1 \notin \sigma(A)$;*
(3) $EF \approx \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ *for some A with $0,1 \notin \sigma(A)$.*

In this case, the operator A in (2) *or* (3) *is unique up to similarity.*

The proofs of (1) ⇒ (2) and (3) ⇒ (2) of the preceding theorem depend, respectively, on the following two lemmas.

LEMMA 2.7. *If E and F are idempotents with 0 and ± 1 not in $\sigma(E-F)$, then there exists an operator A, unique up to similarity, with 0 and 1 not in $\sigma(A)$ such that for any invertible operator D commuting with A*

we have

$$(E, F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & D \\ A(I - A)D^{-1} & I - A \end{bmatrix} \right).$$

Proof. By Theorem 2.1,

$$(E, F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M(B, C) \right)$$

for some B and C with $I + CB$ invertible. Let $A = CB(I + CB)^{-1}$ and $X = C \oplus [-A(I - A)D^{-1}(I + CB)]$. Note that

$$(E - F)^{-2} \approx \begin{bmatrix} I + BC & 0 \\ 0 & I + CB \end{bmatrix}$$

and the assumption that $\pm 1 \notin \sigma(E - F)$ imply the invertibility of BC and CB, whence that of B and C. Thus $0, 1 \notin \sigma(A)$ and X is invertible. On the other hand, since D commutes with A, it also commutes with CB. Using this, we can easily check that

$$X \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X$$

and

$$XM(B, C) = \begin{bmatrix} A & D \\ A(I - A)D^{-1} & I - A \end{bmatrix} X.$$

It follows that

$$(E, F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & D \\ A(I - A)D^{-1} & I - A \end{bmatrix} \right).$$

For the uniqueness of A, assume that A' is an operator having the same property as A. Then, in particular,

$$\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & I - A \\ A & I - A \end{bmatrix} \right) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A' & I - A' \\ A' & I - A' \end{bmatrix} \right)$$

whence their respective products

$$\begin{bmatrix} A & I-A \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A' & I-A' \\ 0 & 0 \end{bmatrix}$$

are also similar. However, the former is similar to $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ (via $\begin{bmatrix} I & A^{-1} \\ 0 & I \end{bmatrix}$), and the latter to $\begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix}$ (via $\begin{bmatrix} I & A'^{-1} \\ 0 & I \end{bmatrix}$). Hence

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix}$$

are similar. It follows easily that A and A' are similar. ∎

LEMMA 2.8. *For $j = 1, 2$, let E_j and F_j be idempotents with 0 and ± 1 not in $\sigma(E_j - F_j)$. Then $(E_1, F_1) \approx (E_2, F_2)$ if and only if $E_1 F_1 \approx E_2 F_2$.*

Proof. The necessity is trivial. To prove the sufficiency, we have

$$(E_j, F_j) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_j & I - A_j \\ A_j & I - A_j \end{bmatrix} \right)$$

for some A_j with $0, 1 \notin \sigma(A_j)$, $j = 1, 2$, by Lemma 2.7. Hence

$$E_j F_j \approx \begin{bmatrix} A_j & I - A_j \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Our assumption implies that

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix},$$

whence $A_1 \approx A_2$ as before. If X is an invertible operator implementing the similarity of A_1 and A_2, then

$$\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$$

does the same for

$$\begin{bmatrix} A_1 & I - A_1 \\ A_1 & I - A_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_2 & I - A_2 \\ A_2 & I - A_2 \end{bmatrix}.$$

It follows that $(E_1, F_1) \approx (E_2, F_2)$. ∎

Proof of Theorem 2.6. To complete the proof, we need only show that (2) ⇒ (1) and (3) ⇒ (2).

Assume that (2) holds. Then $E - F$ is similar to

$$\begin{bmatrix} I - A & A - I \\ -A & A - I \end{bmatrix}.$$

Since both A and the "determinant" $(I - A)(A - I) - (A - I)(-A) = A - I$ are invertible, [7, Problem 71] implies the invertibility of

$$\begin{bmatrix} I - A & A - I \\ -A & A - I \end{bmatrix}$$

and hence that of $E - F$. Similar arguments yield the invertibility of $E - F \pm I$. This proves (1).

If (3) holds, then we have

$$EF \approx \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} A & I - A \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & I - A \\ A & I - A \end{bmatrix}.$$

Since 0 and ± 1 are not in the spectrum of

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & I - A \\ A & I - A \end{bmatrix}$$

as above, (2) follows from Lemma 2.8. ∎

For idempotents on a finite-dimensional space, additional equivalent conditions can be added to the list in Theorem 2.6.

THEOREM 2.9. *Let E and F be idempotents on a finite-dimensional space. Then the following statements are equivalent to* (1), (2), *and* (3) *in*

Theorem 2.6:

(4) $(E, F) \approx \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & (A - A^2)^{1/2} \\ (A - A^2)^{1/2} & I - A \end{bmatrix} \right)$ *for some* A *with* $0, 1 \notin \sigma(A)$;

(5) $(E, F) \approx \left(\begin{bmatrix} S & I - S \\ S & I - S \end{bmatrix}, \begin{bmatrix} S & -(I - S) \\ -S & I - S \end{bmatrix} \right)$ *for some* S *with* $0, \frac{1}{2}, 1 \notin \sigma(S)$;

(6) $(E, F) \approx \left(\begin{bmatrix} S & (S - S^2)^{1/2} \\ (S - S^2)^{1/2} & I - S \end{bmatrix}, \begin{bmatrix} S & -(S - S^2)^{1/2} \\ -(S - S^2)^{1/2} & I - S \end{bmatrix} \right)$ *for some* S *with* $0, \frac{1}{2}, 1 \notin \sigma(S)$;

(7) $E - F \approx \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}$ *for some* D *with* $0, \pm 1 \notin \sigma(D)$;

(8) $E + F \approx \begin{bmatrix} I + D' & 0 \\ 0 & I - D' \end{bmatrix}$ *for some* D' *with* $0, \pm 1 \notin \sigma(D')$;

(9) $\operatorname{ran} E \cap \operatorname{ran} F = \ker E \cap \ker F = \operatorname{ran} E \cap \ker F = \ker E \cap \operatorname{ran} F = \{0\}$.

In (4), A *is unique up to similarity. In* (5) *and* (6), *we may further require* S *to satisfy* $\sigma(S) \subseteq \sigma_1 \equiv \{z \in \mathbb{C} : \operatorname{Re} z > \frac{1}{2}$ *or* $\operatorname{Re} z = \frac{1}{2}$ *and* $\operatorname{Im} z > 0\}$, *in which case* S *is unique up to similarity. In* (7) *and* (8), *we may require* D *and* D' *to satisfy* $\sigma(D), \sigma(D') \subseteq \sigma_2 \equiv \{z \in \mathbb{C} : \operatorname{Re} z > 0$ *or* $\operatorname{Re} z = 0$ *and* $\operatorname{Im} z > 0\}$, *in which case* D *and* D' *are unique up to similarity.*

Note that when E and F are projections (even on infinite-dimensional spaces), the condition (9) above defines the notion of subspaces *in generic position* (cf. [6]). Hence what we do here is to develop a series of equivalent conditions for two "generic" idempotents (on finite-dimensional spaces).

We first prove (1) $\Rightarrow$ (5). For convenience, we will let

$$M_1(S) = \begin{bmatrix} S & I - S \\ S & I - S \end{bmatrix} \quad \text{and} \quad M_2(S) = \begin{bmatrix} S & -(I - S) \\ -S & I - S \end{bmatrix}$$

for any operator S.

LEMMA 2.10. *If* E *and* F *are idempotents with* $\sigma((E - F)^2)$ *not containing* 0 *and not surrounding* 1, *then there exists an operator* S *with* $0, \frac{1}{2}, 1 \notin \sigma(S)$ *and* $\sigma(S) \cap \sigma(I - S) = \varnothing$ *such that* $(E, F) \approx (M_1(S), M_2(S))$. *If* E *and* F *are acting on a finite-dimensional space, then we may further require that* $\sigma(S) \subseteq \sigma_1$, *in which case* S *is unique up to similarity.*

Proof. Assume that

$$X^{-1}FX = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{for some invertible } X.$$

Since $(E - F)^2 + (E + F - I)^2 = I$, our assumption on the spectrum of $(E - F)^2$ implies that $0, \pm 1 \notin \sigma(E + F - I)$. Hence, as proved in [8, Proposition 2], we have

$$X^{-1}[F - (I - E)]^{-1}X = \begin{bmatrix} I & B \\ C & -I \end{bmatrix}.$$

Note that, since $(E + F - I)^2$ is similar to

$$\begin{bmatrix} I & B \\ C & -I \end{bmatrix}^2 = \begin{bmatrix} I + BC & 0 \\ 0 & I + CB \end{bmatrix},$$

we can infer from $\pm 1 \notin \sigma(E + F - I)$ that B and C are invertible. Moreover, that $\sigma((E - F)^2)$ does not surround 1 implies that $\sigma((E + F - I)^2)$ does not surround 0, whence $I + BC$ has a square root D with $0, \pm 1 \notin \sigma(D)$ and $\sigma(D) \cap \sigma(-D) = \varnothing$. Let

$$S = \tfrac{1}{2}(I - D^{-1}) \quad \text{and} \quad Y = \begin{bmatrix} I - D & I + D \\ C & 0 \end{bmatrix}.$$

Then $0, \frac{1}{2}, 1 \notin \sigma(S)$ and $\sigma(S) \cap \sigma(I - S) = \varnothing$. Since both C and $(I - D)C - (I + D)C = -2DC$, the "determinant" of Y, are invertible, [7, Problem 71] implies that Y is invertible. Now it is routine to check that

$$X^{-1}FXY = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Y = YM_2(S)$$

and

$$\begin{aligned} X^{-1}EXY &= X^{-1}(E + F - I)XY + X^{-1}(I - F)XY \\ &= \begin{bmatrix} I & B \\ C & -I \end{bmatrix}^{-1} Y + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Y \\ &= \begin{bmatrix} I - B(I + CB)^{-1}C & B(I + CB)^{-1} \\ (I + CB)^{-1}C & I - (I + CB)^{-1} \end{bmatrix} Y \\ &= \begin{bmatrix} D^{-2} & D^{-2}B \\ CD^{-2} & CD^{-2}B \end{bmatrix} Y \\ &= YM_1(S). \end{aligned}$$

This proves $(E, F) \approx (M_1(S), M_2(S))$.

Next assume that E and F are acting on a finite-dimensional space. Then S is similar to $S_1 \oplus S_2$ with $\sigma(S_1) \subseteq \sigma_1$ and $\sigma(S_2) \subseteq \mathbb{C} \setminus \sigma_1$. Hence we have

$$(M_1(S), M_2(S)) \approx \left(M_1\left(\begin{bmatrix} S_1 & 0 \\ 0 & I - S_2 \end{bmatrix}\right), M_2\left(\begin{bmatrix} S_1 & 0 \\ 0 & I - S_2 \end{bmatrix}\right)\right)$$

with $\sigma\left(\begin{bmatrix} S_1 & 0 \\ 0 & I - S_2 \end{bmatrix}\right)$ contained in σ_1 and not containing 0, $\frac{1}{2}$, and 1.

For the uniqueness, assume that $(M_1(S), M_2(S)) \approx (M_1(S'), M_2(S'))$ for some operator S' with $\sigma(S') \subseteq \sigma_1$. Taking the sum of each pair, we obtain $2S \oplus 2(I - S) \approx 2S' \oplus 2(I - S')$. The conditions on $\sigma(S)$ and $\sigma(S')$ ensure that $S \approx S'$. ■

The model in Lemma 2.10 is more suitable for considering sums and differences of two idempotents, as the following lemma shows.

LEMMA 2.11. *For $j = 1, 2$, let E_j and F_j be idempotents on a finite-dimensional space with 0 and ± 1 not in $\sigma(E_j - F_j)$. Then the following statements are equivalent*:

(1) $(E_1, F_1) \approx (E_2, F_2)$;
(2) $E_1 + F_1 \approx E_2 + F_2$;
(3) $E_1 - F_1 \approx E_2 - F_2$.

Proof. We need only check (2) ⇒ (1) and (3) ⇒ (1).

To prove (2) ⇒ (1), we have, by Lemma 2.10, $(E_j, F_j) \approx (M_1(S_j), M_2(S_j))$ for some S_j with $\sigma(S_j) \subseteq \sigma_1$, $j = 1, 2$. Hence $E_j + F_j \approx 2S_j \oplus 2(I - S_j)$. Our assumption (2) implies that $S_1 \oplus (I - S_1) \approx S_2 \oplus (I - S_2)$. Since $\sigma(S_1) \cap \sigma(I - S_2) = \sigma(S_2) \cap \sigma(I - S_1) = \varnothing$, we infer that $S_1 \approx S_2$, whence $(E_1, F_1) \approx (E_2, F_2)$.

If (3) holds, then $E_1 + (I - F_1) \approx E_2 + (I - F_2)$. Since 0 and ± 1 are not in $\sigma(E_j - (I - F_j))$, $j = 1, 2$, by the identity $(E_j - F_j)^2 + (E_j + F_j - I)^2 = I$ and our assumption on $\sigma(E_j - F_j)$, statement (1) follows from the implication (2) ⇒ (1). ■

The next lemma is a generalization of Lemma 2.10 for finite-dimensional idempotents.

LEMMA 2.12. *If E and F are idempotents on a finite-dimensional space with $\sigma(E - F)$ not containing 0 and ± 1, then there exists an operator S with $0, \frac{1}{2}, 1$ not in $\sigma(S)$ such that for any invertible operator G*

commuting with S we have

$$(E, F) \approx \left(\begin{bmatrix} S & G \\ S(I-S)G^{-1} & I-S \end{bmatrix}, \begin{bmatrix} S & -G \\ -S(I-S)G^{-1} & I-S \end{bmatrix} \right).$$

Moreover, we may require that S satisfy $\sigma(S) \subseteq \sigma_1$, in which case S is unique up to similarity.

Proof. By Lemma 2.10, we have

$$(E, F) \approx \left(\begin{bmatrix} S & I-S \\ S & I-S \end{bmatrix}, \begin{bmatrix} S & -(I-S) \\ -S & (I-S) \end{bmatrix} \right)$$

for some operator S with $0, \frac{1}{2}, 1 \notin \sigma(S)$ and $\sigma(S) \subseteq \sigma_1$. Let

$$E_1 = \begin{bmatrix} S & G \\ S(I-S)G^{-1} & I-S \end{bmatrix} \quad \text{and} \quad F_1 = \begin{bmatrix} S & -G \\ -S(I-S)G^{-1} & I-S \end{bmatrix}.$$

Then

$$E_1 - F_1 = 2\begin{bmatrix} 0 & G \\ S(I-S)G^{-1} & 0 \end{bmatrix}.$$

This latter matrix is similar to

$$2\begin{bmatrix} (S-S^2)^{1/2} & 0 \\ 0 & -(S-S^2)^{1/2} \end{bmatrix}$$

by Lemma 1.8. Thus $\sigma(E_1 - F_1) = \{\pm 2\lambda : \lambda \in \sigma((S - S^2)^{1/2})\}$. From our assumption on $\sigma(S)$, we easily infer that 0 and ± 1 are not in $\sigma(E_1 - F_1)$. Since $E + F \approx 2S \oplus 2(I - S) = E_1 + F_1$, Lemma 2.11 implies that $(E, F) \approx (E_1, F_1)$, as asserted. The uniqueness of S can be proved as before. ∎

The next lemma complements Theorem 2.1 and Corollary 2.2 in the finite-dimensional case.

LEMMA 2.13. *Let E and F be idempotents on a finite-dimensional space.*

(1) *$E - F$ is invertible if and only if* $\operatorname{ran} E \cap \operatorname{ran} F = \ker E \cap \ker F = \{0\}$.

(2) *$\pm 1 \notin \sigma(E - F)$ if and only if* $\operatorname{ran} E \cap \ker F = \ker E \cap \operatorname{ran} F = \{0\}$.

Proof. (1): Assume that $E - F$ is invertible. If $x \in \operatorname{ran} E \cap \operatorname{ran} F$, then $x = Ex = Fx$. Hence $(E - F)x = 0$, and therefore $x = 0$. If $y \in \ker E \cap \ker F$, then $Ey = Fy = 0$. We infer as above that $y = 0$.

Conversely, assume that $(E - F)x = 0$. Then $Ex = Fx$ belongs to $\operatorname{ran} E \cap \operatorname{ran} F = \{0\}$. We obtain that $x \in \ker E \cap \ker F = \{0\}$. It follows that $E - F$ is one-to-one and hence invertible.

(2): This follows from (1) on replacing F there by $I - F$ and noting that $(E - F)^2 + (E + F - I)^2 = I$. ■

Proof of Theorem 2.9. Since on a finite-dimensional space every invertible operator has a square root, (1) ⇒ (4) follows from Lemma 2.7 on letting $D = (A - A^2)^{1/2}$. Similarly, (1) ⇒ (5) is proved in Lemma 2.10, and (1) ⇒ (6) follows on letting $G = (A - A^2)^{1/2}$ in Lemma 2.12. (4) ⇒ (3) is obvious. (5) or (6) implies (7) by Lemma 1.8, while the implications (5) ⇒ (8), (6) ⇒ (8), and (7) ⇒ (1) are all obvious. (8) implies that 0, 1, and 2 are not in $\sigma(E + F)$, whence (1) follows by the identity $(E - F)^2 + (E + F - I)^2 = I$. Finally, the equivalence of (1) and (9) is a consequence of Lemma 2.13. This completes the proof. ■

Note added in proof. After this paper was accepted, we discovered that Proposition 1.11 (2) had been proved before by C. Apostol (cf. C. Apostol, L. A. Fialkow, D. A. Herrero & D. Voiculescu, Approximation of Hilbert space operators, Vol. II, Pitman, Boston, 1984, pp. 71–73). In fact, he constructed an operator T with $T^3 = 0$ such that T is not similar to λT for any scalar $\lambda \neq 1$.

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