J Comb Optim (2014) 27:328–344
DOI 10.1007/s10878-012-9528-1

On the maximum number of fault-free mutually independent Hamiltonian cycles in the faulty hypercube

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Published online: 29 June 2012

Abstract Hsieh and Yu (2007) first claimed that an injured n-dimensional hypercube Q_n contains $(n-1-f)$-mutually independent fault-free Hamiltonian cycles, where $f \leq n-2$ denotes the total number of permanent edge-faults in Q_n for $n \geq 4$, and edge-faults can occur everywhere at random. Later, Kueng et al. (2009a) presented a formal proof to validate Hsieh and Yu's argument. This paper aims to improve this mentioned result by showing that up to $(n-f)$-mutually independent fault-free Hamiltonian cycles can be embedded under the same condition. Let F denote the set of f faulty edges. If all faulty edges happen to be incident with an identical vertex $\mathbf{s}$, i.e., the minimum degree of the survival graph $Q_n - F$ is equal to $n-f$, then $Q_n - F$ contains at most $(n-f)$-mutually independent Hamiltonian cycles starting from $\mathbf{s}$. From such a point of view, the presented result is optimal. Thus, not only does our improvement increase the number of mutually independent fault-free Hamiltonian cycles by one, but also the optimality can be achieved.

Keywords Interconnection network · Graph · Hypercube · Fault tolerance · Hamiltonian cycle

This work is supported in part by the National Science Council of the Republic of China under Contract NSC 98-2218-E-468-001-MY3.

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1 Introduction

In many parallel computer systems, processors are connected on the basis of *interconnection networks* such as meshes, hypercubes, star graphs, bubble-sort networks, etc. For the sake of simplicity, the underlying topology of an interconnection network is usually represented by a *graph*, whose vertices and edges correspond to processors and connection links, respectively. Hence, we use the terms, graph and network, interchangeably. Throughout this paper, we concentrate on loopless undirected graphs. Some important graph-theory notations and definitions are introduced below. For those not defined here, we follow the standard terminology given by Bondy and Murty (2008).

A graph G consists of a nonempty vertex set $V(G)$ and an edge set $E(G)$, which is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of elements in $V(G)\}$. Two vertices u and v of G are *adjacent if* $(u, v) \in E(G)$. The *neighborhood* of vertex v in graph G, denoted by $N_G(v)$, is defined as $\{u \mid u \in V(G), (v, u) \in E(G)\}$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of $V(G)$. The subgraph *induced* by S is the maximal subgraph of G with vertex set S that contains precisely those edges of G joining two vertices in S. We use $G - S$ to denote the subgraph of G induced by $V(G) - S$. Analogously, let F be a nonempty subset of $E(G)$. We use $G - F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F$. The *degree* of a vertex u in G is the number of edges incident with u. A graph G is *k-regular* if the degree of every vertex is equal to k. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint partite sets $V_0(G)$ and $V_1(G)$ such that every edge joins a vertex in $V_0(G)$ and a vertex in $V_1(G)$.

A *path* P of length k, $k \geq 1$, from vertex x to vertex y in a graph G is an ordered sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$. Moreover, a path of length 0, consisting of a single vertex x, is denoted by $\langle x\rangle$. For convenience, we write P as $\langle v_1, v_2, \ldots, v_i, R, v_j, \ldots, v_{k+1}\rangle$, where $i \leq j$, if $R = \langle v_i, \ldots, v_j\rangle$ is a part of P. The ith vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. We use $\ell(P)$ to denote the *length* of P. To emphasize the start and end vertices of P, we also write P as $P[x, y]$. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k, $k \geq 3$, is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A path (respectively, cycle) in the graph G is a *Hamiltonian path* (respectively, *Hamiltonian cycle*) of G if it traverses every vertex of G. A bipartite graph is *Hamiltonian laceable* (Simmons 1978) if there exists a Hamiltonian path joining any two vertices that are in different partite sets. Moreover, a Hamiltonian laceable graph H is *hyper-Hamiltonian laceable* (Lewinter and Widulski 1997) if for any vertex $v \in V_i(H)$ with $i \in \{0, 1\}$, there exists a Hamiltonian path in $H - \{v\}$ joining any two vertices of $V_{1-i}(H)$.

The *n-dimensional hypercube* (or *n-cube* for short), $n \geq 1$, is one of the most popular network topologies discovered for parallel and distributed computation. Not only is it ideally suited to both special-purpose and general-purpose tasks, but it can efficiently simulate many other networks (Leighton 1992). Thus, many attractive properties of hypercubes have been extensively addressed by researchers (Akers and Krishnameurthy 1989; Castañeda and Gotchev 2010; Chang et al. 2004;

Dvořák and Koubek 2009, 2010; Fink and Gregor 2011; Johnsson and Ho 1989; Kueng et al. 2009b; Kung et al. 2009; Leighton 1992; Leu and Kuo 1999; Tsai et al. 2002; Yang et al. 1994). The formal definition of an n-cube is given below. For the sake of clarity, let a boldface letter $\mathbf{u}$ denote an n-bit binary string $b_n \cdots b_i \cdots b_1$. For $1 \leq i \leq n$, we use $(\mathbf{u})^i$ to denote the binary string $b_n \cdots \bar{b}_i \cdots b_1$. Moreover, we use $(\mathbf{u})_i$ to denote the ith bit b_i of $\mathbf{u}$. The *Hamming weight* of $\mathbf{u}$, denoted by $w_H(\mathbf{u})$, is $|\{i \mid (\mathbf{u})_i = 1, 1 \leq i \leq n\}|$. The n-cube Q_n contains 2^n vertices, each of which is labeled by an n-bit binary string. For the purpose of notation consistency, its vertices are also denoted by boldface letters in the rest of this paper. Two vertices $\mathbf{u}$ and $\mathbf{v}$ of Q_n are adjacent if and only if $\mathbf{v} = (\mathbf{u})^i$ for some i, and edge $(\mathbf{u}, (\mathbf{u})^i)$ is called i*-dimensional*. Clearly, Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{\mathbf{u} \in V(Q_n) \mid w_H(\mathbf{u}) \text{ is even}\}$ and $V_1(Q_n) = \{\mathbf{u} \in V(Q_n) \mid w_H(\mathbf{u}) \text{ is odd}\}$.

Sun et al. (2006) first addressed the problem of finding mutually independent Hamiltonian cycles on the n-cube for $n \geq 3$. Later, Hsieh and Yu (2007) claimed that an injured n-cube contains $(n-1-f)$-mutually independent fault-free Hamiltonian cycles, and Kueng et al. (2009a) gave a formal proof to validate this claim, where $f \leq n-2$ is the total number of permanent edge-faults that can occur everywhere at random to injure the n-cube. To be precise, we have to introduce the definition of mutually independent Hamiltonian cycles in advance. Let G be a graph with N vertices. A Hamiltonian cycle C of G is represented by $\langle u_1, u_2, \ldots, u_N, u_1\rangle$, where u_1 is referred to as the start vertex of C. Naturally every vertex of C can serves as the start one. Two Hamiltonian cycles of G, namely $C_1 = \langle u_1, u_2, \ldots, u_N, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_N, v_1\rangle$, are *internally independent* if $u_1 = v_1$ and $u_i \neq v_i$ for $2 \leq i \leq N$. A set $\{C_1, C_2, \ldots, C_m\}$ of m Hamiltonian cycles of G is m*-mutually independent* if and only if any two of them are internally independent for $m \geq 2$. The concept of mutually independent Hamiltonian cycles can be applied in many different areas like those introduced in Hsieh and Yu (2007), Kueng et al. (2008), Kung et al. (2011), Lin et al. (2012), Shih et al. (2010a, 2010b), Su et al. (2011a, 2011b), Sun et al. (2006). This paper aims to improve the mentioned result (Hsieh and Yu 2007; Kueng et al. 2009a) by showing that Q_n has up to $(n-f)$-mutually independent fault-free Hamiltonian cycles, starting from any vertex, when $f \leq n-2$ edges are faulty.

The rest of this paper is organized as follows. The basic properties of hypercubes are introduced in Sect. 2. Our main theorem is presented in Sect. 3. Finally, some concluding remarks are given in Sect. 4.

2 Preliminaries

By definition, the n-cube Q_n has a recursive construction; that is, it can be decomposed into two $(n-1)$-dimensional hypercubes. Let $Q_n^{d,j}$ denote the subgraph of Q_n induced by $\{\mathbf{u} \in V(Q_n) \mid (\mathbf{u})_d = j\}$ for $1 \leq d \leq n$ and $j \in \{0, 1\}$. Obviously, $Q_n^{d,j}$ is isomorphic to Q_{n-1}. Then the d*-partition* of Q_n decomposes Q_n along the dth dimension into $Q_n^{d,0}$ and $Q_n^{d,1}$. The set of crossing edges between $Q_n^{d,0}$ and $Q_n^{d,1}$, denoted by $E_c^d = \{(\mathbf{u}, \mathbf{v}) \in E(Q_n) \mid \mathbf{u} \in V(Q_n^{d,0}), \mathbf{v} \in V(Q_n^{d,1})\}$, consists of all d-dimensional edges in Q_n. It is known that Q_n is vertex-transitive and edge-transitive (Saad and Shultz 1988).

A Hamiltonian graph G is said to be *f-edge-fault-tolerant Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subseteq E(G)$ with $|F| \leq f$. A Hamiltonian laceable graph G is said to be *f-edge-fault-tolerant Hamiltonian laceable* if $G - F$ remains Hamiltonian laceable for every $F \subseteq E(G)$ with $|F| \leq f$. Similarly, a hyper-Hamiltonian laceable graph G is said to be *f-edge-fault-tolerant hyper-Hamiltonian laceable* if $G - F$ remains hyper-Hamiltonian laceable for every $F \subseteq E(G)$ with $|F| \leq f$.

Lemma 1 (Tsai et al. 2002) *Let $n \geq 3$. Then Q_n is $(n-2)$-edge-fault-tolerant Hamiltonian and $(n-2)$-edge-fault-tolerant Hamiltonian laceable.*

Lemma 2 (Tsai et al. 2002) *Let $n \geq 3$. Then Q_n is $(n-3)$-edge-fault-tolerant hyper-Hamiltonian laceable.*

Lemma 3 (Sun et al. 2006) *Let $n \geq 4$. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two arbitrary vertices in different partite sets of Q_n. Then $Q_n - \{\mathbf{x}, \mathbf{y}\}$ is Hamiltonian laceable.*

The proof of the next lemma is presented in Appendix A.

Lemma 4 *Let $n \geq 4$. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are any two adjacent vertices in Q_n. Then $Q_n - \{\mathbf{x}, \mathbf{y}\}$ is $(n-3)$-edge-fault-tolerant Hamiltonian laceable.*

Two Hamiltonian paths of a graph G, represented by $P_1 = \langle u_1, u_2, \ldots, u_{|V(G)|}\rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_{|V(G)|}\rangle$, are *internally independent* if $u_1 = v_1$, $u_{|V(G)|} = v_{|V(G)|}$, and $u_i \neq v_i$ for every $1 < i < |V(G)|$; P_1 and P_2 are *fully independent* if $u_i \neq v_i$ for every $1 \leq i \leq |V(G)|$. A set $\{P_i \mid 1 \leq i \leq m\}$ of m Hamiltonian paths of G are *m-mutually fully independent* if $m = 1$ or its any two Hamiltonian paths are fully independent for $m \geq 2$.

Lemma 5 (Sun et al. 2006) *Let Q_n be an n-cube for $n \geq 2$. Suppose that $\{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n) \mid \mathbf{w}_i \in V_0(Q_n), \mathbf{b}_i \in V_1(Q_n), 1 \leq i \leq n-1\}$ consists of $n-1$ distinct edges with no shared endpoints. Then Q_n contains $(n-1)$-mutually fully independent Hamiltonian paths joining $\mathbf{w}_i$ and $\mathbf{b}_i$ for $1 \leq i \leq n-1$.*

The next lemma plays an important role in deriving the main result of this paper and was proved by Kueng et al. (2009a). To make the paper self-contained, its proof is given in Appendix B.

Lemma 6 (Kueng et al. 2009a) *Let F be a set of f edges in Q_n, where $n \geq 3$ and $f \leq n-2$. Suppose that $A = \{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n) \mid \mathbf{w}_i \in V_0(Q_n), \mathbf{b}_i \in V_1(Q_n), 1 \leq i \leq n-1-f\}$ consists of $n-1-f$ distinct edges with no shared endpoints. Then $Q_n - F$ contains $(n-1-f)$-mutually fully independent Hamiltonian paths $P_1[\mathbf{w}_1, \mathbf{b}_1], \ldots, P_{n-1-f}[\mathbf{w}_{n-1-f}, \mathbf{b}_{n-1-f}]$.*

3 Mutually independent Hamiltonian cycles

The *mutually independent Hamiltonicity* of a graph G, denoted by $\mathcal{IHC}(G)$, is defined as the maximum integer m such that for any vertex $v \in V(G)$, there exist m-mutually independent Hamiltonian cycles of G starting from v.

Theorem 1 (Sun et al. 2006) *$\mathcal{IHC}(Q_n) = n - 1$ if $n \leq 3$, and $\mathcal{IHC}(Q_n) = n$ if $n \geq 4$.*

The last lemma and theorem show that Q_n contains $(n - f)$-mutually independent fault-free Hamiltonian cycles, where f denotes the total number of faulty edges in Q_n for $f \leq n - 2$ and $n \geq 4$.

Lemma 7 *Suppose that F denotes any set of f edges in Q_4. Then $Q_4 - F$ contains $(4 - f)$-mutually independent Hamiltonian cycles starting from any vertex if $f \leq 2$.*

Proof Let $\mathbf{s}$ be any vertex of Q_4 in $V_i(Q_4)$ for any $i \in \{0, 1\}$. If $f = 0$, then Theorem 1 has ensured that Q_4 has 4-mutually independent Hamiltonian cycles starting from $\mathbf{s}$. Thus, we consider $f \in \{1, 2\}$ only. Because the hypercube is edge-transitive, we assume that F contains a 4-dimensional edge. Then Q_4 can be partitioned along the fourth dimension into $Q_4^{4,0}$ and $Q_4^{4,1}$. For the sake of convenience, we define some notations first: $F_0 = F \cap E(Q_4^{4,0})$, $F_1 = F \cap E(Q_4^{4,1})$, $F_c = F \cap E_c^4$, $f_0 = |F_0|$, $f_1 = |F_1|$, $f_c = |F_c|$, and $\delta = 4 - f$. Since $f_c \geq 1$, we have $f_0 \leq 1$ and $f_1 \leq 1$.

Case 1: $f = 2$. Thus, $\delta = 2$. Without loss of generality, we assume that $\mathbf{s}$ is in $Q_4^{4,0}$.

Subcase 1.1: $(\mathbf{s}, (\mathbf{s})^4)$ is fault-free. Since Q_4 has eight 4-dimensional edges, we can find two vertex-disjoint edges $(\mathbf{u}, (\mathbf{u})^4)$ and $(\mathbf{v}, (\mathbf{v})^4)$, both of which are fault-free such that $\mathbf{u} \in V_{1-i}(Q_4^{4,0})$ and $\mathbf{v} \in V_i(Q_4^{4,1})$. By Lemma 1, Q_3 is 1-edge-fault-tolerant Hamiltonian laceable. Hence, there exist two Hamiltonian paths H and T of $Q_4^{4,0} - F_0$ joining pairs $\mathbf{s}, \mathbf{u}$ and $(\mathbf{v})^4, \mathbf{s}$, respectively. Similarly, there exist two Hamiltonian paths P and R of $Q_4^{4,1} - F_1$ joining pairs $(\mathbf{u})^4, (\mathbf{s})^4$ and $(\mathbf{s})^4, \mathbf{v}$, respectively. Let $C_1 = \langle \mathbf{s}, H, \mathbf{u}, (\mathbf{u})^4, P, (\mathbf{s})^4, \mathbf{s} \rangle$ and $C_2 = \langle \mathbf{s}, (\mathbf{s})^4, R, \mathbf{v}, (\mathbf{v})^4, T, \mathbf{s} \rangle$. Then C_1 and C_2 are 2-mutually independent fault-free Hamiltonian cycles starting from $\mathbf{s}$. See Fig. 1(a).

Subcase 1.2: $(\mathbf{s}, (\mathbf{s})^4)$ is faulty.

Condition 1.2.1: $f_0 = f_1 = 0$. Let $\mathbf{x} \in V_i(Q_4^{4,0}) - \{\mathbf{s}\}$ such that $(\mathbf{x}, (\mathbf{x})^4)$ is fault-free, and let $\mathbf{b}$ be a neighbor of $\mathbf{s}$ in $Q_4^{4,0}$ such that edges $(\mathbf{s}, \mathbf{b})$ and $(\mathbf{b}, (\mathbf{b})^4)$ are fault-free. By Lemma 2, Q_3 is hyper-Hamiltonian laceable. Thus, there exists a Hamiltonian path P of $Q_4^{4,0} - \{\mathbf{b}\}$ joining $\mathbf{x}$ and $\mathbf{s}$. By Lemma 1, there exists a Hamiltonian path R of $Q_4^{4,1}$ joining $(\mathbf{b})^4$ and $(\mathbf{x})^4$. For clarity, P and R are represented by $\langle \mathbf{x}, P', \mathbf{v}, \mathbf{s} \rangle$ and $\langle (\mathbf{b})^4, R', \mathbf{y}, (\mathbf{x})^4 \rangle$, respectively, where $\mathbf{v}$ (respectively, $\mathbf{y}$) is a neighbor of $\mathbf{s}$ (respectively, $(\mathbf{x})^4$). Because $Q_4^{4,1}$ is 1-edge-fault-tolerant Hamiltonian laceable, there exists a Hamiltonian path H of $Q_4^{4,1} - \{((\mathbf{x})^4, \mathbf{y})\}$ joining $(\mathbf{x})^4$ and $(\mathbf{b})^4$. Then we set $C_1 = \langle \mathbf{s}, \mathbf{b}, (\mathbf{b})^4, R', \mathbf{y}, (\mathbf{x})^4, \mathbf{x}, P', \mathbf{v}, \mathbf{s} \rangle$ and $C_2 = \langle \mathbf{s}, \mathbf{v}, rev(P'), \mathbf{x}, (\mathbf{x})^4, H, (\mathbf{b})^4, \mathbf{b}, \mathbf{s} \rangle$, where $rev(P')$ is the reverse of P'. As a result, C_1 and C_2 are 2-mutually independent fault-free Hamiltonian cycles starting from $\mathbf{s}$. See Fig. 1(b).

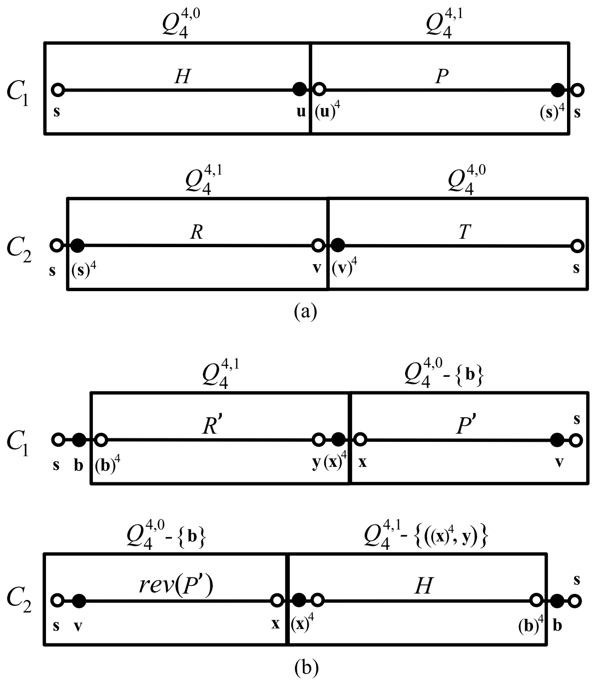

Fig. 1 Illustration for Lemma 7

Table 1 2-mutually independent fault-free Hamiltonian cycles, C_1 and C_2

F	C_1 and C_2
{(0, 8), (8, 12)} or {(0, 8), (10, 14)} or {(0, 8), (13, 15)} or {(0, 8), (10, 11)}	⟨0, 1, 5, 4, 6, 7, 3, 11, 15, 14, 12, 13, 9, 8, 10, 2, 0⟩ ⟨0, 2, 10, 8, 9, 11, 15, 14, 12, 13, 5, 1, 3, 7, 6, 4, 0⟩
{(0, 8), (8, 10)} or {(0, 8), (12, 14)} or {(0, 8), (11, 15)} or {(0, 8), (12, 13)}	⟨0, 1, 3, 2, 6, 7, 5, 13, 15, 14, 10, 11, 9, 8, 12, 4, 0⟩ ⟨0, 4, 12, 8, 9, 13, 15, 14, 10, 11, 3, 1, 5, 7, 6, 2, 0⟩
{(0, 8), (8, 9)} or {(0, 8), (9, 10)}	⟨0, 4, 5, 1, 3, 7, 6, 14, 15, 11, 9, 13, 12, 8, 10, 2, 0⟩ ⟨0, 2, 10, 8, 12, 14, 15, 11, 9, 13, 5, 4, 6, 7, 3, 1, 0⟩
{(0, 8), (9, 13)}	⟨0, 4, 6, 2, 3, 7, 5, 13, 15, 11, 10, 14, 12, 8, 9, 1, 0⟩ ⟨0, 1, 9, 11, 15, 14, 10, 8, 12, 13, 5, 7, 3, 2, 6, 4, 0⟩
{(0, 8), (14, 15)}	⟨0, 2, 6, 4, 5, 7, 3, 11, 15, 13, 12, 14, 10, 8, 9, 1, 0⟩ ⟨0, 1, 9, 8, 10, 11, 15, 13, 12, 14, 6, 2, 3, 7, 5, 4, 0⟩

Condition 1.2.2: $f_0 = 0$ and $f_1 = 1$. With symmetry, we assume that $\mathbf{s} = 0000$. Table 1 shows that there are 2-mutually independent fault-free Hamiltonian cycles starting from 0000, in which all binary strings are decimalized for the purpose of saving space.

Condition 1.2.3: $f_0 = 1$ and $f_1 = 0$. By symmetry, we assume that $\mathbf{s} = 0000$. Table 2 shows that there are 2-mutually independent fault-free Hamiltonian cycles starting from 0000, in which all binary strings are decimalized for the purpose of saving space.

Table 2 2-mutually independent fault-free Hamiltonian cycles, C_1 and C_2

F	C_1 and C_2
$\{(0,4),(0,8)\}$ or $\{(2,6),(0,8)\}$ or $\{(5,7),(0,8)\}$ or $\{(1,3),(0,8)\}$ or $\{(2,3),(0,8)\}$	$\langle 0,1,5,4,6,7,3,11,15,14,12,13,9,8,10,2,0\rangle$ $\langle 0,2,10,14,15,13,12,8,9,11,3,7,6,4,5,1,0\rangle$
$\{(0,2),(0,8)\}$ or $\{(4,6),(0,8)\}$ or $\{(3,7),(0,8)\}$ or $\{(4,5),(0,8)\}$	$\langle 0,1,3,2,6,7,5,13,15,14,10,11,9,8,12,4,0\rangle$ $\langle 0,4,12,14,15,11,10,8,9,13,5,7,6,2,3,1,0\rangle$
$\{(0,1),(0,8)\}$ or $\{(1,5),(0,8)\}$ or $\{(6,7),(0,8)\}$	$\langle 0,4,5,7,3,1,9,8,12,13,15,11,10,14,6,2,0\rangle$ $\langle 0,2,6,14,10,8,12,13,15,11,9,1,3,7,5,4,0\rangle$

Table 3 3-mutually independent fault-free Hamiltonian cycles, C_1, C_2, and C_3

s	C_1, C_2, C_3
$0000_2 = 0_{10}$	$\langle 0,1,5,4,6,7,3,11,15,14,12,13,9,8,10,2,0\rangle$ $\langle 0,2,10,14,15,13,12,8,9,11,3,1,5,7,6,4,0\rangle$ $\langle 0,4,6,7,3,2,10,14,12,8,9,11,15,13,5,1,0\rangle$
$0100_2 = 4_{10}$	$\langle 4,0,1,3,2,6,7,15,11,10,14,12,8,9,13,5,4\rangle$ $\langle 4,5,13,15,11,10,14,12,8,9,1,0,2,3,7,6,4\rangle$ $\langle 4,6,2,10,14,12,8,9,13,15,11,3,7,5,1,0,4\rangle$
$0110_2 = 6_{10}$	$\langle 6,2,0,1,5,7,3,11,9,13,15,14,10,8,12,4,6\rangle$ $\langle 6,4,12,14,10,8,9,13,15,11,3,2,0,1,5,7,6\rangle$ $\langle 6,7,3,11,15,13,12,14,10,8,9,1,5,4,0,2,6\rangle$
$0111_2 = 7_{10}$	$\langle 7,3,1,5,4,0,2,10,11,15,13,9,8,12,14,6,7\rangle$ $\langle 7,5,13,15,14,10,11,9,8,12,4,6,2,0,1,3,7\rangle$ $\langle 7,6,4,0,2,3,1,5,13,9,8,12,14,10,11,15,7\rangle$

Case 2: $f = 1$. Because the hypercube is edge-transitive, we assume that $F = \{(0000, 1000)\}$. With symmetry, we consider $\mathbf{s} \in \{0000, 0100, 0110, 0111\}$ only. Table 3 shows that there are 3-mutually independent fault-free Hamiltonian cycles starting from **s**, in which all binary strings are decimalized for the purpose of saving space. □

Theorem 2 *Let $n \geq 4$. Suppose that F denotes any set of f edges in Q_n. Then $Q_n - F$ contains $(n-f)$-mutually independent Hamiltonian cycles starting from any vertex if $f \leq n-2$.*

Proof Let **s** be any vertex of Q_n. If $f = 0$, then Theorem 1 has ensured that Q_n has n-mutually independent Hamiltonian cycles starting from **s**. Thus, we consider $1 \leq f \leq n-2$ below. Without loss of generality, we assume that F contains a d-dimensional edge for $1 \leq d \leq n$. Then, Q_n can be partitioned along the dth dimension into $Q_n^{d,0}$ and $Q_n^{d,1}$. Furthermore, we assume that $\mathbf{s} \in V_0(Q_n^{d,0})$. For the sake of convenience, we define some notations in advance: $F_0 = F \cap E(Q_n^{d,0})$, $F_1 = F \cap E(Q_n^{d,1})$, $F_c = F \cap E_c$, $f_0 = |F_0|$, $f_1 = |F_1|$, $f_c = |F_c|$, and $\delta = n - f$. Since $f_c \geq 1$, we have $f_0 \leq f - 1$ and $f_1 \leq f - 1$.

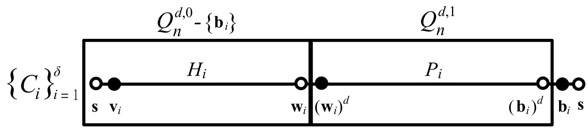

Fig. 2 Illustration for Subcase 1.1 of Theorem 2

The proof proceeds by induction on n. Lemma 7 is the induction basis. For $n \geq 5$, the inductive hypothesis is that the theorem statement holds for Q_k, $4 \leq k \leq n-1$. Since $f_0 \leq f-1 \leq n-3$, $Q_n^{d,0} - F_0$ has $(n-1-f_0)$-mutually independent Hamiltonian cycles starting from $\mathbf{s}$. Without loss of generality, these Hamiltonian cycles can be represented by $\langle \mathbf{s}, \mathbf{v}_i, H_i, \mathbf{w}_i, \mathbf{b}_i, \mathbf{s}\rangle$ for $1 \leq i \leq n-1-f_0$. Let $F_x = F_c \cap \{(\mathbf{w}_i, (\mathbf{w}_i)^d), (\mathbf{b}_i, (\mathbf{b}_i)^d) \mid 1 \leq i \leq n-1-f_0\}$ and $f_x = |F_x|$. Obviously, there exist at least $n-1-f_0-f_x$ integers in $\{1, 2, \ldots, n-1-f_0\}$, say $1, 2, \ldots, n-1-f_0-f_x$, such that $(\mathbf{w}_i, (\mathbf{w}_i)^d)$ and $(\mathbf{b}_i, (\mathbf{b}_i)^d)$ are fault-free for all $i \in \{1, 2, \ldots, n-1-f_0-f_x\}$. For ease of presentation, let $\delta_0 = n-1-f_0-f_x$ and $\delta_1 = n-2-f_1$.

Case 1: $(\mathbf{s}, (\mathbf{s})^d)$ is faulty. Thus, we have $f_x \leq f_c - 1$ and $\delta_0 = n-1-(f_0+f_x) \geq n-1-(f_0+f_c-1) \geq n-f = \delta$. Since $\delta_1 = n-2-f_1 \geq n-2-(f-1) = \delta-1$, we consider the following two subcases.

Subcase 1.1: Suppose that $\delta_1 \geq \delta$. By Lemma 6, $Q_n^{d,1} - F_1$ has δ-mutually fully independent Hamiltonian paths P_i, $1 \leq i \leq \delta$, joining $(\mathbf{w}_i)^d$ and $(\mathbf{b}_i)^d$. Let $C_i = \langle \mathbf{s}, \mathbf{v}_i, H_i, \mathbf{w}_i, (\mathbf{w}_i)^d, P_i, (\mathbf{b}_i)^d, \mathbf{b}_i, \mathbf{s}\rangle$ for each $i \in \{1, 2, \ldots, \delta\}$. Then $\{C_1, C_2, \ldots, C_\delta\}$ is a set of δ-mutually independent Hamiltonian cycles starting from $\mathbf{s}$ in $Q_n - F$. See Fig. 2 for illustration.

Subcase 1.2: Suppose that $\delta_1 = \delta - 1$. Hence, we have $f_1 = f-1$ and $F_c = \{(\mathbf{s}, (\mathbf{s})^d)\}$; that is, $Q_n^{d,0}$ is fault-free. By Lemma 6, $Q_n^{d,1} - F_1$ has $(\delta-1)$-mutually fully independent Hamiltonian paths P_i, $1 \leq i \leq \delta-1$, joining $(\mathbf{w}_i)^d$ and $(\mathbf{b}_i)^d$. Let $C_i = \langle \mathbf{s}, \mathbf{v}_i, H_i, \mathbf{w}_i, (\mathbf{w}_i)^d, P_i, (\mathbf{b}_i)^d, \mathbf{b}_i, \mathbf{s}\rangle$ for $1 \leq i \leq \delta-1$.

Condition 1.2.1: $f = n-2$. Thus, we have $\delta = 2$. Let $\mathbf{x}$ be a vertex in $V_1(Q_n^{d,1})$ such that $d_{Q_n}(\mathbf{x}, (\mathbf{w}_1)^d) \geq 4$. Since $d_{Q_n}((\mathbf{s})^d, (\mathbf{w}_1)^d) = 2$, we have $\mathbf{x} \neq (\mathbf{s})^d$. Furthermore, since $F_c = \{(\mathbf{s}, (\mathbf{s})^d)\}$, edge $(\mathbf{x}, (\mathbf{x})^d)$ is fault-free. By Lemma 1, $Q_n^{d,1} - F_1$ has a Hamiltonian path R joining $(\mathbf{v}_2)^d$ and $\mathbf{x}$. By Lemma 3, $Q_n^{d,0} - \{\mathbf{s}, \mathbf{v}_2\}$ has a Hamiltonian path T joining $(\mathbf{x})^d$ to $\mathbf{b}_2$. Let $C_2 = \langle \mathbf{s}, \mathbf{v}_2, (\mathbf{v}_2)^d, R, \mathbf{x}, (\mathbf{x})^d, T, \mathbf{b}_2, \mathbf{s}\rangle$. Then $\{C_1, C_2\}$ is a set of 2-mutually independent Hamiltonian cycles starting from $\mathbf{s}$ in $Q_n - F$. See Fig. 3(a).

Condition 1.2.2: $f \leq n-3$. Let $Y_n = \{\mathbf{u} \in V(Q_n^{d,1}) \mid d_{Q_n^{d,1}}(\mathbf{u}, P_1(3)) = 3\} \cup \{\mathbf{u} \in V(Q_n^{d,1}) \mid d_{Q_n^{d,1}}(\mathbf{u}, P_2(3)) = 3\}$. Then, we have $|Y_n| \geq \binom{n-1}{3} + 1 > \delta$ for $n = 5$, and $|Y_n| \geq \binom{n-1}{3} > \delta$ for $n \geq 6$. Thus, we can choose a vertex $\mathbf{y}$ in Y_n such that $\mathbf{y} \notin \{P_i(2) \mid 1 \leq i \leq \delta-1\} \cup \{(\mathbf{v}_\delta)^d\}$. Without loss of generality, we assume that $d_{Q_n^{d,1}}(\mathbf{y}, P_1(3)) = 3$. If $(\mathbf{y}, (\mathbf{s})^d)$ is a fault-free edge, then let $\mathbf{z} = (\mathbf{s})^d$; otherwise, let $\mathbf{z} \in N_{Q_n^{d,1}}(\mathbf{y}) - \{(\mathbf{w}_i)^d \mid 1 \leq i \leq \delta-1\}$ such that $(\mathbf{y}, \mathbf{z})$ is a fault-free edge. Since $|N_{Q_n^{d,1}}(\mathbf{y}) - \{\mathbf{z}\}| = n-2 > |\{P_i(3) \mid 2 \leq i \leq \delta-1\}| + f_1 = (n-f-2) + (f-1) = n-3$, we can find a vertex $\mathbf{x}$ in $N_{Q_n^{d,1}}(\mathbf{y}) - \{\mathbf{z}\} - \{P_i(3) \mid 2 \leq i \leq \delta-1\}$ such that $(\mathbf{x}, \mathbf{y})$ is a fault-free edge. It follows from Lemma 4 that $Q_n^{d,1} - \{\mathbf{x}, \mathbf{y}\}$ is $(n-4)$-edge-fault-

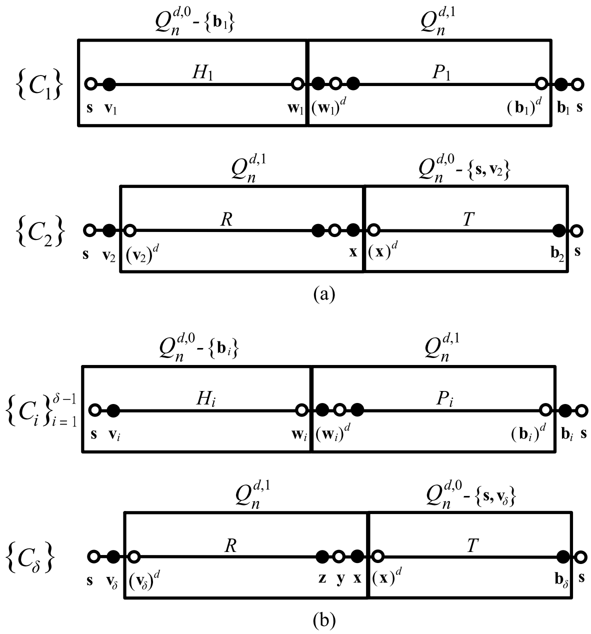

Fig. 3 Illustration for Subcase 1.2 of Theorem 2

tolerant Hamiltonian laceable. Since $f_1 = f - 1 \leq n - 4$, $Q_n^{d,1} - \{\mathbf{x}, \mathbf{y}\}$ has a fault-free Hamiltonian path R joining $(\mathbf{v}_\delta)^d$ and $\mathbf{z}$. By Lemma 3, $Q_n^{d,0} - \{\mathbf{s}, \mathbf{v}_\delta\}$ has a Hamiltonian path T joining $(\mathbf{x})^d$ to $\mathbf{b}_\delta$. Let $C_\delta = \langle \mathbf{s}, \mathbf{v}_\delta, (\mathbf{v}_\delta)^d, R, \mathbf{z}, \mathbf{y}, \mathbf{x}, (\mathbf{x})^d, T, \mathbf{b}_\delta, \mathbf{s}\rangle$. Then $\{C_1, C_2, \ldots, C_\delta\}$ is a set of δ-mutually independent Hamiltonian cycles starting from $\mathbf{s}$ in $Q_n - F$. See Fig. 3(b).

Case 2: $(\mathbf{s}, (\mathbf{s})^d)$ is fault-free.

Subcase 2.1: Suppose that $f_0 + f_x \leq f - 1$ and $f_1 \leq f - 2$. Thus, we have $\delta_0 \geq \delta$ and $\delta_1 \geq \delta$. By Lemma 6, $Q_n^{d,1} - F_1$ has δ-mutually fully independent Hamiltonian paths P_i, $1 \leq i \leq \delta$, joining $(\mathbf{w}_i)^d$ and $(\mathbf{b}_i)^d$. Let $C_i = \langle \mathbf{s}, \mathbf{v}_i, H_i, \mathbf{w}_i, (\mathbf{w}_i)^d, P_i, (\mathbf{b}_i)^d, \mathbf{b}_i, \mathbf{s}\rangle$ for each $i \in \{1, 2, \ldots, \delta\}$. Then $\{C_1, C_2, \ldots, C_\delta\}$ is a set of δ-mutually independent Hamiltonian cycles starting from $\mathbf{s}$ in $Q_n - F$.

Subcase 2.2: Suppose that $f_0 + f_x = f$ or $f_1 = f - 1$. Thus, we have $\delta_0 = \delta - 1$ or $\delta_1 = \delta - 1$. It follows from Lemma 6 that $Q_n^{d,1} - F_1$ has $(\delta - 1)$-mutually fully independent Hamiltonian paths P_i, $1 \leq i \leq \delta - 1$, joining $(\mathbf{w}_i)^d$ and $(\mathbf{b}_i)^d$.

Condition 2.2.1: $f \leq n - 3$. Let $C_i = \langle \mathbf{s}, \mathbf{v}_i, H_i, \mathbf{w}_i, (\mathbf{w}_i)^d, P_i, (\mathbf{b}_i)^d, \mathbf{b}_i, \mathbf{s}\rangle$ for $1 \leq i \leq \delta - 1$. Obviously, we can choose two vertices $\mathbf{x}$ and $\mathbf{x}'$ in $V_0(Q_n^{d,1}) - \{(\mathbf{b}_\delta)^d\} - \{P_i(2) \mid 1 \leq i \leq \delta - 1\}$ such that $(\mathbf{x}, (\mathbf{x})^d) \notin F$ and $(\mathbf{x}', (\mathbf{x}')^d) \notin F$. Since $|\{(\mathbf{w}_i)^d \mid 1 \leq i \leq \delta - 1\} \cup \{(\mathbf{s})^d\}| + f_1 \leq (n - f) + (f - 1) = n - 1 < n < |N_{Q_n^{d,1}}(\mathbf{x}) \cup N_{Q_n^{d,1}}(\mathbf{x}')|$, there exists at least one vertex $\mathbf{y}$ in $N_{Q_n^{d,1}}(\mathbf{x}) \cup N_{Q_n^{d,1}}(\mathbf{x}')$ such that $(\mathbf{x}, \mathbf{y})$ or $(\mathbf{x}', \mathbf{y})$ is fault-free, and $\mathbf{y} \notin \{(\mathbf{w}_i)^d \mid 1 \leq i \leq \delta - 1\} \cup \{(\mathbf{s})^d\}$. Without loss of generality, we assume that $\mathbf{y} \in N_{Q_n^{d,1}}(\mathbf{x})$. Since $f_1 \leq f - 1 \leq n - 4$, it follows from Lemma 2 that $Q_n^{d,1} - F_1 - \{\mathbf{x}\}$ has a Hamiltonian path R joining $(\mathbf{s})^d$ and $\mathbf{y}$, and $Q_n^{d,0} - F_0 - \{\mathbf{s}\}$ has a Hamiltonian path T joining $(\mathbf{x})^d$ and $\mathbf{b}_\delta$. Let $C_\delta = \langle \mathbf{s}, (\mathbf{s})^d, R, \mathbf{y}, \mathbf{x}, (\mathbf{x})^d, T, \mathbf{b}_\delta, \mathbf{s}\rangle$.

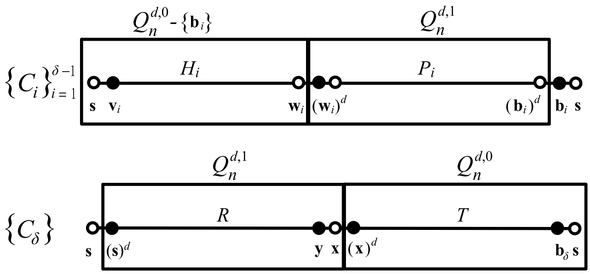

Fig. 4 Illustration for Condition 2.2.1 of Theorem 2

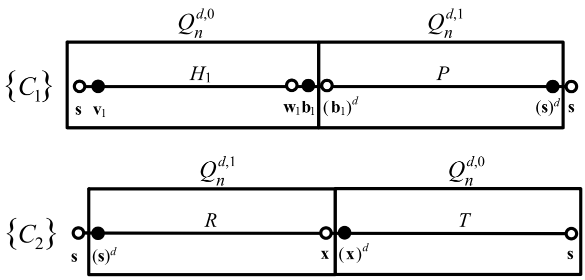

Fig. 5 Illustration for Condition 2.2.2 of Theorem 2

Then $\{C_1, C_2, \ldots, C_\delta\}$ is a set of δ-mutually independent Hamiltonian cycles starting from $\mathbf{s}$ in $Q_n - F$. See Fig. 4 for illustration.

Condition 2.2.2: $f = n - 2$. Thus, $\delta = 2$. Let $\mathbf{x} \in V_0(Q_n^{d,1}) - \{(\mathbf{b}_1)^d\}$ such that $(\mathbf{x}, (\mathbf{x})^d)$ is fault-free. By Lemma 1, $Q_n^{d,1} - F_1$ has a Hamiltonian path P joining $(\mathbf{b}_1)^d$ and $(\mathbf{s})^d$. Similarly, $Q_n^{d,1} - F_1$ has a Hamiltonian path R joining $(\mathbf{s})^d$ and $\mathbf{x}$, and $Q_n^{d,0} - F_0$ has a Hamiltonian path T joining $(\mathbf{x})^d$ and $\mathbf{s}$. Let $C_1 = \langle \mathbf{s}, \mathbf{v}_1, H_1, \mathbf{w}_1, \mathbf{b}_1, (\mathbf{b}_1)^d, P_1, (\mathbf{s})^d, \mathbf{s}\rangle$ and $C_2 = \langle \mathbf{s}, (\mathbf{s})^d, R, \mathbf{x}, (\mathbf{x})^d, T, \mathbf{s}\rangle$. Then $\{C_1, C_2\}$ is a set of 2-mutually independent Hamiltonian cycles starting from $\mathbf{s}$ in $Q_n - F$. See Fig. 5 for illustration.

The proof is completed. □

4 Conclusion

In this paper, we improve the result of finding mutually independent fault-free Hamiltonian cycles in a faulty hypercube, as previously addressed by Hsieh and Yu (2007) and Kueng et al. (2009a). Let F denote the set of f faulty edges in n-cube Q_n. Then we show that $Q_n - F$ has $(n - f)$-mutually independent Hamiltonian cycles starting from any vertex if $f \leq n - 2$. When all faulty edges happen to be incident with an identical vertex $\mathbf{s}$, i.e., the minimum degree of the survival graph $Q_n - F$ is equal to $n - f$, then $Q_n - F$ contains no more than $(n - f)$-mutually independent Hamiltonian cycles starting from $\mathbf{s}$. From such a point of view, the presented result is optimal.

Appendix A: Proof of Lemma 4

Proof The proof proceeds by induction on n. Our computer program verifies that $Q_4 - \{\mathbf{x}, \mathbf{y}\}$ is 1-edge-fault-tolerant Hamiltonian laceable. Please refer to the data reported by Kung (2012). For $n \geq 5$, we assume that $(\mathbf{x}, \mathbf{y})$ is r-dimensional with $1 \leq r \leq n$. Let $F \subset E(Q_n - \{\mathbf{x}, \mathbf{y}\})$ with $1 \leq |F| \leq n-3$. If there exists a d-partition of Q_n, $d \in \{1, 2, \ldots, n\} - \{r\}$, such that at least one faulty edge is d-dimensional, then Q_n is partitioned along the dth dimension; otherwise, every faulty edge is r-dimensional, and Q_n is partitioned into $Q_n^{d,0}$ and $Q_n^{d,1}$ with any $d \in \{1, 2, \ldots, n\} - \{r\}$. Without loss of generality, we assume that $(\mathbf{x}, \mathbf{y})$ is in $Q_n^{d,0}$. Then, the inductive hypothesis is that $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\}$ is $(n-4)$-edge-fault-tolerant Hamiltonian laceable for $n \geq 5$.

Let $F_i = F \cap E(Q_n^{d,i})$ for $i \in \{0, 1\}$. The following two cases show that $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$ is Hamiltonian laceable.

Case 1: $|F_0| \leq n-4$. Let $\mathbf{s} \in V_i(Q_n)$ and $\mathbf{t} \in V_{1-i}(Q_n)$ for any $i \in \{0, 1\}$.

Suppose that both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_n^{d,0}$. By the inductive hypothesis, $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\} - F_0$ is Hamiltonian laceable. Hence, there exists a Hamiltonian path P in $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\} - F_0$ joining $\mathbf{s}$ and $\mathbf{t}$. Clearly, there are $\lceil \frac{\ell(P)}{2} \rceil$ vertex-disjoint edges on P. Since $\lceil \frac{\ell(P)}{2} \rceil = \lceil \frac{2^{n-1}-3}{2} \rceil = 2^{n-2} - 1 > n-3 \geq |F|$ for $n \geq 5$, there exists an edge $(\mathbf{u}, \mathbf{v})$ on P such that $\{(\mathbf{u}, (\mathbf{u})^d), (\mathbf{v}, (\mathbf{v})^d)\} \cap F = \emptyset$. Accordingly, P can be represented as $\langle \mathbf{s}, P_1, \mathbf{u}, \mathbf{v}, P_2, \mathbf{t} \rangle$. By Lemma 1, there exists a Hamiltonian path R in $Q_n^{d,1} - F_1$ joining $(\mathbf{u})^d$ and $(\mathbf{v})^d$. Then $\langle \mathbf{s}, P_1, \mathbf{u}, (\mathbf{u})^d, R, (\mathbf{v})^d, \mathbf{v}, P_2, \mathbf{t} \rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 6(a).

Suppose that both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_n^{d,1}$. Obviously, it follows from Lemma 1 that there exists a Hamiltonian path R in $Q_n^{d,1} - F_1$ joining $\mathbf{s}$ and $\mathbf{t}$. Since $\lceil \frac{\ell(R)}{2} \rceil = \lceil \frac{2^{n-1}-1}{2} \rceil = 2^{n-2} > n-1 \geq |\{\mathbf{x}, \mathbf{y}\}| + |F|$ for $n \geq 5$, there exists an edge $(\mathbf{u}, \mathbf{v})$ on R such that $\{(\mathbf{v})^d, (\mathbf{u})^d\} \cap \{\mathbf{x}, \mathbf{y}\} = \emptyset$ and $\{(\mathbf{u}, (\mathbf{u})^d), (\mathbf{v}, (\mathbf{v})^d)\} \cap F = \emptyset$. Therefore, R can be represented as $\langle \mathbf{s}, R_1, \mathbf{u}, \mathbf{v}, R_2, \mathbf{t} \rangle$. By the inductive hypothesis, there exists a Hamiltonian path P in $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\} - F_0$ joining $(\mathbf{u})^d$ and $(\mathbf{v})^d$. Then $\langle \mathbf{s}, R_1, \mathbf{u}, (\mathbf{u})^d, P, (\mathbf{v})^d, \mathbf{v}, R_2, \mathbf{t} \rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 6(b) for illustration.

Suppose that $\mathbf{s}$ is in $Q_n^{d,0}$ and $\mathbf{t}$ is in $Q_n^{d,1}$. Let $\mathbf{b} \in V_{1-i}(Q_n^{d,0}) - \{\mathbf{x}, \mathbf{y}\}$ such that $(\mathbf{b}, (\mathbf{b})^d) \notin F$. It follows from the inductive hypothesis that there exists a Hamiltonian path P in $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\} - F_0$ joining $\mathbf{s}$ and $\mathbf{b}$. By Lemma 1, there exists a Hamiltonian path R in $Q_n^{d,1} - F_1$ joining $(\mathbf{b})^d$ and $\mathbf{t}$. As a result, $\langle \mathbf{s}, P, \mathbf{b}, (\mathbf{b})^d, R, \mathbf{t} \rangle$ is a Hamiltonian path in $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 6(c) for illustration.

Case 2: $|F_0| = n-3$. This case implies that every faulty edge is r-dimensional. Similarly, let $\mathbf{s} \in V_i(Q_n)$ and $\mathbf{t} \in V_{1-i}(Q_n)$ for any $i \in \{0, 1\}$.

Subcase 2.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_n^{d,0}$. Let $(\mathbf{u}, \mathbf{v}) \in F$ and $F' = F - \{(\mathbf{u}, \mathbf{v})\}$. By the inductive hypothesis, there exists a Hamiltonian path P in $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\} - F'$ joining $\mathbf{s}$ and $\mathbf{t}$. If $(\mathbf{u}, \mathbf{v}) \in E(P)$, then path P can be represented as $\langle \mathbf{s}, P_1, \mathbf{u}, \mathbf{v}, P_2, \mathbf{t} \rangle$; otherwise, P is written as $\langle \mathbf{s}, T_1, \mathbf{p}, \mathbf{q}, T_2, \mathbf{t} \rangle$, where $(\mathbf{p}, \mathbf{q})$ is any edge of P. Clearly, $Q_n^{d,1}$ has Hamiltonian paths R and H joining pairs $(\mathbf{u})^d$, $(\mathbf{v})^d$ and $(\mathbf{p})^d$, $(\mathbf{q})^d$, respectively. Then either $\langle \mathbf{s}, P_1, \mathbf{u}, (\mathbf{u})^d, R, (\mathbf{v})^d, \mathbf{v}, P_2, \mathbf{t} \rangle$ or $\langle \mathbf{s}, T_1, \mathbf{p}, (\mathbf{p})^d, H, (\mathbf{q})^d, \mathbf{q}, T_2, \mathbf{t} \rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 7(a) for illustration.

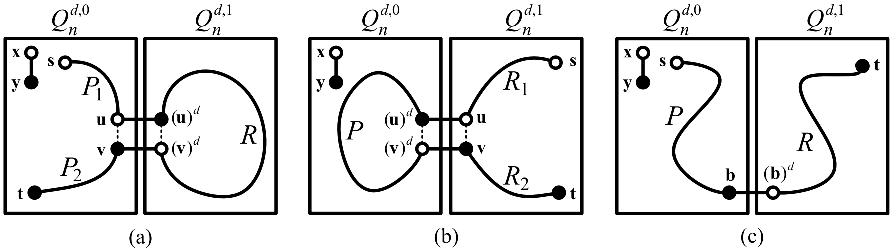

Fig. 6 Illustration for Case 1 of Lemma 4

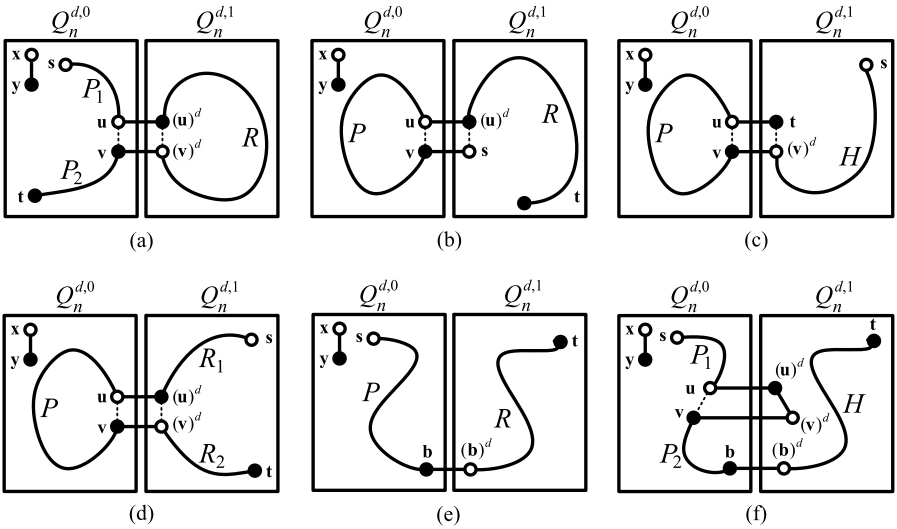

Fig. 7 Illustration for Case 2 of Lemma 4

Subcase 2.2: Both **s** and **t** are in $Q_n^{d,1}$. Since $|F_0| = |F| = n - 3 \geq 2$ for $n \geq 5$, let $(\mathbf{u}, \mathbf{v})$ be a faulty edge such that $\{(\mathbf{u})^d, (\mathbf{v})^d\} \neq \{\mathbf{s}, \mathbf{t}\}$. It is noticed that $(\mathbf{u}, \mathbf{v})$ is an r-dimensional edge. Without loss of generality, we assume that $\mathbf{u} \in V_i(Q_n^{d,0})$, and let $b \in \{1, 2, \ldots, n\} - \{d, r\}$. By the inductive hypothesis, there exists a fault-free Hamiltonian path P in $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\}$ joining **v** and **u**.

Condition 2.2.1: $(\mathbf{v})^d = \mathbf{s}$. By Lemma 2, $Q_n^{d,1} - \{\mathbf{s}\}$ has a Hamiltonian path R joining $(\mathbf{u})^d$ and **t**. Hence, $\langle \mathbf{s}, \mathbf{v}, P, \mathbf{u}, (\mathbf{u})^d, R, \mathbf{t} \rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 7(b).

Condition 2.2.2: $(\mathbf{u})^d = \mathbf{t}$. Lemma 2 ensures that $Q_n^{d,1} - \{\mathbf{t}\}$ has a Hamiltonian path H joining **s** and $(\mathbf{v})^d$. Hence, $\langle \mathbf{s}, H, (\mathbf{v})^d, \mathbf{v}, P, \mathbf{u}, \mathbf{t} \rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 7(c).

Condition 2.2.3: $(\mathbf{v})^d \neq \mathbf{s}$ and $(\mathbf{u})^d \neq \mathbf{t}$. Let $F' = \{((\mathbf{u})^d, ((\mathbf{u})^d)^k) \mid 1 \leq k \leq n, k \neq d, k \neq r, k \neq b\}$. Then it follows from Lemma 1 that $Q_n^{d,1} - F'$ has a Hamiltonian path R joining **s** and **t**. By the hypercube's definition, edge $((\mathbf{u})^d, (\mathbf{v})^d)$ is on path R. Accordingly, R can be represented as either $\langle \mathbf{s}, R_1, (\mathbf{u})^d, (\mathbf{v})^d, R_2, \mathbf{t} \rangle$

or $\langle \mathbf{s}, R_1, (\mathbf{v})^d, (\mathbf{u})^d, R_2, \mathbf{t}\rangle$. As a result, either $\langle \mathbf{s}, R_1, (\mathbf{u})^d, \mathbf{u}, P, \mathbf{v}, (\mathbf{v})^d, R_2, \mathbf{t}\rangle$ or $\langle \mathbf{s}, R_1, (\mathbf{v})^d, \mathbf{v}, P, \mathbf{u}, (\mathbf{u})^d, R_2, \mathbf{t}\rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 7(d) for illustration.

Subcase 2.3: $\mathbf{s}$ is in $Q_n^{d,0}$ and $\mathbf{t}$ is in $Q_n^{d,1}$. Because every faulty edge is r-dimensional, we can find a faulty edge $(\mathbf{u}, \mathbf{v})$ such that $\mathbf{t} \notin \{(\mathbf{u})^d, (\mathbf{v})^d\}$. Let $\mathbf{b} \in V_{1-i}(Q_n^{d,0}) - \{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{b}$ is not incident to any faulty edge. We denote $F - \{(\mathbf{u}, \mathbf{v})\}$ by F'. By the inductive hypothesis, there exists a Hamiltonian path P in $Q_n^{d,0} - \{\mathbf{x}, \mathbf{y}\} - F'$ joining $\mathbf{s}$ and $\mathbf{b}$.

Condition 2.3.1: $(\mathbf{u}, \mathbf{v}) \notin E(P)$. Then, Lemma 1 ensures that there exists a Hamiltonian path R in $Q_n^{d,1}$ joining $(\mathbf{b})^d$ and $\mathbf{t}$. Thus, $\langle \mathbf{s}, P, \mathbf{b}, (\mathbf{b})^d, R, \mathbf{t}\rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig. 7(e).

Condition 2.3.2: $(\mathbf{u}, \mathbf{v}) \in E(P)$. Then, path P can be represented as $\langle \mathbf{s}, P_1, \mathbf{u}, \mathbf{v}, P_2, \mathbf{b}\rangle$. By the inductive hypothesis, there exists a Hamiltonian path H in $Q_n^{d,1} - \{(\mathbf{u})^d, (\mathbf{v})^d\}$ joining $(\mathbf{b})^d$ and $\mathbf{t}$. Therefore, $\langle \mathbf{s}, P_1, \mathbf{u}, (\mathbf{u})^d, (\mathbf{v})^d, \mathbf{v}, P_2, \mathbf{b}, (\mathbf{b})^d, H, \mathbf{t}\rangle$ is a Hamiltonian path of $Q_n - \{\mathbf{x}, \mathbf{y}\} - F$. See Fig 7(f). □

Appendix B: Proof of Lemma 6

Proof The proof proceeds by induction on n. Suppose that $f = 0$. Then this case follows from Lemma 5. Suppose that $f = n-2$. Then we have $\delta = n - 1 - (n-2) = 1$. By Lemma 1, $Q_n - F$ has a Hamiltonian path joining any two vertices in different partite sets. Moreover, this theorem is trivial for Q_3, as the induction basis. In what follows, we consider $1 \le f \le n-3$ and $n \ge 4$. The inductive hypothesis is that the theorem statement is true for Q_{n-1}.

Since $\delta + f = n - 1 < n$, there must exist an integer d in $\{1, 2, \ldots, n\}$ such that $A \cup F$ contains no d-dimensional edges. Since Q_n is edge-transitive, we can assume $d = n$. Then we partition Q_n into $\{Q_n^{n,0}, Q_n^{n,1}\}$ along the nth dimension. Thus, each edge of $A \cup F$ is in either $Q_n^{n,0}$ or $Q_n^{n,1}$. For the sake of convenience, we define some notations to be used later: $F_0 = F \cap E(Q_n^{n,0})$, $F_1 = F \cap E(Q_n^{n,1})$, $F_c = F \cap E_c^n$, $f_0 = |F_0|$, $f_1 = |F_1|$, and $\delta = n - 1 - f$.

Let $r_0 = |\{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n^{n,0}) \mid 1 \le i \le \delta\}|$ and $r_1 = |\{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n^{n,1}) \mid 1 \le i \le \delta\}|$. Clearly, $r_0 + r_1 = \delta$. Without loss of generality, we assume $\{(\mathbf{w}_1, \mathbf{b}_1), \ldots, (\mathbf{w}_{r_0}, \mathbf{b}_{r_0})\} \subset E(Q_n^{n,0})$ if $r_1 = 0$, and $\{(\mathbf{w}_{r_0+1}, \mathbf{b}_{r_0+1}), \ldots, (\mathbf{w}_\delta, \mathbf{b}_\delta)\} \subset E(Q_n^{n,1})$ if $r_1 > 0$. Since $n - 1 = \delta + f = r_0 + r_1 + f_0 + f_1$, we have $r_i + f_j \le n - 1$ for any $i, j \in \{0, 1\}$. Then we have to take the following cases into account.

Case 1: Suppose that $r_i + f_j \le n-2$ for any $i, j \in \{0, 1\}$. Since $r_0 + f_0 \le n-2$, $r_0 \le n - 2 - f_0 = (n-1) - 1 - f_0$. By the inductive hypothesis, $Q_n^{n,0} - F_0$ has r_0-mutually fully independent Hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i]$, $1 \le i \le r_0$, if $r_0 > 0$. Obviously, $H_i[\mathbf{w}_i, \mathbf{b}_i]$ can be represented as $\langle \mathbf{w}_i, H_i', \mathbf{u}_i, \mathbf{b}_i\rangle$, where $\mathbf{u}_i$ is some vertex adjacent to $\mathbf{b}_i$. Similarly, $Q_n^{n,1} - F_1$ has r_1-mutually fully independent Hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, \mathbf{b}_i\rangle$, $r_0 + 1 \le i \le \delta$, if $r_1 > 0$.

For $r_0 > 0$, we construct r_0 paths in $Q_n^{n,1} - F_1$ to incorporate the previously established r_0 paths of $Q_n^{n,0} - F_0$. Since $r_0 + f_1 \le n - 2$, we have $r_0 \le n - 2 - f_1$. By the inductive hypothesis, $Q_n^{n,1} - F_1$ also contains r_0-mutually

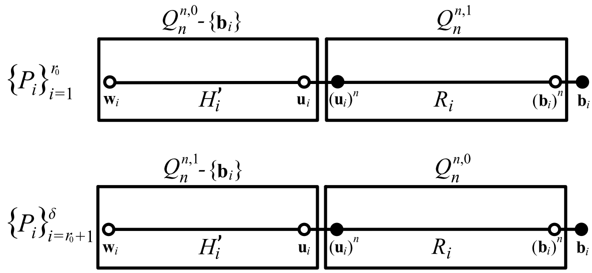

Fig. 8 Illustration for Case 1 of Lemma 6, particularly when $0 < r_0 < \delta$

fully independent Hamiltonian paths $R_1[(\mathbf{u}_1)^n, (\mathbf{b}_1)^n], \ldots, R_{r_0}[(\mathbf{u}_{r_0})^n, (\mathbf{b}_{r_0})^n]$. Similarly, $Q_n^{n,0} - F_0$ also contains r_1-mutually fully independent Hamiltonian paths $R_{r_0+1}[(\mathbf{u}_{r_0+1})^n, (\mathbf{b}_{r_0+1})^n], \ldots, R_\delta[(\mathbf{u}_\delta)^n, (\mathbf{b}_\delta)^n]$ if $r_1 > 0$. Accordingly, we set $P_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, (\mathbf{u}_i)^n, R_i, (\mathbf{b}_i)^n, \mathbf{b}_i \rangle$ for every $1 \le i \le \delta$. Thus, $\{P_1, P_2, \ldots, P_\delta\}$ turns out to be a set of δ-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 8 for illustration.

Case 2: Suppose that $r_i + f_i = n-1$ for some $i \in \{0, 1\}$. Without loss of generality, we assume that $r_0 + f_0 = n - 1$. Since $r_0 = n - 1 - f_0 \ge n - 1 - f = \delta$, we must have $r_0 = \delta$ and $f_0 = f \le n - 3$. It is noticed that $r_0 - 1 = \delta - 1 = n - 2 - f = (n-1) - 1 - f_0$. By the inductive hypothesis, $Q_n^{n,0} - F_0$ has $(r_0 - 1)$-mutually fully independent Hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i]$, $2 \le i \le r_0$. Again, $H_i[\mathbf{w}_i, \mathbf{b}_i]$ can be represented as $\langle \mathbf{w}_i, H_i', \mathbf{u}_i, \mathbf{b}_i \rangle$, where $\mathbf{u}_i$ is some vertex adjacent to $\mathbf{b}_i$.

Subcase 2.1: Suppose $n = 4$. Thus, we have $r_0 = 2$. By Lemma 2, $Q_4^{4,0} - F_0$ has a Hamiltonian path $H_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, \mathbf{u}_1, H_1', (\mathbf{b}_1)^j, \mathbf{b}_1 \rangle$, where $\mathbf{u}_1$ is a vertex adjacent to $\mathbf{w}_1$, and j is some integer of $\{1, 2, 3\}$. Let $X = \{((\mathbf{u}_1)^4, (\mathbf{u}_2)^4)\}$. Similarly, there exist two Hamiltonian paths $R_1[(\mathbf{w}_1)^4, (\mathbf{u}_1)^4]$ and $R_2[(\mathbf{u}_2)^4, (\mathbf{b}_2)^4]$ in $Q_4^{4,1} - X$. Obviously, we have $R_1(7) \ne R_2(1)$ and $R_1(8) \ne R_2(2)$. Then we set $P_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, (\mathbf{w}_1)^4, R_1, (\mathbf{u}_1)^4, \mathbf{u}_1, H_1', (\mathbf{b}_1)^j, \mathbf{b}_1 \rangle$ and $P_2[\mathbf{w}_2, \mathbf{b}_2] = \langle \mathbf{w}_2, H_2', \mathbf{u}_2, (\mathbf{u}_2)^4, R_2, (\mathbf{b}_2)^4, \mathbf{b}_2 \rangle$. Consequently, P_1 and P_2 are 2-mutually fully independent Hamiltonian paths in $Q_4 - F$. See Fig. 9(a) for illustration.

Subcase 2.2: Suppose $n \ge 5$. We first consider $f_0 \le n - 4$. By the inductive hypothesis, $Q_n^{n,1}$ has $(r_0 - 1)$-mutually fully independent Hamiltonian paths $R_i[(\mathbf{u}_i)^n, (\mathbf{b}_i)^n]$, $2 \le i \le r_0$. Then we can choose an integer j in $\{1, 2, \ldots, n-1\}$ satisfying conditions $(\mathbf{b}_1)^j \ne \mathbf{w}_1$ and $((\mathbf{b}_1)^j)^n \notin \{R_i(2^{n-1} - 1) \mid 2 \le i \le r_0\}$. Since $r_0 = n - 1 - f \le n - 2$, such an integer exists. By Lemma 2, $Q_n^{n,0} - F_0 - \{\mathbf{b}_1\}$ has a Hamiltonian path $H_1[\mathbf{w}_1, (\mathbf{b}_1)^j] = \langle \mathbf{w}_1, \mathbf{u}_1, H_1', (\mathbf{b}_1)^j \rangle$, where $\mathbf{u}_1$ is some vertex adjacent to $\mathbf{w}_1$. By Lemma 3, there exists a Hamiltonian path $R_1[(\mathbf{w}_1)^n, (\mathbf{u}_1)^n]$ in $Q_n^{n,1} - \{(\mathbf{b}_1)^n, ((\mathbf{b}_1)^j)^n\}$. Then we set $P_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, (\mathbf{w}_1)^n, R_1, (\mathbf{u}_1)^n, \mathbf{u}_1, H_1', (\mathbf{b}_1)^j, ((\mathbf{b}_1)^j)^n, (\mathbf{b}_1)^n, \mathbf{b}_1 \rangle$ and $P_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, (\mathbf{u}_i)^n, R_i, (\mathbf{b}_i)^n, \mathbf{b}_i \rangle$ for $2 \le i \le r_0$. As a result, $\{P_1, P_2, \ldots, P_{r_0}\}$ turns out to be a set of r_0-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 9(b).

Next, we consider $f_0 = n - 3$. Thus, we have $r_0 = 2$. By Lemma 1, $Q_n^{n,0} - F_0$ has a Hamiltonian path $H_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, \mathbf{u}_1, H_1', (\mathbf{b}_1)^j, \mathbf{b}_1 \rangle$, where $\mathbf{u}_1$ is a vertex adjacent to $\mathbf{w}_1$, and j is some integer of $\{1, 2, \ldots, n-1\}$. By Lemma 3, there exists a Hamiltonian path $R_1[(\mathbf{w}_1)^n, (\mathbf{u}_1)^n]$ in $Q_n^{n,1} - \{(\mathbf{b}_1)^n, ((\mathbf{b}_1)^j)^n\}$.

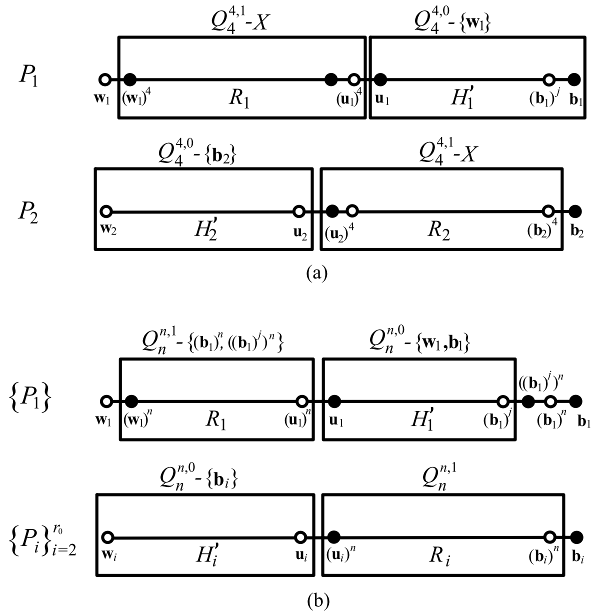

Fig. 9 Illustration for Case 2 of Lemma 6

By the inductive hypothesis, $Q_n^{n,1} - \{((\mathbf{b}_2)^n, ((\mathbf{b}_1)^j)^n)\}$ has a Hamiltonian path $R_2[(\mathbf{u}_2)^n, (\mathbf{b}_2)^n]$. Obviously, we have $R_2(2^{n-1} - 1) \neq ((\mathbf{b}_1)^j)^n$. Again, we set $P_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, (\mathbf{w}_1)^n, R_1, (\mathbf{u}_1)^n, \mathbf{u}_1, H_1', (\mathbf{b}_1)^j, ((\mathbf{b}_1)^j)^n, (\mathbf{b}_1)^n, \mathbf{b}_1\rangle$ and $P_2[\mathbf{w}_2, \mathbf{b}_2] = \langle \mathbf{w}_2, H_2', \mathbf{u}_2, (\mathbf{u}_2)^n, R_2, (\mathbf{b}_2)^n, \mathbf{b}_2\rangle$. Hence, P_1 and P_2 are fully independent Hamiltonian paths in $Q_n - F$. Also see Fig. 9(b).

Case 3: Suppose that $r_i + f_{1-i} = n - 1$ for some $i \in \{0, 1\}$. Without loss of generality, we assume $r_1 + f_0 = n - 1$. Since $r_1 = n - 1 - f_0 \geq n - 1 - f = \delta$, we have $r_1 = \delta$ and $F_0 = F$. By the inductive hypothesis, $Q_n^{n,1}$ has $(r_1 - 1)$-mutually fully independent Hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, \mathbf{b}_i\rangle$, where $\mathbf{u}_i$ is some vertex adjacent to $\mathbf{b}_i$ for $1 \leq i \leq r_1 - 1$. Since $r_1 - 1 = \delta - 1 = n - 2 - f = (n - 1) - 1 - f_0$, $Q_n^{n,0} - F_0$ has $(r_1 - 1)$-mutually fully independent Hamiltonian paths $R_i[(\mathbf{u}_i)^n, (\mathbf{b}_i)^n]$, $1 \leq i \leq r_1 - 1$. Then we set $P_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, (\mathbf{u}_i)^n, R_i, (\mathbf{b}_i)^n, \mathbf{b}_i\rangle$ for $1 \leq i \leq r_1 - 1$. Next, we choose a vertex $\mathbf{v}$ of $V_0(Q_n^{n,0})$ and construct a Hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^n, \mathbf{v}]$ in $Q_n^{n,0} - F_0$ such that $\mathbf{v} \neq R_i(2)$ and $R_{r_1}(2^{n-1} - 1) \neq (\mathbf{u}_i)^n$ for every $1 \leq i \leq r_1 - 1$. How can we do that? We distinguish the following subcases.

Subcase 3.1: Suppose that $n \neq 5$ or $f > 1$. Obviously, vertices $(\mathbf{u}_1)^n, \ldots, (\mathbf{u}_{r_1-1})^n$ have at most $(r_1 - 1)(n - 1)$ neighboring vertices in $Q_n^{n,0}$. Since $|V_0(Q_n^{n,0})| = 2^{n-2} > (r_1 - 1)(n - 1) = (n - 2 - f)(n - 1)$, we can choose a vertex $\mathbf{v}$ other than neighbors of $(\mathbf{u}_1)^n, \ldots, (\mathbf{u}_{r_1-1})^n$. Obviously, we have $\mathbf{v} \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Lemma 1, there exists a Hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^n, \mathbf{v}]$ in $Q_n^{n,0} - F_0$. Because $\mathbf{v}$ is not adjacent to any vertex of $\{(\mathbf{u}_1)^n, \ldots, (\mathbf{u}_{r_1-1})^n\}$, we have $R_{r_1}(2^{n-1} - 1) \neq (\mathbf{u}_i)^n$ for every $1 \leq i \leq r_1 - 1$. By Lemma 2, there exists a Hamiltonian path $H_{r_1}[(\mathbf{v})^n, \mathbf{b}_{r_1}]$ in $Q_n^{n,1} - \{\mathbf{w}_{r_1}\}$. Then we set $P_{r_1} = \langle \mathbf{w}_{r_1}, (\mathbf{w}_{r_1})^n, R_{r_1}, \mathbf{v}, (\mathbf{v})^n, H_{r_1}, \mathbf{b}_{r_1}\rangle$. Consequently,

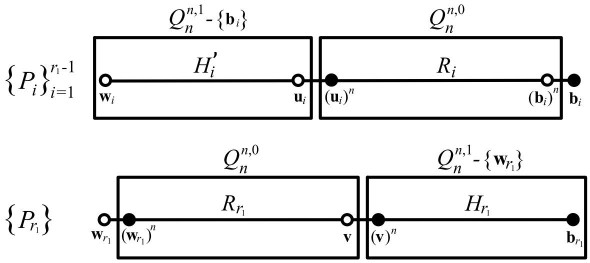

Fig. 10 Illustration for Case 3 of Lemma 6

$\{P_1, P_2, \ldots, P_{r_1}\}$ is a set of r_1-mutually fully independent Hamiltonian paths in $Q_n - F$. Fig. 10 illustrates this subcase.

Subcase 3.2: Suppose that $n = 5$ and $f = 1$. Accordingly, we have $r_1 = 3$.

Condition 3.2.1: Vertices $(\mathbf{u}_1)^n$ and $(\mathbf{u}_2)^n$ have at least one common neighbor. Since $|V_0(Q_n^{n,0})| = 2^{n-2} = 8 > 7 = (r_1 - 1)(n - 1) - 1$, we still can choose a vertex $\mathbf{v}$ from $V_0(Q_n^{n,0})$ other than neighbors of $(\mathbf{u}_1)^n$ and $(\mathbf{u}_2)^n$. Obviously, we have $\mathbf{v} \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Lemma 1, there exists a Hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^n, \mathbf{v}]$ of $Q_n^{n,0} - F_0$ such that $R_{r_1}(2^{n-1} - 1) \neq (\mathbf{u}_i)^n$ for every $1 \leq i \leq r_1 - 1$. By Lemma 2, there exists a Hamiltonian path $H_{r_1}[(\mathbf{v})^n, \mathbf{b}_{r_1}]$ in $Q_n^{n,1} - \{\mathbf{w}_{r_1}\}$. Similarly, we set $P_{r_1} = \langle \mathbf{w}_{r_1}, (\mathbf{w}_{r_1})^n, R_{r_1}, \mathbf{v}, (\mathbf{v})^n, H_{r_1}, \mathbf{b}_{r_1} \rangle$. Then, $\{P_1, P_2, P_3\}$ turns out be a set of 3-mutually fully independent Hamiltonian paths in $Q_n - F$. Also see Fig. 10.

Condition 3.2.2: Vertices $(\mathbf{u}_1)^n$ and $(\mathbf{u}_2)^n$ have no common neighbors. Then we set the vertex $\mathbf{v}$ to be the one adjacent to $(\mathbf{u}_1)^n$ and not identical to $R_1(2)$. Obviously, we have $\mathbf{v} \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Lemma 1, $Q_n^{n,0} - F_0 - \{(\mathbf{v}, (\mathbf{u}_1)^n)\}$ remains Hamiltonian laceable. Thus, there exists a Hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^n, \mathbf{v}]$ of $Q_n^{n,0} - F_0 - \{(\mathbf{v}, (\mathbf{u}_1)^n)\}$ such that $R_{r_1}(2^{n-1} - 1) \neq (\mathbf{u}_i)^n$ for every $1 \leq i \leq r_1 - 1$. By Lemma 2, there exists a Hamiltonian path $H_{r_1}[(\mathbf{v})^n, \mathbf{b}_{r_1}]$ in $Q_n^{n,1} - \{\mathbf{w}_{r_1}\}$. Similarly, we set $P_{r_1} = \langle \mathbf{w}_{r_1}, (\mathbf{w}_{r_1})^n, R_{r_1}, \mathbf{v}, (\mathbf{v})^n, H_{r_1}, \mathbf{b}_{r_1} \rangle$. Then, $\{P_1, P_2, P_3\}$ is a set of 3-mutually fully independent Hamiltonian paths in $Q_n - F$. Also see Fig. 10. □

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