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# Improved inventory models with service level and lead time

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## Abstract

This paper explores the mixed inventory backorder and lost sales problem in which both the lead time and order quantity are treated as decision variables. In a recent paper on *Computers and Operations Research*, Ouyang and Wu considered this problem. However, their algorithms might not find the optimal solution due to flaws in their solution procedure. We develop some lemmas to reveal the parameter effects and then present two complete procedures for finding the optimal solution for the models. The savings are illustrated by solving the same examples from Ouyang and Wu's paper to demonstrate the superiority of our revised algorithms.

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*Keywords:* Inventory; Lead time; Backorder; Lost sales; Distribution-free approach

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## 1. Introduction

Today, the just-in-time (JIT) production system is in vogue. It emphasizes high quality, low stock and short lead time. Many business scholars have focused on “high quality, low stock and short lead time as competitive business goals. The ultimate goal of JIT is a smooth, rapid flow of materials through the system. The idea is to make the process time as short as possible using resources in the best-possible way. Sometimes, the lead time is too long prolonging the process. In this situation lead time reduction is important and a serial improvement objective. Continual improvement can reduce internal production time and satisfy the exterior market. The business service level and competitiveness can then be raised. The issue of lead time reduction in inventory management has thus become a matter of great interest.

Recent studies by scholars on how to control lead time are reviewed in this paper. In the traditional inventory model, as described by Silver and Peterson [1], lead time is considered as a predetermined

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constant or a stochastic parameter. Liao and Shyu [2] stated that lead time is negotiable and can be decomposed into several components, each having a different piecewise linear crash cost function for lead time reduction. Ben-Daya and Raouf [3] extended Liao and Shyu's [2] work to consider both lead time and order quantity as decision variables. Moon and Gallego [4] assumed unfavorable lead time demand distribution and solved both the continuous review and periodic review models with a mixture of backorders and lost sales using the minmax distribution-free approach. Ouyang et al. [5] generalized Ben-Daya and Raouf's [3] assumption that shortages were allowed and constructed variable lead time from a mixed inventory model with backorders and lost sales. Moon and Choi [6] and Lan et al. [7] pointed out the problem in Ouyang et al.'s. [5] method. They found individual optimal order quantities and optimal lead time for a mixed inventory model, and developed a simplified solution procedure. Ouyang and Wu [8] extended the Ouyang et al. [5] article. They relaxed the assumption about the cumulative lead time distribution demand and applied the minimax distribution-free procedure to determine the optimal order quantity and optimal lead time. Wu and Tsai [9] considered that the lead time demands from different customers are not identical. They developed a mixed inventory model with backorders and lost sales for variable lead time demand with a mixed normal distribution. Pan and Hsiao [10] presented inventory models with backorder discount and variable lead time to ensure that customers would be willing to wait for backorders.

This article will study the same inventory model as Ouyang and Wu [11] who considered both lead time and order quantity as decision variables for a mixed inventory model. Ouyang and Wu [11] thought that it is often difficult to determine the stock-out cost value in inventory systems. Therefore, they replaced the stock-out cost with a service level condition. In their paper, first they assumed that the lead time demand follows a normal distribution. They then relaxed the assumption about the lead time distribution demand function and applied the minmax distribution-free procedure to solve the problem. However, there are critical flaws in their solution algorithms under different assumptions. The model studied in this article is the same used by Ouyang and Wu [11]. We will point out the questionable algorithms in their model. Their algorithms are complicated and cannot obtain the optimal solution, as demonstrated by their examples. We will construct correct and efficient algorithms to find the optimum order quantity and reorder point simultaneously when the lead time probability distribution is normal or free. We developed lemmas to reveal the parameter effects and illustrate our improvement by solving the same examples.

## 2. Notation and assumptions

We use the same notations and assumptions as Ouyang and Wu [11].

Notations:

$A$	fixed ordering cost per order
$D$	average demand per year
$h$	inventory holding cost per item per year
$L$	length of lead time
$Q$	order quantity
$X$	the lead time demand which has a distribution function $F$ with finite mean $\mu L$ and standard derivation $\sigma\sqrt{L}$ ( $> 0$ )

- $x^+$  maximum value of  $x$  and 0, i.e.  $x^* = \max\{x, 0\}$
- $\alpha$  proportion of demands that are not met from stock so  $1 - \alpha$  is the service level
- $\beta$  fraction of the demand during the stock-out period that will be backordered
- $C(L)$  Lead time crashing cost
- $EAC(Q, L)$  total expected annual cost

**Assumptions.** (1) The reorder point  $r =$  expected demand during lead time + safety stock ( $SS$ ), and  $SS = k\sigma\sqrt{L}$ , that is,  $r = \mu L + k\sigma\sqrt{L}$ , where  $k$  is the safety factor and satisfies  $P(X > r) = q$ ,  $q$  representing the allowable stock-out probability during  $L$ .

(2)  $B(r) = E[X - r]^+$  is the expected demand shortage at the end of cycle. Hence,  $\beta B(r)$  are the backordered quantities and  $(1 - \beta)B(r)$  are the lost sales. Therefore, the total demand during lead time period equals  $\mu L - (1 - \beta)B(r)$  and the expected net inventory level just before the order arrives is  $r - (\mu L - (1 - \beta)B(r))$ . Moreover, the expected net inventory level at the beginning of the cycle is  $Q + r - (\mu L - (1 - \beta)B(r))$ , so the expected holding cost per cycle is

$$\frac{Q}{D} \left[ \frac{Q}{2} + k\sigma\sqrt{L} + (1 - \beta)B(r) \right].$$

(3) If  $X$  has a normal distribution function  $F(x)$ , according to Ravindran et al. [12], then  $B(r) = \sigma\sqrt{L}\psi(k)$ , where  $\psi(k) = \phi(k) - k[1 - \Phi(k)] > 0$ , and  $\phi, \Phi$  are the standard normal probability density function and distribution function, respectively.

(4) Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point  $r$ .

(5) The lead time  $L$  has  $n$  mutually independent components. The  $i$ th component has a minimum duration  $a_i$ , and normal duration  $b_i$ , and a crash cost per unit time  $c_i$ . Further, we assume that  $c_1 \leq c_2 \leq \dots \leq c_n$ .

(6) The lead time components are crashed one at a time starting with the least  $c_i$  component and so on.

(7) If we let  $L_0 = \sum_{j=1}^n b_j$  and  $L_i$  be the length of lead time with components  $1, 2, \dots, i$  crash to their minimum durations, then  $L_i = \sum_{j=i+1}^n b_j + \sum_{j=1}^i a_j$ . The lead time crash cost  $C(L)$  per cycle for a given  $L \in [L_i, L_{i-1}]$ , is given by  $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$ .

### 3. Normal distribution model

The total expected annual cost is the sum of the ordering cost, holding cost and lead time crash cost, subject to a service level constraint. Hence, the problem is

$$\text{Min } EAC(Q, L) = A \frac{D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} + (1 - \beta)B(r) \right] + \frac{D}{Q} C(L) \tag{1}$$

$$\text{s.t. } \frac{B(r)}{Q} \leq \alpha, \tag{2}$$

where  $0 < Q < \infty$  and  $L \in [L_i, L_{i-1}]$  for  $i = 1, 2, \dots, n$ .

*3.1. Review of normal distribution model of Ouyang and Wu*

Ouyang and Wu considered that the lead time demand  $X$  has a normal distribution function  $F(x)$ . They then solved the following problem:

$$\text{Min } EAC(Q, L) = A \frac{D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} + (1 - \beta)\sigma\sqrt{L}\psi(k) \right] + \frac{D}{Q} C(L) \tag{3}$$

$$\text{s.t. } \frac{\sigma\sqrt{L}\psi(k)}{Q} \leq \alpha, \tag{4}$$

where  $0 < Q < \infty$  and  $L \in [L_i, L_{i-1}]$  for  $i = 1, 2, \dots, n$ .

They ignored the service level constraint and took the partial derivatives of  $EAC(Q, L)$  with respect to  $Q$  and  $L$  in each time interval  $(L_i, L_{i-1})$ , and they found

$$\frac{\partial}{\partial Q} EAC(Q, L) = \frac{h}{2} - \frac{AD}{Q^2} - \frac{D}{Q^2} C(L), \tag{5}$$

$$\frac{\partial}{\partial L} EAC(Q, L) = \frac{hk\sigma}{2\sqrt{L}} + \frac{h(1 - \beta)\sigma\psi(k)}{2\sqrt{L}} - c_i \frac{D}{Q}, \tag{6}$$

$$\frac{\partial^2}{\partial Q^2} EAC(Q, L) = \frac{2AD}{Q^3} + \frac{2D}{Q^3} C(L) > 0 \tag{7}$$

and

$$\frac{\partial^2}{\partial L^2} EAC(Q, L) = - \left[ \frac{hk\sigma}{4\sqrt{L}^3} + \frac{h(1 - \beta)\sigma\psi(k)}{4\sqrt{L}^3} \right] < 0. \tag{8}$$

Ouyang and Wu derived that for fixed  $L \in (L_i, L_{i-1})$ ,  $EAC(Q, L)$  is convex in  $Q$  and for fixed  $Q$ ,  $EAC(Q, L)$  is concave in  $L \in (L_i, L_{i-1})$ . Setting Eq. (5) to zero and solving for  $Q$ , they obtained

$$Q = \left[ \frac{2D}{h} [A + C(L)] \right]^{1/2}. \tag{9}$$

Ouyang and Wu [11] established an iterative algorithm to find the optimal lead time and optimal order quantity. To save space, we do not quote their algorithm, but directly offer the improved algorithm.

*3.2. Modification for normal distribution model*

First, we review the normal distribution model and then point out the error in the Ouyang and Wu algorithm [11]. From Eq. (8), we know that for fixed  $Q$ ,  $EAC(Q, L)$  is concave in  $L \in [L_i, L_{i-1}]$ . Hence, the problem is reduced to consider

$$\text{Min } EAC(Q, L_i) = A \frac{D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L_i} + (1 - \beta)\sigma\sqrt{L_i}\psi(k) \right] + \frac{D}{Q} C(L_i), \tag{10}$$

Table 1  
Lead time data

Lead time component, $i$	Normal duration, $b_i$ (days)	Minimum duration, $a_i$ (days)	$b_i - a_i$ (weeks)	Unit crashing cost, $c_i$ (\$/week)
1	20	6	2	2.8
2	20	6	2	8.4
3	16	9	1	35

where

$$\frac{\sigma}{\alpha} \sqrt{L_i} \psi(k) \leq Q \quad \text{and} \quad i = 0, 1, \dots, n. \tag{11}$$

Using Eq. (7), for a given  $i = 0, 1, \dots, n$ ,  $EAC(Q, L_i)$  is convex in  $[(\sigma/\alpha)\sqrt{L_i}\psi(k), \infty)$ . From Eq. (9), we know that

$$Q_i = \left[ \frac{2D}{h} [A + C(L_i)] \right]^{1/2} \tag{12}$$

is the minimum point without the constraint in Eq. (4). Hence, we have that (a) if  $(\sigma/\alpha)\sqrt{L_i}\psi(k) \leq Q_i$ , then  $Q_i$  is the local minimum of  $EAC(Q, L_i)$  and (b) if  $(\sigma/\alpha)\sqrt{L_i}\psi(k) > Q_i$ , then  $(\sigma/\alpha)\sqrt{L_i}\psi(k)$  is the local minimum of  $EAC(Q, L_i)$ .

Therefore, the optimal order quantity of  $EAC(Q, L_i)$  for  $Q \in [(\frac{\sigma}{\alpha}\sqrt{L_i}\psi(k), \infty)$  is

$$\max \left\{ Q_i, \frac{\sigma}{\alpha} \sqrt{L_i} \psi(k) \right\}. \tag{13}$$

### 3.3. Modified algorithm for normal distribution model

Now, we offer our improved algorithm.

- (1) *Step 1:* For each  $L_i$ ,  $i = 0, 1, \dots, n$ , compute  $Q_i$ , using Eq. (12).
- (2) *Step 2:* Let  $x_i = \max\{Q_i, (\sigma/\alpha)\sqrt{L_i}\psi(k)\}$  for  $i = 0, 1, \dots, n$ .
- (3) *Step 3:* If  $EAC(x_s, L_s) = \min_{i=0,1,\dots,n} EAC(x_i, L_i)$ , then the optimal solution is  $(x_s, L_s)$ .

From our algorithm, this inventory model always has feasible solutions. Conversely, in Step 5 of Algorithm 1 by Ouyang and Wu [11], they predicted that this inventory model sometimes did not have feasible solutions. Hence, their algorithm involves errors.

### 3.4. Numerical example

We use Example 1 from Ouyang and Wu [11] with the following data:  $D=600$  unit/year,  $A=\$200$  per order,  $h=\$20$ ,  $\sigma=7$  units/week,  $\beta=1$ , the service level  $1 - \alpha=0.985$ ,  $q=0.2$  (in this situation, the safety factor  $k=0.845$ ) and the lead time has three components with the data shown in Table 1. The results from our algorithm are listed in Table 3 and we have the optimal quantity  $Q^* = 125.9663$  units, optimal lead time  $L^* = 6$  and the minimum total expected annual cost  $EAC(Q^*, L^*) = \$2528.75$ .

Table 2

Summary of the optimal solution from Ouyang and Wu [11] ( $L_i$  in week)

$i$	$L_i$	$C(L_i)$	$B(r)$	$Q_i$	$EAC(Q_i, L_i)$	$B(r)/Q_i$
0	8	0	2.1818	110	2525.21	0.0198
1	6	5.6	1.8895	111	2511.13	0.0170
2	4	22.4	1.5428	116	2546.94	0.0133
3	3	57.4	1.3361	124	2690.39	0.0108

Table 3

Summary of the optimal solution using the proposed method ( $L_i$  in week)

$i$	$\frac{B(r)}{x} = \frac{\sigma}{x} \sqrt{L_i} \psi(k)$	$Q_i$	$x_i$	$EAC(x_i, L_i)$
0	145.4533	109.5445	145.4533	2614.14
1	125.9663	111.0675	125.9663	2528.75
2	102.8510	115.5162	115.5162	2546.94
3	89.0716	124.2739	124.2739	2690.39

We reproduced Table 2 from Ouyang and Wu [11] as our Table 3. Even though  $EAC(Q_1, L_1)$  is the minimum among  $EAC(Q_i, L_i)$  for  $i = 0, 1, 2, 3$ , however,  $B(r)/Q_1 = 0.017 > \alpha = 0.015$ , so  $Q_1$  and  $L_1$  are not feasible solutions under the service level constraint. They obtained the optimal quantity  $Q^* = 116$  units; optimal lead time  $L^* = 4$  and the minimum total expected annual cost  $EAC(Q^*, L^*) = \$2546.94$ . Therefore, using our algorithm, we have saved \$18.19.

#### 4. Distribution-free model

The information about the form of the probability distribution of lead time is often limited in practice. This makes the probability distribution of lead time not easy to know. We can derive mean and standard derivation from some related information. For facing this kind of problem, many scholars have used minmax distribution-free approach to get optimal order quantity and optimal lead time in some references and had good outcomes.

##### 4.1. Review of Distribution-free model from Ouyang and Wu

Ouyang and Wu [11] used a minimax distribution-free procedure to solve the inventory model with distribution function  $F(x)$  of lead time demand that has mean  $\mu L$ , standard derivation  $\sigma \sqrt{L}$  and unknown probability distribution. According to Gallego and Moon [13] and assumption (1)  $r - \mu L = k\sigma \sqrt{L}$ , they asserted that

$$B(r) \leq \frac{\sigma}{2} (\sqrt{1+k^2} - k) \sqrt{L}. \quad (14)$$

Ouyang and Wu [11] changed the problem in Eq. (3) to consider

$$\text{Min } EAC(Q, L) = A \frac{D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} + \frac{(1-\beta)\sigma}{2} \sqrt{L}(\sqrt{1+k^2}-k) \right] + \frac{D}{Q} C(L) \quad (15)$$

$$\text{s.t. } \frac{1}{2Q} \sigma\sqrt{L}(\sqrt{1+k^2}-k) \leq \alpha, \quad (16)$$

where  $0 < Q < \infty$  and  $L \in [L_i, L_{i-1}]$  for  $i = 1, 2, \dots, n$ .

They ignored the service level constraint and used the first and second partial derivative of  $EAC(Q, L)$  with respect to  $Q$  and  $L$ . Comparing Eqs. (3) and (15), the only difference is  $\psi(k)$  and  $(\sqrt{1+k^2}-k)/2$ . Hence, the monotonic and concave properties of  $EAC(Q, L)$  in Eq. (3) still hold for  $EAC(Q, L)$  in Eq. (15).

Using Proposition 2 from Ouyang and Wu [11], they derived that

$$q \equiv P(X > r) \leq \frac{1}{1+k^2}. \quad (17)$$

Hence, they obtained the range for  $k$  as  $0 \leq k \leq \sqrt{q^{-1}-1}$ . Moreover, they selected a number, say  $N$ , that is large enough to partition the interval  $[0, \sqrt{q^{-1}-1}]$  into  $N$  equal subintervals and let  $k_j = (j/N)\sqrt{q^{-1}-1}$  for  $j = 0, 1, \dots, N$ , and they then established an algorithm to obtain a suitable  $k_j$  and the optimal  $Q$  and  $L$ . To save space, their algorithm is not repeated here.

#### 4.2. Modification for Distribution-free model

Ouyang and Wu [11] could not manage the problem for  $0 \leq k \leq \sqrt{q^{-1}-1}$ . They used the discrete numbers  $k_j = (j/N)\sqrt{q^{-1}-1}$  for  $j = 0, 1, \dots, N$ . However, this is not trivial work to choose a proper number for  $N$ . In Ouyang and Wu [11], they took  $N = 200$ ; however, this number is not big enough to insure the existence of a minimum solution.

Now, we consider the distribution-free model as follows:

$$\text{Min } EAC(Q, L, k) = A \frac{D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} + \frac{(1-\beta)\sigma}{2} \sqrt{L}(\sqrt{1+k^2}-k) \right] + \frac{D}{Q} C(L) \quad (18)$$

$$\text{s.t. } \frac{1}{2\alpha} \sigma\sqrt{L}(\sqrt{1+k^2}-k) \leq Q, \quad (19)$$

$$\text{and } 0 \leq k \leq \sqrt{q^{-1}-1} \quad (20)$$

with  $L \in [L_i, L_{i-1}]$  for  $i = 1, 2, \dots, n$ .

Similar to Eq. (8), we find

$$\frac{\partial^2}{\partial L^2} EAC(Q, L, k) = \frac{-h\sigma}{4\sqrt{L}^3} \left[ k + \frac{(1-\beta)}{2} (\sqrt{1+k^2}-k) \right] < 0.$$

We reduce the problem to the following:

$$\text{Min } EAC(Q, L_i, k) = A \frac{D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L_i} + \frac{(1-\beta)\sigma}{2} \sqrt{L_i}(\sqrt{1+k^2}-k) \right] + \frac{D}{Q} C(L_i) \quad (21)$$

$$\text{s.t. } \frac{1}{2\alpha} \sigma \sqrt{L_i} (\sqrt{1+k^2} - k) \leq Q, \tag{22}$$

$$\text{and } 0 \leq k \leq \sqrt{q^{-1} - 1} \tag{23}$$

with  $i = 0, 1, \dots, n$ .

For a given  $i$ , we obtain the partial derivatives with respect to  $Q$  and  $k$  as follows:

$$\frac{\partial}{\partial k} EAC(Q, L_i, k) = h\sigma \sqrt{L_i} \left[ 1 + \frac{1-\beta}{2} \left( \frac{k}{\sqrt{1+k^2}} - 1 \right) \right], \tag{24}$$

$$\frac{\partial^2}{\partial k^2} EAC(Q, L_i, k) = \frac{(1-\beta)h\sigma \sqrt{L_i}}{2\sqrt{(1+k^2)^3}} > 0, \tag{25}$$

$$\frac{\partial}{\partial Q} EAC(Q, L_i, k) = \frac{h}{2} - \frac{D[A + C(L_i)]}{Q^2} \tag{26}$$

and

$$\frac{\partial^2}{\partial Q^2} EAC(Q, L_i, k) = \frac{2D[A + C(L_i)]}{Q^3} > 0. \tag{27}$$

Since  $\beta$  is the backordered fraction, we have  $0 \leq \beta \leq 1$ . Moreover, using the inequality  $-1 \leq k/\sqrt{1+k^2} - 1 < 0$ , it shows

$$-1/2 \leq \frac{1-\beta}{2} \left( \frac{k}{\sqrt{1+k^2}} - 1 \right) \leq 0.$$

Hence,

$$\frac{\partial}{\partial k} EAC(Q, L_i, k) > 0 \tag{28}$$

First, we solve Eq. (26), and then we know that  $(\partial/\partial Q)EAC(Q, L_i, k) = 0$  occurs at  $\sqrt{(2D/h)[A + C(L_i)]}$ . To simplify the notation, we assume

$$a_i = \sqrt{\frac{2D}{h} [A + C(L_i)]}. \tag{29}$$

Here, we consider constraint (22). Let

$$g(k) = \frac{\sigma \sqrt{L_i}}{2\alpha} (\sqrt{1+k^2} - k). \tag{30}$$

Furthermore, we assume that for a fixed  $k$ ,

$$Q^*(k) = \text{min point of } Q \text{ for } EAC(Q, L_i, k) \text{ subject to } g(k) \leq Q. \tag{31}$$

From

$$g'(k) = \frac{\sigma \sqrt{L_i}}{2\alpha} \left( \frac{k}{\sqrt{1+k^2}} - 1 \right) < 0 \text{ and } g''(k) = \frac{\sigma \sqrt{L_i}}{2\alpha \sqrt{(1+k^2)^3}} > 0,$$



we get that  $g(k)$  decreases and is concave up for  $k \in [0, \sqrt{q^{-1} - 1}]$ . Now, we derive several lemmas to develop our theorem. Based on Eq. (28),  $EAC(Q, L_i, k)$  is convex in  $Q$ . Therefore, we have the following Lemma 1.

**Lemma 1.** For a number  $k$  with  $k \in [0, \sqrt{q^{-1} - 1}]$ , we have  $Q^*(k) = \max\{g(k), a_i\}$ .

We compare  $g(k)$  with  $a_i$ . Solving  $g(k) = a_i$ , we obtain  $k = \sigma\sqrt{L_i}/4a_i\alpha - a_i\alpha/\sigma\sqrt{L_i}$ . To simplify the expression, we assume

$$b_i = \frac{\sigma\sqrt{L_i}}{4a_i\alpha} - \frac{a_i\alpha}{\sigma\sqrt{L_i}}, \tag{32}$$

that is, we have  $g(b_i) = a_i$ .

In the sequel, we distinguish among three cases as follows:

Case 1:  $0 \leq b_i \leq \sqrt{q^{-1} - 1}$ . Case 2:  $b_i < 0$ . Case 3:  $b_i > \sqrt{q^{-1} - 1}$ .

**Lemma 2.** For Case 1, we show that the minimum value of  $EAC(Q, L_i, k)$ , subject to  $g(k) \leq Q$  and  $b_i \leq k \leq \sqrt{q^{-1} - 1}$ , occurs at  $k = b_i$  and  $Q = a_i$ .

**Proof.** Since  $g(k)$  decreases for  $k \in [b_i, \sqrt{q^{-1} - 1}]$  then  $\max\{g(k), a_i\} = a_i$ . Using Lemma 1, to solve  $\min EAC(Q, L_i, k)$ , subject to  $g(k) \leq Q$  and  $b_i \leq k \leq \sqrt{q^{-1} - 1}$ , is equivalent to solving  $\min EAC(Q = a_i, L_i, k)$ , subject to  $b_i \leq k \leq \sqrt{q^{-1} - 1}$ . Using Eq. (28),  $EAC(Q = a_i, L_i, k)$  increases on  $k$ , so minimum value occurs at  $k = b_i$ .  $\square$

**Lemma 3.** For Case 1, we prove that the optimal point  $(Q^*, k^*)$  attaining the minimum value of  $EAC(Q, L_i, k)$ , subject to  $g(k) \leq Q$  and  $0 \leq k \leq b_i$ , can be divided into the following three scenarios:

- (1) If  $d_i \geq 0$  then  $k^* = 0$  and  $Q^* = g(0)$ .
- (2) If  $d_i < 0$  and  $c_i \leq -d_i$  then  $k^* = b_i$  and  $Q^* = g(b_i) = a_i$ .
- (3) If  $d_i < 0$  and  $c_i > -d_i$  then  $k^* = \min\{-d_i/\sqrt{c_i^2 - d_i^2}, b_i\}$  and  $Q^* = g(k^*)$ , with

$$c_i = \frac{2D\alpha}{\sigma\sqrt{L_i}} [A + C(L_i)] + h\sigma\sqrt{L_i} \left( \frac{1}{4\alpha} + \frac{1 - \beta}{2} \right),$$

and

$$d_i = \frac{2D\alpha}{\sigma\sqrt{L_i}} [A + C(L_i)] + h\sigma\sqrt{L_i} \left( 1 - \frac{1}{4\alpha} - \frac{1 - \beta}{2} \right).$$

**Proof.** When  $k \in [0, b_i]$ ,  $g(k)$  decreases, then  $\max\{g(k), a_i\} = g(k)$ . Using Lemma 1, to solve  $\min EAC(Q, L_i, k)$  under  $g(k) \leq Q$  and  $0 \leq k \leq b_i$  is equivalent to solving  $\min EAC(Q = g(k), L_i, k)$ , subject to  $0 \leq k \leq b_i$ . If we assume

$$c_i = \frac{2D\alpha}{\sigma\sqrt{L_i}} [A + C(L_i)] + h\sigma\sqrt{L_i} \left( \frac{1}{4\alpha} + \frac{1 - \beta}{2} \right)$$

and

$$d_i = \frac{2D\alpha}{\sigma\sqrt{L_i}} [A + C(L_i)] + h\sigma\sqrt{L_i} \left( 1 - \frac{1}{4\alpha} - \frac{1 - \beta}{2} \right)$$

then we can rewrite  $EAC(Q = g(k), L_i, k)$  as

$$c_i\sqrt{1+k^2} + d_ik. \tag{33}$$

Motivated by Eq. (33), we assume

$$h(k) = c_i\sqrt{1+k^2} + d_ik. \tag{34}$$

We have  $h'(k) = c_ik/\sqrt{1+k^2} + d_i$  and  $h''(k) = -c_i/\sqrt{(1+k^2)^3} > 0$ . Consequently, we divide this into three scenarios:

- (1) If  $d_i \geq 0$ , it shows  $h'(k) > 0$  and  $h(k)$  increases so  $k^* = 0$  and  $Q^* = g(0)$ .
- (2) If  $d_i < 0$  and  $c_i \leq -d_i$ , then  $h'(k) < 0$  and  $h(k)$  decreases so  $k^* = b_i$  and  $Q^* = g(b_i) = a_i$ .
- (3) If  $d_i < 0$  and  $c_i > -d_i$ , solving  $h'(k) = 0$  then  $k = -d_i/\sqrt{c_i^2 - d_i^2}$ . We have  $h''(k) > 0$ , so we know  $k^* = \min\{-d_i/\sqrt{c_i^2 - d_i^2}, b_i\}$  and  $Q^* = g(k^*)$ .  $\square$

Combining Lemmas 2 and 3, we get the following:

**Lemma 4.** *For Case 1 with  $0 \leq b_i \leq \sqrt{q^{-1} - 1}$ , we show that*

- (1) *If  $d_i \geq 0$ , then  $k^* = 0$  and  $Q^* = g(0)$ .*
- (2) *If  $d_i < 0$  and  $c_i \leq -d_i$ , then  $k^* = b_i$  and  $Q^* = g(b_i) = a_i$ .*
- (3) *If  $d_i < 0$  and  $c_i > -d_i$ , then  $k^* = \min\{-d_i/\sqrt{c_i^2 - d_i^2}, b_i\}$  and  $Q^* = g(k^*)$ .*

**Lemma 5.** *For Case 2, when  $b_i < 0$ , then  $k^* = 0$  and  $Q^* = g(b_i) = a_i$ .*

**Proof.** Eq. (32) shows that  $1 < 2a_i\alpha/\sigma\sqrt{L_i}$ .

Since  $\sqrt{1+k^2} - k = 1/(\sqrt{1+k^2} + k) < 1$  we get that  $g(k) < a_i$  and  $\max\{g(k), a_i\} = a_i$  for  $k \in [0, \sqrt{q^{-1} - 1}]$ . Based on Lemma 1,  $Q^* = a_i$  for  $k \in [0, \sqrt{q^{-1} - 1}]$ . Using Eq. (28), we derive  $k^* = 0$ .  $\square$

**Lemma 6.** *For Case 3, when  $b_i > \sqrt{q^{-1} - 1}$ , we have the same results as Lemma 3.*

**Proof.** Using Eq. (30),  $g(k)$  decreases. Recalling that  $g(b_i) = a_i$ , for  $k \in [0, \sqrt{q^{-1} - 1}]$ , we obtain  $g(k) > a_i$ . Based on Lemma 1, with  $k \in [0, \sqrt{q^{-1} - 1}]$ , then  $Q^*(k) = g(k)$ . Therefore, to solve  $\min EAC(Q, L_i, k)$  subject to  $g(k) \leq Q$  and  $0 \leq k \leq \sqrt{q^{-1} - 1}$  is equivalent to solving  $\min EAC(Q = g(k), L_i, k)$ , subject to  $0 \leq k \leq \sqrt{q^{-1} - 1}$ . Thus, for Case 3, we have the same results as Lemma 3.  $\square$

Finally, putting Lemmas 4–6 together, we derive the main theorem.

**Theorem.** *We prove that*

- (1) *If  $b_i < 0$ , then  $k^* = 0$  and  $Q^* = g(b_i) = a_i$ .*
- (2) *If  $0 \leq b_i$  and  $d_i \geq 0$ , then  $k^* = 0$  and  $Q^* = g(0)$ .*

- (3) If  $0 \leq b_i, d_i < 0$  and  $c_i \leq -d_i$ , then  $k^* = b_i$  and  $Q^* = g(b_i) = a_i$ .
- (4) If  $0 \leq b_i, d_i < 0$  and  $c_i > -d_i$ , then  $k^* = \min\{-d_i/\sqrt{c_i^2 - d_i^2}, b_i\}$  and  $Q^* = g(k^*)$ .

#### 4.3. Modified algorithm for Distribution-free model

Based on our Theorem, a revised and simplified algorithm is produced.

(1) Step 1: Let

$$a_i = \sqrt{\frac{2D}{h} [A + C(L_i)]}, \quad b_i = \frac{\sigma\sqrt{L_i}}{4a_i\alpha} - \frac{a_i\alpha}{\sigma\sqrt{L_i}}, \quad g(k) = \frac{\sigma\sqrt{L_i}}{2\alpha} (\sqrt{1 + k^2} - k),$$

$$y_i = \frac{2D\alpha}{\sigma\sqrt{L_i}} [A + C(L_i)], \quad z_i = h\sigma\sqrt{L_i} \left( \frac{1}{4\alpha} + \frac{1 - \beta}{2} \right), \quad c_i = y_i + z_i$$

and  $d_i = y_i + h\sigma\sqrt{L_i} - z_i$ .

- (2) Step 2: If  $b_i < 0$ , then  $k^* = 0$  and  $Q^* = a_i$ .
- (3) Step 3: If  $0 \leq b_i$  and  $d_i \geq 0$ , then  $k^* = 0$  and  $Q^* = g(0)$ .
- (4) Step 4: If  $0 \leq b_i, d_i < 0$  and  $c_i \leq -d_i$ , then  $k^* = b_i$  and  $Q^* = a_i$ .
- (5) Step 5: If  $0 \leq b_i, d_i < 0$  and  $c_i > -d_i$ , then  $k^* = \min\{-d_i/\sqrt{c_i^2 - d_i^2}, b_i\}$  and  $Q^* = g(k^*)$ .

#### 4.4. Numerical example

Example 2 from Ouyang and Wu [11] is used as our second example. The data are shown in Example 1 except that the probability distribution for the lead time demand is free. They solved the problem, when  $q = 0.2$  and took  $N = 200$ . Using their algorithm, constraint (16) does not hold for  $i = 0, 1$ . Hence, they did not have minimum solutions for  $i = 0, 1$ . We quote their results for  $i = 2, 3$  in Table 4. Ouyang and Wu [11] accepted that the optimal order quantity  $Q^* = 116$ , optimal lead time  $L^* = 4$  weeks, the suitable safety factor  $k^* = 1.89$  and the minimum total expected annual cost  $EAC(Q^*, L^*) = \$2839.06$ .

Based on our algorithm, the results are listed in Table 5. We have  $Q^* = 143.1506$ ,  $L^* = 4$ ,  $k^* = 1.4766$  and  $EAC(Q^*, L^*) = \$2777.12$ .

Comparing Tables 4 and 5, we know that for  $i = 0, 1$ , the algorithm by Ouyang and Wu [11] cannot find any solutions. Likewise, for  $i = 2, 3$ , the algorithm by Ouyang and Wu [11] cannot find the minimum solutions. Therefore, we may conclude that the Algorithm by Ouyang and Wu [11] cannot attain the optimal value.

Table 4  
Summary of the optimal solution from Ouyang and Wu [11]

$i$	$L_i$	$C(L_i)$	$Q_i$	$k_{s(i)}$	$EAC(Q_i, L_i)$	$\sigma\sqrt{L_i}(\sqrt{1 + k_{s(i)}^2} - k_{s(i)})/2Q_i$
2	4	22.4	116	1.89	2839.06	0.015
3	3	57.4	124	1.48	2843.62	0.015

Table 5  
Summary of the optimal solution from the proposed method

$i$	$a_i$	$b_i$	$c_i$	$d_i$	$-d_i/\sqrt{c_i^2 - d_i^2}$	$k^*$	$Q^*$	$EAC(Q^*, k^*)$
0	109.5445	2.9293	6781	-6022	1.9309	1.9309	160.7542	3118.63
1	111.0675	2.4758	5931	-5157	1.7596	1.7596	151.0649	2930.66
2	115.5162	1.8962	4953	-4101	1.4766	1.4766	143.1506	2777.12
3	124.2739	1.4723	4424	-3417	1.2162	1.2162	144.8213	2809.53

## 5. Conclusion

Good inventory management is often the mark of a well-run organization. Inventory levels must be planned carefully to balance the costs and reasonable levels for good customer service. Controlling lead time properly and taking the optimal order quantity are very important in attaining the minimum total expected annual cost. In the above discussions we pointed out two questionable procedures in the paper by Ouyang and Wu [11]. We offered two simplified algorithms to replace the algorithms by Ouyang and Wu. Their methods are too complicated and lose control of the computation procedure. Our refined algorithms are easy to use and mathematically sound.

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