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An inequality for regular near polygons

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Abstract

Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ. We show

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}.$$

We show the following (i)–(iii) are equivalent. (i) Equality is attained above; (ii) Γ is Q-polynomial with respect to θ_1; (iii) Γ is a dual polar graph or a Hamming graph.

MSC: 05E30

Keywords: Near polygon; Distance-regular graph; Q-polynomial; Dual polar graph; Hamming graph

1. Introduction

Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$ (see Section 2 for formal definitions). Suppose the intersection numbers $a_1 > 0$ and $c_2 > 1$. It was shown by Brouwer, Cohen and Neumaier that if Γ has classical parameters $(d, q, 0, \beta)$ then Γ is a Hamming graph or a dual polar graph [2, Theorem 9.4.4]. The same conclusion was obtained by the second author under the assumption that Γ is Q-polynomial and has

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diameter $d \geq 4$ [10, Corollary 5.7]. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ denote the eigenvalues of Γ. It is known that $\theta_0 = k$, where k denotes the valency of Γ. By [2, Proposition 4.4.6(i)],

$$\theta_d \geq -\frac{k}{a_1 + 1},$$

with equality if and only if Γ is a near $2d$-gon. We now state our result.

Theorem 1.1. *Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k, and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ. Then*

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{1.1}$$

Moreover, the following (i)–(iii) *are equivalent.*

(i) *Equality is attained in* (1.1);

(ii) *Γ is Q-polynomial with respect to θ_1;*

(iii) *Γ is a dual polar graph or a Hamming graph.*

2. Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [1] or Brouwer et al. [2] for more background information.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂ and diameter $d := \max\{\partial(x, y) \mid x, y \in X\}$. For $x \in X$ and for all integers i, set

$$\Gamma_i(x) := \{y \mid y \in X, \partial(x, y) = i\}.$$

Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever $|\Gamma_1(x)| = k$ for all $x \in X$. Pick an integer i $(0 \leq i \leq d)$. For $x \in X$ and for $y \in \Gamma_i(x)$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y), \tag{2.1}$$

$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y), \tag{2.2}$$

$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y). \tag{2.3}$$

The graph Γ is said to be *distance-regular* whenever for all integers i $(0 \leq i \leq d)$, and for all $x, y \in X$ with $\partial(x, y) = i$, the numbers

$$c_i := |C(x, y)|, \qquad a_i := |A(x, y)|, \qquad b_i := |B(x, y)| \tag{2.4}$$

are independent of x and y. We call the c_i, a_i, b_i the *intersection numbers* of Γ. We observe $c_0 = 0, a_0 = 0, b_d = 0$ and $c_1 = 1$. For the rest of this paper we assume Γ is distance-regular with diameter $d \geq 3$. We observe Γ is regular with valence $k = b_0$ and that [2, p. 126]

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq d). \tag{2.5}$$

We recall the Bose–Mesner algebra of Γ. For $0 \leq i \leq d$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ which has xy entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the *ith distance matrix* of Γ. Observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{d} A_i = J$; (aiii) $A_i^t = A_i$ $(0 \leq i \leq d)$; (aiv) there exist constants p_{ij}^h $(0 \leq i, j \leq d)$ such that $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$, where I denotes the identity matrix and J denotes the all ones matrix. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. Let $\mathbf{M}$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by A. Using (ai)–(aiv) we find $A_0, A_1, \ldots, A_d$ form a basis of $\mathbf{M}$. We call $\mathbf{M}$ the *Bose–Mesner algebra* of Γ. By [1, p. 59, 64], $\mathbf{M}$ has a second basis $E_0, E_1, \ldots, E_d$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $\sum_{i=0}^{d} E_i = I$; (eiii) $E_i^t = E_i$ $(0 \leq i \leq d)$; (eiv) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq d)$. We call $E_0, E_1, \ldots, E_d$ the *primitive idempotents* for Γ. Since $E_0, E_1, \ldots, E_d$ form a basis for $\mathbf{M}$ there exist real scalars $\theta_0, \theta_1, \ldots, \theta_d$ such that $A = \sum_{i=0}^{d} \theta_i E_i$. By this and (eiv) we find $AE_i = \theta_i E_i$ $(0 \leq i \leq d)$. Observe $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct since A generates $\mathbf{M}$. We assume the E_i are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_d$. We call θ_i the *eigenvalue* of Γ corresponding to E_i. By [1, p. 197] we have $\theta_0 = k$ and $-k \leq \theta_i \leq k$ $(0 \leq i \leq d)$. We call θ_0 the *trivial* eigenvalue.

Let θ denote an eigenvalue of Γ and let E denote the corresponding primitive idempotent. Since $E \in \mathbf{M}$, there exist real numbers $\sigma_0, \sigma_1, \ldots, \sigma_d$ such that

$$E = m|X|^{-1} \sum_{i=0}^{d} \sigma_i A_i, \tag{2.6}$$

where $m = \mathrm{rank}\, E$. We have $\sigma_0 = 1$ and

$$c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i \qquad (0 \leq i \leq d), \tag{2.7}$$

where $\sigma_{-1}, \sigma_{d+1}$ denote indeterminates [1, p. 191]. The sequence $\sigma_0, \sigma_1, \ldots, \sigma_d$ is called the *cosine sequence* associated with θ. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with the eigenvalue k. Comparing (2.5) and (2.7) we find $\sigma_i = 1$ $(0 \leq i \leq d)$. By the *trivial cosine sequence* of Γ we mean the cosine sequence associated with k. Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the corresponding cosine sequence. By (2.7),

$$\sigma_1 = \theta k^{-1}, \tag{2.8}$$

$$\sigma_2 = \frac{\theta^2 - a_1 \theta - k}{k b_1}. \tag{2.9}$$

Combining (2.5), (2.8) and (2.9) we find

$$(\sigma_1 - \sigma_2) b_1 = (\theta + 1)(\sigma_0 - \sigma_1). \tag{2.10}$$

Set $V = \mathbb{R}^X$ (column vectors). We define the inner product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V).$$

For each $x \in X$ set

$$\hat{x} = (0, 0, \ldots, 1, 0, \ldots, 0)^t,$$

where the 1 is in coordinate x. We observe $\{\hat{x} \mid x \in X\}$ is an orthonormal basis for V. By (2.6), for $x, y \in X$ we have

$$\langle E\hat{x}, E\hat{y}\rangle = m|X|^{-1}\sigma_i, \tag{2.11}$$

where $i = \partial(x, y)$.

By a *clique* in Γ we mean a nonempty set consisting of mutually adjacent vertices of Γ. A given clique in Γ is said to be *maximal* whenever it is not properly contained in a clique. The graph Γ is said to be a *near polygon* whenever

(i) Each maximal clique has cardinality $a_1 + 2$.
(ii) For all maximal cliques ℓ and for all $x \in X$, either

  (iia) $\partial(x, y) = d$ for all $y \in \ell$, or
  (iib) there exists an integer i $(0 \le i \le d-1)$ and a unique $z \in \ell$ such that $\partial(x, z) = i$ and $\partial(x, y) = i + 1$ for all $y \in \ell - \{z\}$.

We give an alternate description of a near polygon. Let $K_{1,2,1}$ denote the graph with 4 vertices s, x, y, s' such that $\partial(s, x) = \partial(s, y) = \partial(x, y) = \partial(x, s') = \partial(y, s') = 1$ and $\partial(s, s') = 2$. Then by [2, Theorem 6.4.1] Γ is a near polygon if and only if both the following (i′)–(ii′) hold.

(i′) Γ does not contain an induced $K_{1,2,1}$ subgraph;
(ii′)

$$a_i = a_1 c_i \qquad (0 \le i \le d-1). \tag{2.12}$$

Assume Γ is a near polygon. Then

$$a_d \ge a_1 c_d. \tag{2.13}$$

Moreover $a_d = a_1 c_d$ if and only if no maximal clique satisfies (iia) above [2, Theorem 6.4.1]. In this case we call Γ a *near* $2d$*-gon*. Otherwise we call Γ a *near* $(2d+1)$*-gon*. Assume Γ is a near polygon. The Hoffman bound states that

$$\theta_d \ge -\frac{k}{a_1 + 1}, \tag{2.14}$$

with equality if and only if Γ is a near $2d$-gon [2, Proposition 4.4.6(i)].

Definition 2.1. Let Γ denote a distance-regular graph with diameter $d \ge 3$. We say Γ has *classical parameters* (d, q, α, β) whenever the intersection numbers are given by

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (1 \le i \le d), \tag{2.15}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \le i \le d), \tag{2.16}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}. \tag{2.17}$$

We give two examples of near polygon distance-regular graphs with classical parameters (d, q, α, β).

Example 2.2 (The Hamming Graph $H(d, n)$ $(d \geq 3, n \geq 2)$ [4–6, 8]). X is the set of all d-tuples of elements from the set $\{1, 2, \ldots, n\}$, $xy \in R$ iff x, y differ in exactly 1 coordinate $(x, y \in X)$,

$$q = 1, \qquad \alpha = 0, \qquad \beta = n - 1,$$
$$c_i = i, \qquad b_i = (d - i)(n - 1), \qquad a_i = (n - 2)i \qquad (0 \leq i \leq d),$$
$$\theta_i = (d - i)(n - 1) - i \qquad (0 \leq i \leq d).$$

Example 2.3 (The Dual Polar Graphs [3, 7]). Let U denote a finite vector space with one of the following nondegenerate forms:

name	$\dim(U)$	field	form	ϵ
$B_d(p^n)$	$2d + 1$	$GF(p^n)$	quadratic	1
$C_d(p^n)$	$2d$	$GF(p^n)$	symplectic	1
$D_d(p^n)$	$2d$	$GF(p^n)$	quadratic (Witt index d)	0
${}^2D_{d+1}(p^n)$	$2d + 2$	$GF(p^n)$	quadratic (Witt index d)	2
${}^2A_{2d}(p^n)$	$2d + 1$	$GF(p^{2n})$	Hermitean	$\frac{3}{2}$
${}^2A_{2d-1}(p^n)$	$2d$	$GF(p^{2n})$	Hermitean	$\frac{1}{2}$

where $d \geq 3$, p is prime and $n \in \mathbb{N} \backslash \{0\}$.

A subspace of U is called *isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is d.

X is the set all maximal isotropic subspaces of U,

$$xy \in R \text{ iff } \dim(x \cap y) = d - 1 \qquad (x, y \in X),$$
$$\alpha = 0, \qquad \beta = q^{\epsilon},$$
$$c_i = \frac{q^i - 1}{q - 1}, \qquad a_i = \frac{q^{i+\epsilon} - q^i - q^{\epsilon} + 1}{q - 1} \qquad (0 \leq i \leq d),$$
$$b_i = \frac{q^{i+\epsilon}(q^{d-i} - 1)}{q - 1} \qquad (0 \leq i \leq d - 1),$$
$$\theta_i = \frac{q^{d+\epsilon-i} - q^{\epsilon} - q^i + 1}{q - 1} \qquad (0 \leq i \leq d),$$

where

$$q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n} \text{ respectively.}$$

Note that the dual polar graphs on $B_d(p^n)$ and $C_d(p^n)$ are isomorphic if and only if p is equal to 2 [2, p. 277].

The following three theorems will be used in the proof of our results.

Theorem 2.4 ([9, Theorem 4.1]). *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let q denote a real number at least 1. Then the following conditions* (i), (ii) *are equivalent.*

(i) *Γ has a nontrivial cosine sequence $\sigma_0, \sigma_1, \ldots, \sigma_d$ such that $\sigma_{i-1} - q\sigma_i$ is independent of i $(1 \leq i \leq d)$.*
(ii) *The intersection numbers of Γ are such that $qc_i - b_i - q(qc_{i-1} - b_{i-1})$ is independent of i $(1 \leq i \leq d)$.*

Furthermore, if (i), (ii) *hold, then*

$$c_3 \geq (c_2 - q)(1 + q + q^2). \tag{2.18}$$

Theorem 2.5 ([9, Theorem 4.2]). *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let q denote a real number at least 1. Then the following conditions* (i), (ii) *are equivalent.*

(i) *Statements* (i), (ii) *hold in Theorem 2.4, and $c_3 = (c_2 - q)(1 + q + q^2)$.*
(ii) *There exists $\alpha, \beta \in \mathbb{R}$ such that Γ has classical parameters (d, q, α, β).*

Theorem 2.6 ([2, Theorem 9.4.4]). *Let Γ denote a distance-regular graph with diameter $d \geq 3$ with classical parameters $(d, q, 0, \beta)$. Assume the intersection numbers $a_1 > 0$ and $c_2 > 1$. Suppose Γ is a near polygon. Then Γ is a dual polar graph or a Hamming graph.*

3. The inequality

In this section we obtain the inequality in Theorem 1.1.

Lemma 3.1. *Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k, and intersection numbers $a_1 > 0, c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ. Then*

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{3.1}$$

Proof. Abbreviate $E = E_1$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with θ_1. Fix any two vertices $x, y \in X$ with $\partial(x, y) = 2$. We consider the vectors

$$u = \sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w}, \tag{3.2}$$

$$v = E\hat{x} - E\hat{y}. \tag{3.3}$$

By the Cauchy-Schwartz inequality,

$$\|u\|^2 \|v\|^2 \geq \langle u, v \rangle^2. \tag{3.4}$$

We compute the terms in (3.4). Using (2.11), (3.2) and (3.3) we find

$$\|v\|^2 = 2m|X|^{-1}(\sigma_0 - \sigma_2), \tag{3.5}$$

$$\langle u, v\rangle = 2ma_2|X|^{-1}(\sigma_1 - \sigma_2). \tag{3.6}$$

We now compute $\|u\|^2$. To do this we first discuss the distances between vertices in $A(x, y)$ and vertices in $A(y, x)$. We claim that for all $z \in A(x, y)$, z is adjacent to $c_2 - 1$ vertices in $A(y, x)$ and is at distance 2 from the remaining $a_2 - c_2 + 1$ vertices in $A(y, x)$. To see this fix $z \in A(x, y)$. Then $\ell := A(x, z) \cup \{x, z\}$ is a maximal clique; hence there exists a unique vertex $s \in \ell$ with $\partial(s, y) = 1$. That is $s \in C(x, y) \cap C(z, y)$. Observe $|C(x, y) \cap C(z, y)| = 1$, since any other $s' \in C(x, y) \cap C(z, y)$ will cause either $xss'y$ or $sxzs'$ to be a $K_{1,2,1}$ subgraph. Hence there are $c_2 - 1$ vertices in $C(z, y) \cap A(y, x)$. Observe for $w \in A(y, x)$ we have $\partial(w, x) = 2$ and $\partial(w, s) \le 2$ so $\partial(w, z) \le 2$. We have now proved the claim. Using the claim and applying (2.11) we find

$$\begin{aligned}\|u\|^2 &= \Bigg\|\sum_{z\in A(x,y)} E\hat{z}\Bigg\|^2 + \Bigg\|\sum_{w\in A(y,x)} E\hat{w}\Bigg\|^2 - 2\Bigg\langle \sum_{z\in A(x,y)} E\hat{z}, \sum_{w\in A(y,x)} E\hat{w}\Bigg\rangle \\ &= 2ma_2|X|^{-1}(\sigma_0 + (a_1 - c_2)\sigma_1 + (c_2 - a_1 - 1)\sigma_2). \end{aligned} \tag{3.7}$$

Evaluating (3.4) using (3.5)–(3.7) we routinely find

$$(\sigma_0 + (a_1 - c_2)\sigma_1 + (c_2 - a_1 - 1)\sigma_2)(\sigma_0 - \sigma_2) \ge a_2(\sigma_1 - \sigma_2)^2. \tag{3.8}$$

Evaluating (3.8) using (2.8), (2.9) and (2.12) we obtain

$$(\theta_1 - k)^2(\theta_1(a_1 + 1) + k)(k - \theta_1(c_2 - 1) - a_1 - c_2) \ge 0. \tag{3.9}$$

Clearly $(\theta_1 - k)^2 > 0$. By (2.14) and since $\theta_1 > \theta_d$ we find $\theta_1(a_1 + 1) + k > 0$. Evaluating (3.9) using these comments we find

$$k - \theta_1(c_2 - 1) - a_1 - c_2 \ge 0$$

and (3.1) follows. □

Remark 3.2. Referring to Examples 2.2 and 2.3, the eigenvalue θ_1 satisfies (3.1) with equality.

We comment on the proof of Lemma 3.1.

Lemma 3.3. *With the notation of Lemma* 3.1, *the following* (i)–(iii) *are equivalent.*

(i) *Equality is attained in* (3.1).

(ii) *For all* $x, y \in X$ *such that* $\partial(x, y) = 2$,

$$\sum_{z\in A(x,y)} E\hat{z} - \sum_{w\in A(y,x)} E\hat{w} \in \mathrm{Span}(E\hat{x} - E\hat{y}). \tag{3.10}$$

(iii) *There exist* $x, y \in X$ *such that* $\partial(x, y) = 2$ *and*

$$\sum_{z\in A(x,y)} E\hat{z} - \sum_{w\in A(y,x)} E\hat{w} \in \mathrm{Span}(E\hat{x} - E\hat{y}). \tag{3.11}$$

Here $E = E_1$.

Proof. Observe from the proof of Lemma 3.1 that equality is attained in (3.1) if and only if equality is attained in (3.4). We claim $v \neq 0$. This will follow from (3.5) provided we can show $\sigma_0 \neq \sigma_2$. Suppose $\sigma_0 = \sigma_2$. Setting $\theta = \theta_1$ and $\sigma_2 = \sigma_0$ in (2.10) and simplifying the result we find $\theta_1 = -b_1 - 1$. This is inconsistent with (2.14) and $\theta_1 > \theta_d$. We have now shown $\sigma_0 \neq \sigma_2$ and it follows $v \neq 0$. We now see that equality is attained in (3.4) if and only if $u \in \mathrm{Span}(v)$. The result follows. □

4. The case of equality

In this section we consider the case of equality in (3.1).

Lemma 4.1. *Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 > 0, c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the corresponding cosine sequence. Suppose equality holds in (3.1). Then $\sigma_{i-1} - q\sigma_i$ is independent of i $(1 \leq i \leq d)$, where $q = c_2 - 1$.*

Proof. Setting $c_2 = q + 1$ in (3.1) and using $k - a_1 - 1 = b_1$ we find $\theta_1 + 1 = b_1 q^{-1}$. In particular $\theta_1 \neq -1$. Observe $\sigma_1 \neq \sigma_2$; otherwise $\sigma_0 = \sigma_1$ by (2.10) forcing $\theta_1 = k$ by (2.8), a contradiction. Evaluating (2.10) using $\theta_1 + 1 = b_1 q^{-1}$ we find

$$\frac{\sigma_0 - \sigma_1}{\sigma_1 - \sigma_2} = q. \tag{4.1}$$

Fix two vertices $x, y \in X$ with $\partial(x, y) = 2$. Abbreviate $E = E_1$. By Lemma 3.3 there exists $\lambda \in \mathbb{R}$ such that

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} = \lambda(E\hat{x} - E\hat{y}). \tag{4.2}$$

Fix an integer i $(1 \leq i \leq d - 1)$ and pick $u \in X$ with $\partial(u, x) = i - 1$ and $\partial(u, y) = i + 1$. Taking the inner product of $E\hat{u}$ with both sides of (4.2) and using the fact that Γ is a near polygon, we find

$$a_2(\sigma_i - \sigma_{i+1}) = \lambda(\sigma_{i-1} - \sigma_{i+1}). \tag{4.3}$$

Setting $i = 1$ in (4.3) we find $a_2(\sigma_1 - \sigma_2) = \lambda(\sigma_0 - \sigma_2)$. From (4.1) we find $\sigma_0 - \sigma_2 = (\sigma_1 - \sigma_2)(1 + q)$. By these comments $\lambda = a_2/(q + 1)$. Evaluating (4.3) using this we find

$$\sigma_{i-1} - q\sigma_i = \sigma_i - q\sigma_{i+1} \qquad (1 \leq i \leq d - 1).$$

From this we find $\sigma_{i-1} - q\sigma_i$ is independent of i for $1 \leq i \leq d$. □

Lemma 4.2. *Let Γ denote a near polygon distance-regular graph with $d \geq 3$ and intersection numbers $a_1 > 0, c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ and assume equality holds in (3.1). Then Γ has classical parameters $(d, q, 0, \beta)$.*

Proof. Let the scalar q be as in Lemma 4.1. By Lemma 4.1 we have Theorem 2.4(i) and hence Theorem 2.4(ii). Applying Theorem 2.4(ii) with $i = 2, 3$ we find

$$qc_2 - b_2 - q(qc_1 - b_1) = qc_3 - b_3 - q(qc_2 - b_2). \tag{4.4}$$

Simplifying (4.4) using (2.5) and $c_2 = q + 1$, $a_2 = a_1 c_2$ we obtain

$$(a_1 + 1 + q)(1 + q + q^2 - c_3) = a_3 - a_1 c_3. \tag{4.5}$$

By (2.12) we have $a_3 = a_1 c_3$ if $d > 3$, and by (2.13) we have $a_3 \geq a_1 c_3$ if $d = 3$. In any case $a_3 \geq a_1 c_3$ so the right-hand side of (4.5) is nonnegative. Also $a_1 + 1 + q > 0$ since $q = c_2 - 1$. Evaluating (4.5) using these comments we find

$$c_3 \leq 1 + q + q^2. \tag{4.6}$$

By (2.18) and using $c_2 = 1 + q$ we find $c_3 \geq 1 + q + q^2$. Now $c_3 = 1 + q + q^2$ and so $c_3 = (c_2 - q)(1 + q + q^2)$. Applying Theorem 2.5 we find there exist real numbers α, β such that Γ has classical parameters (d, q, α, β). By (2.15) we find $c_2 = (1 + q)(1 + \alpha)$. By the construction $c_2 = q + 1$. Comparing these equations we find $\alpha = 0$. □

Proof of Theorem 1.1. The inequality (1.1) is from (3.1).

(i) $\Longrightarrow$ (iii). By Lemma 4.2, Γ has classical parameters $(d, q, 0, \beta)$. By this and Theorem 2.6 we find Γ is a dual polar graph or a Hamming graph.

(iii) $\Longrightarrow$ (ii). This is immediate from [2, Corollary 8.5.3].

(ii) $\Longrightarrow$ (i). Lemma 3.3(ii) holds by [9, Theorem 3.3], so Lemma 3.3(i) holds and the result follows.

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