QUALITY AND RELIABILITY ENGINEERING INTERNATIONAL
Qual. Reliab. Engng. Int. 2005; **21**:29–42
Published online 15 November 2004 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/qre.605

Research

Capability Testing Based on C_{pm} with Multiple Samples

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Numerous process capability indices have been proposed in the manufacturing industry to provide unitless measures on process performance, which are effective tools for quality improvement and assurance. Most existing methods for capability testing are based on the distribution frequency approaches. Recently, Bayesian approaches have been proposed for testing capability indices C_p and C_{pm} but restricted to cases with one single sample. In this paper, we consider estimating and testing capability index C_{pm} based on multiple samples. We propose accordingly a Bayesian procedure for testing C_{pm}. Based on the Bayesian procedure, we develop a simple but practical procedure for practitioners to use in determining whether their manufacturing processes are capable of reproducing products satisfying the preset capability requirement. A process is capable if all the points in the credible interval are greater than the pre-specified capability level. To make the proposed Bayesian approach practical for in-plant applications, we tabulate the minimum values of $C^*(p)$ for which the posterior probability p reaches various desirable confidence levels.

KEY WORDS: process capability indices; Bayesian approach; credible interval; unbiased estimator; posterior probability; multiple samples

1. INTRODUCTION

Process capability indices, which establish the relationships between the actual process performance and the manufacturing specifications, have been the focus of recent research in quality assurance and capability analysis literature. Those capability indices, quantifying process precision, process accuracy, and process performance, are important for any production improvement activities and quality program implementation. The first process capability index appearing in the literature is the precision index C_p, which is defined by Kane[1] as

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

where USL is the upper specification limit, LSL is the lower specification limit, and σ is the process standard deviation. The numerator of C_p gives the range over which the process measurements are predefined.

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Received 15 February 2003
Revised 12 May 2003

The denominator gives the range over which the process is actually varying. The index C_{p} was designed to measure the magnitude of the overall process variation relative to the manufacturing tolerance, and is used for processes that are normally distributed, and are in statistical control. Clearly, the index only measures the precision of a process (product quality consistency), and does not take into account whether the process is centered.

In order to reflect the impact of the deviation of the process mean μ from the center point M of the specification limits on the process capability, several indices have been proposed, including

$$C_{\mathrm{PU}} = \frac{\mathrm{USL} - \mu}{3\sigma}, \quad C_{\mathrm{PL}} = \frac{\mu - \mathrm{LSL}}{3\sigma}$$

$$C_{\mathrm{pk}} = \min\left\{\frac{\mathrm{USL} - \mu}{3\sigma}, \frac{\mu - \mathrm{LSL}}{3\sigma}\right\}$$

However, both C_{p} and C_{pk} are independent of the target value T. Neither C_{p} nor C_{pk} takes the closeness of the process output to the target value T (on-target issues) into consideration. Taking the targeting as well as the process spread into consideration, a modification of C_{p} incorporating the Taguchi loss function, which has been referred to as C_{pm}, is introduced independently by Hsiang and Taguchi[2] and Chan *et al.*[3]. The process capability index C_{pm} is defined as

$$C_{\mathrm{pm}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\tau} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}$$

where $\mathrm{USL} - \mathrm{LSL}$ is the allowable tolerance range of the process, d is the half-interval length, and τ is the measure of the average product deviation from the target value T. The term $\tau^2 = \sigma^2 + (\mu - T)^2 = E[(X - T)^2]$ incorporates two variation components: (1) variation to the process mean and (2) deviation of the process mean from the target. The process capability index C_{pm}, sometimes called the Taguchi index, emphasizes the ability of clustering around the target, which therefore reflects the degrees of process targeting. The capability index C_{pm} is not primarily designed to provide an exact measure of the number of conforming items, i.e. the process yield. However, we note that $E[(X - T)^2]$ is the expected loss, where the process loss of a characteristic X missing the target is often assumed to be well approximated by the symmetric squared error loss function, loss $(X) = (X - T)^2$. Hence, the capability index C_{pm} may be termed as a loss-based index. The indices C_{p} and C_{pk} have been referred to as the first-generation process capability indices, and the index C_{pm} is often called the second-generation process capability indices.

Pearn *et al.*[4] proposed the process capability index C_{pmk}, which combines the merits of the three earlier indices C_{p}, C_{pk} and C_{pm}. The index C_{pmk} alerts the user if the process variance increases and/or the process mean deviates from its target value. The index C_{pmk}, referred to as the third-generation process capability index, has been defined as follows:

$$C_{\mathrm{pmk}} = \min\left\{\frac{\mathrm{USL} - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - \mathrm{LSL}}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}$$

Note that the indices presented above are designed to monitor the performance for only normal and near-normal processes with symmetric tolerances. These indices have been shown to be inappropriate for cases with asymmetric tolerances. In practice, the process mean μ and the process variance σ^2 are unknown. In order to calculate the index value, sample data must be collected and a great degree of uncertainty may be introduced into the capability assessments due to sampling error. The approach of simply looking at the calculated values of the estimated indices and then making a decision on whether the given process is capable is highly unreliable, since this ignores the sampling error. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their sampling distributions, learning that capability measures must be reported in confidence intervals or via capability testing. Statistical properties of the estimators of those indices under various process conditions have been investigated extensively, see Chan *et al.*[3], Pearn *et al.*[4], Bordignon and Scagliarini[5], Borges and Ho[6], Chang *et al.*[7], Hoffman[8], Nahar *et al.*[9], Noorossana[10], Pearn *et al.*[11], Pearn and Lin[12], and Zimmer *et al.*[13]. Kotz and Johnson[14] presented a thorough review for the development of process capability indices in the past 10 years.

Existing research for capability testing has focused on the traditional frequency approaches. However, the sampling distributions are usually so complicated that this makes establishing the exact confidence interval very difficult. An alternative is to consider the Bayesian approach where we can specify a prior distribution for the parameter of interest, obtain the posterior distribution for the parameter, and then make inferences about the parameter using its posterior distribution given the observations. It is not difficult to obtain the posterior distribution when a prior distribution is given, even when the form of the posterior distribution is complicated, as one could always use numerical methods or Monte Carlo methods (Kalos and Whitlock[15]) to obtain an approximate point estimate or interval estimate. This is the advantage of the Bayesian approach over the traditional distribution frequency approach.

2. *ESTIMATION OF C_{pm}*

The process mean μ and the process variance σ^2 must be estimated form the sample. Thus, the estimated index $\hat{C}_{pm}$ is obtained by replacing μ and σ^2 by their estimators. Chan *et al.*[3] and Boyles[16] proposed two different estimators of C_{pm} respectively defined as follows:

$$\hat{C}_{pm(CCS)} = \frac{d}{3\sqrt{s^2 + (\bar{x} - T)^2}} \quad \text{and} \quad \hat{C}_{pm(B)} = \frac{d}{3\sqrt{s_n^2 + (n/(n-1))(\bar{x} - T)^2}}$$

where

$$\bar{x} = \sum_{i=1}^{n} x_i/n, \quad s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1) \quad \text{and} \quad s_n^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/n$$

In fact, the two estimators, $\hat{C}_{pm(CCS)}$ and $\hat{C}_{pm(B)}$, are asymptotically equivalent. Note that $\bar{x}$ and s_n^2 are the maximum likelihood estimators (MLEs) of μ and σ^2, respectively. Hence, the estimated $\hat{C}_{pm(B)}$ is the MLE of C_{pm}. Further, the term $s_n^2 + (\bar{x} - T)^2$ in the denominator of $\hat{C}_{pm(B)}$ is the uniformly minimum variance unbiased estimator (UMVUE) of the term $\sigma^2 + (\mu - T)^2$ in the denominator of C_{pm}, where

$$s_n^2 + (\bar{x} - T)^2 = \sum_{i=1}^{n} (x_i - T)^2/n \quad \text{and} \quad \tau^2 = \sigma^2 + (\mu - T)^2 = E[(x - T)^2]$$

Therefore, for reliability purposes, it is reasonable to use $\hat{C}_{pm(B)}$.

Under the assumption of normality, Kotz and Johnson[17] obtained the rth moment, and calculated the first two moments, the mean, and the variance of $\hat{C}_{pm}$. Zimmer and Hubele[18] provided tables of exact percentiles for the sampling distribution of the estimator $\hat{C}_{pm}$. Zimmer *et al.*[19] proposed a graphical procedure to obtain exact confidence intervals for C_{pm}, where the parameter $\xi = (\mu - T)/\sigma$ is assumed to be a known constant. Using a method similar to that presented in Vännman[20], Lin and Pearn[21] obtained an exact form of the cumulative distribution function of $\hat{C}_{pm}$. Under the assumption of normality, the cumulative distribution function of $\hat{C}_{pm}$ can be expressed in terms of a mixture of the chi-square distribution and the normal distribution, for $x > 0$, where $b = d/\sigma$, $\xi = (\mu - T)/\sigma$, $G(\cdot)$ is the cumulative distribution function of the chi-square distribution χ^2_{n-1}, and $\phi(\cdot)$ is the probability density function of the standard normal distribution $N(0, 1)$.

$$F_{\hat{C}_{pm}}(x) = 1 - \int_0^{b\sqrt{n}/(3x)} G\left(\frac{b^2 n}{9x^2} - t^2\right) [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]\, dt$$

2.1. Estimation of C_{pm} for multiple samples

For a single sample, Boyles[16] showed that $\hat{\tau}^2 = s_n^2 + (\bar{x} - T)^2$ is the unbiased estimator of $\sigma^2 + (\mu - T)^2$. Therefore, for cases where the data are collected as multiple samples, we consider m samples each of size n_i and suggest the following estimator of C_{pm}, where $\bar{x}_i$ is the ith sample mean, and s_i is the ith sample standard deviation:

$$\hat{C}^*_{pm} = \frac{d}{3\hat{\tau}'^2}, \quad \hat{\tau}'^2 = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - T)^2}{\sum_{i=1}^m n_i} \tag{1}$$

First, by taking the expectation of the numerator of $\hat{\tau}'^2$, we obtain

$$\begin{aligned} E\left(\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - T)^2\right) &= E\left(\sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij}^2\right) - 2T \times E\left(\sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij}\right) + E\left(\sum_{i=1}^m \sum_{j=1}^{n_i} T^2\right) \\ &= \sum_{i=1}^m \sum_{j=1}^{n_i} E(x_{ij}^2) - 2T \times \sum_{i=1}^m \sum_{j=1}^{n_i} E(x_{ij}) + \sum_{i=1}^m n_i T^2 \\ &= \sum_{i=1}^m n_i(\mu^2 + \sigma^2) - 2T \times \sum_{i=1}^m n_i \mu + \sum_{i=1}^m n_i T^2 \\ &= \sum_{i=1}^m n_i[\sigma^2 + (\mu - T)^2] \end{aligned}$$

Thus, the estimator $\hat{\tau}'^2$, such that $E(\hat{\tau}'^2) = \sigma^2 + (\mu - T)^2$, is the unbiased estimator of $\sigma^2 + (\mu - T)^2$. However, for multiple control samples, we need to consider the variation between and within multiple samples. Thus, we define the ratio of total within sample variation (SSW) and total sum of square variation (SST) as

$$\gamma = \frac{\text{SSW}}{\text{SST}} = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2}{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2} = \frac{\sum_{i=1}^m (n_i - 1)s_p^2}{\sum_{i=1}^m (n_i - 1)s_p^2 + \sum_{i=1}^m n_i(\bar{x}_i - \bar{\bar{x}})^2} \tag{2}$$

where

$$s_p^2 = \sum_{i=1}^m (n_i - 1)s_i^2 \Big/ \sum_{i=1}^m (n_i - 1)$$

is the pooled variance of these samples. The total sample variation about target value T can be decomposed as

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - T)^2 &= \sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 + \sum_{i=1}^m \sum_{j=1}^{n_i} (\bar{x}_i - \bar{\bar{x}})^2 + \sum_{i=1}^m \sum_{j=1}^{n_i} (\bar{\bar{x}} - T)^2 \\ &= \sum_{i=1}^m (n_i - 1)s_p^2 + \frac{1-\gamma}{\gamma} \sum_{i=1}^m (n_i - 1)s_p^2 + \sum_{i=1}^m n_i \delta^2 s_p^2 \\ &= \left(\frac{1}{\gamma} \sum_{i=1}^m (n_i - 1) + \sum_{i=1}^m n_i \delta^2\right) s_p^2 \end{aligned}$$

Thus, the generation of the estimator of C_{pm} for multiple samples defined in (1) can be rewritten as

$$\hat{C}^*_{pm} = \frac{d}{3s_p\sqrt{\sum_{i=1}^m (n_i - 1)/(\gamma \sum_{i=1}^m n_i) + \delta^2}}, \quad \delta = \frac{|\bar{\bar{x}} - T|}{s_p} \tag{3}$$

For the single sample, that is, $m = 1$, $\gamma = 1$, and $s_p = s$, the estimator of C_{pm}, $\hat{C}^*_{pm} = d/(3s\sqrt{(n-1)/n + \delta^2})$, which can be reduced to the estimator $\hat{C}_{pm}$ defined in Boyles[16].

3. BAYESIAN APPROACH FOR TESTING C_{pm}

A Bayesian procedure for assessing process capability was proposed in Cheng and Spiring[22] for the index C_p under the assumption that the process mean μ is equal to the target value T. However, the restriction of $\mu = T$ is not a practical assumption for many industrial applications. Shiau *et al.*[23] proposed a Bayesian procedure for the general situation without the restriction on the process mean. However, the research work focused on cases with one single sample. A common practice of process capability estimation in the manufacturing industry is to first implement a daily-based or weekly-based sample data collection plan for monitoring/controlling the process stability, then to analyze the past 'in control' data. Therefore, it is practical to develop a procedure for assessing process capability for cases with multiple samples. In addition, for practitioners' convenience, we provide a simple but practical procedure for computing the posterior probability.

A $100p\%$ credible interval is the Bayesian analogue of the classical $100p\%$ confidence interval, where p is the confidence level for the interval. The credible interval covers $100p\%$ of the posterior distribution of the parameter[24]. Assuming that the m samples are random samples taken from an independent and identically distributed (i.i.d.) normal distribution with mean μ and variance σ^2 ($N(\mu, \sigma^2)$). The measures of the ith sample $x_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ with sample size n_i. Then, the likelihood function for μ and σ is

$$L(\mu, \sigma|\mathbf{x}) = (2\pi\sigma^2)^{-\sum_{i=1}^m n_i/2} \exp\left\{-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \mu)^2}{2\sigma^2}\right\}$$

The first step for the Bayesian approach is to find an appropriate prior. Usually, when there is little or no prior information, or there is only one parameter, one of the most widely used non-informative priors is the so-called reference prior, which is a non-informative prior that maximizes the difference between information (entropy) on the parameter provided by the prior and by the posterior. In other words, the reference prior allows the prior to provide as little information as possible about the parameter (see Bernardo and Smith[25] for more details). Therefore, in this paper we adopt the following non-informative reference prior:

$$\pi(\mu, \sigma) = 1/\sigma, \quad 0 < \sigma < \infty$$

3.1. *Posterior probability*

The posterior probability density function (PDF) of (μ, σ), $f(\mu, \sigma|x)$, may be expressed as follows:

$$f(\mu, \sigma|\mathbf{x}) \propto L(\mu, \sigma|\mathbf{x}) \times \pi(\mu, \sigma) \propto \sigma^{-(\sum_{i=1}^m n_i + 1)} \exp\left\{-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \mu)^2}{2\sigma^2}\right\}$$

since

$$\int_0^\infty \int_{-\infty}^\infty \sigma^{-(\sum_{i=1}^m n_i + 1)} \exp\left\{-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \mu)^2}{2\sigma^2}\right\} \mathrm{d}\mu\, \mathrm{d}\sigma$$
$$= \int_0^\infty \sigma^{-(\sum_{i=1}^m n_i + 1)} \exp\left(-\frac{1}{\beta\sigma^2}\right)$$
$$\times \left[\int_{-\infty}^\infty \exp\left(-\frac{\sum_{i=1}^m n_i(\mu - \bar{\bar{x}})^2}{2\sigma^2}\right) \mathrm{d}\mu\right] \mathrm{d}\sigma = \sqrt{\frac{\pi}{2\sum_{i=1}^m n_i}}\Gamma(\alpha)\beta^\alpha$$

In order to satisfy the integration property, the probability over PDF is 1, so that

$$f(\mu,\sigma|\mathbf{x}) = \frac{2\sqrt{\sum_{i=1}^{m} n_i}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^{\alpha}}\sigma^{-(\sum_{i=1}^{m} n_i+1)} \exp\left(-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right)$$

$$\alpha = \left(\sum_{i=1}^{m} n_i - 1\right)/2, \quad \beta = \left[\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2 \Big/ 2\right]^{-1} \tag{4}$$

$$\bar{\bar{x}} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij} \Big/ \sum_{i=1}^{m} n_i = \sum_{i=1}^{m} n_i \bar{x}_i \Big/ \sum_{i=1}^{m} n_i \quad \text{for } -\infty < \mu < \infty,\ 0 < \sigma < \infty$$

As we mentioned earlier, it is natural to consider the quantity $p = \Pr\{\text{process is capable}|\mathbf{x}\}$ in the Bayesian approach. Since the index C_{pm} is the focus in this paper, we are interested in finding the posterior probability $p = \Pr\{C_{pm} > \omega|\mathbf{x}\}$ for some fixed positive number ω. Therefore, given a pre-specified precision level $\omega > 0$; the posterior probability based on index C_{pm} that a process is capable, is given as the following, where $\Phi(\cdot)$ is the cumulative distribution of the standard normal distribution, and γ and δ are defined as in (2) and (3):

$$p = \Pr\{C_{pm} > \omega|\mathbf{x}\} = \int_0^t \frac{1}{\Gamma(\alpha)y^{\alpha+1}} \exp\left(-\frac{1}{y}\right)[\Phi(b_1(y)+b_2(y)) - \Phi(b_1(y)-b_2(y))]\,\mathrm{d}y$$

$$t = \frac{2}{\sum_{i=1}^{m}(n_i-1)}\left(\frac{\hat{C}_{pm}^*}{\omega}\right)^2\left(\frac{\sum_{i=1}^{m}(n_i-1)}{\sum_{i=1}^{m} n_i} + \gamma\delta^2\right)$$

$$b_1(y) = \sqrt{\frac{2\gamma\sum_{i=1}^{m} n_i}{\sum_{i=1}^{m}(n_i-1)y}}\,\delta \tag{5}$$

$$b_2(y) = \sqrt{\sum_{i=1}^{m} n_i}\left(\frac{t}{y} - 1\right)^{1/2}$$

The derivations of (5) are given in Appendix A. Note that the posterior probability p depends on m, n_i, γ, ω and $\hat{C}_{pm}^*$ only through m, n_i, γ, δ and $\hat{C}_{pm}^*/\omega$. C^* is denoted by $C^* = \hat{C}_{pm}^*/\omega$. From expression (5) we can see that it is rather complicated to compute p without advanced computer programming skills. However, by noticing that there is a one-to-one correspondence between p and C^* when m and n_i are given, and the fact that γ, δ and $\hat{C}_{pm}^*$ can be calculated from the process data, we find that the minimum value of C^* required to ensure the posterior probability p reaching a certain desirable level can be useful in assessing process capability. This minimum value is denoted by $C^*(p)$. Thus, the value $C^*(p)$ satisfies

$$p = \Pr\{C_{pm} > \omega|\mathbf{x}\} = \Pr\left\{C_{pm} > \frac{\hat{C}_{pm}^*}{C^*(p)}\middle|\mathbf{x}\right\}$$

4. *THE TEST PROCEDURE*

A $100p\%$ credible interval for C_{pm} is $[\hat{C}_{pm}^*/C^*(p), \infty)$, where p is a number between 0 and 1, say 0.95, for a 95% confidence interval. This means that the posterior probability that the credible interval contains C_{pm} is p.

In our Bayesian approach we say that the process is capable in a Bayesian sense if all the points in this credible interval are greater than a pre-specified value of ω, say 1.00 or 1.33. When this happens, we have $p = \Pr\{C_{pm} > \omega|\mathbf{x}\}$. In other words, to see if a process is capable (with capability level ω and confidence level p), we only need to check if $\hat{C}^*_{pm} > C^*(p) \times \omega$.

Therefore, for users' convenience in applying our Bayesian procedure, we tabulate the minimum values of $C^*(p)$ for various values of $\gamma = 0.7(0.1)1.0$ and $\delta = 0(0.5)2.0$ with $n = 5(5)20$, $m = 2(2)10$ in Table I(a)–(d) and Table II(a)–(d) for $p = 0.95, 0.99$, respectively. For example, if $\omega = 1.33$ is the minimum capability requirement, then for $p = 0.95$, with $m = 10$ of each sample size $n_i = n = 15$ and $\gamma = 0.9$, $\delta = 0.5$ we can find $C^*(p) = 1.1082$ from Table I(c). Thus, the minimum value $\hat{C}^*_{pm}$ required for the process to be capable is $C^*(p) \times \omega = 1.1082 \times 1.33 = 1.4739$. That is, if $\hat{C}^*_{pm}$ is greater than 1.4739, we say that the process is capable in a Bayesian sense. The computer program for computing the required minimum values of $C^*(p)$ is available from the authors.

From these tables we observe that for each fixed p, m, n and γ the value of $C^*(p)$ decreases as δ increases. This phenomenon can be explained by the relationship of $\hat{C}^*_{pm}$ in (3). For a fixed $\hat{C}^*_{pm}$, s_p becomes smaller when δ becomes larger, and a smaller s_p means that it is plausible that the underlying process is tighter (i.e. with smaller σ). Since the estimation is usually more accurate with the data drawn from a tighter process, it is then plausible that the estimate $\hat{C}^*_{pm}$ is more accurate with a smaller s_p and the required minimum value $C^*(p)$ is smaller, since we need only a smaller $C^*(p)$ to account for the smaller uncertainty in the estimation. Intuitively, if the estimation error in our estimate is potentially large, then it is reasonable that we need a large $\hat{C}^*_{pm}$ to be able to claim that the process is capable, and this means that the corresponding minimum value $C^*(p)$ should be large as well. Thus, the value of $C^*(p)$ decreases as δ increases, and this pattern is consistent with Shiau *et al.*[23]. Alternatively, according to the definition of γ, as (2) becomes larger, the variation between these multiple samples will become smaller when the other conditions are fixed. And the smaller the variation is between these multiple samples, the more stable the process. Thus, we only need a smaller $C^*(p)$ to assess the process capability. Another observation from the tables is that the value of $C^*(p)$ decreases as n and/or m increases for fixed δ, γ and p. This can also be explained by the same reasoning as above, since the estimation will be more accurate with a larger sample size.

As a result, to judge if a given process meets the capability requirement, we first determine the pre-specified value ω, the capability requirement, and the α-risk or the confidence level p for the interval. Checking the appropriate table (or run the program), we may obtain the critical value $C^*(p)$ based on given values of p, m sub-samples of size n_i and γ calculated from samples. If the estimated value $\hat{C}^*_{pm}$ is greater than the critical value $C^*(p) \times \omega$, then we may conclude that the process meets the capability requirement ($C_{pm} > \omega$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_{pm} \leq \omega$. In the following, we present a simple step-by-step procedure for testing the process precision. The practitioners can use the procedure on their in-plant applications to obtain reliable decisions.

Step 1. Decide the definition of 'capable' (ω, normally set to 1.00 or 1.33), and the confidence level p for the interval (normally set to 0.99, 0.975 or 0.95). The chance of true C_{pm} lying in this interval is p.

Step 2. Calculate the value of the estimator $\hat{C}^*_{pm}$, γ and δ based on m multiple control samples of each sample size n_i.

Step 3. Check the table and find the critical value $C^*(p)$ based on given values of p, m subgroups of each sample size n_i, and γ which is calculated in Step 2.

Step 4. Conclude that the process is capable ($C_{pm} > \omega$) if the $\hat{C}^*_{pm}$ value is greater than the critical value $C^*(p) \times \omega$. Otherwise, we do not have enough information to conclude that the process is capable.

Table I. Critical values $C^*(p)$ for multiple samples with $m = 2(2)10$, $\delta = 0.5(0.5)2.0$, $\gamma = 0.7(0.1)1.0$, $p = 0.95$

	0				0.5				1.0				1.5				2.0			
δ	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1
(a) $n = 5$																				
$m = 2$	1.8318	1.8318	1.8318	1.8318	1.7410	1.7302	1.7180	1.7097	1.3495	1.3293	1.3120	1.2971	1.4430	1.4207	1.4012	1.3841	1.5571	1.5559	1.5368	1.5194
$m = 4$	1.4464	1.4464	1.4464	1.4464	1.4059	1.4009	1.3961	1.3914	1.2079	1.1966	1.1870	1.1786	1.2585	1.2466	1.2361	1.2268	1.3275	1.3169	1.3072	1.2983
$m = 6$	1.3264	1.3264	1.3264	1.3264	1.3006	1.2973	1.2941	1.2910	1.1600	1.1516	1.1444	1.1381	1.1975	1.1887	1.1810	1.1741	1.2471	1.2396	1.2327	1.2264
$m = 8$	1.2654	1.2654	1.2654	1.2654	1.2465	1.2440	1.2416	1.2393	1.1344	1.1274	1.1215	1.1162	1.1650	1.1579	1.1516	1.1459	1.2050	1.1990	1.1935	1.1884
$m = 10$	1.2276	1.2276	1.2276	1.2276	1.2127	1.2107	1.2008	1.2068	1.1178	1.1118	1.1066	1.1021	1.1443	1.1381	1.1327	1.1278	1.1783	1.1733	1.1686	1.1643
(b) $n = 10$																				
$m = 2$	1.4464	1.4464	1.4464	1.4464	1.4099	1.4053	1.4008	1.3966	1.3364	1.3262	1.3169	1.3082	1.2690	1.2571	1.2466	1.2372	1.2180	1.2065	1.1966	1.1880
$m = 4$	1.2654	1.2654	1.2654	1.2654	1.2485	1.2462	1.2440	1.2419	1.2100	1.2043	1.1990	1.1941	1.1712	1.1642	1.1579	1.1522	1.1406	1.1335	1.1274	1.1221
$m = 6$	1.2016	1.2016	1.2016	1.2016	1.1906	1.1890	1.1875	1.1861	1.1632	1.1591	1.1552	1.1515	1.1343	1.1289	1.1241	1.1198	1.1109	1.1054	1.1007	1.0966
$m = 8$	1.1674	1.1674	1.1674	1.1674	1.1592	1.1580	1.1569	1.1557	1.1375	1.1341	1.1309	1.1279	1.1137	1.1093	1.1053	1.1017	1.0942	1.0896	1.0857	1.0822
$m = 10$	1.1455	1.1455	1.1455	1.1455	1.1390	1.1380	1.1371	1.1361	1.1207	1.1178	1.1151	1.1125	1.1002	1.0963	1.0929	1.0897	1.0832	1.0792	1.0758	1.0727
(c) $n = 15$																				
$m = 2$	1.3264	1.3264	1.3264	1.3264	1.3040	1.3011	1.2982	1.2955	1.2553	1.2482	1.2417	1.2356	1.2075	1.1988	1.1911	1.1842	1.1699	1.1614	1.1539	1.1474
$m = 4$	1.2016	1.2016	1.2016	1.2016	1.1909	1.1895	1.1880	1.1866	1.1643	1.1602	1.1564	1.1528	1.1357	1.1304	1.1256	1.1213	1.1123	1.1069	1.1022	1.0980
$m = 6$	1.1554	1.1554	1.1554	1.1554	1.1484	1.1474	1.1464	1.1454	1.1291	1.1261	1.1232	1.1205	1.1075	1.1034	1.0997	1.0963	1.0893	1.0851	1.0814	1.0781
$m = 8$	1.1302	1.1302	1.1302	1.1302	1.1249	1.1241	1.1233	1.1225	1.1094	1.1069	1.1045	1.1023	1.0914	1.0880	1.0849	1.0821	1.0762	1.0727	1.0695	1.0668
$m = 10$	1.1138	1.1138	1.1138	1.1138	1.1096	1.1089	1.1082	1.1076	1.0964	1.0943	1.0922	1.0902	1.0808	1.0778	1.0751	1.0727	1.0675	1.0644	1.0616	1.0592
(d) $n = 20$																				
$m = 2$	1.2654	1.2654	1.2654	1.2654	1.2493	1.2472	1.2451	1.2430	1.2122	1.2066	1.2015	1.1967	1.1741	1.1671	1.1608	1.1551	1.1434	1.1364	1.1302	1.1248
$m = 4$	1.1674	1.1674	1.1674	1.1674	1.1596	1.1585	1.1574	1.1563	1.1388	1.1355	1.1324	1.1295	1.1155	1.1111	1.1071	1.1035	1.0960	1.0915	1.0875	1.0840
$m = 6$	1.1302	1.1302	1.1302	1.1302	1.1250	1.1242	1.1234	1.1226	1.1097	1.1072	1.1049	1.1026	1.0919	1.0885	1.0854	1.0826	1.0767	1.0731	1.0700	1.0672
$m = 8$	1.1095	1.1095	1.1095	1.1095	1.1056	1.1050	1.1044	1.1038	1.0933	1.0912	1.0893	1.0874	1.0784	1.0755	1.0729	1.0706	1.0656	1.0626	1.0599	1.0576
$m = 10$	1.0961	1.0961	1.0961	1.0961	1.0929	1.0924	1.0919	1.0914	1.0824	1.0806	1.0789	1.0773	1.0695	1.0669	1.0646	1.0626	1.0582	1.0556	1.0532	1.0511

Table II. Critical values $C^*(p)$ for multiple samples with $m = 2(2)10$, $\delta = 0.5(0.5)2.0$, $\gamma = 0.7(0.1)1.0$, $p = 0.99$

	0				0.5				1.0				1.5				2.0			
δ	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1
(a) $n = 5$																				
$m = 2$	2.3202	2.3202	2.3202	2.3202	2.1772	2.1600	2.1436	2.1277	1.1977	1.8838	1.8534	1.8257	1.7039	1.6682	1.6371	1.6097	1.5549	1.5221	1.4945	1.4706
$m = 4$	1.6688	1.6688	1.6688	1.6688	1.6085	1.6011	1.5938	1.5869	1.4907	1.4747	1.4602	1.4468	1.3866	1.3687	1.3529	1.3889	1.3104	1.2935	1.2790	1.2663
$m = 6$	1.4803	1.4803	1.4803	1.4803	1.4431	1.4383	1.4336	1.4290	1.3640	1.3529	1.3427	1.3333	1.2903	1.2774	1.2659	1.2557	1.2349	1.2225	1.2118	1.2024
$m = 8$	1.3870	1.3870	1.3870	1.3870	1.3603	1.3567	1.3532	1.3498	1.2995	1.2907	1.2726	1.2751	1.2406	1.2301	1.2209	1.2126	1.1956	1.1854	1.1767	1.1670
$m = 10$	1.3302	1.3302	1.3302	1.3302	1.3093	1.3065	1.3037	1.3001	1.2592	1.2518	1.245	1.2386	1.2093	1.2003	1.1924	1.1853	1.1707	1.1619	1.1543	1.1477
(b) $n = 10$																				
$m = 2$	1.6688	1.6688	1.6688	1.6688	1.6145	1.6077	1.6011	1.5946	1.5042	1.4888	1.4747	1.4617	1.4025	1.3845	1.3687	1.3545	1.3256	1.3083	1.2935	1.2805
$m = 4$	1.3870	1.3870	1.3870	1.3870	1.3631	1.3599	1.3567	1.3536	1.3069	1.2985	1.2907	1.2835	1.2498	1.2394	1.2301	1.2218	1.2047	1.1944	1.1854	1.1776
$m = 6$	1.2913	1.2913	1.2913	1.2913	1.2761	1.2740	1.2718	1.2697	1.2366	1.2305	1.2248	1.2194	1.1943	1.1865	1.1795	1.1732	1.1601	1.1522	1.1454	1.1393
$m = 8$	1.2408	1.2408	1.2408	1.2408	1.2297	1.2280	1.2264	1.2247	1.1984	1.1934	1.1888	1.1844	1.1638	1.1573	1.1515	1.1463	1.1355	1.1289	1.1232	1.1181
$m = 10$	1.2088	1.2088	1.2088	1.2088	1.2000	1.1986	1.1973	1.1959	1.1737	1.1695	1.1655	1.1618	1.1440	1.1384	1.1333	1.1288	1.1193	1.1136	1.1086	1.1042
(c) $n = 15$																				
$m = 2$	1.4803	1.4803	1.4803	1.4803	1.4480	1.4437	1.4396	1.4356	1.3762	1.3657	1.3560	1.3470	1.3052	1.2923	1.2809	1.2707	1.2495	1.2368	1.2258	1.2162
$m = 4$	1.2913	1.2913	1.2913	1.2913	1.2767	1.2746	1.2725	1.2705	1.2382	1.2322	1.2265	1.2213	1.1964	1.1886	1.1816	1.1753	1.1623	1.1544	1.1475	1.1414
$m = 6$	1.2322	1.2322	1.2322	1.2322	1.2137	1.2123	1.2109	1.2095	1.1861	1.1816	1.1774	1.1735	1.1546	1.1486	1.1433	1.1384	1.1283	1.1222	1.1169	1.1121
$m = 8$	1.1864	1.1864	1.1864	1.1864	1.1793	1.1782	1.1771	1.1760	1.1571	1.1535	1.1500	1.1468	1.1311	1.1262	1.1217	1.1177	1.1092	1.1040	1.0995	1.0956
$m = 10$	1.1627	1.1627	1.1627	1.1627	1.1570	1.1561	1.1551	1.1542	1.1382	1.1351	1.1321	1.1293	1.1157	1.1114	1.1075	1.1040	1.0965	1.0920	1.0881	1.0846
(d) $n = 20$																				
$m = 2$	1.3870	1.3870	1.3870	1.3870	1.3643	1.3613	1.3582	1.3553	1.3101	1.3019	1.2943	1.2872	1.2540	1.2436	1.2344	1.2261	1.2089	1.1985	1.1895	1.1816
$m = 4$	1.2408	1.2408	1.2408	1.2408	1.2303	1.2287	1.2271	1.2256	1.2003	1.1955	1.1910	1.1867	1.1664	1.1600	1.1542	1.1490	1.1381	1.1315	1.1258	1.1207
$m = 6$	1.1864	1.1864	1.1864	1.1864	1.1794	1.1784	1.1773	1.1762	1.1576	1.1540	1.1506	1.1473	1.1318	1.1268	1.1224	1.1183	1.1099	1.1047	1.1002	1.0962
$m = 8$	1.1565	1.1565	1.1565	1.1565	1.1513	1.1504	1.1495	1.1487	1.1336	1.1306	1.1278	1.1251	1.1122	1.1080	1.1043	1.1009	1.0938	1.0894	1.0856	1.0822
$m = 10$	1.1371	1.1371	1.1371	1.1371	1.1329	1.1321	1.1314	1.1307	1.1178	1.1153	1.1128	1.1105	1.0992	1.0956	1.0923	1.0893	1.0831	1.0792	1.0759	1.0729

Table III. Some recommended minimum capability requirements for special processes

Process type	Capability requirement	NCs (ppm)
Existing processes	1.33	66.07
New processes	1.50	6.80
Existing processes with safety, strength, or critical parameters	1.50	6.80
New processes with safety, strength, or critical parameters	1.67	0.54

5. APPLICATION EXAMPLE

Peripheral devices such as drivers, printers, and CD-ROMs are connected to the host through a special bus called SCSI (Small Computer System Interface). The fast edge rated signals that are transmitted through the SCSI cable generate ringing on the bus. This will slow down communication between host and peripherals. The SCSI standard recommends proper resistor (Thevenin) termination at host and peripheral locations to eliminate transmission line effects. Dual Thevenin Termination Networks offer high integration and performance in a miniature QSOP or SOIC package, which saves spacious board space, provides manufacturing cost reduction and reliability efficiencies. A terminating resistor is used to reduce or eliminate unwanted reflections on a transmission line. It can perform this function only when its resistance value matches the characteristic impedance of the transmission line. The resistors used for terminating the transmission lines should be noiseless, stable and functional at high frequencies. Unlike thin film-based resistor networks, conventional thick film resistors used for terminating transmission lines are not stable over temperature and time and impose system performance limitations at very high frequencies.

5.1. *Capability requirement*

In the industry, some minimum capability requirements for special types of processes have been recommended. In particular, it is recommended that there be a minimum process capability of 1.33 for existing processes, and 1.50 for new processes; 1.50 also for existing processes on safety, strength, or critical parameters; and 1.67 for new processes on safety, strength, or critical parameters. The recommended guidelines for minimum quality requirements and the corresponding parts per million (ppm) of non-conformities (NCs) for those processes are summarized in Table III.

The integrated passive networks are manufactured using advanced thin film technologies including ultra-stable and self-passivating tantalum nitride resistors, gold interconnect metallization and reliable MNOS capacitors to achieve excellent uniformity, performance and reliability. Thin film resistor technology is the preferred solution for all applications that require low noise, long-term stability and excellent performance at very high frequencies. To illustrate the application of assessing process capability for multiple control samples, we consider a real example taken from an electronic component manufacturer, located on the Science-Based Industrial Park in Taiwan, developing passive and active components for the personal computers, telecommunications, industrial controls, automotive parts, and avionics. The factory manufactures various types of resistors. For a particular model of the resistors investigated, the target value is set to $T = 10.0$ mil, and the tolerance of thickness is 2.0 mil, that is, the lower and upper specification limit are set to $\text{LSL} = 8.0$ mil and $\text{USL} = 12.0$ mil. If the characteristic data do not fall within the tolerance (LSL, USL), the lifetime or reliability of the resistors will be discounted. The collected sample data (10 samples each of size 15), which are under statistical control, are displayed in Table IV.

We now apply the Bayesian procedure in the following. A $100p\%$ credible interval means the posterior probability that the true PCI lying in this interval is p. Let p be a high probability, say, 0.95. Suppose for this particular process under consideration to be capable, the process index must reach at least a certain

Table IV. The 10 samples each of 15 observations

Samples									
1	2	3	4	5	6	7	8	9	10
10.21	9.66	9.80	9.48	10.74	10.71	10.00	10.09	10.58	10.23
10.19	10.36	9.96	9.91	9.72	10.36	10.12	10.12	10.42	10.44
9.88	10.55	10.04	9.94	10.34	10.17	10.29	9.99	9.58	9.86
10.73	10.31	9.99	9.93	10.88	10.53	9.62	10.57	10.44	10.16
10.59	9.72	10.35	10.08	10.48	10.15	9.98	10.50	10.39	10.14
10.21	10.00	9.94	9.59	10.01	10.09	10.00	9.43	10.87	9.99
10.61	10.34	10.96	10.01	10.71	10.14	10.12	10.60	9.56	11.12
10.68	9.77	10.33	9.85	10.15	9.76	9.97	9.86	10.26	10.10
9.86	10.12	10.39	10.50	10.46	10.15	10.56	9.90	10.16	10.00
10.69	10.40	10.63	9.77	10.38	10.36	10.60	9.84	10.46	9.97
10.12	11.11	9.13	9.97	10.39	10.28	9.76	10.31	9.83	10.50
10.62	10.25	10.57	10.03	10.33	10.05	9.78	10.03	10.09	10.47
9.73	11.03	10.24	10.02	10.33	9.50	9.74	9.53	10.43	10.30
10.35	10.23	10.65	10.37	10.15	10.29	10.48	9.72	10.38	10.17
10.51	9.98	10.70	9.81	10.26	10.29	9.79	10.56	10.27	10.04

Table V. The calculated sample mean and the sample variance for the 10 samples

Sample i	1	2	3	4	5	6	7	8	9	10
$\bar{x}_i$	10.332	10.255	10.245	9.951	10.354	10.188	10.053	10.070	10.247	10.233
s_i^2	0.110	0.178	0.207	0.066	0.085	0.083	0.096	0.141	0.129	0.097

level ω, say, 1.33. That is, the requirement for the process yield is no more than 2700 ppm. From the process data, we compute the lower bound of the credible interval for the index. The Bayesian testing procedure is simple. That is, if $\hat{C}^*_{\rm pm} > C^*(p) \times \omega$, then we say that the process is capable in a Bayesian sense.

The calculated sample mean $\bar{x}_i$ and the sample variance s_i^2 for the ten sub-samples of size 15 are tabulated in Table V. Thus,

$$\bar{\bar{x}} = 10.1929$$

$$s_{\rm p}^2 = \sum_{i=1}^{m} s_i^2 \Big/ m = 0.1192$$

$$\gamma = \frac{m(n-1)s_{\rm p}^2}{m(n-1)s_{\rm p}^2 + n\sum_{i=1}^{m}(\bar{x}_i - \bar{\bar{x}})^2} = 0.8816$$

$$\delta = \frac{|\bar{\bar{x}} - T|}{s_{\rm p}} = 0.5587$$

$$\hat{C}^*_{\rm pm} = \frac{d}{3s_{\rm p}\sqrt{(n-1)/(\gamma n) + \delta^2}} = 1.6489$$

Next, we check the tables or run the computer software to obtain the critical value $\hat{C}^*(p) \times \omega = 1.1069 \times 1.33 = 1.4722$ based on $p = 0.95$, $m = 10$, $n = 15$. Since the sample estimator $\hat{C}^*_{\rm pm}$ from the samples, 1.6489, is greater than the critical value $C^* = \hat{C}^*(p) \times \omega = 1.4722$, we may conclude, with 95% confidence level, that the process meets the capability requirement '$C_{\rm pm} > 1.33$' in this case.

6. CONCLUSIONS

Using process capability indices to quantify manufacturing process precision and performance is an essential part of implementing a quality improvement program. Most existing tests of the capability indices are obtained from the distributional frequency approaches. Statistical properties of the estimated C_{pm} based on one single sample have been investigated extensively. But, the properties of the estimated C_{pm} based on multiple samples have been comparatively neglected. In this paper, we have considered the problem of estimating and testing process capability based on multiple samples. We accordingly proposed a Bayesian procedure for capability testing. Based on these multiple control samples, we also developed a simple step-by-step procedure. The practitioners can use the proposed procedure to determine whether their manufacturing processes are capable of reproducing products satisfying the preset precision requirements. A process is capable if all the points in the credible interval are greater than the pre-specified capability level ω. To make this Bayesian procedure practical for in-plant applications, we tabulated the minimum values of $C^*(p)$ for which the posterior probability p reaches various desirable confidence levels.

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments and careful readings, which improved the paper.

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APPENDIX A. DERIVATION OF EXPRESSION (5)

For the multiple control samples, given a pre-specified capability level $\omega > 0$, the posterior probability based on index C_{pm} that a process is capable is given as

$$p = \Pr\{C_{pm} > \omega|\mathbf{x}\} = \Pr\left\{\frac{\text{USL} - \text{LSL}}{6\tau} > \omega \,\middle|\, \mathbf{x}\right\} = \Pr\left\{\tau < \frac{\text{USL} - \text{LSL}}{6\omega} \,\middle|\, \mathbf{x}\right\}$$

$$= \Pr\left\{\sigma^2 + (\mu - T)^2 < \left(\frac{\text{USL} - \text{LSL}}{6\omega}\right)^2 \,\middle|\, \mathbf{x}\right\} = \int_0^{((\text{USL}-\text{LSL})/6\omega)} \int_{T-\sqrt{a^2-\sigma^2}}^{T+\sqrt{a^2-\sigma^2}} f(\mu, \sigma|\mathbf{x})\, \mathrm{d}\mu\, \mathrm{d}\sigma$$

Denote $a = (\text{USL} - \text{LSL})/6\omega$ and $g(\sigma) = \sqrt{a^2 - \sigma^2}$. Then

$$p = \int_0^a \int_{T-g(\sigma)}^{T+g(\sigma)} f(\mu, \sigma|\mathbf{x})\, \mathrm{d}\mu\, \mathrm{d}\sigma$$

$$= \int_0^a \int_{T-g(\sigma)}^{T+g(\sigma)} \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i + 1)} \exp\left(-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \mu)^2}{2\sigma^2}\right) \mathrm{d}\mu\, \mathrm{d}\sigma$$

$$= \int_0^a \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i + 1)} \exp\left(-\frac{1}{\beta\sigma^2}\right) \int_{T-g(\sigma)}^{T+g(\sigma)} \exp\left(-\frac{\sum_{i=1}^m n_i(\mu - \bar{\bar{x}})^2}{2\sigma^2}\right) \mathrm{d}\mu\, \mathrm{d}\sigma$$

$$= \int_0^a \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \left[\Phi\left(\frac{T - \bar{\bar{x}} + g(\sigma)}{\sigma\Big/\sqrt{\sum_{i=1}^m n_i}}\right) - \Phi\left(\frac{T - \bar{\bar{x}} - g(\sigma)}{\sigma\Big/\sqrt{\sum_{i=1}^m n_i}}\right)\right] \mathrm{d}\sigma \qquad \text{(A1)}$$

where

$$\alpha = \left(\sum_{i=1}^m n_i - 1\right)\Big/2, \quad \beta = \left[\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2 \Big/ 2\right]^{-1}, \quad \bar{\bar{x}} = \left[\sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij} \Big/ \sum_{i=1}^m n_i\right]$$

for $-\infty < \mu < \infty$, $0 < \sigma < \infty$, and $\Phi(\cdot)$ is the cumulative distribution of the standard normal distribution.

By changing the variables, we let $y = \beta\sigma^2$. Then, $\mathrm{d}y = 2\beta\sigma\, \mathrm{d}\sigma$, and

$$s_p/\sigma = \sqrt{2\gamma \Big/ \left[\sum_{i=1}^m (n_i - 1)y\right]}$$

Thus, the posterior probability p for multiple control samples, which are given in (A1), can be simplified to

$$\begin{aligned} p &= \Pr\{C_{\mathrm{pm}} > \omega|\mathbf{x}\} \\ &= \int_0^t \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times [\Phi\,(b_1(y)+b_2(y)) - \Phi\,(b_1(y)-b_2(y))]\;\mathrm{d}y \end{aligned}$$

where

$$b_1(y) = \frac{T-\bar{\bar{x}}}{\sigma\Big/\sqrt{\sum_{i=1}^m n_i}} = \sqrt{\sum_{i=1}^m n_i}\left(\frac{T-\bar{\bar{x}}}{s_{\mathrm{p}}}\right)\left(\frac{s_{\mathrm{p}}}{\sigma}\right) = \sqrt{\frac{2\gamma\sum_{i=1}^m n_i}{\sum_{i=1}^m (n_i-1)y}}\delta$$

$$\begin{aligned} b_2(y) &= \frac{g(\sigma)}{\sigma\Big/\sqrt{\sum_{i=1}^m n_i}} = \sqrt{\sum_{i=1}^m n_i}\left(\frac{g(\sigma)}{\sigma}\right) = \sqrt{\sum_{i=1}^m n_i}\left(\frac{a^2-\sigma^2}{\sigma^2}\right)^{1/2} \\ &= \sqrt{\sum_{i=1}^m n_i}\left(\frac{a^2}{\sigma^2}-1\right)^{1/2} = \sqrt{\sum_{i=1}^m n_i}\left(\frac{\beta a^2}{y}-1\right)^{1/2} = \sqrt{\sum_{i=1}^m n_i}\left(\frac{t}{y}-1\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} t = \beta a^2 &= \frac{2\gamma a^2}{\sum_{i=1}^m (n_i-1)s_{\mathrm{p}}^2} = \frac{2\gamma}{\sum_{i=1}^m (n_i-1)s_{\mathrm{p}}^2}\left(\frac{\mathrm{USL}-\mathrm{LSL}}{6\omega}\right)^2 = \frac{2\gamma}{\sum_{i=1}^m (n_i-1)\omega^2}\left(\frac{\mathrm{USL}-\mathrm{LSL}}{6s_{\mathrm{p}}}\right)^2 \\ &= \frac{2\gamma}{\sum_{i=1}^m (n_i-1)\omega^2}\left(\frac{\mathrm{USL}-\mathrm{LSL}}{6\hat{\sigma}'}\right)^2\left(\frac{\hat{\tau}'}{s_{\mathrm{p}}}\right)^2 = \frac{2\gamma}{\sum_{i=1}^m (n_i-1)}\left(\frac{\hat{C}_{\mathrm{pm}}^*}{\omega}\right)^2\left(\frac{\hat{\tau}'}{s_{\mathrm{p}}}\right)^2 \\ &= \frac{2}{\sum_{i=1}^m (n_i-1)}\left(\frac{\hat{C}_{\mathrm{pm}}^*}{\omega}\right)^2\left(\frac{\sum_{i=1}^m (n_i-1)}{\sum_{i=1}^m n_i} + \gamma\delta^2\right) \end{aligned}$$

Therefore, the posterior probability p for multiple control samples, which is given in (5), can be derived.

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