New Solvers for Higher Dimensional Poisson Equations by Reduced B-Splines

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We use higher dimensional B-splines as basis functions to find the approximations for the Dirichlet problem of the Poisson equation in dimension two and three. We utilize the boundary data to remove unnecessary bases. Our method is applicable to more general linear partial differential equations. We provide new basis functions which do not require as many B-splines. The number of new bases coincides with that of the necessary knots. The reducing process uses the boundary conditions to redefine a basis without extra artificial assumptions on knots which are outside the domain. Therefore, more accuracy would be expected from our method. The approximation solutions satisfy the Poisson equation at each mesh point and are solved explicitly using tensor product of matrices. Numer Methods Partial Differential Eq 30: 393–405, 2014

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I. INTRODUCTION

In this article, we provide a new computing method for higher dimensional Poisson equations by using reduced B-splines as basis functions. In Viswanadham [1], they employed a collection method for the fifth-order boundary value problem of one dimensional (1D) differential equation. Koch–Schmidt [2] gave a precise definition of the n-dimensional B-splines. In Sections I and II, we adopt their n-dimensional B-splines and use the collection method as in Viswanadham [1] to reduce the involved bases by boundary data one-by-one and solve the Poisson equations in $\mathbb{R}^2$ and $\mathbb{R}^3$. Here, we emphasize that the reducing process is applicable to any order linear partial differential equation.

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First, we summarize the B-splines of four different expressions [3–5]. With the help of those B-splines, we are able to reduce the numbers of basis functions of B-splines by given boundary conditions. The reducing process is similar to that of [1] if we are solving the k-th-order differential equations or partial differential equations. This method provides better error estimates, in particular, when the given data of Neumann values or higher order differentials on boundary are involved.

In Section III, we continue to consider 3D B-spline functions, based on Koch–Schmidt [2]. Then, we apply 3D B-splines as a basis to a Poisson equation in a cubic domain. Some numerical experiments are given in last section of this article. We start from 1D B-splines. Let Δx be the mesh length and $x_i = a + i\Delta x, i \in I$. Define the j-th B-spline of degree $k-1$ recursively by

$$B_j^1(x) := \begin{cases} 1, & \text{if } x \in [x_j, x_{j+1}), \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

and

$$B_j^k(x) := \frac{x - x_j}{x_{j+k-1} - x_j} B_j^{k-1}(x) + \frac{x_{j+k} - x}{x_{j+k} - x_{j+1}} B_{j+1}^{k-1}(x). \tag{1.2}$$

The above is usually referred to as the Cox-de Boor recursion formula [5].

Each $B_j^k(x)$ is a polynomial and has compact support on $[x_j, x_{j+k}]$. Moreover, $B_j^k(x)$ has first derivative for $k \geq 3$ and second derivative for $k \geq 4$ and so on. $B_j^k(x)$ can be expressed by divided difference as follows. Define the divided difference recursively by

$$[x_j] f = f(x_j), \quad j \in I, \tag{1.3}$$

$$[x_j, \ldots, x_{j+k}] f = \frac{[x_{j+1}, \ldots, x_{j+k}] f - [x_j, \ldots, x_{j+k-1}] f}{x_{j+k} - x_j}, \quad \text{for } j, k \in I. \tag{1.4}$$

Therefore by induction, we have

$$\begin{aligned} B_j^k(x) &= (x_{j+k} - x_j)[x_j, \ldots, x_{j+k}](\cdot - x)_+^{k-1}, \quad \forall x \in \mathbb{R}, \\ &= k\Delta x \sum_{\ell=j}^{j+k} \frac{(x_\ell - x)_+^{k-1}}{g'_{j,k}(x_\ell)}, \\ &= \frac{1}{(k-1)!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} (x_{\ell+j} - x)_+^{k-1}, \quad \text{if } \Delta x = 1, \end{aligned} \tag{1.5}$$

where

$$y_+ = \begin{cases} y, & \text{if } y \geq 0, \\ 0, & \text{if } y < 0, \end{cases}$$

and

$$g_{j,k}(x) = (x - x_j)(x - x_{j+1}) \cdots (x - x_{j+k}), \quad \forall j, k,$$
$$g'_{j,k}(x_\ell) = \Delta x^k (-1)^{k+j-\ell} (k + j - \ell)! (\ell - j)!, \quad \ell \geq j.$$

The functions generated by higher dimensional B-splines are defined by

$$u\left(x^1, x^2, \ldots, x^n\right) = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_n} \alpha_{j_1 j_2 \cdots j_n} B_{j_1}^{k_1}\left(x^1\right) B_{j_2}^{k_2}\left(x^2\right) \cdots B_{j_n}^{k_n}\left(x^n\right), \tag{1.6}$$

where $B_{j_1}^{k_1}\left(x^1\right), B_{j_2}^{k_2}\left(x^2\right), \ldots, B_{j_n}^{k_n}\left(x^n\right)$ are the 1D B-splines with $(k_1 - 1), (k_2 - 1), \ldots, (k_n - 1)$ degrees, respectively. Without loss of generality, let Ω be a n-dimensional cube, $(a, b)^n$. Let $h = \frac{b-a}{N}$ be the mesh length and $x_j^k = a + jh$ for each $k = 1, \ldots, n$. A mesh in $\overline{\Omega}$ is

$$\begin{aligned}
\Omega_d &:= \left\{\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_n}^n\right) | i_1, i_2, \ldots, i_n = 0, \ldots, N\right\}. \\
\Omega_d^o &:= \left\{\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_n}^n\right) | i_1, i_2, \ldots, i_n = 1, \ldots, N-1\right\}, \\
\partial\Omega_d &:= \Omega_d - \Omega_d^o,
\end{aligned} \tag{1.7}$$

In this article, we adopt the B-splines of fourth degree for a second-order differential equation. Hence, for simplicity, we redenote the third degree B-spline functions by

$$B_{j+2}(x) := B_j^4(x), \quad \text{for } j = -3, \ldots, N-1. \tag{1.8}$$

Assume the space of B-spline functions generated by $\left\{B_{j_1}\left(x^1\right) B_{j_2}\left(x^2\right) \cdots B_{j_n}\left(x^n\right) \mid j_1, j_2, \ldots, j_n = -1, \cdots N+1\right\}$ to be

$$\mathfrak{M}^n\left(\Omega_d\right) = \left\{\sum_{j} \sum_{j_2} \cdots \sum_{j_n} \alpha_{j_1 j_2 \cdots j_n} B_{j_1}\left(x^1\right) B_{j_2}\left(x^2\right) \cdots B_{j_n}\left(x^n\right), \alpha_{j_1 \cdots j_n} \in \mathbb{R}\right\}. \tag{1.9}$$

II. 2D B-SPLINES FOR THE POISSON EQUATIONS

In this section, we consider $n = 2$. To reduce some generators of B-splines, redefine a new basis

$$\tilde{B}_j(x) := \begin{cases} B_j(x) - \frac{B_{-1}(x)}{B_{-1}(a)} B_j(a), & \text{if } j = -1, 0, 1, \\ B_j(x), & \text{if } j = 2, \ldots, N-2, \\ B_j(x) - \frac{B_{N+1}(x)}{B_{N+1}(b)} B_j(b), & \text{if } j = N-1, N, N+1, \end{cases} \tag{2.1}$$

where $\tilde{B}_{-1}(x) \equiv 0$ and $\tilde{B}_{N+1}(x) \equiv 0$. Hence, the number of B-splines required for a basis is reduced from $N+3$ to be $N+1$. We call $\left\{\tilde{B}_j\right\}$ to be the basis of the reduced B-splines. This implies that if $u_h \in \mathfrak{M}^2\left(\Omega_d\right)$ and $\left(x^1, x^2\right) \in \Omega_d^o$,

$$u_h\left(x^1, x^2\right) = \sum_{j_1=0}^{N} \sum_{j_2=0}^{N} \alpha_{j_1 j_2} \tilde{B}_{j_1}\left(x^1\right) \tilde{B}_{j_2}\left(x^2\right) + H\left(x^1, x^2\right), \tag{2.2}$$

where

$$H\left(x^1, x^2\right) = \begin{cases} g\left(x^1, x^2\right), & \text{if } \left(x^1, x^2\right) \in \partial\Omega_d \\ 0, & \text{if } \left(x^1, x^2\right) \in \Omega_d^o. \end{cases} \tag{2.3}$$

Rearrange the knots to be $\mathbf{z}_i = \left(x_{i_1}^1, x_{i_2}^2\right)$ and define new basis functions

$$\tilde{\mathfrak{B}}_j\left(\mathbf{z}_i\right) = \tilde{B}_{j_1}\left(x_{i_1}^1\right)\tilde{B}_{j_2}\left(x_{i_2}^2\right) + \chi_{\left(x_{j_1}^1, x_{j_2}^2\right)}\left(x_{i_1}^1, x_{i_2}^2\right), \tag{2.4}$$

where j, j_1, j_2, $\left(1 \le j \le (N+1)^2,\ 0 \le j_1, j_2 \le N\right)$, satisfy $j = j_1 + 1 + j_2(N+1)$,

$$\begin{cases} j_1 = j - 1 - \lceil \frac{j-1}{N+1} \rceil (N+1), \\ j_2 = \lceil \frac{j-1}{N+1} \rceil (N+1), \end{cases}$$

and $\lceil x \rceil$ is the greatest integer which is smaller than or equal to x,

$$\chi_{\left(x_{j_1}^1, x_{j_2}^2\right)}\left(x_{i_1}^1, x_{i_2}^2\right) = \begin{cases} 1, & \text{if} \left(x_{i_1}^1, x_{i_2}^2\right) = \left(x_{j_1}^1, x_{j_2}^2\right) \in \partial\Omega_d, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $u_h\left(\mathbf{z}_j\right) := u_h\left(x_{j_1}^1, x_{j_2}^2\right)$ is expressed as

$$u_h\left(x_{i_1}^1, x_{i_2}^2\right) = \sum_{j_1=0}^{N}\sum_{j_2=0}^{N} \alpha_{j_1 j_2}\left(\tilde{B}_{j_1}\left(x_{i_1}^1\right)\tilde{B}_{j_2}\left(x_{i_2}^2\right) + \chi_{\left(x_{j_1}^1, x_{j_2}^2\right)}\left(x_{i_1}^1, x_{i_2}^2\right)\right) \tag{2.5}$$

or

$$u_h\left(\mathbf{z}_j\right) = \sum_{k=1}^{(N+1)^2} \tilde{\alpha}_k \tilde{\mathfrak{B}}_k\left(\mathbf{z}_j\right) \tag{2.6}$$

Assume u_h satisfies the Poisson equation $\Delta u_h = f$ at each mesh point $\mathbf{z}_j \in \Omega_d^o$, then the coefficients $\tilde{\alpha}_k$ of (2.6) satisfy a matrix form

$$\begin{bmatrix} R & P & R & & & & \\ & R & P & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & P & R & \\ & & & & R & P & R \end{bmatrix}_{(N-1)^2 \times (N+1)^2} \cdot \mathfrak{X} = 3h^2 f, \tag{2.7}$$

where R and P are $(N-1)\times(N+1)$ rectangular matrices given by

$$R = \begin{bmatrix} 1 & 1 & 1 & & & & \\ & 1 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 1 & \\ & & & & 1 & 1 & 1 \end{bmatrix}_{(N-1)\times(N+1)}$$

$$P = \begin{bmatrix} 1 & -8 & 1 & & & & \\ & 1 & -8 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -8 & 1 & \\ & & & & 1 & -8 & 1 \end{bmatrix}_{(N-1)\times(N+1)},$$

and $\mathfrak{X} = \left[(X_1)^t, (X_2)^t, \ldots, (X_{N+1})^t\right]^t$, $X_j = \left[\tilde{\alpha}_{(j-1)(N+1)+1}, \ldots, \tilde{\alpha}_{(j-1)(N+1)+N+1}\right]$.

Using the given boundary values, it is sufficient to consider interior mesh points in Ω_d^o. Hence, removing the first and last elements of X_j, we get

$$Y_j := \left[\tilde{\alpha}_{j(N+1)+2}, \tilde{\alpha}_{j(N+1)+3}, \ldots, \tilde{\alpha}_{j(N+1)+N}\right]^t, \quad j = 1, 2, \ldots, N-1. \tag{2.8}$$

That is

$$X_{j+1} = \left[\tilde{\alpha}_{j(N+1)+1},\, Y_j,\, \tilde{\alpha}_{(j+1)(N+1)}\right].$$

Let $F_1 := \left[F_1^1, F_1,^2, \ldots, F_1^{N-1}\right]^t$ be a $(N-1)\times 1$ matrix where

$$F_1^1 = f(a+h, a+h) - \frac{1}{3h^2}\left(\sum_{\ell=0}^{2} g(a+\ell h, a) + g(a+h, a) + g(a+2h, a)\right),$$

$$F_1^k = f(a+kh, a+h) - \frac{1}{3h^2}\sum_{\ell=0}^{2} g(a+(k-1+\ell)h, a), \quad k = 2, \ldots, N-2,$$

$$F_1^{N-1} = f(b-h, a+h) - \frac{1}{3h^2}\left(\sum_{\ell=0}^{2} g(b-\ell h, a) + g(b, a+h) + g(b, a+2h)\right).$$

To reduce the number of basis functions in (2.6), we start from

$$(TY_1 + SY_2) = 3h^2 F_1, \tag{2.9}$$

where T and S are $(N-1)\times(N-1)$ tridiagonal matrices,

$$T := \begin{bmatrix} -8 & 1 & 0 & \cdots & & \\ 1 & -8 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -8 & 1 \\ & & & 1 & -8 \end{bmatrix}_{(N-1)\times(N-1)} \quad \text{and} \quad S := \begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1 & 1 \\ & & & 1 & 1 \end{bmatrix}_{(N-1)\times(N-1)}. \tag{2.10}$$

On the other side of the boundary, it satisfies

$$(SY_{N-2} + TY_{N-1}) = 3h^2 F_{N-1},$$

where $F_{N-1} := \left[F_{N-1}^1, F_{N-1},^2, \ldots, F_{N-1}^{N-1}\right]^t$ and

$$F_{N-1}^1 = f(a+h, b-h) - \frac{1}{3h^2} \cdot \left(g(a+h, b) + g(a+2h, b) + \sum_{\ell=0}^{2} g(a, b-\ell h) \right),$$

$$F_{N-1}^k = f(a+kh, b-h) - \frac{1}{3h^2} \sum_{\ell=0}^{2} g(a + (k-1+\ell)h, b), \quad k = 2, \ldots, N-2,$$

$$F_{N-1}^{N-1} = f(b-h, b-h) - \frac{1}{3h^2} \cdot \left(g(b, b-h) + g(b, b-2h) + \sum_{\ell=0}^{2} g(b-\ell h, b) \right).$$

For $j = 2, \ldots, N-2$, it is

$$\left(SY_{j-1} + TY_j + SY_{j+1}\right) = 3h^2 F_j \tag{2.11}$$

where $F_j := \left[F_j^1, F_j^2, \ldots, F_j^{N-1}\right]^t$ and

$$F_j^1 = f\left(a+h, y_j\right) - \frac{1}{3h^2} \cdot \sum_{\ell=-1}^{1} g\left(a, y_{j+\ell}\right),$$

$$F_j^\ell = f\left(a+\ell h, y_j\right), \quad \ell = 2, \ldots, N-2,$$

$$F_j^{N-1} = f\left(b-h, y_j\right) - \frac{1}{3h^2} \sum_{\ell=-1}^{1} g\left(b, y_{j+\ell}\right).$$

Thus, we rewrite the system (2.7) as

$$\mathbb{A} \cdot Y = 3h^2 F, \tag{2.12}$$

where

$$\begin{aligned}
\mathbb{A} &:= S \otimes S - 9I \otimes I, \\
Y &:= \left[Y_1^t, Y_2^t, \ldots, Y_{N-1}^t\right]^t, \\
F &:= \left[(F_1)^t, (F_2)^t, \ldots, \left(F_{(N-1)}\right)^t\right]^t, \\
F_j &:= \left(F_j^1, F_j^2, \ldots, F_j^{N-1}\right).
\end{aligned} \tag{2.13}$$

If we set $\tilde{F}\left(x_i^1, x_j^2\right) = F_j^i$, $i, j = 1, 2, \ldots, (N-1)$ and $\tilde{F} = f$ on $\partial\Omega_d$, then for $\left(x_i^1, x_j^2\right) \in \Omega_d$,

$$\tilde{F}\left(x_i^1, x_j^2\right) = f\left(x_i^1, x_j^2\right) - \sum_{(j_1, j_2) \in \Lambda} \alpha_{j_1 j_2} \left(\tilde{B}_{j_1, x^1 x^1}\left(x_{i_1}^1\right) \tilde{B}_{j_2}\left(x_{i_2}^2\right) + \tilde{B}_{j_1}\left(x_{i_1}^1\right) \tilde{B}_{j_2, x^2 x^2}\left(x_{i_2}^2\right) \right). \tag{2.14}$$

where $\Lambda = \{(0, j_2), (j_1, 0), (N, j_2), (j_1, N) \mid j_1, j_2 = 0, 1, \ldots, N\}$. The matrix S is decomposed by

$$S = QDQ^t, \tag{2.15}$$

where Q^t is the transpose matrix of Q and

$$D = \mathrm{diag}\left(1 + 2\cos\frac{\pi}{N}, 1 + 2\cos\frac{2\pi}{N}, \ldots, 1 + 2\cos\frac{(N-1)\pi}{N}\right),$$

$$V_k = \frac{1}{\sqrt{\frac{N}{2}}}\left[\sin\frac{k\pi}{N}, \sin\frac{2k\pi}{N}, \ldots, \sin\frac{(N-1)k\pi}{N}\right]^t, \quad k = 1, \ldots, N-1,$$

$$Q = [V_1|V_2|\ldots|V_{N-1}]. \tag{2.16}$$

and $Q's$ column vectors are V_j, $j = 1, \ldots, N-1$ satisfying $Q = Q^t = Q^{-1}$.

To solve (2.12), we use the fact that for any $n \times n$ matrices A, B, C, and D,

$$(A \otimes B)(C \otimes D) = (AB) \otimes (CD). \tag{2.17}$$

where $A \otimes B = \left[a_{ij}B\right]_{n^2 \times n^2}$. Let I be the $(N-1) \times (N-1)$ identity matrix. Hence, the matrix $\mathbb{A}$ of (2.13) is expressed as, [13],

$$\begin{aligned} \mathbb{A} &= \left(QDQ^t\right) \otimes \left(QDQ^t\right) - 9\left(QIQ^t\right) \otimes \left(QIQ^t\right), \\ &= (Q \otimes Q)(J)\left(Q^t \otimes Q^t\right). \end{aligned} \tag{2.18}$$

where $J := (D \otimes D - 9I \otimes I)$. It shows that $\mathbb{A}$ is invertible by $-11 < J < -6$. Conclusively, we obtain the following theorem.

Theorem 2.1. *If the function u_h defined in (2.5) or (2.6) in $\Omega_d \subset \mathbb{R}^2$ satisfies*

$$\begin{cases} \Delta u_h = f, & \text{in } \Omega_d^o, \\ u_h = g, & \text{on } \partial\Omega_d, \end{cases} \tag{2.19}$$

then the coefficients $\tilde{\alpha}_k$, $k \neq j(N+1), j(N+1)+1, j = 1, \ldots, N+1$ satisfy (2.12) *and* (2.13). *From* (2.12)*, they are determined by the matrix equation*

$$Y = 3h^2 (Q \otimes Q)(D \otimes D - 9I \otimes I)^{-1}\left(Q^t \otimes Q^t\right)F \tag{2.20}$$

where $(D \otimes D - 9I \otimes I)^{-1}$ is diagonal and

$$\left(\left(Q^t \otimes Q^t\right)F\right)_i = \sum_{j=1}^{N-1} Q_{ij} Q F_j^t$$

The above Theorem 2.1 gives us a new solver and the solution u_h of (2.5) is solved by (2.20). The approximation u_h satisfies $\Delta u_h = 0$ at every mesh point in Ω_d^o and $u_h = g$ on $\partial\Omega_d$. For each mesh length $h > 0$, let u_h and v satisfy $\Delta u_h = f$ in Ω_d^o with $u_h = g$ on $\partial\Omega_d$ and $\Delta v = f$ in Ω^o with $v = g$ on $\partial\Omega$, respectively. An immediate application of Theorem 2.1 is thoroughly explained in Froese–Oberman, [6], to the Monge–Ampére equation as follows. Let $\lambda_1(u)$ and $\lambda_2(u)$ be the eigenvalues of D^2u where u satisfies $\det D^2u = f(u)$ in Ω and $u = g$ on $\partial\Omega$. Then, u satisfies

$$\Delta u = \sqrt{\lambda_1^2(u) + \lambda_2^2(u) + 2f(u)}. \tag{2.21}$$

Then, one can carry out the following iterations:

1. Choose the first guess u^0.
2. $\Delta u^1 = \sqrt{2f(u^0)}$.
3. $\Delta u^{n+1} = \sqrt{2f(u^n) + \lambda_1^2(u^n) + \lambda_2^2(u^n)}$.

In next section, we consider the 3D Poisson equation and in Section IV, we provide some 3D numerical experiments which seem to indicate that u_h converges to v in L^∞-norm as $h \to 0$.

III. 3D B-SPLINES FOR THE POISSON EQUATIONS

In this section, we consider the Dirichlet problem

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

in a 3D cube, $\Omega = (a,b)^3$. Let $\mathfrak{M}^3(\Omega_d)$ be the space generated by $\{B_i(x^1)B_j(x^2)B_k(x^3) \mid i,j,k = -1,\ldots,N+1\}$. For $u_h \in \mathfrak{M}^3(\Omega_d)$ and $(x^1,x^2,x^3) \in \Omega_d^o$, u_h has the form

$$u_h(x^1,x^2,x^3) = \sum_{j_1=-1}^{N+1}\sum_{j_2=-1}^{N+1}\sum_{j_1=-1}^{N+1} \alpha_{j_1j_2j_3} B_{j_1}(x^1) B_{j_2}(x^2) B_{j_3}(x^3), \quad \alpha_{j_1j_2j_3} \in \mathbb{R}. \tag{3.2}$$

Using the similar process in Section II with the boundary values to remove some basis functions, u becomes

$$u_h(x^1,x^2,x^3) = \sum_{j_1=0}^{N}\sum_{j_2=0}^{N}\sum_{j_3=0}^{N} \alpha_{j_1j_2j_3} \tilde{B}_{j_1}(x^1) \tilde{B}_{j_2}(x^2) \tilde{B}_{j_3}(x^3) + H(x^1,x^2,x^3), \tag{3.3}$$

where $\tilde{B}_j$'s are defined in (2.1) and

$$H(x^1,x^2,x^3) = \begin{cases} g(x^1,x^2,x^3), & \text{on } \partial\Omega_d, \\ 0, & \text{in } \Omega^o. \end{cases}$$

Hence, the number of the new basis functions is $(N+1)^3$ which is the same as that of the knots. As in Section 2, define

$$\mathcal{B}_{(j_1,j_2,j_3)}(x_{i_1}^1, x_{i_2}^2, x_{i_3}^3) = \tilde{B}_{j_1}(x_{i_1}^1) \tilde{B}_{j_2}(x_{i_2}^2) \tilde{B}_{j_3}(x_{i_3}^3) + \chi_{(x_{j_1}^1, x_{j_2}^2, x_{j_3}^3)}(x_{i_1}^1, x_{i_2}^2, x_{i_3}^3) \tag{3.4}$$

where

$$\chi_{\left(x^1_{j_1},x^2_{j_2},x^3_{j_3}\right)}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)=\begin{cases}1, & \text{if}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)=\left(x^1_{j_1},x^2_{j_2},x^3_{j_3}\right)\in\partial\Omega_d\\ 0, & \text{others,}\end{cases}$$

and rewrite (3.3) to be

$$u_h\left(x^1,x^2,x^3\right)=\sum_{j_1=0}^{N}\sum_{j_2=0}^{N}\sum_{j_3=0}^{N}\alpha_{j_1j_2j_3}\mathcal{B}_{(j_1,j_2,j_3)}\left(x^1,x^2,x^3\right). \tag{3.5}$$

Since $w\left(x^1,x^2,x^3\right)=u_h\left(x^1,x^2,x^3\right)$ on $\partial\Omega_d$, we have

$$\alpha_{j_1j_2j_3}=w\left(x^1_{j_1},x^2_{j_2},x^3_{j_3}\right),\ \text{for those}\ \ (j_1,j_2,j_3)\ \text{such that}\ \left(x^1_{j_1},x^2_{j_2},x^3_{j_3}\right)\in\partial\Omega_d. \tag{3.6}$$

Assume the function u_h of (3.5) satisfies $\Delta u_h=f$ at each mesh point in Ω^o_d. Thus,

$$\sum_{j_1=0}^{N}\sum_{j_2=0}^{N}\sum_{j_3=0}^{N}\alpha_{j_1j_2j_3}\,\Delta\mathcal{B}_{(j_1,j_2,j_3)}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)=f\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right).$$

By (2.1) and (3.6), one may remove the B-splines at boundary points so that

$$\sum_{j_1=1}^{N-1}\sum_{j_2=1}^{N-1}\sum_{j_3=1}^{N-1}\alpha_{j_1j_2j_3}\,\Delta\mathcal{B}_{(j_1,j_2,j_3)}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)=\mathfrak{S}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right), \tag{3.7}$$

where

$$\mathfrak{S}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)=f\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)-\sum_{\left(x^1_{j_1},x^2_{j_2},x^3_{j_3}\right)\in\partial\Omega_d}\alpha_{j_1j_2j_3}\,\Delta\mathcal{B}_{(j_1,j_2,j_3)}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right),$$

for $\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right)\in\Omega^o_d$.

The indices j and (j_1,j_2,j_3) have the following relation

$$\begin{aligned}
j_3&=\left\lceil\frac{j-1}{(N-1)^2}\right\rceil+1,\\
j_2&=\left\lceil\frac{j-1-(j_3-1)(N-1)^2}{(N-1)}\right\rceil+1,\\
j_1&=j-1-(j_3-1)(N-1)^2-(j_2-1)(N-1)
\end{aligned}$$

or $j=1+j_1+(j_2-1)(N-1)+(j_3-1)(N-1)^2$. Thus, rearrange the indices of the knots and the basis functions to be

$$\begin{aligned}
\xi_i&:=\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right),\\
\tilde{\mathcal{B}}_j(\xi_i)&:=\mathcal{B}_{(j_1,j_2,j_3)}\left(x^1_{i_1},x^2_{i_2},x^3_{i_3}\right),\\
\tilde{\alpha}_j&:=\alpha_{j_1j_2j_3},
\end{aligned} \tag{3.8}$$

Write (3.7) in a matrix form

$$\mathbb{A} \cdot \mathbb{Y} = 12h^2 \mathbb{F}, \tag{3.9}$$

where

$$\begin{aligned}
\mathbb{A} &:= \left[\Delta \mathfrak{B}_j(\xi_i)\right]_{(N-1)^3 \times (N-1)^3}, \qquad \Delta \mathfrak{B}_j(\xi_i) = \tilde{\mathcal{B}}_{j,xx}(\xi_i) + \tilde{\mathcal{B}}_{j,yy}(\xi_i) + \tilde{\mathcal{B}}_{j,zz}(\xi_i),\\
\mathbb{Y} &:= \left[\alpha_1, \alpha_2, \ldots, \alpha_{(N-1)^3}\right]^t,\\
\mathbb{F} &:= \left[\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{(N-1)^3}\right]^t,\\
\mathfrak{S}_j &:= \mathfrak{S}\left(x^1_{j_1}, x^2_{j_2}, x^3_{j_3}\right), \qquad \text{given in } (3.7).
\end{aligned}$$

More precisely, by (2.15), $\mathbb{A}$ is

$$\begin{aligned}
\mathbb{A} &= S \otimes (S \otimes S + S \otimes I + I \otimes S - 3I \otimes I)\\
&\quad + I \otimes (S \otimes S - 3S \otimes I - 3I \otimes S - 27I \otimes I)\\
&= S \otimes S \otimes S + S \otimes S \otimes I + S \otimes I \otimes S + I \otimes S \otimes S\\
&\quad - 3S \otimes I \otimes I - 3I \otimes S \otimes I - 3I \otimes I \otimes S - 27I \otimes I \otimes I\\
&= (Q \otimes Q \otimes Q) \cdot \mathfrak{J} \cdot \left(Q^t \otimes Q^t \otimes Q^t\right),
\end{aligned} \tag{3.10}$$

where $\mathfrak{J}$ is

$$\begin{aligned}
\mathfrak{J} &= D \otimes D \otimes D + D \otimes D \otimes I + D \otimes I \otimes D + I \otimes D \otimes D\\
&\quad - 3D \otimes I \otimes I - 3I \otimes D \otimes I - 3I \otimes I \otimes D - 27I \otimes I \otimes I\\
&= \frac{1}{3}(D + 3I) \otimes (D \otimes D - 9I \otimes I) + \frac{1}{3}(D \otimes D - 9I \otimes I) \otimes (D + 3I)\\
&\quad + \frac{1}{3}(D \otimes (D + 3I) \otimes D - 3I \otimes (D + 3I) \otimes 3I)\\
&< 0
\end{aligned}$$

and diagonal so that $\mathbb{A}$ is invertible. We then have the following theorem.

Theorem 3.1. *If the function u_h defined in (3.3) or (3.5) in $\Omega_d \subset \mathbb{R}^3$ satisfies*

$$\begin{cases} \Delta u_h = f, & \text{in } \Omega_d^o,\\ u_h = g, & \text{on } \partial\Omega_d, \end{cases} \tag{3.11}$$

then the coefficients $\tilde{\alpha}_k$ of (3.5), $\left(\text{for those } k,\ \xi_k \in \Omega_d^0\right)$, are solved. We obtain $\mathbb{Y}$ of (3.9) to be

$$\mathbb{Y} = 12h^2 \, (Q \otimes Q \otimes Q)\, \mathfrak{J}^{-1} \left(Q^t \otimes Q^t \otimes Q^t\right) \mathbb{F}. \tag{3.12}$$

Therefore, the coefficients $\alpha_{j_1 j_2 j_3}$ in (3.5) are obtained and so is u_h.

Figure 1 and Figure 2 are the applications of Theorem 3.1.

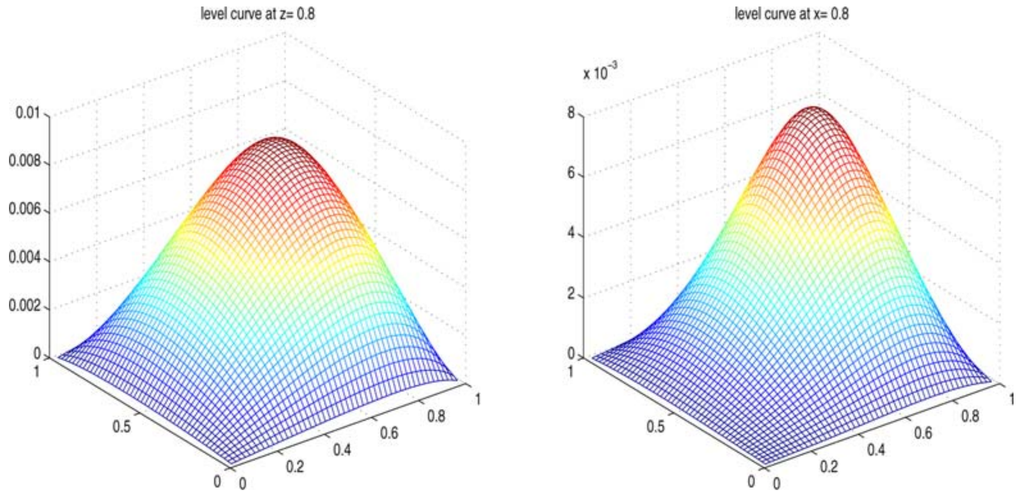

FIG. 1. $\Delta u = -\sqrt{x}yz^2$, in Ω_d and $u=0$, on $\partial\Omega_d$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

In general, if the Dirichlet problem is considered in $\mathbb{R}^n$, then u_h is expressed as

$$u_h\left(x^1, x^2, \ldots, x^n\right) = \sum_{j_1=0}^{N}\sum_{j_2=0}^{N}\cdots\sum_{j_n=0}^{N}\alpha_{j_1 j_2 \cdots j_n}\tilde{B}_{j_1}\left(x^1\right)\tilde{B}_{j_2}\left(x^2\right)\ldots\tilde{B}_{j_n}\left(x^n\right) + H\left(x^1, x^2, \ldots, x^n\right), \tag{3.13}$$

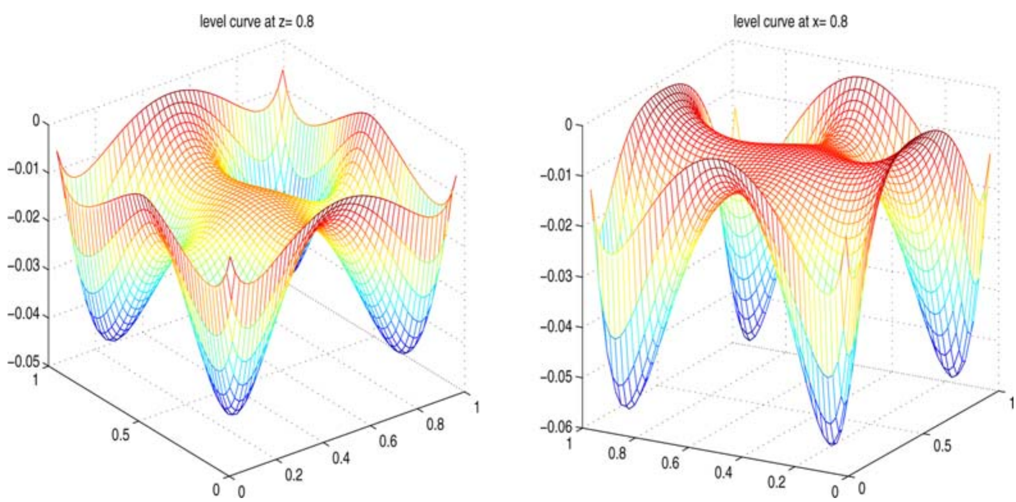

FIG. 2. $\Delta u = 10000|x - \frac{1}{2}|\left(y - \frac{1}{2}\right)^2\left(z - \frac{1}{2}\right)^4$, in Ω_d and $u=0$, on $\partial\Omega_d$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

TABLE I. $E(h_n) \approx Ch_n^p,\ p \approx \frac{\ell og\left(\frac{E(h_n)}{E(h_{n+10})}\right)}{(\ell og\, h_n - \ell og\, h_{n+10})}$.

nn	$u = \sin(x^2+y^2+z^2)$		$u = \exp(x^2+y^2+z^2)$		$u = \ell og\ (1+x)+y^2+z^2$	
	$E(h_n)$	p	$E(h_n)$	p	$E(h_n)$	P
21	0.0045000	2.00000000	0.10734043	1.73413895	0.00140000	1.90991457
31	0.0020000	2.07811698	0.05313695	1.80659547	0.00064537	1.93634167
41	0.0011000	1.87439991	0.03159969	1.84763056	0.00036973	1.95078806
51	0.0007240	1.99318316	0.02092324	1.87416638	0.00023924	1.95983435
61	0.0005034	1.99513004	0.01486723	1.89277279	0.00016736	1.96571129
71	0.0003701	1.99605764	0.01110491	1.90655710	0.00012361	1.97085003
81	0.0002835	1.99673012	0.00860894	1.91718560	9.5000E-05	1.97403236
91	0.0002241	1.99733907	0.00686880	1.92563411	7.5300E-05	1.97690051
101	0.0001815	1.99763120	0.00560749	1.93251278	6.1100E-05	1.97903341
111	0.0001501	1.99778545	0.00466420	1.93822300	5.0600E-05	1.98090702
121	0.0001261	1.99869334	0.00394034	1.94303982	4.2600E-05	1.98217388
131	0.0001075	1.99800958	0.00337279	1.94715811	3.6400E-05	1.98363335
141	0.0000927	1.99862803	0.00291958	1.95071981	3.1400E-05	1.98519207
151	0.0000807	1.99846868	0.00255194	1.95383078	2.7400E-05	1.98538757

The above table shows that the convergence rate of our solver in L^∞-norm is about $h^{1.95}$ at least. The values p's seem to close to 2 when the mesh length h tends to 0. Note that the method we provided in this article is working for any linear partial differential equation with high accuracy. One may refer to [8] in which perturbation problems of some differential equations were approximated by the reduced B-splines with very good accuracy of convergence.

where $\tilde{B}_{j_k}$ are defined in (2.1) and

$$H\left(x^1, x^2, \ldots, x^n\right) = \begin{cases} g\left(x^1, x^2, \ldots, x^n\right), & \text{if } \left(x^1, x^2, \ldots, x^n\right) \in \partial\Omega_d, \\ 0, & \text{if } \left(x^1, x^2, \ldots, x^n\right) \in \Omega_d^o. \end{cases}$$

Following the similar process in this section, one is able to obtain the approximation u_h for the solutions of Poisson equation in $\mathbb{R}^n$.

IV. NUMERICAL EXPERIMENT

In this section, we provide some numerical experiments and use the results to make a prediction of the convergent rate of our solver. Note that in general, we cannot expect a solution to be smooth everywhere on the boundary because our boundary is not smooth. Let $h > 0$ be the mesh length and Ω_d the mesh domain in $\overline{\Omega}$, where $\Omega = \prod_{i=1}^n (a_i, b_i)$. Let u_h and v satisfy $\Delta u_h = f$ on $\Omega_d^o = \Omega \cap \Omega_d$ with $u_h = f$ on $\partial\Omega_d$ and $\Delta v = f$. in Ω^o with $v = f$ on $\partial\Omega$, respectively. We extend the mesh function u_h to be a piecewise linear function on $\overline{\Omega}$. The following numerical computations seem to indicate that $u_h \to v$, as $h \to 0$ uniformly on $\overline{\Omega}$. In what follows, let $h_n = 1/(n-1)$ be the mesh length and $E(h) = ||v - u_h||_{\infty,\Omega_d}$. The following is a heuristic way to find the error for numerical simulation. Let $E(h) \approx C \cdot h^p$ where C depends on given data only. Then,

$$p \approx \frac{\ell og\left[\frac{E(h)}{E\left(\frac{h}{2}\right)}\right]}{\ell og\, 2} \tag{4.1}$$

We conclude that if p turns out to be a mess, one could stipulate that $E(h) = C \cdot h^p |\ell\text{og}\, h|$ instead, where C depends on given data only. Employing our solver for three functions, the numerical results are obtained as shown in Table I.

Remark. Note that the solutions in these examples will be smooth (see Chapter 4 of [7]) away from the corners but not necessarily smooth at the corners. In Example 1, the solution will be Lipschitz continuous (see Chapter 4 of [7]) but not continuously differentiable at all corners.

References

1. K. N. S. K. Viswanadham, P. M. Krishna, and C. P. Rao, Numerical solution of fifth order boundary value problems by collocation method with sixth order B-splines, Int J Appl Sci Eng 8 (2010), 119–125.
2. K. R. Koch and M. Schmidt, N-dimensional B-spline surface estimated by lofting for locally improving IRI, J Geodetic Sci 1 (2011), 41–45.
3. N. Caglar and H. Caglar, B-spline solution of singular boundary value problems, Appl Math Comput 182 (2006), 1509–1513.
4. P. Massopust, Interpolation and approximation with splines and fractals, Chapter 2, Oxford University Press, 2010.
5. G. Nurnberger, Approximation by spline functions, pp94, Springer-Verlag, 1989.
6. B. D. Froese, A. M. Oberman, Convergent finite difference solvers for viscosity solutions of the elliptic Monge-Ampere equation in dimensions two or higher, SIAM J. Numer. Anal. 49 (2011), 1692–1714.
7. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd Ed., Springer Verlag, 1983.
8. H. J. Chien, H. J. Kuo, and C. T. Wang, Higher dimensional B-splines and their applications, Matimyas Matematika 34 (2011), 17–26.